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A Dynamic Model with Friction and Adhesion with Applications to Rocks

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Dynamic frictional contact with adhesion of a viscoelastic body and a foundation is formulated as a hemivariational inequality. This may model the dynamics of rock layers. The normal stress–displacement relation on the contact boundary is nonmonotone and nonconvex because of the adhesion process. A sequence of regularized problems is considered, the necessary a priori estimates are obtained, and the existence of a weak solution for the hemivariational inequality is established by passing to the limit as the regularization parameter vanishes.

Key Words: dynamic frictional contact; rocks; hemivariational inequality; adhesion; normal compliance; viscoelastic body.

1. INTRODUCTION

The aim of this paper is to study a dynamic contact problem involving the unilateral phenomena of coupled adhesion and friction. The setting we employ and the result we obtain are very general, but our particular interest lies in the frictional contact between rocks which involves adhe-

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sion or bonding. Adhesion and friction are highly nonlinear processes due to the nonmonotone stress–strain relationship which contains vertical jumps that correspond to abrupt stiffness changes. To accommodate such stress–strain laws, the theory of generalized gradients of Clarke [2] has been recently extended and applied in contact mechanics by Panagiotopoulos [15]. This approach allows for the rigorous formulation of mathematical models for these phenomena through variational and hemivariational inequalities, which we use in this work.

Contact problems involving both adhesion and friction effects have been studied mostly in special cases: in problems involving constitutive relations with uncoupled shear and normal stress, or in problems with given normal stress. However, interactions between normal and tangential contact forces are often present in problems arising in applications, such as in contact of rocks. A general static problem of frictional contact with adhesion of rocks has been recently studied in [5]. There, a model for the process has been developed and the existence of its weak solutions established by using the theory of hemivariational inequalities. Here we extend their results to the dynamic case.

In this paper we establish the existence of weak solutions for a specific problem. However, the constitutive relation which we employ is not convex, and this approach can be extended to other dynamic problems in mechanics with nonmonotone and nonconvex constitutive relations.

General problems of adhesion were considered by Frémond and coworkers in [3, 4, 16] where the model was derived from thermodynamical considerations. Friction, however, was not taken into account. There, a bonding field was introduced to describe the adhesion and an equation for its evolution was derived. A one-dimensional, quasi-static, and frictionless contact problem with adhesion, using the bonding field, has been investigated in [6]. The quasi-static problem with friction and adhesion, using the bonding field, has been modeled and investigated recently in [17, 18].

Recent results on dynamic frictional problems without adhesion can be found in [1, 9–11, 13] and in the references therein.

We use a graph to model the contact. It describes the adhesion and allows for interpenetration of surface asperities, as in the *normal compliance condition*, see, e.g., [1, 7, 8, 10, 13, 19]. The graph has a vertical segment related to the sudden debonding when all the bonds are severed. This leads to the use of the generalized subgradient theory, since the graph is not convex. The rest of the paper is structured as follows. The classical model, its weak formulation, and the statement of our results are given in Section 2. The material is assumed to be viscoelastic and linear, for the sake of simplicity. We employ the normal compliance condition for the compressive part of the contact, and model the adhesion with a graph which has a vertical segment at the yield point where debonding takes place. In Section 3, we consider a sequence of approximate problems in which the vertical segment in the adhesion condition is replaced with a tilted segment. This approximation may be useful in constructing numerical algorithms for the problem. We use the recent theory of [9] to obtain the existence of the unique solution for each approximate problem. A priori estimates on the approximate solutions are derived in Section 4. Using these estimates allows us to pass to the limit and obtain a solution of the original problem.

It may be of interest to investigate the dynamic problem when the adhesion is modeled by the bonding function, following Frémond, instead of having one graph for contact and adhesion.

2. CLASSICAL MODEL, WEAK FORMULATION AND RESULTS

In this section, we present the physical setting and formulate the model as a system of differential equations and initial and boundary conditions. Then we introduce a weak formulation, state the assumptions on the data and our main result. Because of adhesion, the contact condition is nonconvex and, therefore, the problem is formulated as a hemivariational inequality (see, e.g., [14] and references therein). For the sake of simplicity, the bulk material is assumed to be linear; the nonlinear effects arise from the contact with the foundation.

The physical setting is depicted in Fig. 1. A viscoelastic body, the rock, occupies (in its reference configuration) the region Ω in $\mathbb{R}^m$ $(m = 2, 3)$. Its boundary is divided into three disjoint parts. On Γ_D the body is clamped; known tractions act on Γ_N; and on Γ_C the body may contact a foundation. We assume that the foundation is soft, of the Winkler type, or is rigid but has a layer of deformable asperities. The reference configuration is assumed to be stress-free and the process isothermal.

Let $f_B = (f_{B1}(x,t), \dots, f_{Bm}(x,t))$ be the (dimensionless) density of applied body forces acting in Ω and let $f_N = (f_{N1}(x,t), \dots, f_{Nm}(x,t))$ be the

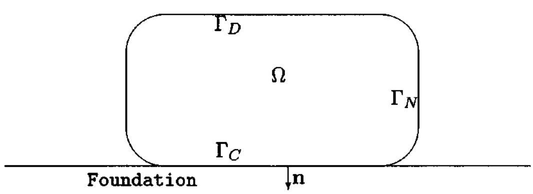

FIG. 1. The physical setting; Γ_C is the contact surface.

tractions applied on Γ_N. For the sake of simplicity, we assume that the density of the material is constant equal to 1. Let $\boldsymbol{u} = (u_1(x,t), \ldots, u_m(x,t))$ and $\sigma = (\sigma_{ij}(x,t))$ for $i, j = 1, \ldots, m$, represent the dimensionless displacement vector and stress tensor, at location x and time t, respectively. The equations of motion take the (dimensionless) form

$$\boldsymbol{u}'' - \operatorname{Div} \sigma = \boldsymbol{f}_B \qquad \text{in } \Omega_T. \tag{2.1}$$

Here and below, $i, j = 1, \ldots, m$; the repeated index convention is employed; the prime represents the time derivative; the portion of a subscript prior to a comma indicates a component and the portion after the comma refers to a partial derivative. We use the Kelvin–Voight stress–strain relation

$$\sigma_{ij} = a_{ijkl} u_{k,l} + b_{ijkl} u'_{k,l} \qquad \text{in } \Omega_T. \tag{2.2}$$

Here, $a = (a_{ijkl})$ and $b = (b_{ijkl})$ are the tensors of elastic and of viscosity coefficients, respectively. This relation holds within linearized elasticity, and we assume small displacements and strains.

The initial conditions are

$$\boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0, \qquad \boldsymbol{u}'(\cdot, 0) = \boldsymbol{v}_0 \qquad \text{in } \Omega. \tag{2.3}$$

To describe the boundary conditions, we introduce the unit outward normal $\boldsymbol{n} = (n_1, \ldots, n_m)$ on Γ. We assume that Γ is Lipschitz, hence $\boldsymbol{n}$ exists at almost every point. We then let $\sigma_n = \sigma_{ij} n_i n_j$ and $u_n = \boldsymbol{u} \cdot \boldsymbol{n}$ be the normal components of σ and $\boldsymbol{u}$ on Γ, and let $\sigma_\tau = \sigma \cdot \boldsymbol{n} - \sigma_n \boldsymbol{n}$, $\boldsymbol{u}_\tau = \boldsymbol{u} - u_n \boldsymbol{n}$ be the tangential vectors. We use the following boundary conditions:

$$\boldsymbol{u} = 0 \qquad \text{on } \Gamma_D, \tag{2.4}$$

$$\sigma \cdot \boldsymbol{n} = \boldsymbol{f}_N \qquad \text{on } \Gamma_N. \tag{2.5}$$

We turn to consider the conditions on the potential contact surface Γ_C, which is where our main interest lies. Physically, the contact surface is assumed to be covered with adhesive material, such as liquid glue, or there is a weak chemical bonding between the materials. This implies that for small tensile contact force there is resistance to separation. Let $g > 0$ be the *bond length*, and then $u_n = -g$ denotes the maximal distance for which bonding still holds, and let $p^* > 0$ denote the *tensile yield limit*, i.e., the maximal tensile force that the bonds can support. For $-g < u_n \leq 0$, there is tensile traction $0 \leq \sigma_n \leq p^*$ on Γ_C. However, when the pulling force at a point exceeds the threshold $\sigma_n = p^*$, the surfaces debond, the connections snap, and the contact at the point is lost. When the normal traction is negative, i.e., compressive, the penetration of the body's surface asperities into the outer surface of the foundation takes place. This represents a foundation with soft surface or the deformation of surface asperities. We assume a general relationship between the normal stress and normal displacement

$$-\sigma_n(u_n,\cdot) \in \mathscr{P}_n(u_n,\cdot) \qquad \text{on } \Gamma_C. \tag{2.6}$$

Here, for almost every $x \in \Gamma_C$, the graph $\mathscr{P}_n(\cdot, x)$ is such that

$$\begin{aligned}
&\mathscr{P}_n(\cdot, x) = 0 \text{ on } (-\infty, -g(x)],\\
&\mathscr{P}_n(-g(x), x) = [-p^*(x), 0],\\
&\mathscr{P}_n(\cdot, x) \text{ is an increasing Lipschitz function on } (-g(x), 0],\\
&\mathscr{P}_n(0, x) = 0,\\
&\mathscr{P}_n(\cdot, x) \text{ is an increasing Lipschitz function on } [0, \infty).
\end{aligned} \tag{2.7}$$

The portion of the graph on $[0, \infty)$ represents the normal compliance of the surfaces (see, e.g., [1, 7, 8, 10, 13, 19] and references therein). The graph is nonconvex, which leads to a hemivariational inequality formulation of the problem.

We note that the dependence of $\mathscr{P}_n$ on x is via g, and below we denote $\mathscr{P}_n(\cdot, x)$ by $\mathscr{P}_n$. A possible choice of the graph, depicted in Fig. 2, is

$$\mathscr{P}_n(\xi, x) = \begin{cases} \frac{1}{\varepsilon}\xi & \text{if } \xi \geq 0,\\ \alpha\xi & \text{if } -g(x) < \xi \leq 0,\\ [-\alpha g(x), 0] & \text{if } \xi = -g(x),\\ 0 & \text{if } \xi < -g(x). \end{cases} \tag{2.8}$$

Here, $\alpha > 0$ is the slope for $-g(x) < \xi \leq 0$, and in the normal compliance portion of the graph the penetration of the foundation is penalized with the coefficient $1/\varepsilon$, for ε positive and small. Then, $p^*(x) = \alpha g(x)$ is the tensile yield limit.

The following graph has been used in [5], where the contact was between two deformable bodies,

$$\mathscr{P}_n(\xi, x) = \begin{cases} 0 & \text{if } |\xi| > g(x),\\ \alpha\xi & \text{if } |\xi| \leq g(x),\\ [-\alpha g(x), 0] & \text{if } \xi = -g(x),\\ [0, \alpha g(x)] & \text{if } \xi = g(x). \end{cases} \tag{2.9}$$

Similar graphs can be found in [14] (and the references therein).

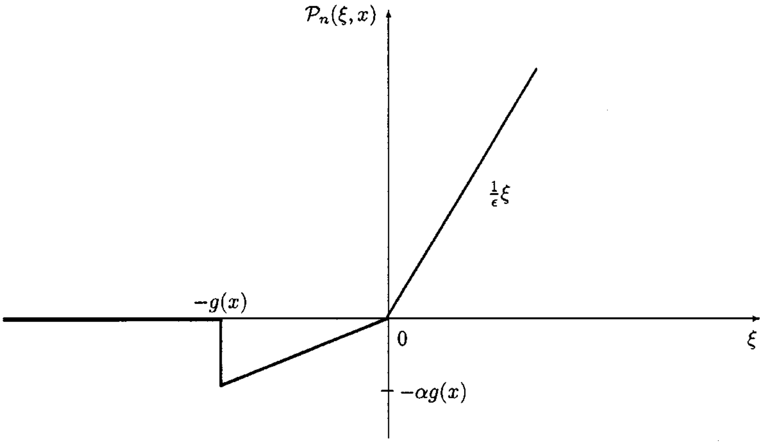

FIG. 2. Normal stress–displacement relationship (2.8).

We note that a different approach to modeling adhesion can be found in [3, 4] (see also the references therein) where a new dependent variable, the *bonding function*, which describes the ratio of active bonds at each point on the surface, was introduced. A differential equation for this variable was derived from a virtual power argument. The steady problem was analyzed in [16].

We turn to the tangential frictional contact condition. The usual Coulomb friction law is

$$|\sigma_\tau| \leq \mu|\sigma_n| \qquad \text{on } \Gamma_C,$$
$$\boldsymbol{u}'_\tau \neq 0 \quad \Rightarrow \quad \frac{\boldsymbol{u}'_\tau}{|\boldsymbol{u}'_\tau|} = -\frac{\sigma_\tau}{\mu|\sigma_n|}.$$

Here μ is the friction coefficient. By convention, $\sigma_\tau = 0$ when there is no contact ($\sigma_n = 0$) and $\boldsymbol{u}'_\tau$ remains undetermined. In the case of adhesion, this condition needs to be modified, since when σ_n is positive the body is pulled away from the foundation and we assume that there is no friction. Therefore, we use the following friction law:

$$|\sigma_\tau| \leq \mu(-\sigma_n)_+ \qquad \text{on } \Gamma_C,$$
$$\boldsymbol{u}'_\tau \neq 0 \text{ and } \sigma_n < 0 \quad \Rightarrow \quad \frac{\boldsymbol{u}'_\tau}{|\boldsymbol{u}'_\tau|} = -\frac{\sigma_\tau}{\mu|\sigma_n|}. \tag{2.10}$$

When the tangential stress is less than the limiting value $\mu(-\sigma_n)_+$, the boundary sticks to the foundation: the part of the boundary where it takes place is called the *stick zone*; when the tangential stress reaches its limiting value, the boundary slips: this is the so-called *slip zone*. The slip is opposite to the shear stress σ_τ.

The classical formulation of the *dynamic viscoelastic frictional contact problem with adhesion* is to find a function $\boldsymbol{u}$ such that (2.1)–(2.6) and (2.10) hold.

It is well known that, generally, there are no classical solutions for the problem because of the regularity ceiling related to possible jumps in the velocity. Therefore, we turn to the weak or variational formulation of the problem. To this end we introduce the following Hilbert spaces:

$$E = \left\{ w \in H^1(\Omega)^m : w = 0 \text{ on } \Gamma_D \right\}, \tag{2.11}$$

$$V = \left\{ \eta \in H^1(\Omega) : \eta = 0 \text{ on } \Gamma_D \cup \Gamma_N \right\}, \tag{2.12}$$

$$\begin{gathered} H = L^2(\Omega), \qquad H^m = \left(L^2(\Omega) \right)^m, \qquad \mathbb{E} = L^2(0,T;E), \\ \mathbb{V} = L^2(0,T;V). \end{gathered} \tag{2.13}$$

Below, we use $\|\cdot\|_E$ and $\|\cdot\|_V$ to denote the norms of E and V, respectively, and $|\cdot|_H$ and $|\cdot|_{H^m}$ denote the norms of H and H^m. Also, $\langle \cdot , \cdot \rangle$ denotes the duality pairing between E' and E, or V' and V, where the meaning is evident from the context.

We now describe the assumptions on the data.

The coefficients of elasticity and viscosity satisfy

$$\begin{aligned} & a_{ijkl} \in L^\infty(\Omega), \qquad b_{ijkl} \in L^\infty(\Omega), \\ & a_{ijkl} = a_{jikl}, \qquad a_{ijkl} = a_{klij}, \qquad a_{ijkl} = a_{ijlk}, \\ & a_{ijkl}\chi_{ij}\chi_{kl} \ge \alpha_1 |\chi_{ij}|^2 \qquad \text{for all symmetric tensors } \chi = (\chi_{ij}); \\ & b_{ijkl} = b_{jikl}, \qquad b_{ijkl} = b_{klij}, \qquad b_{ijkl} = b_{ijlk}, \\ & b_{ijkl}\chi_{ij}\chi_{kl} \ge \alpha_2 |\chi_{ij}|^2 \qquad \text{for all symmetric tensors } \chi = (\chi_{ij}). \end{aligned} \tag{2.14}$$

Here α_1 and α_2 are positive constants.

The body forces satisfy

$$\boldsymbol{f}_B \in \mathbb{E}'. \tag{2.15}$$

The friction coefficient satisfies

$$\mu : \Gamma_C \to (0, +\infty) \qquad \text{and} \qquad 0 < \mu_* \le \mu \le \mu^* \qquad \text{a.e. on } \Gamma_C, \tag{2.16}$$

where μ_* and μ^* are constants.

The boundary and initial data satisfy

$$f_N \in L^2\Big(0, T; \big(L^2(\Gamma_N)\big)^m\Big); \tag{2.17}$$

$$\boldsymbol{u}_0 \in E, \qquad \boldsymbol{v}_0 \in H^m. \tag{2.18}$$

We conclude this section with a brief description of a hemivariational formulation of the problem, similar to the one in [5]. For almost every $x \in \Gamma_C$, let $\beta_n(\cdot, x)$ be the function given by

$$\begin{aligned} &\beta_n(\xi, x) = \mathscr{P}_n(\xi, x) \qquad \text{if } \xi \neq -g, \\ &\beta_n(-g(x), x) = 0. \end{aligned}$$

The graph of β is the one depicted in Fig. 2, but without the vertical segment at $x = -g$. Let $\varphi_n(\cdot, x)$ be the function $\varphi_n(\xi, x) = \int_0^\xi \beta_n(s, x)\, ds$ for $\xi \in \mathbb{R}$. We define the functional Φ_n: $L^1(T_C) \to \mathbb{R}$ as

$$\Phi_n(z) = \int_{\Gamma_C} \varphi_n(z(x), x)\, d\Gamma,$$

where $d\Gamma$ denotes the surface measure on Γ_C. This definition makes sense only when $\varphi_n(z(\cdot), \cdot\,) \in L^1(T_C)$, and Φ_n is a Lipschitz continuous function, but is not necessarily convex.

Now, we may write the contact condition (2.6) as

$$-\sigma_n \in \partial \Phi_n(u_n) \qquad \text{on } \Gamma_C,$$

where $\partial \Phi_n$ represents the generalized subdifferential of Φ_n in the sense of Clarke (see, e.g., [14]). For this reason, the problem is formulated as a hemivariational inequality.

The generalized subdifferential in the sense of Clarke is a generalization of the usual subdifferential of a convex function. The latter is the set of all subgradients of the convex function at each point: when the function is differentiable at a point, its subdifferential contains only the gradient, and when it is not differentiable, the subdifferential contains the slopes of all the supporting lines (i.e., the lines which lie below the graph and touch it at the point only). In the case of a nonconvex function, the generalized subdifferential may contain vertical finite segments, too.

Let $\varphi_T(\eta; z) = \mu|\eta|\,|z|$ and define the functional Φ_T by

$$\Phi_T(\eta; z) = \int_{\Gamma_C} \varphi_T(\eta; z)\, d\Gamma,$$

provided the integral exists. Then, we may rewrite the friction condition (2.10) as

$$-\sigma_T \in \partial_z \Phi_T(\sigma_n; \boldsymbol{u}_T) \qquad \text{on } \Gamma_C,$$

where $\partial_z \Phi_T(\eta; z)$ is the subdifferential of Φ_T with respect to its second variable. Since Φ_T is convex, this is the usual subdifferential.

3. WEAK FORM OF THE PROBLEM

In this section, we derive an abstract form of the problem. To that end let $p^+: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$p^+(\xi) = \begin{cases} \mathscr{P}_n(\xi) & \text{if } \xi > 0, \\ 0 & \text{if } \xi \le 0. \end{cases}$$

It is Lipschitz continuous and monotone increasing, and there exists a constant $K > 0$ such that

$$|p^+(\xi_1) - p^+(\xi_2)| \le K|\xi_1 - \xi_2| \qquad \text{for } \xi_1, \xi_2 \in \mathbb{R}. \tag{3.1}$$

Thus, (2.10) can be written as

$$|\sigma_\tau| \le \mu p^+(u_n), \qquad \boldsymbol{v}_\tau \neq \boldsymbol{0} \quad \Rightarrow \quad \frac{\boldsymbol{v}_\tau}{|\boldsymbol{v}_\tau|} = \frac{-\sigma_\tau}{\mu p^+(u_n)}, \tag{3.2}$$

where $\boldsymbol{v}_\tau = \boldsymbol{u}'_\tau$. Similarly, for almost every $x \in \Gamma_C$, let $p^-(\cdot, x): \mathbb{R} \to \mathbb{R}$ be the graph

$$p^-(\xi, x) = \begin{cases} 0 & \text{if } \xi > 0, \\ \mathscr{P}_n(\xi, x) & \text{if } \xi \le 0. \end{cases}$$

As usual, derivation of the abstract problem involves integration by parts. Let $\boldsymbol{w} \in \mathbb{E}$ and $\boldsymbol{v} = \boldsymbol{u}'$, then we integrate by parts in the balance of momentum equation (2.1); taking into account (2.4)–(2.6), we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \boldsymbol{v}' \cdot \boldsymbol{w}\, dx\, dt \\
&\quad = \int_0^T \int_{\Gamma_C \cup \Gamma_N} (\sigma_\tau + \sigma_n \boldsymbol{n}) \cdot \boldsymbol{w}\, d\Gamma\, dt - \int_0^T \int_\Omega \sigma : \nabla \boldsymbol{w}\, dx\, dt \\
&\qquad + \int_0^T \int_\Omega \boldsymbol{f}_B \cdot \boldsymbol{w}\, dx\, dt + \int_0^T \int_{\Gamma_N} \boldsymbol{f}_N \cdot \boldsymbol{w}\, d\Gamma\, dt \\
&\quad \in \int_0^T \int_{\Gamma_C} -\mathscr{P}_n(u_n, x) \boldsymbol{n} \cdot \boldsymbol{w}_n\, d\Gamma\, dt \\
&\qquad + \int_0^T \int_{\Gamma_C} \sigma_\tau \cdot \boldsymbol{w}_\tau\, d\Gamma\, dt + \int_0^T \int_{\Gamma_N} \boldsymbol{f}_N \cdot \boldsymbol{w}\, d\Gamma\, dt \\
&\qquad - \int_0^T \int_\Omega \sigma : \nabla \boldsymbol{w}\, dx\, dt + \int_0^T \int_\Omega \boldsymbol{f}_B \cdot \boldsymbol{w}\, dx\, dt.
\end{aligned}$$

Now, it follows from (3.2) that regardless of whether $\boldsymbol{w}_\tau \neq \boldsymbol{0}$ or not, there exists an element $\boldsymbol{z} \in L^\infty(0, T; L^\infty(\Gamma_C)^m)$ such that

$$\int_0^T \int_{\Gamma_C} \sigma_\tau \cdot \boldsymbol{w}_\tau \, d\Gamma \, dt = -\int_0^T \int_{\Gamma_C} \mu p^+(u_n) \boldsymbol{z} \cdot \boldsymbol{w}_\tau \, d\Gamma, \tag{3.3}$$

and

$$\int_0^T \int_{\Gamma_C} \boldsymbol{z} \cdot \boldsymbol{w}_\tau \, d\Gamma \leq \int_0^T \int_{\Gamma_C} (|\boldsymbol{v}_\tau + \boldsymbol{w}_\tau| - |\boldsymbol{v}_\tau|) \, d\Gamma. \tag{3.4}$$

Thus, there exists $\boldsymbol{z} \in L^\infty(0, T; L^\infty(\Gamma_C)^m)$ satisfying (3.4) such that

$$\begin{aligned} &\int_0^T \int_\Omega \boldsymbol{v}' \cdot \boldsymbol{w} \, dx \, dt + \int_0^T \int_{\Gamma_C} \mathscr{P}_n(u_n, x) \boldsymbol{n} \cdot \boldsymbol{w}_n \, d\Gamma \, dt \\ &\quad + \int_0^T \int_{\Gamma_C} \mu z p^+(\mu_n) \cdot \boldsymbol{w}_\tau \, d\Gamma \\ &\quad + \int_0^T \int_\Omega \sigma : \nabla \boldsymbol{w} \, dx \, dt \ni \int_0^T \int_{\Gamma_N} \boldsymbol{f}_N \cdot \boldsymbol{w} \, d\Gamma \, dt + \int_0^T \int_\Omega \boldsymbol{f}_B \cdot \boldsymbol{w} \, dx \, dt. \end{aligned} \tag{3.5}$$

Therefore, using (2.2), we define the viscosity, elasticity, and normal compliance operators A, B, P^+: $E \to E'$, respectively, by

$$\langle A\boldsymbol{u}, \boldsymbol{w} \rangle = \int_\Omega a_{ijkl} u_{k,l} w_{i,j} \, dx, \tag{3.6}$$

$$\langle B\boldsymbol{u}, \boldsymbol{w} \rangle = \int_\Omega b_{ijkl} u_{k,l} w_{i,j} \, dx, \tag{3.7}$$

$$\langle P^+(\boldsymbol{u}), \boldsymbol{w} \rangle = \int_{\Gamma_C} p^+(u_n) w_n \, d\Gamma, \tag{3.8}$$

for all $\boldsymbol{u}$, $\boldsymbol{w} \in E$. It follows from (2.14) that there exists $\eta > 0$ such that, for all $\boldsymbol{u} \in E$,

$$\langle A\boldsymbol{u}, \boldsymbol{u} \rangle \geq \eta \big(\|\boldsymbol{u}\|_E^2 - |\boldsymbol{u}|_{H^m}^2 \big), \tag{3.9}$$

$$\langle B\boldsymbol{u}, \boldsymbol{u} \rangle \geq \eta \big(\|\boldsymbol{u}\|_E^2 - |\boldsymbol{u}|_{H^m}^2 \big). \tag{3.10}$$

We note that the operators A, B, and P^+ extend, in a natural way, to operators defined on $\mathbb{E}$ into $\mathbb{E}'$. With a slight abuse of notation, we use below the same symbol to denote both the original operators and their extensions.

Next, let $\boldsymbol{f} \in \mathbb{E}'$ be given by

$$\langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\mathbb{E}', \mathbb{E}} = \int_0^T \int_\Omega \boldsymbol{f}_B \cdot \boldsymbol{w} \, dx \, dt + \int_0^T \int_{\Gamma_N} \boldsymbol{f}_N \cdot \boldsymbol{w} \, d\Gamma \, dt. \tag{3.11}$$

Finally, let $\mathscr{P}(\mathbb{E}')$ be the set of all subsets of $\mathbb{E}'$. We consider the friction operator Q mapping $\mathbb{E}$ into $\mathscr{P}(\mathbb{E}')$, defined as follows: $\boldsymbol{v}^* \in Q(\boldsymbol{v}) \subseteq \mathbb{E}'$ means that there exists $\boldsymbol{z} \in L^\infty(0,T; L^\infty(\Gamma_C)^m)$ satisfying

$$\int_0^T \int_{\Gamma_C} \boldsymbol{z} \cdot \boldsymbol{w}_T \, d\Gamma \, dt \leq \int_0^T \int_{\Gamma_C} (|\boldsymbol{v}_T + \boldsymbol{w}_T| - |\boldsymbol{v}_T|) \, d\Gamma \, dt, \tag{3.12}$$

such that

$$\langle \boldsymbol{v}^*, \boldsymbol{w} \rangle = \int_0^T \int_{\Gamma_C} \mu p^+(u_n) \boldsymbol{z} \cdot \boldsymbol{w}_T \, d\Gamma \, dt \qquad \forall \boldsymbol{w} \in E. \tag{3.13}$$

We have now all the ingredients needed to state the weak formulation of the problem and our main result in this work.

THEOREM 3.1. *Let* (2.7), (2.14)–(2.18) *hold. Then there exists a triplet* $\{\xi, \boldsymbol{u}, \boldsymbol{v}\}$ *such that*

$$\xi \in L^\infty(0,T; L^\infty(\Gamma_C)), \qquad \boldsymbol{v} \in \mathbb{E}, \qquad \boldsymbol{v}' \in \mathbb{E}', \tag{3.14}$$

$$\xi(x,t) \in p^-(u_n(x,t), x) \qquad \text{a.e. on } \Gamma_C \times (0,T), \tag{3.15}$$

$$\boldsymbol{v}' + B\boldsymbol{v} + A\boldsymbol{u} + P^+(\boldsymbol{u}) + \Lambda_n^* \xi + Q(\boldsymbol{v}) \ni \boldsymbol{f}, \tag{3.16}$$

$$\boldsymbol{u}(t) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}(s) \, ds \qquad \text{a.e. } t \in (0,T), \tag{3.17}$$

$$\boldsymbol{v}(0) = \boldsymbol{v}_0. \tag{3.18}$$

Here, γ_n is the map from E into $L^2(\Gamma_C)$ defined by $\gamma_n \boldsymbol{u} = u_n$, γ_n^* is its adjoint map, and

$$\Lambda_n^* \xi = \int_0^T \int_{\Gamma_C} \gamma_n^* \xi \, d\Gamma \, dt.$$

We note that ξ represents the tension due to adhesion, P^+ represents the compressive part of the normal contact traction, and Q represents the friction.

4. APPROXIMATE PROBLEM

In this section, we consider a regularized version of the problem where the vertical segment of the adhesion part in the graph $\mathscr{P}_n$ is replaced by segments with decreasing slopes. We use the results of [9] to show that each one of the approximate problems has a unique solution.

Let $\delta > 0$ and let $p_\delta^-(\cdot, x)$: $\mathbb{R} \to \mathbb{R}$ be, for $p^*(x) = \alpha g(x)$ and almost every $x \in \Gamma_C$, the piecewise linear approximation of $p^-(\cdot, x)$ given by

$$p_\delta^-(\xi, x) = \begin{cases} -\frac{1}{\delta} p^*(x)(\xi + g(x) + \delta) & \text{if } -\delta - g(x) \leq \xi \leq -g(x), \\ p^-(\xi, x) & \text{otherwise}. \end{cases}$$

Thus $p_\delta^-(\cdot, x) = p^-(\cdot, x)$ except on the interval $[-\delta - g(x), -g(x)]$, where $p_\delta^-(\cdot, x)$ is a linear function. Clearly, p_δ^- is Lipschitz continuous and there exists $K_\delta > 0$ such that

$$|p_\delta^-(\xi_1, x) - p_\delta^-(\xi_2, x)| \leq K_\delta |\xi_1 - \xi_2| \qquad \forall \xi_1, \xi_2 \in \mathbb{R}, \tag{4.1}$$

where $K_\delta \to \infty$ as $\delta \to 0^+$. The function $\mathscr{P}_n^\delta(\xi, x)$ and the modified part p_δ^- are depicted in Fig. 3.

We associate with the function p_δ^- the operator P_δ^-: $E \to E'$, given by

$$\langle P_\delta^-(\boldsymbol{u}), \boldsymbol{w} \rangle = \int_{\Gamma_C} p_\delta^-(u_n(x), x) w_n(x)\, d\Gamma, \tag{4.2}$$

for all $\boldsymbol{u}, \boldsymbol{w} \in E$. The operator P_δ^- extends naturally to an operator from $\mathbb{E}$ into $\mathbb{E}'$.

The following nonlinear evolution inclusion is the abstract form of the approximate problem.

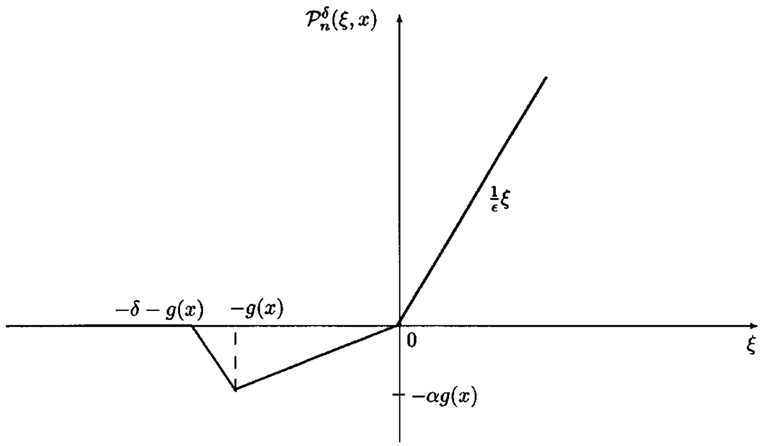

FIG. 3. The approximate function $\mathscr{P}_n^\delta(\xi, x)$.

PROBLEM $\mathscr{P}_\delta$. Find a pair $\{\boldsymbol{u}_\delta, \boldsymbol{v}_\delta\}$ such that

$$\boldsymbol{v}_\delta \in \mathbb{E}, \qquad \boldsymbol{v}_\delta(0) = \boldsymbol{v}_0, \qquad \boldsymbol{v}'_\delta \in \mathbb{E}', \tag{4.3}$$

$$\boldsymbol{v}'_\delta + B\boldsymbol{v}_\delta + A\boldsymbol{u}_\delta + P^+(\boldsymbol{u}_\delta) + P^-_\delta(\boldsymbol{u}_\delta) + Q(\boldsymbol{v}_\delta) \ni \boldsymbol{f}, \tag{4.4}$$

$$\boldsymbol{u}_\delta(t) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}_\delta(s)\, ds \qquad \text{a.e. } t \in (0,T). \tag{4.5}$$

We now establish the existence of the solution $\{\boldsymbol{u}_\delta, \boldsymbol{v}_\delta\}$ of the approximate problem $\mathscr{P}_\delta$, for each $\delta > 0$, and obtain a priori estimates independent of δ.

We remark that the approximate problem has some interest on its own. It has better mathematical properties than the idealized problem, and, indeed, the solution is more regular and is unique. For this reason it may be used as a basis for numerical approximations of the problem.

To prove the existence and uniqueness of the solution for Problem (4.3)–(4.5), we need the following two results due to Lions [12] and Simon [20], respectively.

THEOREM 4.1. *Let $p \geq 1$, $q > 1$, and let $W \subseteq U \subseteq Y$ be Banach spaces with compact inclusion map i: $W \to U$ and continuous inclusion map i: $U \to Y$. Then the set*

$$S_R = \big\{\boldsymbol{u} \in L^p(0,T;W)\colon \boldsymbol{u}' \in L^q(0,T;Y),$$
$$\|\boldsymbol{u}\|_{L^p(0,T;W)} + \|\boldsymbol{u}'\|_{L^q(0,T;Y)} < R\big\},$$

is precompact in $L^p(0,T;U)$.

THEOREM 4.2. *Let $q > 1$ and W, U, and Y be as in Theorem 4.1. Then the set*

$$S_{RT} = \big\{\boldsymbol{u}\colon \|\boldsymbol{u}(t)\|_W + \|\boldsymbol{u}'\|_{L^q(0,T;Y)} \leq R,\, t \in [0,T]\big\},$$

is precompact in $C(0,T;U)$.

In order to use Theorems 4.1 and 4.2, we introduce a Banach space U such that $E \subseteq U$, the embedding $E \to U$ is compact, and the trace map $U \to L^2(\Gamma_C)^m$ is continuous. We denote by $\|\cdot\|_U$ the norm on U.

For technical reasons, we change the independent variable and use $\boldsymbol{y}(t)e^{\lambda t} = \boldsymbol{v}(t)$, for $\lambda \geq 0$. Then, Problem (4.3)–(4.5) written in terms of $\boldsymbol{y}$ is

$$\boldsymbol{y} \in \mathbb{E}, \qquad \boldsymbol{y}(0) = \boldsymbol{v}_0, \qquad \boldsymbol{y}' \in \mathbb{E}', \tag{4.6}$$

$$\boldsymbol{y}' + \lambda\boldsymbol{y} + B\boldsymbol{y} + e^{-\lambda(\cdot)}A\boldsymbol{u} + e^{-\lambda(\cdot)}P^+(\boldsymbol{u}) + e^{+\lambda(\cdot)}P^-_\delta(\boldsymbol{u})$$
$$+ e^{-\lambda(\cdot)}Q\big(e^{\lambda(\cdot)}\boldsymbol{y}\big) \ni \boldsymbol{f}, \tag{4.7}$$

$$\boldsymbol{u}(t) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}(s)\, ds \qquad \text{a.e. } t \in (0,T). \tag{4.8}$$

We define the Banach space $\mathbb{X}$, endowed with the norm $\|\cdot\|_{\mathbb{X}}$, as follows:

$$\mathbb{X} = \{ \boldsymbol{y} \in \mathbb{E} \colon \boldsymbol{y}' \in \mathbb{E}' \}, \qquad \|\boldsymbol{y}\|_{\mathbb{X}} = \|\boldsymbol{y}\|_{\mathbb{E}} + \|\boldsymbol{y}'\|_{\mathbb{E}'}. \tag{4.9}$$

Let also $\mathscr{P}(\mathbb{X}')$ be the set of all subsets of the dual space $\mathbb{X}'$.

PROPOSITION 4.3. *The operator* $Q_\lambda \colon \mathbb{X} \to \mathscr{P}(\mathbb{X}')$ *defined by* $Q_\lambda(\boldsymbol{y}) = e^{-\lambda(\cdot)} Q(e^{\lambda(\cdot)} \boldsymbol{y})$ *is pseudomonotone and bounded.*

The proof of this proposition is accomplished through the following lemmas.

LEMMA 4.4. *If* $\boldsymbol{v}^k \to \boldsymbol{v}$ *weakly in* $\mathbb{X}$, *then* $\boldsymbol{v}^k \to \boldsymbol{v}$ *in* $L^2(0,T;(L^2(\Gamma_C))^m)$.

Proof. If $\boldsymbol{v}^k$ fails to converge to $\boldsymbol{v}$ in $L^2(0,T;L^2(\Gamma_C)^m)$, there exist an $\varepsilon > 0$ and a subsequence, still denoted by $\boldsymbol{v}_k$, such that $\|\boldsymbol{v}^k - \boldsymbol{v}\|_{L^2(0,T;U)} \geq \varepsilon$. Then we can extract a further subsequence such that $\boldsymbol{v}^k \to \boldsymbol{w}$ strongly in $L^s(0,T;U)$, for some $\boldsymbol{w}$. But the weak convergence of $\boldsymbol{v}^k$ to $\boldsymbol{v}$ in $\mathbb{X}$ implies the weak convergence of $\boldsymbol{v}^k$ to $\boldsymbol{v}$ in $L^2(0,T;U)$. Hence $\boldsymbol{w} = \boldsymbol{v}$, which contradicts the assumption that $\|\boldsymbol{v}^k - \boldsymbol{v}\|_{L^2(0,T;U)} \geq \varepsilon$.

LEMMA 4.5. *If* $\boldsymbol{y}^k \to \boldsymbol{y}$ *weakly in* $\mathbb{X}$, *then*

$$p^+\big(u_n^k\big) \to p^+(u_n) \qquad \text{in } L^2\big(0,T;L^2(\Gamma_C)\big). \tag{4.10}$$

Proof. It follows from (3.1) that

$$|p^+\big(u_n^k\big) - p^+(u_n)| \leq K|u_n^k - u_n|. \tag{4.11}$$

Now,

$$|u_n^k(t) - u_n(t)|_{L^2(\Gamma_C)} \leq \int_0^t |v_n^k(s) - v_n(s)|_{L^2(\Gamma_C)}\, ds$$

$$= \int_0^t e^{\lambda s} |y_n^k(s) - y_n(s)|_{L^2(\Gamma_C)}\, ds,$$

and using the Jensen inequality, we obtain

$$\|u_n^k - u_n\|^2_{L^2(0,T;L^2(\Gamma_C))} \leq C_{T\lambda} \int_0^T \int_0^T |y_n^k(s) - y_n(s)|^2_{L^2(\Gamma_C)}\, ds\, dt, \tag{4.12}$$

where $C_{T\lambda}$ is a positive constant which depends on T and λ. We deduce from Lemma 4.4 that $y_n^k \to y_n$ strongly in $L^2(0,T;L^2(\Gamma_C))$, and this together with (4.11) and (4.12) yield the result.

LEMMA 4.6. *Let* $\boldsymbol{y}^k \to \boldsymbol{y}$ *weakly in* $\mathbb{X}$ *and* $\boldsymbol{z}^k \to \boldsymbol{z}$ *weak* in* $L^\infty(0,T;L^\infty(\Gamma_C)^m)$. *Then*

$$\int_0^T \int_{\Gamma_C} \mu p^+\big(u_n^k\big) \boldsymbol{z}^k \cdot \xi \, d\Gamma\, dt \to \int_0^T \int_{\Gamma_C} \mu p^+(u_n) \boldsymbol{z} \cdot \xi \, d\Gamma\, dt, \tag{4.13}$$

for all $\xi \in L^2(0,T;L^2(\Gamma_C)^m)$.

Proof. We argue by contradiction. Let $\varepsilon > 0$. If (4.13) does not hold, then there exist $\xi \in L^2(0,T; L^2(\Gamma_C)^m)$ and two sequences $\{\boldsymbol{y}^k\}$ and $\{\boldsymbol{z}^k\}$ such that $\boldsymbol{y}^k \to \boldsymbol{y}$ weakly in $\mathbb{X}$, $\boldsymbol{z}^k \to \boldsymbol{z}$ weak* in $L^\infty(0,T; L^\infty(\Gamma_C)^m)$ and

$$\left| \int_0^T \int_{\Gamma_C} \mu p^+\left(u_n^k\right) \boldsymbol{z}^k \cdot \xi \, d\Gamma \, dt - \int_0^T \int_{\Gamma_C} \mu p^+(u_m) \boldsymbol{z} \cdot \xi \, d\Gamma \, dt \right| \geq 2\varepsilon. \quad (4.14)$$

Since $L^\infty(0,T; L^\infty(\Gamma_C)^m)$ is dense in $L^s(0,T; L^2(\Gamma_C)^m)$, we may assume that (4.14) holds for some $\xi \in L^\infty(0,T; L^\infty(\Gamma_C)^m)$ with ε in place of 2ε. However, it follows from Lemma 4.5 that

$$\int_0^T \int_{\Gamma_C} \mu p^+\left(u_n^k\right) \boldsymbol{z}^k \cdot \xi \, d\Gamma \, dt \to \int_0^T \int_{\Gamma_C} \mu p^+(u_n) \boldsymbol{z} \cdot \xi \, d\Gamma \, dt,$$

and so (4.14) cannot hold for all k. This contradiction proves the lemma.

Proof of Proposition 4.3. It is clear that Q_λ is bounded, and it is straightforward to show that $Q_\lambda(\boldsymbol{y})$ is convex. We now show that $Q_\lambda(\boldsymbol{y})$ is closed. Let W be a weakly open set in $\mathbb{X}'$ and let $W_\lambda = e^{\lambda(\cdot)}$. Assume that $\boldsymbol{y}^k \to \boldsymbol{y}$ weakly in $\mathbb{X}$, $Q_\lambda(\boldsymbol{y}) \subseteq W$, and let $(\boldsymbol{y}^k)^* \in Q_\lambda(\boldsymbol{y}^k) \setminus W$ for all k. Then $\boldsymbol{v}^k \to \boldsymbol{v}$ weakly in $\mathbb{X}$, W_λ is a weakly open set in $\mathbb{X}'$ containing $Q(\boldsymbol{v})$, and $(\boldsymbol{v}^k)^* = e^{\lambda(\cdot)}(\boldsymbol{y}^k)^* \in Q(\boldsymbol{v}^k) \setminus W_\lambda$ for all k. Next, let $\{\boldsymbol{z}^k\}$ be a sequence in $L^\infty(0,T; L^\infty(\Gamma_C)^m)$, satisfying (3.12) and (3.13), such that, possibly for a subsequence, $\boldsymbol{z}^k \to \boldsymbol{z}$ weak* in $L^\infty(0,T; L^\infty(\Gamma_C)^m)$. It follows from Lemma 4.4 that $\boldsymbol{z}$ satisfies (3.12). Now, we obtain from Lemma 4.6 that $(\boldsymbol{v}^k)^* \to \boldsymbol{v}^*$ weakly in $\mathbb{E}'$, and thus

$$\langle \boldsymbol{v}^*, \boldsymbol{w} \rangle = \int_0^T \int_{\Gamma_C} \mu p^+(u_n) \boldsymbol{z} \cdot \boldsymbol{w}_T \, d\Gamma \, dt, \qquad \boldsymbol{w} \in \mathbb{E}.$$

Then, by the definition of Q, we obtain that $\boldsymbol{v}^* \in Q(\boldsymbol{v}) \subseteq W_\lambda$. This is a contradiction to the assumption that $(\boldsymbol{v}^k)^* \notin W_\lambda$, for all k. Hence $Q(\boldsymbol{v}^k) \subseteq W_\lambda$ for all sufficiently large k. This argument also shows that $Q_\lambda(\boldsymbol{y})$ is closed.

It remains to verify the limit condition for pseudomonotone operators. To that end, assume that $\boldsymbol{y}^k \to \boldsymbol{y}$ weakly in $\mathbb{X}$ and let $(\boldsymbol{y}^k)^* \in Q_\lambda(\boldsymbol{y}^k)$, for all k. We show that if $\boldsymbol{w} \in \mathbb{X}$, then

$$\liminf_{k \to \infty} \langle (\boldsymbol{y}^k)^*, \boldsymbol{y}^k - \boldsymbol{w} \rangle \geq \langle \boldsymbol{y}^*(\boldsymbol{w}), \boldsymbol{y} - \boldsymbol{w} \rangle, \qquad \boldsymbol{y}^*(\boldsymbol{w}) \in Q_\lambda(\boldsymbol{y}).$$

We choose a subsequence $\boldsymbol{y}^k$ (which depends on $\boldsymbol{w}$) such that

$$\lim_{k \to \infty} \langle (\boldsymbol{y}^k)^*, \boldsymbol{y}^k - \boldsymbol{w} \rangle = \liminf_{k \to \infty} \langle (\boldsymbol{y}^k)^*, \boldsymbol{y}^k - \boldsymbol{w} \rangle.$$

Let $(\boldsymbol{v}^k)^* = e^{\lambda(\cdot)}(\boldsymbol{y}^k)^* \in Q(\boldsymbol{v}^k)$ and let $\boldsymbol{z}^k \in L^\infty(0,T; L^\infty(\Gamma_C)^m)$ be a related function satisfying (3.12) and (3.13), for all k. We extract a further subsequence, if necessary, such that

$$\boldsymbol{z}^k \to \boldsymbol{z} \qquad \text{weak* in } L^\infty\big(0,T; L^\infty(\Gamma_C)^m\big).$$

Then $\boldsymbol{z}$ satisfies (3.12) by Lemma 4.4. It follows from Lemma 4.6 that if we define $\boldsymbol{y}^*(\boldsymbol{w})$ by

$$\langle \boldsymbol{y}^*(\boldsymbol{w}), b\rangle = \int_0^T \int_{\Gamma_C} e^{-\lambda t}\mu p^+(u_n)\boldsymbol{z}\cdot b_T \, d\Gamma\, dt,$$

for $b \in \mathbb{E}$, we obtain

$$\begin{aligned}
&\liminf_{k\to\infty} \langle (\boldsymbol{y}^k)^*, \boldsymbol{y}^k - \boldsymbol{w}\rangle \\
&\quad = \lim_{k\to\infty} \langle (\boldsymbol{y}^k)^*, \boldsymbol{y}^k - \boldsymbol{w}\rangle \\
&\quad = \lim_{k\to\infty} \int_0^T \int_{\Gamma_C} e^{-\lambda t}\mu p^+\big(u_n^k\big)\boldsymbol{z}^k \cdot \big(\boldsymbol{y}_T^k - \boldsymbol{w}_T\big)\, d\Gamma\, dt, \\
&\quad = \int_0^T \int_{\Gamma_C} e^{-\lambda t}\mu p^+(u_n)\boldsymbol{z}\cdot(\boldsymbol{y}_T - \boldsymbol{w}_T)\, d\Gamma\, dt, \\
&\quad = \langle \boldsymbol{y}^*(\boldsymbol{w}), \boldsymbol{y} - \boldsymbol{w}\rangle.
\end{aligned}$$

This completes the proof of Proposition 4.3

LEMMA 4.7. *If $\boldsymbol{v}^k \to \boldsymbol{v}$ weakly in $\mathbb{X}$, then $P^+(\boldsymbol{u}^k) \to P^+(\boldsymbol{u})$ in $\mathbb{E}'$.*

Proof. Let $\boldsymbol{w} \in \mathbb{E}$. Using (3.1), we obtain

$$\begin{aligned}
&\big|\langle P^+(\boldsymbol{u}^k) - P^+(\boldsymbol{u}), \boldsymbol{w}\rangle\big| \\
&\quad \le K\int_0^T \int_{\Gamma_C} |u_n^k - u_n|\,|w_n|\, d\Gamma\, dt, \\
&\quad \le K\int_0^T \left(\int_{\Gamma_C} |u_n^k - u_n|^2\, d\Gamma\right)^{1/2} \left(\int_{\Gamma_C} |w_n|^2\, d\Gamma\right)^{1/2} dt, \\
&\quad \le K\|u_n^k - u_n\|_{L^2(0,T;L^2(\Gamma_C))}\|\boldsymbol{w}\|_{\mathbb{E}}.
\end{aligned}$$

Thus,

$$\|P^+(\boldsymbol{u}^k) - P^+(\boldsymbol{u})\|_{\mathbb{E}'} \le K\|\gamma \boldsymbol{u}^k - \gamma \boldsymbol{u}\|_{L^2(0,T;L^2(\Gamma_C)^m)},$$

and the desired result follows from Lemma 4.4.

It is easy to check that for each $\lambda \geq 0$, the operator $\boldsymbol{y} \mapsto e^{-\lambda(\cdot)} A\boldsymbol{u}$ is monotone. In fact,

$$\begin{aligned}
&\langle e^{-\lambda(\cdot)} A(\boldsymbol{u}^1 - \boldsymbol{u}^2), \boldsymbol{y}^1 - \boldsymbol{y}^2 \rangle \\
&\quad = \frac{1}{2} \int_0^T e^{-2\lambda} \frac{d}{dt} \langle A(\boldsymbol{u}^1 - \boldsymbol{u}^2), \boldsymbol{u}^1 - \boldsymbol{u}^2 \rangle \, dt \\
&\quad = \frac{1}{2} e^{-2\lambda T} \langle A\big(\boldsymbol{u}^1(T) - \boldsymbol{u}^2(T)\big), \boldsymbol{u}^1(T) - \boldsymbol{u}^2(T) \rangle \\
&\qquad + \lambda \int_0^T \langle A(\boldsymbol{u}^1 - \boldsymbol{u}^2), \boldsymbol{u}^1 - \boldsymbol{u}^2 \rangle e^{-2\lambda t} \, dt. \qquad (4.15)
\end{aligned}$$

Next, $\boldsymbol{y}^k \to \boldsymbol{y}$ weakly in $\mathbb{X}$ if and only if $\boldsymbol{v}^k \to \boldsymbol{v}$ weakly in $\mathbb{X}$ and Lemma 4.7 implies that the operator $\boldsymbol{y} \mapsto e^{-\lambda(\cdot)} P^+(\boldsymbol{u})$ is completely continuous. Similar considerations show that the operator $y \mapsto e^{-\lambda(\cdot)} P_\delta^-(\boldsymbol{u})$ is completely continuous. Thus, if we let

$$\begin{aligned}
\mathscr{A}_\lambda \boldsymbol{y} &= \lambda \boldsymbol{y} + B\boldsymbol{y} + e^{-\lambda(\cdot)} A\boldsymbol{u} + e^{-\lambda(\cdot)} P^+(\boldsymbol{u}) + e^{-\lambda(\cdot)} P_\delta^-(\boldsymbol{u}) \\
&\quad + e^{-\lambda(\cdot)} Q\big(e^{\lambda(\cdot)} \boldsymbol{y}\big), \qquad (4.16)
\end{aligned}$$

then $\mathscr{A}_\lambda$ is a sum of pseudomonotone bounded operators. Consequently, $\mathscr{A}_\lambda \colon \mathbb{X} \to \mathscr{P}(\mathbb{X}')$ is pseudomonotone and bounded. The last three terms of (4.16) have the property that if $\boldsymbol{v}^*$ is either equal to or an element of any one of these terms, then

$$|\langle \boldsymbol{v}^*, \boldsymbol{y} \rangle| \leq C \|\boldsymbol{y}\|_U^2 + C,$$

where C is a constant independent of $\boldsymbol{y}$ and λ. Therefore, using the inequality, $\|\boldsymbol{y}\|_U^2 \leq \varepsilon \|\boldsymbol{y}\|_E^2 + C_\varepsilon |\boldsymbol{y}\|_{H^m}^2$, which results from the compactness of the embedding of E into U, choosing ε small enough, and then choosing λ large enough, we find in addition that $\mathscr{A}_\lambda$ is coercive. Thus, by the existence theorem of [9], we conclude that the system (4.6)–(4.8) has a solution, and consequently, there exists a solution of Problem (4.3)–(4.5).

We have the following theorem.

THEOREM 4.8. *For each $\delta > 0$, there exists a unique solution of Problem $\mathscr{P}_\delta$.*

Proof. It remains to verify the uniqueness. Assume that $\boldsymbol{v}^1$ and $\boldsymbol{v}^2$ solve Problem $\mathscr{P}_\delta$. Let $\boldsymbol{u}^i(t) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}^i(s)\, ds$, let $(\boldsymbol{v}^i)^* \in Q(\boldsymbol{v}^i)$, and denote by $\boldsymbol{z}^i$ the element of $L^\infty(0,T; L^\infty(\Gamma_C)^m)$ that satisfies (3.12) and

(3.13), for $i = 1, 2$. From (3.13) we obtain

$$\int_0^t \langle (\boldsymbol{v}^1)^* - (\boldsymbol{v}^2)^*, \boldsymbol{v}^1 - \boldsymbol{v}^2 \, ds$$

$$\geq \int_0^t \int_{\Gamma_C} \left(\mu p^+ (u_n^1) \boldsymbol{z}^1 - \mu p^+ (u_n^2) \boldsymbol{z}^2 \right) \left(\boldsymbol{v}_T^1 - \boldsymbol{v}_T^2 \right) d\Gamma \, ds$$

$$\geq \int_0^t \int_{\Gamma_C} \mu \boldsymbol{z}^1 \left(p^+ (u_n^1) - p^+ (u_n^2) \right) \cdot \left(\boldsymbol{v}_T^1 - \boldsymbol{v}_T^2 \right) d\Gamma \, ds$$

$$+ \int_0^t \int_{\Gamma_C} \mu p^+ (u_n^1) (\boldsymbol{z}^1 - \boldsymbol{z}^2) \cdot \left(\boldsymbol{v}_T^1 - \boldsymbol{v}_T^2 \right) d\Gamma \, ds. \tag{4.17}$$

Using (3.12) for z^1 and z^2, we find that the last term on the right-hand side is nonnegative. Therefore,

$$\int_0^t \langle (\boldsymbol{v}^1)^* - (\boldsymbol{v}^2)^*, \boldsymbol{v}^1 - \boldsymbol{v}^2 \rangle \, ds$$

$$\geq -C \int_0^t \|\boldsymbol{u}^1(s) - \boldsymbol{u}^2(s)\|_U \|\boldsymbol{v}^1(s) - \boldsymbol{v}^2(s)\|_U \, ds,$$

where C is a constant which may depend on $\boldsymbol{z}^1$, μ, T, and K. Using the definitions of $\boldsymbol{u}^1$ and $\boldsymbol{u}^2$, in terms of $\boldsymbol{v}^1$ and $\boldsymbol{v}^2$, we may write

$$\int_0^t \langle (\boldsymbol{v}^1)^* - (\boldsymbol{v}^2)^*, \boldsymbol{v}^1 - \boldsymbol{v}^2 \, ds \geq -C \int_0^t \|\boldsymbol{v}^1(s) - \boldsymbol{v}^2(s)\|_U^2 \, ds. \tag{4.18}$$

From (4.4), (4.15), (4.18), (3.9), (3.10), and after adjusting the constant C to depend on δ, we obtain that

$$\frac{1}{2} |\boldsymbol{v}^1(t) - \boldsymbol{v}^2(t)|_{H^m}^2 + \eta \int_0^t \left(\|\boldsymbol{v}^1(s) - \boldsymbol{v}^2(s)\|_E^2 - \|\boldsymbol{v}^1(s) - \boldsymbol{v}^2(s)|_{H^m}^2 \right) ds$$

$$- C \int_0^t \|\boldsymbol{v}^1(s) - \boldsymbol{v}^2(s)\|_U^2 \, ds \leq 0. \tag{4.19}$$

Using the inequality $\|\boldsymbol{u}\|_U \leq \varepsilon \|\boldsymbol{u}\|_E + C_\varepsilon \|\boldsymbol{u}|_{H^m}$ for ε such that $0 < \varepsilon < \eta$, we find

$$|\boldsymbol{v}^1(t) - \boldsymbol{v}^2(t)|_{H^m}^2 + \int_0^t \|\boldsymbol{v}^1(s) - \boldsymbol{v}^2(s)\|_E^2 \, ds$$

$$\leq C \int_0^t |\boldsymbol{v}^1(s) - \boldsymbol{v}^2(s)|_{H^m}^2 \, ds,$$

where C depends on η, δ, z^1, μ, T, and K. Then, by Gronwall's inequality we find that $\boldsymbol{v}^1 = \boldsymbol{v}^2$, which proves the uniqueness of the solution, and therefore, the theorem.

5. ESTIMATES AND THE LIMIT

In this section, we prove Theorem 3.1. To that end, we establish estimates on the solutions of the approximate problems $\mathscr{P}_\delta$ leading to the following theorem.

THEOREM 5.1. *There exists a constant*, C, *independent of* δ, *such that*

$$|\boldsymbol{v}_\delta(t)|_{H^m}^2 + \int_0^t \|\boldsymbol{v}_\delta(s)\|_E^2 \, ds + \|\boldsymbol{u}_\delta(t)\|_E^2 + \int_{\Gamma_C} \Phi(u_{\delta n}(\cdot, t)) \, d\Gamma \le C. \quad (5.1)$$

Proof. To simplify the notation, we omit the subscript δ in this proof. We apply (4.4) to $\boldsymbol{v}$ and integrate from 0 to t. We consider the resulting nonlinear terms first,

$$\begin{aligned}\int_0^t \langle P^+(\boldsymbol{u}), \boldsymbol{v}\rangle \, ds &= \int_0^t \int_{\Gamma_C} p^+(u_n(x,s)) v_n(x,s) \, d\Gamma \, ds \\ &= \int_{\Gamma_C} \int_0^t p^+(u_n(x,s)) v_n(x,s) \, ds \, d\Gamma \\ &= \int_{\Gamma_C} \Phi(u_n(x,t)) - \Phi(u_{0n}(x)) \, d\Gamma,\end{aligned}$$

where Φ is the indefinite integral of p^+, i.e., $d\Phi/dt = p^+$. Therefore,

$$\int_0^t \langle P^+(\boldsymbol{u}), \boldsymbol{v}\rangle \, ds \ge \int_{\Gamma_C} \Phi(u_n(t,x)) \, d\Gamma - C. \quad (5.2)$$

Here and below, C denotes a generic constant which is independent of δ. Then,

$$\begin{aligned}\int_0^t \langle P_\delta^-(\boldsymbol{u}), \boldsymbol{v}\rangle \, ds &= \int_0^t \int_{\Gamma_C} p_\delta^-(u_n(x), x) v_n(x) \, d\Gamma \, ds, \\ &\ge -C - \int_0^t \|\boldsymbol{v}\|_U^2 \, ds. \quad (5.3)\end{aligned}$$

Next, we consider the term involving $Q(\boldsymbol{v})$. Let $\boldsymbol{v}^*$ be the element of $Q(\boldsymbol{v})$ for which equality occurs in (4.4), and let $\boldsymbol{z} \in L^\infty(0, T; L^\infty(\Gamma_C)^m)$ be the function satisfying (3.12) and (3.13); then

$$\int_0^t \langle \boldsymbol{v}^*, \boldsymbol{v} \rangle \, ds = \int_0^t \int_{\Gamma_C} \mu p^+(u_n) \boldsymbol{z} \cdot \boldsymbol{v}_T \, d\Gamma \, ds.$$

It follows from (3.12) that $\|\boldsymbol{z}\|_{L^\infty(0,T;\, L^\infty(\Gamma_C)^m)} \leq 1$. Since p^+ is Lipschitz and equals zero at $\xi = 0$, we obtain that

$$\int_0^t \langle \boldsymbol{v}^*, \boldsymbol{v} \rangle \, ds \geq -C \int_0^t \|\boldsymbol{u}\|_U \|\boldsymbol{v}\|_U \, ds. \tag{5.4}$$

From estimates (5.2)–(5.4), (4.4), and (2.14), it results that

$$\begin{aligned} &\frac{1}{2} |\boldsymbol{v}(t)|^2_{H^m} + \alpha_2 \int_0^t \|\boldsymbol{v}(s)\|^2_E \, ds + \frac{\alpha_1}{2} \|\boldsymbol{u}(t)\|^2_E - C \int_0^t \|\boldsymbol{v}\|^2_U \, ds \\ &\quad + \int_{\Gamma_C} \Phi(u_n(t,x)) - \Phi(u_{0n}(x)) \, d\Gamma - C \int_0^t \|\boldsymbol{u}(s)\|^2_U \, ds \\ &\quad \leq C + C \int_0^t |\boldsymbol{v}(s)|^2_{H^m} \, ds + \frac{a_1}{2} |\boldsymbol{u}(t)|^2_{H^m}, \end{aligned} \tag{5.5}$$

where α_1 and α_2 are the positive constants appearing in (2.14). On the other hand, we have

$$|\boldsymbol{u}(t)|^2_{H^m} \leq C + \int_0^t |\boldsymbol{v}(s)|^2_{H^m} \, ds, \tag{5.6}$$

and

$$\int_0^t \|\boldsymbol{u}(s)\|^2_U \, ds \leq C + \int_0^t \int_0^s \|\boldsymbol{v}(r)\|^2_U \, dr \, ds. \tag{5.7}$$

Using now Gronwall's inequality, it follows from (5.5)–(5.7) that

$$\begin{aligned} &|\boldsymbol{v}(t)|^2_{H^m} + \int_0^t \|\boldsymbol{v}(s)\|^2_E \, ds + \|\boldsymbol{u}(t)\|^2_E + \int_{\Gamma_C} \Phi(u_n(t,x)) \, d\Gamma \\ &\quad - C \int_0^t \|\boldsymbol{v}\|^2_U \, ds \leq C. \end{aligned} \tag{5.8}$$

Finally, we use the compactness of the embedding of E into U and apply Gronwall's inequality again and obtain (5.1) from (5.8).

We use estimate (5.1) to pass to the limit when $\delta \to 0$ and thus obtain the existence of a solution for problem (3.15)–(3.18).

For each $\delta > 0$, let $\{\boldsymbol{u}_\delta, \boldsymbol{v}_\delta\}$ denote the unique solution of Problem (4.4)–(4.5). Using the estimate (5.1), (4.4), and the boundedness of the operators, Theorems 4.1 and 4.2 imply that there exists a subsequence of $\{\boldsymbol{u}_\delta, \boldsymbol{v}_\delta\}$ such that

$$\boldsymbol{v}'_\delta \to \boldsymbol{v}' \qquad \text{weakly in } \mathbb{E}', \tag{5.9}$$

$$\boldsymbol{v}_\delta \to \boldsymbol{v} \qquad \text{weakly in } \mathbb{E}, \tag{5.10}$$

$$\boldsymbol{u}_\alpha \to \boldsymbol{u} \qquad \text{strongly in } C(0,T;U), \tag{5.11}$$

$$u_{n\delta} \to u_n \qquad \text{strongly in } L^2(\Gamma_C \times (0,T)), \tag{5.12}$$

$$u_{n\delta}(x,t) \to u_n(x,t) \qquad \text{a.e. in } \Gamma_C \times (0,T), \tag{5.13}$$

$$\boldsymbol{v}_\delta \to \boldsymbol{v} \qquad \text{strongly in } L^2(0,T;U), \tag{5.14}$$

$$p_\delta^-(u_{\delta n}, \cdot) \to \xi \qquad \text{weak* in } L^\infty(\Gamma_C \times (0,T)). \tag{5.15}$$

Let $\boldsymbol{v}_\delta^*$ denote the element of $Q(\boldsymbol{v}_\delta)$ which yields equality in (4.4); thus,

$$\boldsymbol{v}'_\delta + B\boldsymbol{v}_\delta + A\boldsymbol{u}_\delta + P^+(\boldsymbol{u}_\delta) + P_\delta^-(\boldsymbol{u}_\delta) + \boldsymbol{v}_\delta^* = \boldsymbol{f},$$

and let $\boldsymbol{z}_\delta \in L^\infty(0,T; L^\infty(\Gamma_C)^m)$ be the function in the definition of $Q(\boldsymbol{v}_\delta)$, (3.12), and (3.13). Furthermore, we may also assume that

$$\boldsymbol{z}_\delta \to \boldsymbol{z} \qquad \text{weak* in } L^\infty\big(0,T; L^\infty(\Gamma_C)^m\big). \tag{5.16}$$

Using (5.11), (5.16), and the definition of $Q(\boldsymbol{v}_\delta)$ in (3.13), we conclude that

$$\boldsymbol{v}_\delta^* \to \boldsymbol{v}^* \qquad \text{weakly in } \mathbb{E}',$$

where

$$\langle \boldsymbol{v}^*, \boldsymbol{w}\rangle = \int_0^T \int_{\Gamma_C} \mu p^+(u_n)\boldsymbol{z}\cdot \boldsymbol{w}_T \, d\Gamma\, dt, \qquad \boldsymbol{w} \in \mathbb{E},$$

and thus, $\boldsymbol{v}^* \in Q(\boldsymbol{v})$. On the other hand, we obtain from (5.11) and (3.1) that

$$P^+(\boldsymbol{u}_\delta) \to P^+(\boldsymbol{u}) \qquad \text{strongly in } \mathbb{E}'.$$

Let now K be the set

$$K = \{\psi \in L^\infty(\Gamma_C \times (0,T)) : 0 \geq \psi(x,t) \geq -p^*(x) \text{ a.e. on } \Gamma_C \times (0,T)\}.$$

K is a closed and convex subset of $L^\infty(\Gamma_C \times (0,T))$, and from the definition of the function p_δ^-, it follows that $p_\delta^-(u_{\delta n}, \cdot) \in K$, for each δ. Therefore, we obtain from (5.15) that $\xi \in K$. Using (4.2), we have for $\boldsymbol{w} \in \mathbb{E}$,

$$\langle P_\delta^-(\boldsymbol{u}_\delta), \boldsymbol{w}\rangle = \int_0^T \int_{\Gamma_C} p_\delta^-(u_{\delta n}(x,t), x) w_n(x,t) \, d\Gamma\, dt,$$

and using (5.15), we obtain that

$$\langle P_\delta^-(\boldsymbol{u}_\delta), \boldsymbol{w}\rangle \to \int_0^T \int_{\Gamma_C} \xi(x,t) w_n(x,t)\, d\Gamma\, dt.$$

Let us consider now $\xi(x,t)$. Suppose first that (x,t) is a point at which $u_n(x,t) \neq -g(x)$ and is also a point where $u_{\delta n}(x,t) \to u_n(x,t)$. Then $p_\delta^-(u_n(x,t), x) = p^-(u_n(x,t), x)$ for all δ sufficiently small. By the continuity of $p^-(x,\cdot)$ at such points, $p_\delta^-(u_{\delta n}(x,t), x) \to p^-(u_n(x,t), x)$. Consequently, if such points comprise a set S of positive measure, then for almost every point in S, $p^-(u_n(x,t), x) = \xi(x,t)$. On the other hand, the observation that ξ lies in K implies that even if $u_n(x,t) = -g(x)$, $\xi(x,t) \in p^-(u_n(x,t), x)$ almost everywhere. This completes the proof of Theorem 3.1.

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