Available online at www.sciencedirect.com
SCIENCE DIRECT®

J. Math. Anal. Appl. 296 (2004) 8–20

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Periodic boundary value problems for first order impulsive integro-differential equations of mixed type ☆

Zhimin He [a,*], Xiaoming He [b]

[a] *Department of Applied Mathematics, Central South University, Changsha 410083, Hunan, PR China*
[b] *Department of Applied Mathematics, Beijing Institute of Technology, Beijing 100081, PR China*

Received 22 April 2002

Available online 10 June 2004

Submitted by K.A. Ames

Abstract

This paper investigates the existence of minimal and maximal solutions of periodic boundary value problem for first order impulsive integro-differential equations of mixed type by establishing a comparison result and using the method of upper and lower solutions and the monotone iterative technique.

Keywords: Impulsive integro-differential equation; Periodic boundary value problem; Upper and lower solution; Monotone iterative technique

1. Introduction

Differential equations with impulses provide an adequate mathematical model of many evolutionary processes that suddenly change their state at certain moments (see [1,2,4–10]). In this paper, we consider the periodic boundary value problem for first order impulsive

☆ This work was supported by Mathematical Tianyuan Foundation (No. A0324621) of People's Republic of China.
* Corresponding author.
E-mail address: hezhimin@mail.csu.edu.cn (Z. He).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2003.12.047

integro-differential equations of mixed type (PBVP),

$$\begin{cases} x'(t) = f(t, x(t), [Tx](t), [Sx](t)), & t \neq t_k,\ t \in J, \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1, 2, \ldots, p, \\ x(0) = x(2\pi), \end{cases} \tag{1}$$

where $f \in C(J \times R \times R \times R, R)$, $J = [0, 2\pi]$, $I_k \in C(R, R)$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of $x(t)$ at $t = t_k$ $(k = 1, 2, \ldots, p)$, $0 < t_1 < t_2 < \cdots < t_k < \cdots < t_p < 2\pi$,

$$[Tx](t) = \int_0^t K(t, s)x(s)\, ds, \qquad [Sx](t) = \int_0^{2\pi} H(t, s)x(s)\, ds,$$

$K \in C(D, R_+)$, $D = \{(t, s) \in J \times J\colon t \geqslant s\}$, $H \in C(J \times J, R_+)$, $R_+ = [0, +\infty)$.

Monotone iterative technique coupled with the method of upper and lower solutions has been widely used in the treatment of existence results of initial and boundary value problems for nonlinear differential equations in recent years (see [3–11]). The basic idea is that using the upper and lower solutions as an initial iteration one can construct monotone sequences from a corresponding linear problem, and these sequences converge monotonically to the minimal and maximal solutions of the nonlinear problem. When the method is applied to impulsive differential equations, it usually need a suitable impulsive differential inequality as a comparison principle.

The results in the paper are inspired by D.J. Guo, V. Lakshmikantham, and X.Z. Liu in [6], X.Z. Liu and D.J. Guo in [7], Y.B. Chen and W. Zhuang in [11]. In Section 2, we establish a comparison principle, i.e., Lemma 2.2. In Section 3, we discuss the existence and uniqueness of the solutions for a linear periodic boundary value problem for impulsive integro-differential equation, i.e., Lemmas 3.1, 3.2. Finally, by use of the monotone iterative technique and the method of upper and lower solutions, we obtain the existence theorem of extremal solutions for PBVP (1).

2. Preliminaries and comparison principle

Let $PC(J, R) = \{x\colon J \to R,\ x(t)$ is continuous everywhere except some t_k at which $x(t_k^-)$ and $x(t_k^+)$ exist and $x(t_k^-) = x(t_k)\}$. Evidently, $PC(J, R)$ is a Banach space with norm $\|x\|_{PC} = \sup_{t \in J} |x(t)|$. Let $J' = J \backslash \{t_1, t_2, \ldots, t_p\}$, $\Omega = PC(J, R) \cap C^1(J', R)$. A function $x \in \Omega$ is called a solution of PBVP (1) if it satisfies (1).

Let $k_0 = \max\{K(t, s)\colon (t, s) \in D\}$, $h_0 = \max\{H(t, s)\colon (t, s) \in J \times J\}$. We list the following assumptions for convenience:

(A_0) There exist functions $\alpha, \beta \in \Omega$, $\alpha(t) \leqslant \beta(t)$ $(\forall t \in J)$ such that

$$\begin{cases} \alpha'(t) \leqslant f(t, \alpha(t), [T\alpha](t), [S\alpha](t)) - r_\alpha, & t \neq t_k,\ t \in J, \\ \Delta\alpha(t_k) \leqslant I_k(\alpha(t_k)) - l_{\alpha k}, & k = 1, 2, \ldots, p, \end{cases} \tag{2}$$

and

$$\begin{cases} \beta'(t) \geqslant f(t, \beta(t), [T\beta](t), [S\beta](t)) + r_\beta, & t \neq t_k,\ t \in J, \\ \Delta\beta(t_k) \geqslant I_k(\beta(t_k)) + l_{\beta k}, & k = 1, 2, \ldots, p, \end{cases} \tag{3}$$

where for $M > 0$, $N_1 > 0$, $N_2 > 0$, $0 \leqslant L_k < 1$ ($k = 1, 2, \ldots, p$), and r_α, r_β, $l_{\beta k}$, $l_{\alpha k}$ ($k = 1, 2, \ldots, p$) are given by

$$r_\alpha = \begin{cases} 0, & \text{if } \alpha(0) \leqslant \alpha(2\pi), \\ \left(\frac{Mt+1}{2\pi} + \frac{N_1 k_0 t^2}{4\pi} + \pi N_2 h_0\right)[\alpha(0) - \alpha(2\pi)], & \text{if } \alpha(0) > \alpha(2\pi), \end{cases}$$

$$r_\beta = \begin{cases} 0, & \text{if } \beta(0) \geqslant \beta(2\pi), \\ \left(\frac{Mt+1}{2\pi} + \frac{N_1 k_0 t^2}{4\pi} + \pi N_2 h_0\right)[\beta(2\pi) - \beta(0)], & \text{if } \beta(0) < \beta(2\pi), \end{cases}$$

$$l_{\alpha k} = \begin{cases} 0, & \text{if } \alpha(0) \leqslant \alpha(2\pi), \\ \frac{L_k t_k}{2\pi}[\alpha(0) - \alpha(2\pi)], & \text{if } \alpha(0) > \alpha(2\pi), \end{cases}$$

$$l_{\beta k} = \begin{cases} 0, & \text{if } \beta(0) \geqslant \beta(2\pi), \\ \frac{L_k t_k}{2\pi}[\beta(2\pi) - \beta(0)], & \text{if } \beta(0) < \beta(2\pi), \end{cases}$$

that is, $\alpha(t)$ and $\beta(t)$ are lower and upper solutions of PBVP (1), respectively.

(A_1) The function $f \in C(J \times R \times R \times R, R)$ satisfies

$$f(t, u, v, w) - f(t, \bar{u}, \bar{v}, \bar{w}) \geqslant -M(u - \bar{u}) - N_1(v - \bar{v}) - N_2(w - \bar{w}),$$

whenever $\alpha(t) \leqslant \bar{u} \leqslant u \leqslant \beta(t)$, $[T\alpha](t) \leqslant \bar{v} \leqslant v \leqslant [T\beta](t)$, $[S\alpha](t) \leqslant \bar{w} \leqslant w \leqslant [S\beta](t)$, $t \in J$, where $M > 0$, $N_1 > 0$, $N_2 > 0$.

(A_2) The functions $I_k \in C(R, R)$ satisfy

$$I_k(x) - I_k(y) \geqslant -L_k(x - y),$$

whenever $\alpha(t_k) \leqslant y \leqslant x \leqslant \beta(t_k)$ ($k = 1, 2, \ldots, p$), and $0 \leqslant L_k < 1$ ($k = 1, 2, \ldots, p$).

Lemma 2.1 [1]. *Assume that*

(B_0) *the sequence* $\{t_k\}$ *satisfies* $0 \leqslant t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ *with* $\lim_{k\to+\infty} t_k = +\infty$.
(B_1) $m \in PC^1(R_+, R)$ *is left continuous at* t_k *for* $k = 1, 2, \ldots$.
(B_2) *for* $k = 1, 2, \ldots$, $t \geqslant t_0$,

$$m'(t) \leqslant p(t)m(t) + q(t), \quad t \neq t_k, \qquad m\left(t_k^+\right) \leqslant d_k m(t_k) + b_k,$$

where $p, q \in C(R_+, R)$, $d_k \geqslant 0$ *and* b_k *are real constants.*

Then,

$$m(t) \leqslant m(t_0) \prod_{t_0 < t_k < t} d_k \exp\left(\int_{t_0}^{t} p(s)\,ds\right) + \int_{t_0}^{t} \prod_{s < t_k < t} d_k \exp\left(\int_{s}^{t} p(\sigma)\,d\sigma\right) q(s)\,ds$$
$$+ \sum_{t_0 < t_k < t} \prod_{t_k < t_j < t} d_j \exp\left(\int_{t_k}^{t} p(s)\,ds\right) b_k.$$

Lemma 2.2. *Let* $t_0 = 0$, $t_{p+1} = 2\pi$. *Assume that* $m \in \Omega$ *satisfies*

$$\begin{cases} m'(t) \leqslant -Mm(t) - N_1[Tm](t) - N_2[Sm](t) - r_m, & t \neq t_k,\ t \in J, \\ \Delta m(t_k) \leqslant -L_k m(t_k) - l_{mk}, & k = 1, 2, \ldots, p, \end{cases} \tag{4}$$

where constants $M > 0$, $N_1 > 0$, $N_2 > 0$, $0 \leqslant L_k < 1$ $(k = 1, 2, \ldots, p)$, and r_m, l_{mk} $(k = 1, 2, \ldots, p)$ are given by

$$r_m = \begin{cases} 0, & \text{if } m(0) \leqslant m(2\pi), \\ \left(\frac{Mt+1}{2\pi} + \frac{N_1 k_0 t^2}{4\pi} + \pi N_2 h_0\right)[m(0) - m(2\pi)], & \text{if } m(0) > m(2\pi), \end{cases}$$

$$l_{mk} = \begin{cases} 0, & \text{if } m(0) \leqslant m(2\pi), \\ \dfrac{L_k t_k}{2\pi}[m(0) - m(2\pi)], & \text{if } m(0) > m(2\pi). \end{cases}$$

If

$$M^{-1}(N_1 k_0 + N_2 h_0)\left(e^{4\pi M} - 1\right) \leqslant \frac{\{\prod_{0<t_k<2\pi}(1 - L_k)\}^2}{\int_0^{2\pi} \prod_{s<t_k<2\pi}(1 - L_k)\, ds}, \tag{5}$$

then $m(t) \leqslant 0$ for $t \in J$.

Proof.

Case 1. $m(0) \leqslant m(2\pi)$. Let $u(t) = m(t)e^{Mt}$ for $t \in J$. Then $u \in \Omega$ satisfies

$$\begin{cases} u'(t) \leqslant -N_1 \int_0^t k^*(t,s)u(s)\, ds - N_2 \int_0^{2\pi} h^*(t,s)u(s)\, ds, & t \neq t_k,\ t \in J, \\ \Delta u(t_k) \leqslant -L_k u(t_k), & k = 1, 2, \ldots, p, \\ u(0) \leqslant u(2\pi)e^{-2\pi M}, \end{cases} \tag{6}$$

where $k^*(t,s) = K(t,s)e^{M(t-s)}$, $h^*(t,s) = H(t,s)e^{M(t-s)}$. We now prove

$$u(t) \leqslant 0 \quad \text{for } t \in J. \tag{7}$$

Assume that (7) is not true. Then, there are two cases:

(a) there exists $t_1^* \in J$ such that $u(t_1^*) > 0$, and $u(t) \geqslant 0$ for $t \in J$;
(b) there exist t_1^*, $t_2^* \in J$ such that $u(t_1^*) > 0$ and $u(t_2^*) < 0$.

In case (a): (6) implies that

$$\begin{cases} u'(t) \leqslant 0, & t \neq t_k,\ t \in J, \\ \Delta u(t_k) \leqslant 0, & k = 1, 2, \ldots, p. \end{cases}$$

This means that $u(t)$ is nonincreasing in J, and therefore

$$u(0) \geqslant u\left(t_1^*\right) > 0, \tag{8}$$

and

$$u(0) \geqslant u(2\pi) \geqslant u(0)e^{2\pi M} > 0, \tag{9}$$

which contradicts (8).

In case (b): let $\inf_{t\in J} u(t) = -\lambda$. Then $\lambda > 0$, and there exists $t_i < t_0^* \leqslant t_{i+1}$ for some i such that $u(t_0^*) = -\lambda$ or $u(t_i^+) = -\lambda$. We may assume that $u(t_0^*) = -\lambda$ (since, in case of $u(t_i^+) = -\lambda$, the proof is similar). From (6), we have

$$u'(t) \leqslant \lambda N_1 k_0 \int_0^t e^{M(t-s)}\,ds + \lambda N_2 h_0 \int_0^{2\pi} e^{M(t-s)}\,ds \leqslant \lambda M_0, \quad t \neq t_k,\ t \in J,$$

where $M_0 = M^{-1}(N_1k_0 + N_2h_0)(e^{2\pi M} - 1)$.

Consider the inequalities

$$\begin{cases} u'(t) \leqslant \lambda M_0, & t \neq t_k,\ t \in [t_0^*, 2\pi], \\ u(t_k^+) \leqslant (1 - L_k)u(t_k), & k = i+1, i+2, \ldots, p, \end{cases}$$

and Lemma 2.1 implies

$$u(t) \leqslant u\bigl(t_0^*\bigr) \prod_{t_0^* < t_k < t} (1 - L_k) + \int_{t_0^*}^{t} \prod_{s < t_k < t} (1 - L_k)(\lambda M_0)\,ds. \tag{10}$$

Let $t = 2\pi$ in (10), then

$$u(2\pi) \leqslant -\lambda \prod_{t_0^* < t_k < 2\pi} (1 - L_k) + \lambda M_0 \int_{t_0^*}^{2\pi} \prod_{s < t_k < 2\pi} (1 - L_k)\,ds. \tag{11}$$

If $u(2\pi) > 0$, then (11) gives

$$M_0 > \frac{\prod_{t_0^* < t_k < 2\pi}(1 - L_k)}{\int_{t_0^*}^{2\pi} \prod_{s < t_k < 2\pi}(1 - L_k)\,ds} \geqslant \frac{\prod_{0 < t_k < 2\pi}(1 - L_k)}{\int_0^{2\pi} \prod_{s < t_k < 2\pi}(1 - L_k)\,ds}, \tag{12}$$

which contradicts (5). So, we have $u(2\pi) \leqslant 0$, and by (6), $u(0) \leqslant u(2\pi)e^{-2\pi M} \leqslant 0$. Hence $0 < t_1^* < 2\pi$. Let $t_j < t_1^* \leqslant t_{j+1}$ for some j.

We first assume that $t_0^* < t_1^*$. So $i \leqslant j$. Let $t = t_1^*$ in (10), then

$$0 < u(t_1^*) \leqslant -\lambda \prod_{t_0^* < t_k < t_1^*} (1 - L_k) + \int_{t_0^*}^{t_1^*} \prod_{s < t_k < t_1^*} (1 - L_k)(\lambda M_0)\,ds, \tag{13}$$

which gives

$$M_0 > \frac{\prod_{t_0^* < t_k < t_1^*}(1 - L_k)}{\int_{t_0^*}^{t_1^*} \prod_{s < t_k < t_1^*}(1 - L_k)\,ds} \geqslant \frac{\prod_{0 < t_k < 2\pi}(1 - L_k)}{\int_0^{2\pi} \prod_{s < t_k < 2\pi}(1 - L_k)\,ds},$$

and this contradicts (5).

Next we assume that $t_1^* < t_0^*$. So $j \leqslant i$. Consider the inequalities

$$\begin{cases} u'(t) \leqslant \lambda M_0, & t \neq t_k,\ t \in J, \\ u(t_k^+) \leqslant (1 - L_k)u(t_k), & k = 1, 2, \ldots, p, \end{cases}$$

and Lemma 2.1 implies

$$u(t) \leqslant u(0) \prod_{0<t_k<t} (1 - L_k) + \int_0^t \prod_{s<t_k<t} (1 - L_k)(\lambda M_0)\, ds. \tag{14}$$

Let $t = t_1^*$ in (14), then

$$0 < u\big(t_1^*\big) \leqslant u(0) \prod_{0<t_k<t_1^*} (1 - L_k) + \lambda M_0 \int_0^{t_1^*} \prod_{s<t_k<t_1^*} (1 - L_k)\, ds, \tag{15}$$

which implies

$$u(0) \prod_{0<t_k<t_1^*} (1 - L_k) > -\lambda M_0 \int_0^{t_1^*} \prod_{s<t_k<t_1^*} (1 - L_k)\, ds.$$

By (6), we obtain

$$-\lambda M_0 \int_0^{t_1^*} \prod_{0<t_k<t_1^*} (1 - L_k)\, ds < u(2\pi) e^{-2\pi M} \prod_{0<t_k<t_1^*} (1 - L_k). \tag{16}$$

From (11), (16), we have

$$\begin{aligned} -\lambda M_0 \int_0^{t_1^*} \prod_{s<t_k<t_1^*} (1 - L_k)\, ds < e^{-2\pi M} \prod_{0<t_k<t_1^*} (1 - L_k) \Bigg\{ -\lambda \prod_{t_0^*<t_k<2\pi} (1 - L_k) \\ + \lambda M_0 \int_{t_0^*}^{2\pi} \prod_{s<t_k<2\pi} (1 - L_k)\, ds \Bigg\}, \end{aligned}$$

or

$$\begin{aligned} \prod_{0<t_k<t_1^*} (1 - L_k) \prod_{t_0^*<t_k<2\pi} (1 - L_k) < M_0 \prod_{0<t_k<t_1^*} (1 - L_k) \int_{t_0^*}^{2\pi} \prod_{s<t_k<2\pi} (1 - L_k)\, ds \\ + M_0 e^{2\pi M} \int_0^{t_1^*} \prod_{s<t_k<t_1^*} (1 - L_k)\, ds. \end{aligned}$$

Hence

$$\Bigg\{ \prod_{0<t_k<2\pi} (1 - L_k) \Bigg\}^2 \leqslant \prod_{0<t_k<t_1^*} (1 - L_k) \prod_{t_0^*<t_k<2\pi} (1 - L_k) \prod_{0<t_k<2\pi} (1 - L_k)$$

$$
\begin{aligned}
&< M_0 \prod_{0<t_k<t_1^*}(1-L_k) \prod_{0<t_k<2\pi}(1-L_k) \int_{t_0^*}^{2\pi} \prod_{s<t_k<2\pi}(1-L_k)\,ds \\
&\quad + M_0 e^{2\pi M} \prod_{0<t_k<2\pi}(1-L_k) \int_0^{t_1^*} \prod_{s<t_k<t_1^*}(1-L_k)\,ds \\
&< M_0\bigl(e^{2\pi M}+1\bigr) \int_0^{2\pi} \prod_{s<t_k<2\pi}(1-L_k)\,ds,
\end{aligned}
$$

which contradicts (5). This proof is complete in the case $m(0) \leqslant m(2\pi)$.

Case 2. $m(0) > m(2\pi)$. Let $\bar{m}(t) = m(t) + g(t)$, where

$$g(t) = \frac{t}{2\pi}\bigl[m(0) - m(2\pi)\bigr],$$

then

$$g(0) = 0, \qquad g(2\pi) = m(0) - m(2\pi), \quad \text{and} \quad g(t) \geqslant 0 \quad \text{for } t \in J.$$

Hence, we have

$$\bar{m}(0) = m(0) = m(2\pi) + g(2\pi) = \bar{m}(2\pi),$$

and

$$
\begin{aligned}
\bar{m}'(t) &= m'(t) + g'(t) \\
&\leqslant -Mm(t) - N_1[Tm](t) - N_2[Sm](t) \\
&\quad - \left(\frac{Mt+1}{2\pi} + \frac{N_1 k_0 t^2}{4\pi} + \pi N_2 h_0\right)\bigl[m(0) - m(2\pi)\bigr] + \frac{1}{2\pi}\bigl[m(0) - m(2\pi)\bigr] \\
&= -M\bar{m}(t) - N_1[T\bar{m}](t) - N_2[S\bar{m}](t) \\
&\quad + N_1 \int_0^t K(t,s)\frac{s}{2\pi}\bigl[m(0) - m(2\pi)\bigr]\,ds \\
&\quad + N_2 \int_0^{2\pi} H(t,s)\frac{s}{2\pi}\bigl[m(0) - m(2\pi)\bigr]\,ds \\
&\quad - \left(\frac{N_1 k_0 t^2}{4\pi} + \pi N_2 h_0\right)\bigl[m(0) - m(2\pi)\bigr] \\
&\leqslant -M\bar{m}(t) - N_1[T\bar{m}](t) - N_2[S\bar{m}](t), \quad t \neq t_k,\ t \in J,
\end{aligned}
$$

$$
\begin{aligned}
\Delta\bar{m}(t_k) &= \Delta m(t_k) \leqslant -L_k m(t_k) - \frac{L_k t_k}{2\pi}\bigl[m(0) - m(2\pi)\bigr] \\
&= -L_k \bar{m}(t_k), \quad k = 1, 2, \ldots, p.
\end{aligned}
$$

In view of Case 1, we see that $\bar{m}(t) \leqslant 0$ for $t \in J$. Therefore $m(t) \leqslant 0$ for $t \in J$. Thus the proof of Lemma 2.2 is complete. □

Corollary 2.1. *Let* $\delta = \max\{t_k - t_{k-1}\colon\ k = 1, 2, \ldots, p+1\}$ (*where* $t_0 = 0$, $t_{p+1} = 2\pi$). *Assume that* $m \in \Omega$ *satisfies* (4), *and constants* $M > 0$, $N_1 > 0$, $N_2 > 0$, $0 \leqslant L_k < 1$ $(k = 1, 2, \ldots, p)$. *If*

$$M^{-1}(N_1 k_0 + N_2 h_0)\big(e^{4\pi M} - 1\big) \leqslant \frac{\{\prod_{k=1}^{p}(1 - L_k)\}^2}{1 + \sum_{n=1}^{p}\prod_{k=n}^{p}(1 - L_k)}, \tag{17}$$

then $m(t) \leqslant 0$ *for* $t \in J$.

Proof. Assume that inequality (17) holds, we have

$$\begin{aligned}\int_0^{2\pi} \prod_{s<t_k<2\pi}(1 - L_k)\,ds &= \sum_{n=0}^{p}\int_{t_n^+}^{t_{n+1}} \prod_{s<t_k<2\pi}(1 - L_k)\,ds \\ &= \sum_{n=1}^{p}\prod_{k=n}^{p}(1 - L_k)(t_n - t_{n-1}) + (t_{p+1} - t_p) \\ &\leqslant \delta\left\{1 + \sum_{n=1}^{p}\prod_{k=n}^{p}(1 - L_k)\right\}.\end{aligned} \tag{18}$$

Using (17) and (18), we see that inequality (5) holds. So, Lemma 2.2 yields that $m(t) \leqslant 0$ for $t \in J$. □

3. Linear periodic boundary value problems

Consider the following periodic boundary value problem for a linear impulsive integro-differential equation (PBVP):

$$\begin{cases} u'(t) + Mu(t) = -N_1[Tu](t) - N_2[Su](t) + \sigma(t), & t \neq t_k,\ t \in J, \\ \Delta u(t_k) = -L_k u(t_k) + I_k(\eta(t_k)) + L_k\eta(t_k), & k = 1, 2, \ldots, p, \\ u(0) = u(2\pi), \end{cases} \tag{19}$$

where constants $M > 0$, $N_1 > 0$, $N_2 > 0$, and $0 \leqslant L_k < 1$ $(k = 1, 2, \ldots, p)$, $I_k \in C(J, R)$ $(k = 1, 2, \ldots, p)$, $\sigma \in PC(J, R)$, and $\eta \in \Omega$.

Lemma 3.1. $u \in \Omega$ *is a solution of PBVP* (19) *if and only if* $u \in PC(J, R)$ *is a solution of the following impulsive integral equation*:

$$\begin{aligned} u(t) = {} & \int_0^{2\pi} G(t, s)\big\{\sigma(s) - N_1[Tu](s) - N_2[Su](s)\big\}\,ds \\ & + \sum_{0<t_k<2\pi} G(t, t_k)\big(-L_k u(t_k) + I_k\big(\eta(t_k)\big) + L_k\eta(t_k)\big), \quad t \in J, \end{aligned} \tag{20}$$

where

$$G(t,s)=\frac{1}{1-e^{-2\pi M}}\begin{cases} e^{-M(t-s)}, & 0\leqslant s<t\leqslant 2\pi,\\ e^{-M(2\pi+t-s)}, & 0\leqslant t\leqslant s\leqslant 2\pi.\end{cases}$$

Proof. Assume that $u\in\Omega$ is a solution of (19). By the variation of parameters formula, we get

$$u(t)=u(0)e^{-Mt}+\int_0^t e^{-M(t-s)}\bigl\{\sigma(s)-N_1[Tu](s)-N_2[Su](s)\bigr\}\,ds + \sum_{0<t_k<t} e^{-M(t-t_k)}\bigl(-L_k u(t_k)+I_k\bigl(\eta(t_k)\bigr)+L_k\eta(t_k)\bigr). \tag{21}$$

Setting $t=2\pi$ in (21) and using the boundary condition $u(0)=u(2\pi)$, we obtain

$$u(0)=\frac{1}{1-e^{-2\pi M}}\left\{\int_0^{2\pi} e^{-M(2\pi-s)}\bigl(\sigma(s)-N_1[Tu](s)-N_2[Su](s)\bigr)\,ds + \sum_{0<t_k<2\pi} e^{-M(2\pi-t_k)}\bigl(-L_k u(t_k)+I_k\bigl(\eta(t_k)\bigr)+L_k\eta(t_k)\bigr)\right\}. \tag{22}$$

Substituting (22) into (21), we see that $u\in PC(J,R)$ satisfies (20).

If $u\in PC(J,R)$ is a solution of (20), then $u\in C^1(J',R)$ and

$$\begin{cases} u'(t)+Mu(t)=-N_1[Tu](t)-N_2[Su](t)+\sigma(t), & t\neq t_k,\ t\in J,\\ \Delta u(t_k)=-L_k u(t_k)+I_k(\eta(t_k))+L_k\eta(t_k), & k=1,2,\dots,p.\end{cases}$$

Setting $t=0,\,2\pi$ in (20), respectively, we have

$$\begin{aligned} u(2\pi)&=\frac{1}{1-e^{-2\pi M}}\left\{\int_0^{2\pi} e^{-M(2\pi-s)}\bigl(\sigma(s)-N_1[Tu](s)-N_2[Su](s)\bigr)\,ds + \sum_{0<t_k<2\pi} e^{-M(2\pi-t_k)}\bigl(-L_k u(t_k)+I_k\bigl(\eta(t_k)\bigr)+L_k\eta(t_k)\bigr)\right\}\\ &=u(0).\end{aligned}$$

Therefore $u\in\Omega$ is a solution of (19). Thus Lemma 3.1 is proved. □

Lemma 3.2. *Assume that* $M>0$, $N_1>0$, $N_2>0$, *and* $0\leqslant L_k<1$ $(k=1,2,\dots,p)$, $I_k\in C(J,R)$ $(k=1,2,\dots,p)$, $\sigma\in PC(J,R)$, $\eta\in\Omega$, *and the following inequality holds*:

$$2\pi M^{-1}(N_1k_0+N_2h_0)+\frac{1}{1-e^{-2\pi M}}\sum_{k=1}^{p}L_k<1. \tag{23}$$

Then PBVP (19) *possesses a unique solution in* Ω.

Proof. For any $u \in \Omega$, consider the operator F defined by the formula

$$(Fu)(t) = \int_0^{2\pi} G(t,s)\big\{\sigma(s) - N_1[Tu](s) - N_2[Su](s)\big\}\,ds + \sum_{0<t_k<2\pi} G(t,t_k)\big(-L_k u(t_k) + I_k\big(\eta(t_k)\big) + L_k\eta(t_k)\big), \quad t \in J.$$

Then $Fu \in \Omega$, i.e., $F\Omega \subset \Omega$.

For every $u, v \in \Omega$, $t \in J$, we have

$$\begin{aligned}\big|(Fu)(t) - (Fv)(t)\big| &\leqslant \int_0^{2\pi} G(t,s)\big\{N_1\big|[Tu](s) - [Tv](s)\big| \\ &\quad + N_2\big|[Su](s) - [Sv](s)\big|\big\}\,ds \\ &\quad + \sum_{0<t_k<2\pi} G(t,t_k)L_k\big|u(t_k) - v(t_k)\big| \\ &\leqslant \left\{2\pi M^{-1}(N_1k_0 + N_2h_0) + \frac{1}{1-e^{-2\pi M}}\sum_{k=1}^{p} L_k\right\}\|u - v\|_{PC}.\end{aligned}$$

Hence

$$\|Fu - Fv\|_{PC} = \sup_{t\in J}\big|(Fu)(t) - (Fv)(t)\big| \leqslant \alpha\|u - v\|_{PC},$$

where

$$\alpha = 2\pi M^{-1}(N_1k_0 + N_2h_0) + \frac{1}{1-e^{-2\pi M}}\sum_{k=1}^{p} L_k < 1.$$

Thus the operator F is a contraction on Ω. That is, there is a unique element $u \in \Omega$ such that $u = Fu$. This u is the unique solution of PBVP (19). The proof of Lemma 3.2 is complete. □

4. Main result

Theorem 4.1. *Assume that conditions* (A_0)–(A_2) *hold and the inequalities* (5) *and* (23) *hold. Then, there exist monotone sequences* $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ *with* $\alpha_0 = \alpha$, $\beta_0 = \beta$ *such that* $\lim_{n\to\infty}\alpha_n(t) = \rho(t)$, $\lim_{n\to\infty}\beta_n(t) = r(t)$ *uniformly on* J, *and* ρ, r *are the minimal and the maximal solutions of PBVP* (1), *respectively, such that*

$$\alpha_0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_n \leqslant \rho \leqslant x \leqslant r \leqslant \beta_n \leqslant \cdots \leqslant \beta_2 \leqslant \beta_1 \leqslant \beta_0 \quad \text{on } J,$$

where x *is any solution of PBVP* (1) *such that* $\alpha(t) \leqslant x(t) \leqslant \beta(t)$ *on* J.

Proof. Let $[\alpha,\beta]=\{x\in\Omega\colon\ \alpha(t)\leqslant x(t)\leqslant\beta(t),\ t\in J\}$. For any $\eta\in[\alpha,\beta]$, consider PBVP (19) with

$$\sigma(t)=f\big(t,\eta(t),[T\eta](t),[S\eta](t)\big)+M\eta(t)+N_1[T\eta](t)+N_2[S\eta](t).$$

By Lemmas 3.1 and 3.2, PBVP (19) possesses a unique solution $u\in\Omega$. We define an operator A by $u=A\eta$, then the operator A has the following properties:

(i) $\alpha\leqslant A\alpha$, $A\beta\leqslant\beta$;
(ii) A is monotone nondecreasing in $[\alpha,\beta]$, i.e., for any $\eta_1,\eta_2\in[\alpha,\beta]$, $\eta_1\leqslant\eta_2$ implies $A\eta_1\leqslant A\eta_2$.

To prove (i), set $m=\alpha_0-\alpha_1$, where $\alpha_1=A\alpha_0$, then $m(0)-m(2\pi)=\alpha_0(0)-\alpha_0(2\pi)$ since $\alpha_1(0)=\alpha_1(2\pi)$, and

$$\begin{aligned}
m'(t)&=\alpha_0'(t)-\alpha_1'(t)\\
&\leqslant f\big(t,\alpha_0(t),[T\alpha_0](t),[S\alpha_0](t)\big)-r_{\alpha_0}\\
&\quad-\big\{-M\alpha_1(t)-N_1[T\alpha_1](t)-N_2[S\alpha_1](t)\\
&\quad+f\big(t,\alpha_0(t),[T\alpha_0](t),[S\alpha_0](t)\big)+M\alpha_0(t)+N_1[T\alpha_0](t)+N_2[S\alpha_0](t)\big\}\\
&=-Mm(t)-N_1[Tm](t)-N_2[Sm](t)-r_m,\quad t\neq t_k,\ t\in J,\\
\Delta m(t_k)&=\Delta\alpha_0(t_k)-\Delta\alpha_1(t_k)\\
&\leqslant I_k(\alpha_0(t_k))-l_{\alpha_0k}-\big[-L_k\alpha_1(t_k)+I_k\big(\alpha_0(t_k)\big)+L_k\alpha_0(t_k)\big]\\
&=-L_km(t_k)-l_{mk},\quad k=1,2,\dots,p,
\end{aligned}$$

where $r_{\alpha_0}, r_m, l_{\alpha_0k}, l_{mk}$ $(k=1,2,\dots,p)$ are given by

$$r_{\alpha_0}=r_m=\begin{cases}0, & \text{if } m(0)\leqslant m(2\pi),\\ \big(\frac{Mt+1}{2\pi}+\frac{N_1k_0t^2}{4\pi}+\pi N_2h_0\big)[m(0)-m(2\pi)], & \text{if } m(0)>m(2\pi),\end{cases}$$

$$l_{\alpha_0k}=l_{mk}=\begin{cases}0, & \text{if } m(0)\leqslant m(2\pi),\\ \frac{L_kt_k}{2\pi}[m(0)-m(2\pi)], & \text{if } m(0)>m(2\pi).\end{cases}$$

By Lemma 2.2, we get $m(t)\leqslant 0$ on J, i.e., $\alpha\leqslant A\alpha$. Similar arguments show that $A\beta\leqslant\beta$.

To prove (ii), let $\eta_1,\eta_2\in[\alpha,\beta]$ such that $\eta_1\leqslant\eta_2$ on J and set $m=u_1-u_2$, where $u_1=A\eta_1$, $u_2=A\eta_2$. Using (A_1), (A_2), and (19), we get

$$\begin{aligned}
m'(t)&=u_1'(t)-u_2'(t)\\
&=\big\{-Mu_1(t)-N_1[Tu_1](t)-N_2[Su_1](t)+f\big(t,\eta_1(t),[T\eta_1](t),[S\eta_1](t)\big)\\
&\quad+M\eta_1(t)+N_1[T\eta_1](t)+N_2[S\eta_1](t)\big\}\\
&\quad-\big\{-Mu_2(t)-N_1[Tu_2](t)-N_2[Su_2](t)\\
&\quad+f\big(t,\eta_2(t),[T\eta_2](t),[S\eta_2](t)\big)+M\eta_2(t)+N_1[T\eta_2](t)+N_2[S\eta_2](t)\big\}\\
&\leqslant-Mm(t)-N_1[Tm](t)-N_2[Sm](t),\quad t\neq t_k,\ t\in J,
\end{aligned}$$

$$\begin{aligned}\Delta m(t_k) = \Delta u_1(t_k) - \Delta u_2(t_k) &= \bigl[-L_k u_1(t_k) + I_k\bigl(\eta_1(t_k)\bigr) + L_k \eta_1(t_k)\bigr] \\ &\quad - \bigl[-L_k u_2(t_k) + I_k\bigl(\eta_2(t_k)\bigr) + L_k \eta_2(t_k)\bigr] \\ &\leqslant -L_k m(t_k), \quad k = 1, 2, \ldots, p,\end{aligned}$$

and it is clear that $m(0) = m(2\pi)$. In view of Lemma 2.2, we have $m(t) \leqslant 0$ on J, that is, $u_1 \leqslant u_2$ on J.

It is now easy to define the sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that $\alpha_{n+1} = A\alpha_n$, $\beta_{n+1} = A\beta_n$ $(n = 0, 1, 2, \ldots)$. From (i), (ii), we obtain

$$\alpha_0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_n \leqslant \cdots \leqslant \beta_n \leqslant \cdots \leqslant \beta_2 \leqslant \beta_1 \leqslant \beta_0 \quad \text{on } J,$$

and each $\alpha_n, \beta_n \in \Omega$ $(n = 1, 2, \ldots)$ satisfies

$$\begin{aligned}\alpha_n(t) &= \int_0^{2\pi} G(t,s)\bigl\{\sigma_{n-1}(s) - N_1[T\alpha_n](s) - N_2[S\alpha_n](s)\bigr\}\,ds \\ &\quad + \sum_{0<t_k<2\pi} G(t,t_k)\bigl(-L_k\alpha_n(t_k) + I_k\bigl(\alpha_{n-1}(t_k)\bigr) + L_k\alpha_{n-1}(t_k)\bigr), \quad t \in J, \\ \beta_n(t) &= \int_0^{2\pi} G(t,s)\bigl\{\bar{\sigma}_{n-1}(s) - N_1[T\beta_n](s) - N_2[S\beta_n](s)\bigr\}\,ds \\ &\quad + \sum_{0<t_k<2\pi} G(t,t_k)\bigl(-L_k\beta_n(t_k) + I_k\bigl(\beta_{n-1}(t_k)\bigr) + L_k\beta_{n-1}(t_k)\bigr), \quad t \in J,\end{aligned}$$

where

$$\begin{aligned}\sigma_{n-1}(t) &= f\bigl(t, \alpha_{n-1}(t), [T\alpha_{n-1}](t), [S\alpha_{n-1}](t)\bigr) + M\alpha_{n-1}(t) + N_1[T\alpha_{n-1}](t) \\ &\quad + N_2[S\alpha_{n-1}](t), \\ \bar{\sigma}_{n-1}(t) &= f\bigl(t, \beta_{n-1}(t), [T\beta_{n-1}](t), [S\beta_{n-1}](t)\bigr) + M\beta_{n-1}(t) + N_1[T\beta_{n-1}](t) \\ &\quad + N_2[S\beta_{n-1}](t).\end{aligned}$$

Therefore there exist ρ, r such that $\lim_{n\to\infty} \alpha_n(t) = \rho(t)$, $\lim_{n\to\infty} \beta_n(t) = r(t)$ uniformly on J. Clearly ρ, r satisfy PBVP (1). To prove that ρ, r are minimal and maximal solutions of PBVP (1), let $x(t)$ be any solution of PBVP (1) such that $x \in [\alpha, \beta]$. Suppose that there exists a positive integer n such that $\alpha_n(t) \leqslant x(t) \leqslant \beta_n(t)$ on J. Then, setting $m = \alpha_{n+1} - x$, we have

$$\begin{aligned}m'(t) &= \alpha'_{n+1}(t) - x'(t) \\ &= \bigl\{-M\alpha_{n+1}(t) - N_1[T\alpha_{n+1}](t) - N_2[S\alpha_{n+1}](t) \\ &\quad + f\bigl(t, \alpha_n(t), [T\alpha_n](t), [S\alpha_n](t)\bigr) \\ &\quad + M\alpha_n(t) + N_1[T\alpha_n](t) + N_2[S\alpha_n](t)\bigr\} - f\bigl(t, x(t), [Tx](t), [Sx](t)\bigr) \\ &\leqslant -Mm(t) - N_1[Tm](t) - N_2[Sm](t), \quad t \neq t_k,\ t \in J,\end{aligned}$$

$$\begin{aligned}\Delta m(t_k) &= \Delta\alpha_{n+1}(t_k) - \Delta x(t_k)\\ &= \big[-L_k\alpha_{n+1}(t_k) + I_k\big(\alpha_n(t_k)\big) + L_k\alpha_n(t_k)\big] - I_k\big(x(t_k)\big)\\ &\leqslant -L_k m(t_k), \quad k = 1, 2, \dots, p,\\ m(0) &= m(2\pi).\end{aligned}$$

By Lemma 2.2, it follows that $m(t) \leqslant 0$ on J, that is, $\alpha_{n+1}(t) \leqslant x(t)$ on J. Similarly, we obtain $x(t) \leqslant \beta_{n+1}(t)$ on J. Since $\alpha_0(t) \leqslant x(t) \leqslant \beta_0(t)$ on J, by induction we get $\alpha_n(t) \leqslant x(t) \leqslant \beta_n(t)$ on J for every n. Therefore, we obtain $\rho(t) \leqslant x(t) \leqslant r(t)$ on J by taking limit as $n \to \infty$. The proof of Theorem 4.1 is complete. □

References

[1] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

[2] D.D. Bainov, P.S. Simeonov, Impulsive Differential Equations: Periodic Solutions and Applications, Longman, Harlow, 1993.

[3] G.S. Ladde, V. Lakshmikantham, A.S. Vatsala, Monotone Iterative Techniques for Nonlinear Differential Equations, Pitman, London, 1985.

[4] G.S. Ladde, S. Sathanantham, Periodic boundary value problem for impulsive integro-differential equations of Volterra type, J. Math. Phys. Sci. 25 (1991) 119–129.

[5] X.Z. Liu, Iterative methods for solutions of impulsive functional differential systems, Appl. Anal. 44 (1992) 171–182.

[6] D.J. Guo, V. Lakshmikantham, X.Z. Liu, Nonlinear Integral Equations in Abstract Spaces, Kluwer Academic, London, 1996.

[7] X.Z. Liu, D.J. Guo, Periodic boundary value problems for impulsive integro-differential equations of mixed type in Banach spaces, Chinese Ann. Math. Ser. B 19 (1998) 517–528.

[8] Z.L. Wei, Solutions of periodic boundary value problems for first order nonlinear impulsive differential equations in Banach spaces, J. Systems Sci. Math. Sci. 19 (1999) 378–384.

[9] G.S. Hristova, D.D. Bainov, Monotone iterative techniques of V. Lakshmikantham for a boundary value problem for systems of impulsive differential-difference equations, J. Math. Anal. Appl. 197 (1996) 1–13.

[10] G.S. Hristova, D.D. Bainov, Application of the monotone iterative-techniques of V. Lakshmikantham for solving the initial value problem for impulsive differential-difference equations, J. Rocky Mountain Math. 23 (1993) 609–618.

[11] Y.B. Chen, W. Zhuang, On monotone iterative method for periodic boundary value problem for nonlinear integro-differential equations, Nonlinear Anal. 22 (1994) 295–303.