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NOTE

Symmetrically Homoclinic Orbits for Symmetric Hamiltonian Systems

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In this paper, we study the existence of symmetric homoclinic orbits for first order and second order Hamiltonian systems with some symmetric Hamiltonian functions.

Key Words: symmetry; homoclinic orbits; Hamiltonian systems.

In recent years, many authors [1–3, 5–30, 34, 36–46] have used the variational methods to study the existence and the multiplicity of homoclinic orbits for Hamiltonian systems. In this paper, we will study the existence of a symmetric homoclinic orbit for the first order symmetric Hamiltonian system and the existence of infinitely many odd homoclinic orbits for classical Hamiltonian systems with even potentials.

We are given a C^2 map H: $R^{2N} \to R$, and we consider the associated system of ordinary differential equations

$$\begin{aligned} \dot{x}(t) &= JH'(x) \\ x(\pm\infty) &= 0, \end{aligned} \tag{1.1}$$

where J denotes the $2N \times 2N$ matrix

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

with $J^* = J^{-1} = -J$.

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We obtain the following results:

THEOREM 1.1. *Assume H satisfies*

(H1) $H \in C^2(R^{2N}, R)$

(H2) $H(-p, q) = H(p, q), \forall p, q \in R^N$

(H3) $H_q(0,0) = 0$

(H4) $H''(0) = 0$

(H5) $\exists \alpha > 2$ *such that* $\forall x \in R^{2N}$, $\alpha H(x) \leq H'(x)x$

(H6) $\exists k_1 > 0$ *such that* $\forall x \in R^{2N}$, $H(x) \geq k_1 |x|^\alpha$

(H7) $\exists k_2 > 0$ *such that* $\forall x \in R^{2N}$, $|H'(x)| \leq k_2 |x|^{\alpha - 1}$.

Then (1.1) *has at least one homoclinic orbit* $x = (p, q)$ *to the origin which satisfies* $p(-t) = -p(t)$ *and* $q(-t) = q(t)$.

Remark 1. In all published papers, there is a quadratic term for the Hamiltonian function. Here we remove this term.

Remark 2. (H5) implies $H(x) = 0(|x|^2)$ as $|x| \to 0$. (H4) can be canceled out.

THEOREM 1.2. *Assume V satisfies*

(V1) $V \in C^2(R^n, R)$;

(V2) $V(-x) = V(x), \forall x \in R^n$;

(V3) *there is a* $\mu > 2$ *such that* $0 < \mu V(x) \leq x \cdot V'(x), \forall x \in R^n \setminus \{0\}$;

(V4) $V''(0) = 0$.

Then there are infinitely many odd homoclinic orbits for the second order Hamiltonian system:

$$\begin{aligned} \ddot{x} + V'(x) &= 0 \\ x(\pm\infty) &= \dot{x}(\pm\infty) = 0. \end{aligned} \tag{1.2}$$

2. THE PROOF OF THEOREM 1.1

Let $W = W^{1,2}(R, R^{2N})$ be the Sobolev space of R^{2N}-valued functions defined on R:

$$E = \{x = (p, q) \in W \,|\, p(-t) = -p(t), q(-t) = q(t), \forall t \in R\}. \tag{2.1}$$

The functional corresponding to the system (1.1) is defined by

$$f(x) = \int_{-\infty}^{\infty} \tfrac{1}{2}(-J\dot{x}, x)\, dt - \int_{-\infty}^{\infty} H(x)\, dt \qquad \forall x \in E. \tag{2.2}$$

Following the ideas of [35, 31–33], we have

LEMMA 2.1. *Suppose* (H1) *and* (H4) *hold. Then* $f \in C^1(E, R)$, *and* $x = (p, q) \in E$ *is a critical point of* f *restricted on* E *if and only if it is a* $C^1(R, R^n)$*-solution of* (1.1) *such that* p *is odd and* q *even in* t.

Proof. (i) By (H1) and (H4), similar to the proof of Coti Zelati and Rabinowitz [26], $f \in C^1(E, R)$.

(ii) Suppose $x \in E$ is a critical point of f on E. Then there holds

$$\int_{-\infty}^{\infty} (-J\dot{x} \cdot y - H'(x) \cdot y)\, dt = 0, \qquad \forall y \in E. \tag{2.3}$$

By (H1), $H' \in C^1(W^{1,2}, W^{1,2})$. (H2) and (H3) imply $H'(0) = 0$. By $x \in W^{1,2}(R, R^{2N})$ and the regularity theorem on composition mappings, we have $u \equiv H'(x(\cdot)) \in W^{1,2}(R, R^{2N})$ and $u \in E$; that is, $u = (u_1, u_2)$ satisfies

$$u_1(-t) = -u_1(t) \qquad \text{and} \qquad u_2(-t) = u_2(t). \tag{2.4}$$

We consider the boundary value problem of the linear system,

$$\begin{aligned} \dot{z}(t) &= Ju \\ z(\pm\infty) &= 0, \end{aligned} \tag{2.5}$$

which possesses a unique solution $Z(t) \in C^1(R, R^{2n})$ and is given by

$$Z(t) = J \cdot \int_{-\infty}^{t} u(s)\, ds, \qquad \forall t \in R. \tag{2.6}$$

By (2.4) and (2.6) we know that

$$Z(t) = (Z_1(t), Z_2(t)) = \left(-\int_{-\infty}^{t} u_2(s)\, ds, \int_{-\infty}^{t} u_1(s)\, ds \right)$$

satisfies

$$Z_1(-t) = -Z_1(t), \qquad Z_2(-t) = Z_2(t). \tag{2.7}$$

By (2.6) and $u \in W^{1,2}(R, R^{2n})$ we know $Z \in W^{2,2}(R, R^{2n})$.

From (2.5) we obtain that for $\forall y \in E$ there holds

$$\int_{-\infty}^{\infty} \big(-J\dot{Z} \cdot y - H'(x) \cdot y\big)\, dt = 0. \tag{2.8}$$

Combining with (2.3) yields

$$\int_{-\infty}^{\infty} J(\dot{x} - \dot{Z}) \cdot y\, dt = 0, \qquad \forall y \in E. \tag{2.9}$$

By (2.3) and (2.6) we have $x \in W^{2,2}(R, R^{2n})$, $Z \in W^{2,2}(R, R^{2n})$. So

$$\tilde{y} = J(\dot{x} - \dot{Z}) \in W^{1,2}(R, R^{2n}). \tag{2.10}$$

Set $x = (x_1, x_2)$, $Z = (Z_1, Z_2)$; then

$$\tilde{y} = J(\dot{x} - \dot{Z}) = \left(\dot{Z}_2 - \dot{x}_2, \dot{x}_1 - \dot{Z}_1\right) \equiv (\tilde{y}_1, \tilde{y}_2).$$

Then

$$\tilde{y}_1(-t) = -\tilde{y}_1(t), \qquad \tilde{y}_2(-t) = \tilde{y}_2(t). \tag{2.11}$$

Hence $\tilde{y} \in E$.

In (2.9), we can set $y = \tilde{y}$ to obtain

$$\int_{-\infty}^{\infty} |\dot{x} - \dot{Z}|^2 \, dt = 0. \tag{2.12}$$

Hence

$$x(t) - Z(t) \equiv \text{constant}, \qquad \forall t \in R. \tag{2.13}$$

By $x(\pm\infty) = Z(\pm\infty) = 0$, we know

$$x(t) - Z(t) \equiv 0.$$

Thus $x(t) = Z(t) \in C^1(R, R^n)$ and is a solution of (1.1) by (2.5). Now the proof of Theorem 1.1 is similar to that of Hofer and Wysocki [29].

3. THE PROOF OF THEOREM 1.2

Let $W = W^{1,2}(R, R^n)$, which has the usual norm $(\int_{-\infty}^{\infty}(|\dot{q}|^2 + |q|^2))^{1/2}$ which is equivalent to the norm

$$\|q\| = \left(\int_{-\infty}^{\infty} |\dot{q}|^2 \, dt + |q(0)|^2\right)^{1/2}. \tag{3.1}$$

The functional corresponding to the system (1.2) is defined by

$$f(x) = \int_{-\infty}^{\infty} \left[\tfrac{1}{2}|\dot{x}|^2 - V(x)\right] dt, \qquad \forall x \in W. \tag{3.2}$$

Let

$$\tilde{E} = \{x \in W \mid x(-t) = -x(t), \forall t \in R\}. \tag{3.3}$$

Then $\tilde{E}$ is a closed subspace of W and, therefore, is a Hilbert space. By $x(-t) = -x(t)$ we have $x(0) = 0$. Hence we have

$$\|x\| = \left(\int_{-\infty}^{\infty} |\dot{x}|^2 \, dt \right)^{1/2}, \qquad \forall x \in \tilde{E}. \tag{3.4}$$

Following the ideas of [31–33, 35], we have

LEMMA 3.1. *Suppose* (V1), (V2), *and* (V4) *hold. Then* $f \in C^1(\tilde{E}, R)$, *and* $x \in \tilde{E}$ *is a critical point of* f *restricted on* $\tilde{E}$ *if and only if it is an odd* $C^2(R, R^n)$*-solution of* (1.2).

Proof. (i) By (V1), (V4), and [26], we know $f \in C^1(\tilde{E}, R)$.

(ii) Suppose $x \in \tilde{E}$ is a critical point of f on $\tilde{E}$. Then there holds

$$\int_{-\infty}^{\infty} (\dot{x}\dot{y} - V'(x) \cdot y) \, dt = 0, \qquad \forall y \in \tilde{E}. \tag{3.5}$$

By (V1), we have $w \equiv V'(x(\cdot), t) \in C(R, R^n)$. Furthermore, by (V1), $V' \in C^1(R^n \times R, R)$ and $x \in W^{1,2}(R, R^n)$. By (V2), we have $V'(0) = 0$. So by the regular theorem about the composition mapping we have $w \in W^{1,2}(R, R^n)$.

The boundary value problem of the linear system

$$\begin{aligned} \ddot{q} + w &= 0 \\ q(\pm\infty) = \dot{q}(\pm\infty) &= 0 \end{aligned} \tag{3.6}$$

possesses a unique solution $Q \in C^2(R, R^n)$ and

$$\int_{S_1}^{S} \ddot{Q}(\tau) \, d\tau = \int_{S_1}^{S} -w(\tau) \, d\tau, \qquad \forall S, S_1 \in R \tag{3.7}$$

$$\dot{Q}(S) - \dot{Q}(S_1) = -\int_{S_1}^{S} w(\tau) \, d\tau, \qquad \forall S, S_1 \in R. \tag{3.8}$$

Because $\lim_{t_1 \to -\infty} Q(S_1) = 0$, so $\int_{-\infty}^{S} w(\tau) \, d\tau$ exists and

$$-\int_{-\infty}^{S} w(\tau) \, d\tau = \dot{Q}(S) \tag{3.9}$$

$$-\int_{t_1}^{t} \left(\int_{-\infty}^{S} W(\tau) \, d\tau \right) ds = Q(t) - Q(t_1), \qquad \forall t_1, t \in R. \tag{3.10}$$

Because $\lim_{t_1 \to -\infty} Q(t_1) = 0$, so $-\int_{-\infty}^{t}(\int_{-\infty}^{S} w(\tau)\, d\tau)\, ds$ exists and

$$Q(t) = -\int_{-\infty}^{t} \int_{-\infty}^{S} w(\tau)\, d\tau\, ds. \tag{3.11}$$

So $Q \in C^2(R, R^n)$.

Since w is odd, so is Q. By $Q(\pm\infty) = \dot{Q}(\pm\infty) = 0$, we know $Q \in \tilde{E}$.

From (3.6) we obtain that for $\forall y \in \tilde{E}$ there holds

$$\int_{-\infty}^{\infty} \left(\dot{Q}\dot{y} - V'(x) \cdot y\right) dt = 0. \tag{3.12}$$

Combining with (3.5) yields

$$\int_{-\infty}^{\infty} \left(\dot{x} - \dot{Q}\right) \cdot \dot{y}\, dt = 0, \qquad \forall y \in \tilde{E}. \tag{3.13}$$

Letting $y = x - Q$, by the fact $x(0) = Q(0) = 0$ we obtain

$$|x(t) - Q(t)| \le \int_0^{|t|} \left|\dot{x}(s) - \dot{Q}(s)\right| ds \le \sqrt{|t|}\, \|\dot{x} - \dot{Q}\|_{L^2} = 0, \qquad \forall t \in R. \tag{3.14}$$

Thus $x = Q \in C^2(R, R^n)$ and is a solution of (1.2) by (3.6).

Now the proof of Theorem 1.2 follows from Lemma 3.1 and the arguments of Coti Zelati-Rabinowitz [26].

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