

Necessary and sufficient conditions for discrete and differential inclusions of elliptic type

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Abstract

This paper deals for the first time with the Dirichlet problem for discrete (P_D), discrete approximation problem on a uniform grid and differential (P_C) inclusions of elliptic type. In the form of Euler–Lagrange inclusion necessary and sufficient conditions for optimality are derived for the problems under consideration on the basis of new concepts of locally adjoint mappings. The results obtained are generalized to the multidimensional case with a second order elliptic operator.

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1. Introduction

The present paper is devoted to an investigation of problems described by so-called discrete and differential inclusions of elliptic type. The past decade has seen an ever more intensive development of the theory of extremal problems concerned by multivalued mappings with lumped and distributed parameters [4,5,7,9,16,26,28,33–38].

A lot of problems in economic dynamics, as well as classical problems on optimal control in vibrations, chemical, engineering, heat, diffusion processes, differential games, and so on, can be reduced to such investigations. We refer the reader to the survey papers [1–4,10–14,22,26,30,

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33,35–38]. Now let us explain the principal method that we use to obtain mentioned results. The present paper is divided conditionally into six parts.

In Section 2 first are given some suitable definitions and supplementary notions that constitute a certain method which facilitates obtaining necessary and sufficient conditions. Besides, adjoint and locally adjoint multivalued (LAM) functions are defined and the connection between them is established. Then a certain extremal Dirichlet's problem is formulated for elliptic discrete (P_D) and differential (P_C) inclusions with elliptic Laplace's operator.

In Section 3 for problem (P_D) we use one of the constructions of convex and nonsmooth analysis to get necessary and sufficient conditions for optimality. The latter can be reduced to finite-dimensional problems of mathematical programming, namely to minimization of functions on the intersection of the finite number of sets. In the reviewed results the arisen adjoint inclusions are stated in the Euler–Lagrange form [16,28,29]. These results turn out that this form automatically implies the Weierstrass–Pontryagin maximum condition. Note that it happens because the LAM is not the same as in [25,29,30]. Another definition of the LAM is introduced by Mordukhovich and is called coderivative of multifunctions at a given point [29]. Moreover, it appears that the use of the convex upper approximations (CUA) for nonconvex functions and locally tents [30] are very suitable to obtain the optimality conditions for posed problems. Observe that the main successful application of locally approximations and transition to convex approximations of sets is the establishment of necessary conditions for nonconvex optimization problems. In the field of different convex and nonconvex approximations of functions and sets the reader can also consult Clarke [6,7], Demianov [8], Frankowska [13], Mordukhovich [25,28,29], Pshenichnyi [30], Rockafellar [27,31] for related and additional material.

In Section 4 we use difference approximations of partial derivatives and grid functions on a uniform grid to approximate the Dirichlet problem for differential inclusions of elliptic type and to derive the necessary and sufficient conditions for optimality for the discrete-approximation problem. The latter is possible by passing to necessary conditions for an extremum of an discrete elliptic inclusions (P_D) in Section 3. It turns out that the concerned method requires some special equivalence theorems of a LAM, which arose in discrete and discrete approximation problems. These equivalence theorems that we proved allow us to make a bridge between problems (P_D) and (P_C). Obviously, such difference problems, in addition to being of independent interest, can play an important role also in computational procedures.

In Section 5 we are able to use the result in Section 4 to get sufficient conditions for optimality for differential inclusions of elliptic type. The derivation of sufficient conditions is implemented by passing to the formal limit as the discrete steps tend to zero. Of course, by using the suggested methods for ordinary differential inclusions of Mordukhovich [27,29] or Pshenichnyi [30] it can be proved that the obtained sufficient conditions are also necessary for optimality. At the end of Section 5 we consider linear optimal control problem of elliptic type. This example shows that in known problems the adjoint inclusion coincides with the adjoint equation which is traditionally obtained with the help of the Hamiltonian function.

In Section 6 the results obtained are generalized to the multi-dimensional case with a second order elliptic operator (P_M).

Some duality relations and optimality conditions for an extremum of different control problems with partial differential inclusions can be found in [2,4,5,9,17–21].

It must be pointed out that in elliptic differential inclusions the solution is taken in the space of classical solutions. However, as it will be seen from the context, the definition below of the concept of a solution in this or that sense is introduced only for simplicity and does not in any

way restrict the class of problems under consideration. Therefore, at the end of the paper we indicate general ways of extending the results to the case of generalized solutions [24].

2. Necessary concepts and problems statements

Let R^n be the n -dimensional Euclidean space, (u_1, u_2) is a pair of elements $u_1, u_2 \in R^n$ and $\langle u_1, u_2 \rangle$ is their inner product. A multivalued mapping $F: R^{4n} \rightarrow 2^{R^n}$ is convex if its graph $\text{gph } F = \{(u_1, u_2, u_3, u_4, v): v \in F(u_1, u_2, u_3, u_4)\}$ is a convex subset of R^{5n} . It is convex-valued if $F(u)$ is a convex set for each $u = (u_1, u_2, u_3, u_4) \in \text{dom } F = \{u: F(u) \neq \emptyset\}$. F is closed if $\text{gph } F$ is a closed set in R^{5n} .

Let us introduce the notations:

$$M(u, v^*) = \sup_v \{ \langle v, v^* \rangle : v \in F(u) \}, \quad v^* \in R^n,$$

$$F(u, v^*) = \{ v \in F(u) : \langle v, v^* \rangle = M(u, v^*) \}.$$

For convex F we let $M(u, v^*) = -\infty$ if $F(u) = \emptyset$. Let $\text{ri } A$ be the relative interior of a set $A \subset R^n$, i.e., the set of interior points of A with respect to its affine hull $\text{Aff } A$.

The cone $K_A(u_0)$ of tangent directions of the set A at a point $u_0 \in A$ is called a local tent [1] if for each $\bar{u}_0 \in \text{ri } K_A(u_0)$ there exists a convex cone $K \subseteq K_A(u_0)$ and a continuous mapping $\psi(\bar{u})$ defined in a neighborhood of the origin such that

- (1) $\bar{u}_0 \in \text{ri } K$, $\text{Lin } K = \text{Lin } K_A(u_0)$, where $\text{Lin } K$ is the linear span of K ,
- (2) $\psi(\bar{u}) = \bar{u} + r(\bar{u})$, $\|\bar{u}\|^{-1} r(\bar{u}) \rightarrow 0$,
- (3) $u_0 + \psi(\bar{u}) \in A$, $\bar{u} \in K \cap S_\varepsilon(0)$ for some $\varepsilon > 0$, where $S_\varepsilon(0)$ is the ball of radius ε and with center the origin.

For a convex mapping F at a point $(u^0, v^0) \in \text{gph } F$, $u^0 = (u_1^0, u_2^0, u_3^0, u_4^0) \in R^{4n}$, $v^0 \in R^n$,

$$\begin{aligned} K_{\text{gph } F}(u^0, v^0) &= \text{cone}(\text{gph } F - (u^0, v^0)) \\ &= \{ (\bar{u}, \bar{v}) : \bar{u} = \lambda(u - u^0), \bar{v} = \lambda(v - v^0), \lambda > 0, \forall (u, v) \in \text{gph } F \}. \end{aligned}$$

Definition 2.1. For a convex mapping F a multivalued mapping from R^n into R^{4n} defined by

$$F^*(v^*, (u, v)) = \{ u^* : (u^*, -v^*) \in K_{\text{gph } F}^*(u, v) \}$$

is called the locally adjoint mapping (LAM) to F at the point $(u, v) \in \text{gph } F$, where $K_{\text{gph } F}^*(u, v)$ is the dual to the cone $K_{\text{gph } F}(u, v)$.

We refer to [6,25,29,30] for various definitions in this direction. Note that in Definition 4.3 in [29] is used the normal cone construction and LAM to F is called the coderivative of F at a given point.

The function $h(\cdot, u)$ is called a convex upper approximation (CUA) of a function $g(\cdot): R^n \rightarrow R^1 \cup \{\pm\infty\}$ at every fixed point $u \in \text{dom } g = \{u: |g(u)| < +\infty\}$ if

- (1) $h(\bar{u}, u) \geq \Phi(\bar{u}, u)$ for all $\bar{u} \neq 0$, where

$$\Phi(\bar{u}, u) = \sup_{r(\cdot)} \limsup_{\lambda \downarrow 0} \frac{g(u + \lambda \bar{u} + r(\lambda)) - g(u)}{\lambda}.$$

Here the exterior supremum is taken on all $r(\cdot)$ such that $\lambda^{-1}r(\lambda) \rightarrow 0$, where $\lambda \downarrow 0$,
 (2) $h(\cdot, u)$ is a convex closed (lower semicontinuous) positive-homogeneous function.

Further, the set $\partial h(0, u) = \{u^* \in R^n: h(\bar{u}, u) \geq \langle \bar{u}, u^* \rangle, \forall \bar{u} \in R^n\}$ is called the subdifferential of g at a point u and is denoted by $\partial g(u)$. For convex functions continuous at u this definition coincides with the usual definition of subdifferential [14,30]. A function g is said to be proper if it does not take the value $-\infty$ and is not identically equal to $+\infty$.

Let us prove some supplementary results.

Lemma 2.1. *Let $F: R^n \rightarrow 2^{R^n}$ be a convex multivalued mapping. Then*

$$F^*(v^*, (u, v)) = \begin{cases} \partial_u M(u, v^*), & v \in F(u, v^*), \\ \emptyset, & v \notin F(u, v^*). \end{cases}$$

Proof. Note that for convex F support function M is concave on u and convex on v^* function if $F(u)$ is closed. So $\partial_u M(u, v^*) = -\partial_u [-M(u, v^*)]$ is a set of u^* such that

$$M(u_1, v^*) - M(u, v^*) \leq \langle u^*, u_1 - u \rangle \quad (2.1)$$

for all $u_1 \in R^n$. Let $u^* \in F^*(v^*, (u, v))$. By the definition of the dual cone $K_{\text{gph } F}^*(u, v)$ it means that

$$\langle \bar{u}, u^* \rangle - \langle \bar{v}, v^* \rangle \geq 0, \quad (\bar{u}, \bar{v}) \in K_{\text{gph } F}(u, v),$$

or

$$\langle u_1 - u, u^* \rangle - \langle v_1 - v, v^* \rangle \geq 0, \quad (u_1, v_1) \in \text{gph } F. \quad (2.2)$$

If $u_1 = u$, $v_1^* \in F(u)$, this inequality implies $\langle v, v^* \rangle \geq \langle v_1, v^* \rangle$ that is $v \in F(u, v^*)$ and $\langle v, v^* \rangle = M(u, v^*)$. Then it follows from (2.2) that

$$\langle v_1, v^* \rangle - M(u, v^*) \leq \langle u_1 - u, u^* \rangle.$$

The supremum on $v_1 \in F(u_1)$ gives us the inequality

$$M(u_1, v^*) - M(u, v^*) \leq \langle u^*, u_1 - u \rangle$$

or $u^* \in \partial_u M(u, v^*)$. Let now $u^* \in \partial_u M(u, v^*)$, $v \in F(u, v^*)$, then by going in the reverse direction, it is not hard to see that $u^* \in F^*(v^*, (u, v))$. This completes the proof of the lemma. \square

Definition 2.2. The following multivalued mapping defined by

$$F^*(v^*, (u^0, v^0)) = \{u^*: M(u, v^*) - M(u^0, v^*) \leq \langle u^*, u - u^0 \rangle, \forall (u, v) \in \text{gph } F\}, \\ v^0 \in F(u^0, v^*),$$

is called the LAM to nonconvex mapping F at a point $(u^0, v^0) \in \text{gph } F$.

It is clear that for convex F the function $M(\cdot, v^*)$ is concave and by Lemma 2.1 this definition of LAM coincides with the definition of LAM in convex case.

Lemma 2.2. *If for a convex multivalued mapping F the set $F(u)$ is closed, then*

$$\partial_{v^*} M(u, v^*) = F(u, v^*).$$

Proof. The proof of the lemma follows immediately from [30, Theorem 3.11] and so it is omitted. \square

Definition 2.3. Let $O^+ \text{gph } F$ be the recessive cone [31] to a convex mapping F , i.e.,

$$O^+ \text{gph } F = \{(\bar{u}, \bar{v}): (u + \lambda \bar{u}, v + \lambda \bar{v}) \in \text{gph } F, \lambda \geq 0, \forall (u, v) \in \text{gph } F\}.$$

Then multivalued mapping F^* defined by

$$F^*(u^*) = \{u^*: (u^*, -v^*) \in (O^+ \text{gph } F)^*\}$$

is called an adjoint to the convex F .

It is clear that if $\text{gph } F$ is a convex cone, then this definition coincides with the definition of Pshenichnyi [30]. By the standard way it can be proved that the following results are true.

Proposition 2.1. If $\text{gph } F$ is a convex and closed set in R^{2n} then

$$\bigcap_{(u,v) \in \text{gph } F} K_{\text{gph } F}(u, v) = O^+ \text{gph } F$$

holds.

Proposition 2.2. If $\text{gph } F$ is a convex closed set in R^{2n} then

$$\overline{\bigcup_{(u,v) \in \text{gph } F} K_{\text{gph } F}^*(u, v)} = (O^+ \text{gph } F)^*,$$

where the bar denotes closure.

Corollary 2.1. Let F be a convex closed mapping. Then the adjoint mapping and LAM to F are connected with the relation

$$F^*(v^*) = \overline{\bigcup_{(u,v) \in \text{gph } F} F^*(v^*, (u, v))}, \quad v \in F(u, v^*).$$

Proof. By Proposition 2.2, we obtain the required equality at once. It remains to observe only that for pair (u, v) , $v \notin F(u, v^*)$ by Lemma 2.1 $F^*(v^*, (u, v)) = \emptyset$. This completes the proof of the corollary. \square

Let us denote

$$H(u^*, v^*) = \inf\{\langle u, u^* \rangle - \langle v, v^* \rangle: (u, v) \in \text{gph } F\}.$$

It is clear that

$$H(u^*, v^*) = \inf_u \{\langle u, u^* \rangle - M(u, v^*)\}. \quad (2.3)$$

Corollary 2.2. u^* is an element of the LAM F^* , i.e., if and only if the equality $H(u^*, v^*) = \langle u, u^* \rangle - M(u, v^*)$ holds.

Proof. By Lemma 2.1, the inclusion $u^* \in F^*(v^*, (u, v))$ is equivalent to

$$u^* \in \partial_u M(u, v^*), \quad v \in F(u, v^*),$$

and so the inequality (2.1) holds. Rewriting (2.1) in the form

$$\langle u, u^* \rangle - M(u, v^*) \leq \langle u_1, u^* \rangle - M(u_1, v^*)$$

and taking infimum on u_1 , we have the relation

$$H(u^*, v^*) \geq \langle u, u^* \rangle - M(u, v^*).$$

Now comparing this inequality with the reverse inequality following from (2.3) ends the proof of the corollary. \square

Definition 2.4. Multivalued mapping F we call quasisuperlinear if its graph can be represented as $\text{gph } F = \Omega + K$ where Ω is a convex compactum, K is a convex closed cone.

Corollary 2.3. For a convex mapping F we have

$$\text{dom } H := \{(u^*, v^*): H(u^*, v^*) > -\infty\} \subseteq (O^+ \text{gph } F)^*.$$

In the case of quasisuperlinearity $\text{dom } H = K^*$.

Proof. Let assume the contrary: let $(u_0^*, v_0^*) \in \text{dom } H$, but $(u_0^*, v_0^*) \notin (O^+ \text{gph } F)^*$. It means that there exists a pair $(\bar{u}_0, \bar{v}_0) \in O^+ \text{gph } F$ for which $\langle \bar{u}_0, u_0^* \rangle - \langle \bar{v}_0, v_0^* \rangle < 0$. By Definition 2.3 we can write $(u + \lambda \bar{u}_0, v + \lambda \bar{v}_0) \in \text{gph } F$ for all $(u, v) \in \text{gph } F$ and $\lambda > 0$. Then

$$\begin{aligned} \langle u + \lambda \bar{u}_0, u_0^* \rangle - \langle v + \lambda \bar{v}_0, v_0^* \rangle &= \langle u, u_0^* \rangle - \langle v, v_0^* \rangle + \lambda [\langle \bar{u}_0, u_0^* \rangle - \langle \bar{v}_0, v_0^* \rangle] \rightarrow -\infty, \\ \text{when } \lambda &\rightarrow +\infty, \end{aligned}$$

obtained contradiction proves the first statement of the lemma. Furthermore, when F is quasisuperlinear, we get

$$\text{dom } H = \text{dom}(H_\Omega + H_K) = \text{dom } H_\Omega \cap \text{dom } H_K = \text{dom } H_K = K^*,$$

where

$$H_A(u^*, v^*) = \inf_{(u, v) \in A} \{\langle u, u^* \rangle - \langle v, v^* \rangle\}.$$

The lemma is proved. \square

The following example shows that the inverse inclusion generally is not true. In fact, let $F: R^1 \rightarrow 2^{R^1}$ is given as $F(u) = \{v: v \geq u^2\}$, $\text{gph } F = \{(u, v): v \geq u^2\}$. Obviously $O^+ \text{gph } F = \{0\} \times R^{1+}$, where R^{1+} is the positive ordinate. Therefore $(O^+ \text{gph } F)^* = \{(u^*, v^*): u^* \in R^1, v^* \in R^{1+}\}$. Then it is clear that $(1, 0) \notin \text{dom } H$, but $(1, 0) \in (O^+ \text{gph } F)^*$.

Corollary 2.4. Let F be a quasisuperlinear mapping and $M(., v^*)$ be a proper closed function. Then the duality relation

$$\inf_{u^* \in F^*(v^*)} \{\langle u, u^* \rangle - H(u^*, v^*)\} = \sup_{v \in F(u)} \langle v, v^* \rangle$$

holds.

Proof. By the previous Corollary 2.3, $\text{dom } H = K^*$ is correct. Therefore with regard to Theorem 4.4.III of [30] it is not hard to see that

$$\inf_{u^*} \{ \langle u, u^* \rangle - H(u^*, v^*) \} = \inf_{u^*} \{ \langle u, u^* \rangle - H_{\Omega}(u^*, v^*); u^* \in F^*(v^*) \} = \sup_{v \in F(u)} \langle v, v^* \rangle.$$

Remark 2.1. If $\text{gph } F$ is a convex cone, then $H(u^*, v^*) = 0$ for $u^* \in F(v^*)$ and so the equality

$$\inf_{u^* \in F^*(v^*)} \langle u, u^* \rangle = \sup_{v \in F(u)} \langle v, v^* \rangle$$

holds.

In the next section we consider the following optimization problem for discrete elliptic inclusions:

$$\text{minimize} \quad \sum_{x_1=1, \dots, T-1, x_2=1, \dots, L-1} g_{x_1, x_2}(u_{x_1, x_2}) \quad (2.4)$$

$$\text{subject to} \quad u_{x_1+1, x_2} \in F_{x_1, x_2}(u_{x_1-1, x_2}, u_{x_1, x_2-1}, u_{x_1, x_2}, u_{x_1, x_2+1}), \quad (2.5)$$

and

$$\begin{aligned} u_{x_1, 0} &= \alpha_{0x_1}, \quad u_{x_1, L} = \alpha_{Lx_1}, \quad u_{0, x_2} = \beta_{0x_2}, \quad u_{T, x_2} = \beta_{Tx_2}, \\ x_1 &= 1, \dots, T-1; \quad x_2 = 1, \dots, L-1, \end{aligned} \quad (2.6)$$

where $g_{x_1, x_2}: R^n \rightarrow R^1 \cup \{\pm\infty\}$ are functions taking values on the extended line, F_{x_1, x_2} are multivalued mappings, $F_{x_1, x_2}: R^{4n} \rightarrow 2^{R^n}$, and $\alpha_{0x_1}, \alpha_{Lx_1}, \beta_{0x_2}, \beta_{Tx_2}$ are fixed vectors, T, L are some natural numbers. We label this problem (P_D) and call it Dirichlet problem for discrete inclusion of elliptic type.

Let us denote $D = \{(x_1, x_2): x_1 = 0, \dots, T; x_2 = 0, \dots, L, (x_1, x_2) \neq (0, 0), (0, L), (T, 0), (T, L)\}$. Then a set of points $\{u_{x_1, x_2}\}_D = \{u_{x_1, x_2}: (x_1, x_2) \in D\}$ is called a feasible solution for the problem (P_D) if it satisfies the inclusions (2.6). It is easy to see that for each fixed T and L the boundary condition (2.6) enable us to choose some feasible solution, and the number of points to be determined and discrete inclusions are equal. In this sense the name “discrete elliptic inclusions” is justified. The following condition is assumed below for the functions g_{x_1, x_2} and the mappings F_{x_1, x_2} ($x_1 = 1, \dots, T-1; x_2 = 1, \dots, L-1$).

Hypothesis (H1). Assume that in the problem (P_D) the mappings F_{x_1, x_2} are such that the cone of tangent directions $K_{\text{gph } F_{x_1, x_2}}(\tilde{u}_{x_1-1, x_2}, \tilde{u}_{x_1, x_2-1}, \tilde{u}_{x_1, x_2}, \tilde{u}_{x_1, x_2+1}, \tilde{u}_{x_1+1, x_2})$ are local tents, where \tilde{u}_{x_1, x_2} are the points of the optimal solution $\{\tilde{u}_{x_1, x_2}\}_D$. Assume, moreover, that the functions g_{x_1, x_2} admit a CUA $h_{x_1, x_2}(\tilde{u}, \tilde{u}_{x_1, x_2})$ at the points \tilde{u}_{x_1, x_2} that is continuous with respect to \tilde{u} . The latter means that the subdifferentials $\partial g_{x_1, x_2}(\tilde{u}_{x_1, x_2}) := \partial h_{x_1, x_2}(0, \tilde{u}_{x_1, x_2})$ are defined.

The problem (P_D) is said to be convex if the mappings F_{x_1, x_2} are convex and g_{x_1, x_2} are convex proper functions.

Hypothesis (H2). Let the considered problem (P_D) is convex and $\{u_{x_1, x_2}^0\}_D$ is some feasible solution for it. Then suppose that

$$\begin{aligned} (u_{x_1-1, x_2}^0, u_{x_1, x_2-1}^0, u_{x_1, x_2}^0, u_{x_1, x_2+1}^0, u_{x_1+1, x_2}^0) &\in \text{ri gph } F_{x_1, x_2}, \\ u_{x_1, x_2}^0 &\in \text{ri dom } g_{x_1, x_2}, \quad x_1 = 1, \dots, T-1; \quad x_2 = 1, \dots, L-1. \end{aligned}$$

In Section 5 we study the following problem for elliptic differential inclusion:

$$\text{minimize } J(u(\cdot)) := \iint_R g(u(x), x) dx \quad (2.7)$$

$$\text{subject to } \Delta u(x) \in F(u(x), x), \quad x \in R, \quad (2.8)$$

and

$$u(x) = \beta(x), \quad x \in B, \quad (2.9)$$

where Δ is a Laplace's operator:

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2},$$

$F(\cdot, x): R^n \rightarrow 2^{R^n}$ is a multivalued mapping for all fixed $x = (x_1, x_2)$, R is a bounded region of R^2 , a closed piecewise-smooth simple curve B is the boundary of R , $g: R^n \times R \rightarrow R^1$ and β are continuous functions, $dx = dx_1 dx_2$.

We label this continuous problem (P_C) and call it Dirichlet problem for elliptic differential inclusions. The problem is to find a solution $\tilde{u}(\cdot)$ of the boundary value problem (2.8), (2.9) that minimizes $J(u(\cdot))$. Here, a feasible solution is understood to be a classical solution for simplicity of the exposition. At the end of Section 6 we introduce the concept of a generalized solution and show that it is possible to carry over the results obtained in this case.

The subject of the research in Section 6 is the following multidimensional optimal control problem (P_M) for elliptic differential inclusions:

$$\text{minimize } J(u(\cdot)) := \int_G g(u(x), x) dx \quad (2.10)$$

$$\text{subject to } Lu(x) \in F(u(x), x), \quad x \in G, \quad (2.11)$$

and

$$u(x) = \alpha(x), \quad x \in S, \quad (2.12)$$

where $F(\cdot, x): R^1 \rightarrow 2^{R^1}$ is a convex closed multivalued mapping for all n -dimensional vectors $x = (x_1, \dots, x_n)$, G is a bounded set of R^n , a closed piecewise-smooth surface S is the boundary of G , $g: R^1 \times G \rightarrow R^1$ is a continuous and convex on u function, α is continuous and $dx = dx_1 dx_2 \dots dx_n$. L is a second-order elliptic operator:

$$Lu := \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + c(x)u,$$

$$a_{ij}(x) \in C^1(\bar{G}), \quad b_i(x) \in C^1(\bar{G}), \quad c(x) \in C(\bar{G}),$$

where $\|a_{ij}(x)\|$ is a positively definite matrix, $C(\bar{G})$ and $C^1(\bar{G})$ are the spaces of continuous functions and functions having a continuous derivative in G , respectively.

A function $u(x)$ in $C^2(G) \cap C(\bar{G})$, that satisfies the inclusion (2.11) in G and the boundary condition (2.12) on S we call a classical solution of the problem posed, where $C^2(G)$ is the space of functions $u(\cdot)$ having continuous second-order derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$, $i, j = 1, \dots, n$. It is required to find a classical solution $\tilde{u}(\cdot)$ of the boundary value problem (P_M) that minimizes $J(u(\cdot))$.

3. Necessary and sufficient conditions for the Dirichlet problem of discrete elliptic inclusions

At first we consider the convex problem (P_D) .

Theorem 3.1. Assume that F_{x_1, x_2} , $x_1 = 1, \dots, T-1$, $x_2 = 1, \dots, L-1$ are convex multivalued mappings, and g_{x_1, x_2} are convex proper functions continuous at the points of some feasible solution $\{u_{x_1, x_2}^0\}_D$. Then for the $\{\tilde{u}_{x_1, x_2}\}_D$ to be an optimal solution of the problem (P_D) , it is necessary that there exist a number $\lambda = 0$ or 1 and vectors $\{\psi_{x_1, x_2}^*\}$, $\{\eta_{x_1, x_2}^*\}$, $\{\xi_{x_1, x_2}^*\}$, $\{u_{x_1, x_2}^*\}$ simultaneously not all equal to zero such that:

- (i) $(\psi_{x_1, x_2}^*, \xi_{x_1, x_2}^*, u_{x_1-1, x_2}^*, \eta_{x_1, x_2}^*)$
 $\in F_{x_1, x_2}^*(u_{x_1, x_2}^*, (\tilde{u}_{x_1-1, x_2}, \tilde{u}_{x_1, x_2-1}, \tilde{u}_{x_1, x_2}, \tilde{u}_{x_1, x_2+1}, \tilde{u}_{x_1+1, x_2}))$
 $+ \{0\} \times \{0\} \times \{\psi_{x_1+1, x_2}^* + \xi_{x_1, x_2+1}^* + \eta_{x_1, x_2-1}^* - \lambda \partial g_{x_1, x_2}(\tilde{u}_{x_1, x_2})\} \times \{0\};$
- (ii) $\psi_{0, x_2}^* = 0, \quad u_{T, x_2}^* = 0, \quad x_2 = 1, \dots, L-1;$
 $\eta_{x_1, 0}^* = 0, \quad \xi_{x_1, L}^* = 0, \quad x_1 = 1, \dots, T-1.$

Under Hypothesis (H2), the conditions (i) and (ii) are also sufficient for the optimality of $\{\tilde{u}_{x_1, x_2}\}_D$.

Proof. One of the essential points in the proofs is the use of convex programming results. With this goal we form the $(m = 2n(L-1) + n(T-1)(L+1))$ -dimensional vector $w = (u_0, u_1, \dots, u_T)$, where for $x_1 = 1, \dots, T-1$, $(u_{x_1} = (u_{x_1, 0}, u_{x_1, 1}, \dots, u_{x_1, L}) \in R^{n(L+1)})$ -dimensional vector and $u_0 = (u_{0, 1}, \dots, u_{0, L-1}) \in R^{n(L-1)}$, $u_T = (u_{T, 1}, \dots, u_{T, L-1}) \in R^{n(L-1)}$. Let us consider the following convex sets defined in the space R^m :

$$\begin{aligned} M_{x_1, x_2} &= \{w = (u_0, u_1, \dots, u_T): \\ &\quad (u_{x_1-1, x_2}, u_{x_1, x_2-1}, u_{x_1, x_2}, u_{x_1, x_2+1}, u_{x_1+1, x_2}) \in \text{gph } F_{x_1, x_2}\}, \\ x_1 &= 1, \dots, T-1, \quad x_2 = 1, \dots, L-1, \\ H_1 &= \{w = (u_0, \dots, u_T): u_{x_1, 0} = \alpha_{0x_1}, \quad x_1 = 1, \dots, T-1\}, \\ H_2 &= \{w = (u_0, \dots, u_T): u_{0, x_2} = \beta_{0x_2}, \quad x_2 = 1, \dots, L-1\}, \\ H_L &= \{w = (u_0, \dots, u_T): u_{x_1, L} = \alpha_{Lx_1}, \quad x_1 = 1, \dots, T-1\}, \\ H_T &= \{w = (u_0, \dots, u_T): u_{T, x_2} = \beta_{Tx_2}, \quad x_2 = 1, \dots, L-1\}. \end{aligned}$$

Now setting

$$g(w) = \sum_{x_1=1, \dots, T-1; x_2=1, \dots, L-1} g_{x_1, x_2}(u_{x_1, x_2}),$$

we can easily show that the convex problem (P_D) is equivalent to the following convex minimization problem in the space R^m :

$$g(w) \rightarrow \inf, \quad w \in N = \left(\bigcap_{\substack{x_1=1, \dots, T-1 \\ x_2=1, \dots, L-1}} M_{x_1, x_2} \right) \cap H_1 \cap H_2 \cap H_L \cap H_T. \quad (3.1)$$

Then we try to write out the necessary and sufficient conditions [15,18,23,25,30,34] for convex minimization problem (3.1). For this, it is necessary to calculate the dual cones $K_{M_{x_1,x_2}}^*(w)$, $K_{H_1}^*(w)$, $K_{H_2}^*(w)$, $K_{H_L}^*(w)$, $K_{H_T}^*(w)$, $w \in N$.

Lemma 3.1.

$$\begin{aligned} K_{M_{x_1,x_2}}^*(w) = \{w^* = (u_0^*, \dots, u_T^*): \\ (u_{x_1-1,x_2}^*, u_{x_1,x_2-1}^*, u_{x_1,x_2}^*, u_{x_1,x_2+1}^*, u_{x_1+1,x_2}^*) \\ \in K_{F_{x_1,x_2}}^*(u_{x_1-1,x_2}, u_{x_1,x_2-1}, u_{x_1,x_2}, u_{x_1,x_2+1}, u_{x_1+1,x_2}), u_{i,j}^* = 0, \\ (i,j) \neq (x_1-1, x_2), (x_1, x_2-1), (x_1, x_2), (x_1, x_2+1), (x_1+1, x_2)\} \\ x_1 = 1, \dots, T-1, x_2 = 1, \dots, L-1. \end{aligned}$$

Proof. Let $\bar{w} \in K_{M_{x_1,x_2}}^*(w)$, $w \in N$. This means that $w + \lambda \bar{w} \in M_{x_1,x_2}$ for sufficiently small $\lambda > 0$ or in other words,

$$\begin{aligned} (u_{x_1-1,x_2} + \lambda \bar{u}_{x_1-1,x_2}, u_{x_1,x_2-1} + \lambda \bar{u}_{x_1,x_2-1}, u_{x_1,x_2} + \lambda \bar{u}_{x_1,x_2}, u_{x_1,x_2+1} + \lambda \bar{u}_{x_1,x_2+1}, \\ u_{x_1+1,x_2} + \lambda \bar{u}_{x_1+1,x_2}) \in \text{gph } F_{x_1,x_2}. \end{aligned}$$

Thus

$$\begin{aligned} K_{M_{x_1,x_2}}(\bar{w}) = \{\bar{w} = (\bar{u}_0, \dots, \bar{u}_T): \\ (\bar{u}_{x_1-1,x_2}, \bar{u}_{x_1,x_2-1}, \bar{u}_{x_1,x_2}, \bar{u}_{x_1,x_2+1}, \bar{u}_{x_1+1,x_2}) \\ \in K_{F_{x_1,x_2}}(u_{x_1-1,x_2}, u_{x_1,x_2-1}, u_{x_1,x_2}, u_{x_1,x_2+1}, u_{x_1+1,x_2})\}. \end{aligned} \quad (3.2)$$

On the other hand, $w^* \in K_{M_{x_1,x_2}}^*(w)$ is equivalent to the condition

$$\langle \bar{w}, w^* \rangle = \sum_{\substack{x_1=1, \dots, T-1 \\ x_2=1, \dots, L-1}} \langle \bar{u}_{i,j}, u_{i,j}^* \rangle \geq 0, \quad \bar{w} \in K_{M_{x_1,x_2}}(\bar{w}),$$

where the components $\bar{u}_{i,j}$ of the vector \bar{w} (see (3.2)) are arbitrary. Therefore, the last relation is valid only for $u_{i,j}^* = 0$, $(i,j) \neq (x_1-1, x_2), (x_1, x_2-1), (x_1, x_2), (x_1, x_2+1), (x_1+1, x_2)$. This ends the proof of the lemma. \square

It is also easy to show that

$$\begin{aligned} K_{H_1}^*(w) &= \{w^* = (u_0^*, \dots, u_T^*): u_{x_1,x_2}^* = 0, x_1 = 1, \dots, T-1, x_2 \neq 0, u_0^* = u_T^* = 0\}, \\ K_{H_2}^*(w) &= \{w^* = (u_0^*, \dots, u_T^*): u_{x_1}^* = 0, x_1 = 1, \dots, T\}, \\ K_{H_L}^*(w) &= \{w^* = (u_0^*, \dots, u_T^*): u_{x_1,x_2}^* = 0, x_1 = 1, \dots, T-1, x_2 \neq L, u_0^* = u_T^* = 0\}, \\ K_{H_T}^*(w) &= \{w^* = (u_0^*, \dots, u_T^*): u_0^* = 0, u_{x_1}^* = 0, x_1 = 1, \dots, T-1\}. \end{aligned} \quad (3.3)$$

Further, by the hypothesis of the theorem $\tilde{w} = (\tilde{u}_0, \tilde{u}_1, \dots, \tilde{u}_T)$ is a solution of the convex minimization problem (3.1) and $g(\cdot)$ is continuous at the point $w^0 = (u_0^0, \dots, u_T^0)$. Then we can assert the existence of vectors

$$\begin{aligned} w^*(x_1, x_2) \in K_{M_{x_1,x_2}}^*(\tilde{w}), \quad \tilde{w}^* \in K_{H_1}^*(\tilde{w}), \quad \hat{w}^* \in K_{H_2}^*(\tilde{w}), \quad w^{L*} \in K_{H_L}^*(\tilde{w}), \\ w^{T*} \in K_{H_T}^*(\tilde{w}), \quad w^{0*} \in \partial_w g(\tilde{w}) \end{aligned}$$

and of a number $\lambda = 0$ or 1 not equal to zero simultaneously, such that

$$\sum_{\substack{x_1=1,\dots,T-1 \\ x_2=1,\dots,L-1}} w^*(x_1, x_2) + \bar{w}^* + \hat{w}^* + w^{L*} + w^{T*} = \lambda w^{0*}. \quad (3.4)$$

This equality plays a central role in the investigations to follow. Let $[w^*]_{x_1, x_2}$ denotes the components of the vector w^* for the given pair (x_1, x_2) . Then using Lemma 3.1 and the relations (3.3), we get

$$\begin{aligned} & \left[\sum_{\substack{x_1=1,\dots,T-1 \\ x_2=1,\dots,L-1}} w^*(x_1, x_2) + \bar{w}^* + \hat{w}^* + w^{L*} + w^{T*} \right]_{x_1, x_2} \\ &= \begin{cases} u_{0, x_2}^*(1, x_2) + \hat{u}_{0, x_2}^*, & x_1 = 0, x_2 = 1, \dots, L-1, \\ u_{T, x_2}^*(T-1, x_2) + u_{T, x_2}^{T*}, & x_1 = T, x_2 = 1, \dots, L-1, \\ u_{x_1, 0}^*(x_1, 1) + \bar{u}_{x_1, 0}^*, & x_1 = 1, \dots, T-1, x_2 = 0, \\ u_{x_1, L}^*(x_1, L-1) + u_{x_1, L}^{L*}, & x_1 = 1, \dots, T-1, x_2 = L, \end{cases} \end{aligned} \quad (3.5)$$

where it is taken into account that

$$[\hat{w}^*]_{0, x_2} = \hat{u}_{0, x_2}^*, \quad [w^{T*}]_{T, x_2} = u_{T, x_2}^{T*}, \quad [\bar{w}^*]_{x_1, 0} = \bar{u}_{x_1, 0}^*, \quad [w^{L*}]_{x_1, L} = u_{x_1, L}^{L*}.$$

Because of arbitrariness of vectors $\hat{u}_{0, x_2}^*, u_{T, x_2}^{T*}, x_2 = 1, \dots, L-1, \bar{u}_{x_1, 0}^*, u_{x_1, L}^{L*}, x_1 = 1, \dots, T-1$, it follows from the relations (3.4) and (3.5) that the equalities

$$\begin{aligned} u_{0, x_2}^*(1, x_2) + \hat{u}_{0, x_2}^* &= 0, & u_{T, x_2}^*(T-1, x_2) + u_{T, x_2}^{T*} &= 0, \\ u_{x_1, 0}^*(x_1, 1) + \bar{u}_{x_1, 0}^* &= 0, & u_{x_1, L}^*(x_1, L) + u_{x_1, L}^{L*} &= 0, \\ x_1 &= 1, \dots, T-1, & x_2 &= 1, \dots, L-1, \end{aligned}$$

always hold. Thus (3.4) implies

$$\begin{aligned} & u_{x_1, x_2}^*(x_1+1, x_2) + u_{x_1, x_2}^*(x_1, x_2+1) + u_{x_1, x_2}^*(x_1, x_2) + u_{x_1, x_2}^*(x_1, x_2-1) \\ & + u_{x_1, x_2}^*(x_1-1, x_2) = \lambda u_{x_1, x_2}^{0*}, \\ & u_{1, x_2}^*(0, x_2) = 0, \quad u_{x_1, 1}^*(x_1, 0) = 0, \quad u_{x_1, L-1}^*(x_1, L) = 0, \quad u_{T-1, x_2}^*(T, x_2) = 0, \\ & [w^{0*}]_{x_1, x_2} = u_{x_1, x_2}^{0*}, \quad x_1 = 1, \dots, T-1, \quad x_2 = 1, \dots, L-1. \end{aligned} \quad (3.6)$$

Using Lemma 3.1 and Definition 2.1 of a LAM, it can be concluded that

$$\begin{aligned} & (u_{x_1-1, x_2}^*(x_1, x_2), u_{x_1, x_2-1}^*(x_1, x_2), u_{x_1, x_2}^*(x_1, x_2), u_{x_1, x_2+1}^*(x_1, x_2)) \\ & \in F_{x_1, x_2}^*(-u_{x_1+1, x_2}^*(x_1, x_2), (\tilde{u}_{x_1-1, x_2}, \tilde{u}_{x_1, x_2-1}, \tilde{u}_{x_1, x_2}, \tilde{u}_{x_1, x_2+1}, \tilde{u}_{x_1+1, x_2})) \\ & x_1 = 1, \dots, T-1, \quad x_2 = 1, \dots, L-1. \end{aligned} \quad (3.7)$$

Then introducing the new notations

$$\begin{aligned} u_{x_1-1, x_2}^*(x_1, x_2) &= \psi_{x_1, x_2}^*, & u_{x_1, x_2-1}^*(x_1, x_2) &= \xi_{x_1, x_2}^*, & u_{x_1, x_2+1}^*(x_1, x_2) &= \eta_{x_1, x_2}^*, \\ -u_{x_1+1, x_2}^*(x_1, x_2) &= u_{x_1, x_2}^*, \end{aligned}$$

we see from (3.6) and (3.7) that the first part of the theorem is valid. On the other hand, it follows from Hypothesis (H2) that the representation (3.4) holds with parameter $\lambda = 1$ for the

point $w^{0*} \in \partial_w g(\tilde{w}) \cap K_N^*(\tilde{w})$. Hence the conditions (i) and (ii) are sufficient for optimality of $\{\tilde{u}_{x_1, x_2}\}_D$. This completes the proof of the theorem. \square

Now let us try to write the result of Theorem 3.1 in a more symmetrical form. Note that the function M defined in Section 2 is a convex on v^* if the set $F(u_1, u_2, u_3, u_4)$ is convex and closed. Then taking into account Lemmas 2.1 and 2.2, we obtain the following result.

Corollary 3.1. *Let the conditions of the previous Theorem 3.1 are satisfied and in addition $F(u_1, u_2, u_3, u_4)$ is a closed set for every (u_1, u_2, u_3, u_4) . Then for the optimality of $\{\tilde{u}_{x_1, x_2}\}_D$ it is necessary that there exist a number $\lambda = 0$ or 1 and vectors $\{\psi_{x_1, x_2}^*, \{\eta_{x_1, x_2}^*, \{\xi_{x_1, x_2}^*, \{u_{x_1, x_2}^*\}$ simultaneously not equal to zero, such that*

$$\begin{aligned} u_{x_1, x_2}^* &\in \partial_{v^*} M_{x_1, x_2}(\tilde{u}_{x_1-1, x_2}, \tilde{u}_{x_1, x_2-1}, \tilde{u}_{x_1, x_2}, \tilde{u}_{x_1, x_2+1}, u_{x_1, x_2}^*), \\ (\psi_{x_1, x_2}^*, \xi_{x_1, x_2}^*, u_{x_1-1, x_2}^*, \eta_{x_1, x_2}^*) &\in \partial_u M_{x_1, x_2}(\tilde{u}_{x_1-1, x_2}, \tilde{u}_{x_1, x_2-1}, \tilde{u}_{x_1, x_2}, \tilde{u}_{x_1, x_2+1}, u_{x_1, x_2}^*) \\ &\quad + \{0\} \times \{0\} \times \{\psi_{x_1+1, x_2}^* + \xi_{x_1, x_2+1}^* + \eta_{x_1, x_2-1}^* - \lambda \partial g_{x_1, x_2}(\tilde{u}_{x_1, x_2})\} \times \{0\}, \\ \psi_{0, x_2}^* &= 0, \quad u_{T, x_2}^* = 0, \quad x_2 = 1, \dots, L-1, \\ \eta_{x_1, 0}^* &= 0, \quad \xi_{x_1, L}^* = 0, \quad x_1 = 1, \dots, T-1. \end{aligned}$$

If Hypothesis (H2) is fulfilled the conditions (i), (ii) are sufficient for optimality.

Remark 3.1. If in convex problem (P_D) the functions and multi-functions are polyhedral then Hypothesis (H2) is superfluous.

Theorem 3.2. *Assume Hypothesis (H1) for the nonconvex problem (P_D) . Then for $\{\tilde{u}_{x_1, x_2}\}_D$ to be an optimal solution of this nonconvex problem (P_D) it is necessary that there exist a number $\lambda = 0$ or 1 and vectors $\{\psi_{x_1, x_2}^*, \{\eta_{x_1, x_2}^*, \{\xi_{x_1, x_2}^*, \{u_{x_1, x_2}^*\}$ simultaneously not all equal to zero, satisfying the conditions (i), (ii) of Theorem 3.1.*

Proof. In this case Hypothesis (H1) ensures the relation (3.4) for nonconvex problem (P_D) or (3.1). Therefore we get the necessary condition as in Theorem 3.1 by starting from the relation (3.4), written out for the nonconvex problem (P_D) . \square

Remark 3.2. Suppose D_1 is a set of pairs (x_1, x_2) consisting of the integer numbers x_1 and x_2 . Then the set of interior points of D_1 for which the points of the form $(x_1 \pm 1, x_2)$ and $(x_1, x_2 \pm 1)$ belong to this set which we denote by D . And let D has a connectivity property that is all points of D can be connected with some zigzag whose segments are parallel either to axes $0x_1$ or $0x_2$. Moreover, assume that Γ is the set of boundary points of D so that $D_1 = D \cup \Gamma$. Now, instead of (2.6) we consider the following condition:

$$u_{x_1, x_2} = \alpha_{x_1, x_2}, \quad (x_1, x_2) \in \Gamma, \quad (3.8)$$

where α_{x_1, x_2} are a fixed vectors for every (x_1, x_2) . It is understood that for every point belonging to Γ there exists some interior point $(x_1, x_2) \in D$ for which the given boundary point is one of the form $(x_1, x_2 \pm 1)$, $(x_1 \pm 1, x_2)$. In this case the set of points of the form $(x_1 + 1, x_2)$, $(x_1 - 1, x_2)$, $(x_1, x_2 + 1)$, $(x_1, x_2 - 1)$ we call “right”, “left”, “upper”, “lower” sets respectively and denote Q_r, Q_{le}, Q_u, Q_{lo} . Obviously $\Gamma = Q_r \cup Q_{le} \cup Q_u \cup Q_{lo}$.

Then by analogue it can be shown that for problems (2.4), (2.5) and (3.8) the boundary condition (ii) of Theorem 3.1 consists of the following:

$$\begin{aligned}\psi_{x_1, x_2}^* &= 0, & (x_1, x_2) \in Q_{le}; & \quad \eta_{x_1, x_2}^* = 0, & (x_1, x_2) \in Q_{lo}; \\ \xi_{x_1, x_2}^* &= 0, & (x_1, x_2) \in Q_u; & \quad u_{x_1, x_2}^* = 0, & (x_1, x_2) \in Q_r.\end{aligned}$$

4. Approximation of the continuous problem and necessary condition for the discrete approximation problem

In this section we use difference derivatives to approximate the problem (P_C) and with help of Theorems 3.1 and 3.2 we formulate a necessary (and sufficient in the convex case) condition for it. We choose steps δ and h on the x_1 and x_2 axes, respectively, using the grid functions $u_{x_1, x_2} = u_{\delta h}(x_1, x_2)$ on a uniform grid on R .

Let $\Delta u = A_1 u + A_2 u$, where $A_i u = \partial^2 u / \partial x_i^2$ ($i = 1, 2$). We introduce the following difference operators, defined on the three-point models [32], i.e., each of the operators $A_1 u, A_2 u$ we approximate with the $\tilde{A}_1 u$ and $\tilde{A}_2 u$:

$$\begin{aligned}\tilde{A}_1 u(x) &:= \frac{u(x_1 + \delta, x_2) - 2u(x_1, x_2) + u(x_1 - \delta, x_2)}{\delta^2}, \\ \tilde{A}_2 u(x) &:= \frac{u(x_1, x_2 + h) - 2u(x_1, x_2) + u(x_1, x_2 - h)}{h^2}.\end{aligned}$$

The point (x_1, x_2) is called regular [32] if the four points $(x_1 \pm \delta, x_2), (x_1, x_2 \pm h)$ belong to $\bar{R} = R \cup B$. Otherwise the point (x_1, x_2) is nonregular.

The set of regular knot points are denoted by $\omega_{\delta h}^\circ$ and nonregular points by $\omega_{\delta h}^*$. The set of intersection of lines $x_1 = i\delta, x_2 = jh, i, j = 0, \pm 1, \pm 2, \pm 3, \dots$, and arc B are called boundary knot points and denoted by $\gamma_{\delta h}$. Thus according to the set \bar{R} , we have grid $\bar{\omega}_{\delta h} = \omega_{\delta h}^\circ \cup \omega_{\delta h}^* \cup \gamma_{\delta h}$. Assume that $\bar{\omega}_{\delta h}$ is a connected set. According to (2.3), we have $u_{\delta h}(x_1, x_2) = \beta(x_1, x_2), (x_1, x_2) \in \gamma_{\delta h}$.

For nonregular knot points, there are different conditions. For such points we use the value $\beta(\bar{x})$ of the function β where $\bar{x} \in \gamma_{\delta h}$ is a closest knot point for a given nonregular point $u(x) = u_{\delta h}(x) = \beta(\bar{x}), x \in \omega_{\delta h}^*$.

Now with respect to the problem (P_C) , we associate the following difference boundary value problem approximating it:

$$\begin{aligned}(P_A) \quad & \text{minimize} \quad J_{\delta h}(u(x_1, x_2)) := \sum_{(x_1, x_2) \in \bar{\omega}_{\delta h}} \delta h g(u(x_1, x_2), x_1, x_2) \\ & \text{subject to} \quad \tilde{A}_1 u(x) + \tilde{A}_2 u(x) \in F(u(x), x), \quad x = (x_1, x_2) \in \bar{\omega}_{\delta h}, \\ & \quad \text{and} \quad u(x) = \beta(x), \quad x \in \gamma_{\delta h}.\end{aligned}$$

At first for simplicity assume that (P_A) is a discrete approximation problem for problem (P_C) , where $R = (0, 1) \times (0, 1)$ so that

$$\bar{\omega}_{\delta h} = \{(x_1, x_2): x_1 = 0, \delta, \dots, 1; x_2 = 0, h, \dots, 1, (x_1, x_2) \neq (0, 0), (0, 1), (1, 0), (1, 1)\}.$$

Now we reduce the problem (P_A) to a problem of the form (P_D) . To this end we introduce a new mapping $Q(., x): R^{4n} \rightarrow 2^{R^n}$:

$$Q(u_1, u_2, u_3, u_4, x) := 2(1 + \theta)u_3 - u_1 - \theta(u_4 + u_2) + \delta^2 F(u_3, x), \quad \theta = \frac{\delta^2}{h^2}, \quad (4.1)$$

and we rewrite the problem (P_A) as follows:

$$\text{minimize } J_{\delta h}(u(.,.)), \quad (4.2)$$

$$\begin{aligned} \text{subject to } & u(x_1 + \delta, x_2) \in Q(u(x_1 - \delta, x_2), u(x_1, x_2 - h), u(x_1, x_2), u(x_1, x_2 + h), x_1, x_2) \\ & (x_1, x_2) \in \bar{\omega}_{\delta h}, \quad u(x_1, x_2) = \beta(x_1, x_2), \quad (x_1, x_2) \in \gamma_{\delta h}. \end{aligned} \quad (4.3)$$

By Theorem 3.1 for optimality of the feasible solution $\{\tilde{u}(x_1, x_2)\}$, $(x_1, x_2) \in \bar{\omega}_{\delta h}$, in problem (4.2), (4.3) it is necessary that there exist vectors $\{u^*(x_1, x_2)\}$, $\{\psi^*(x_1, x_2)\}$, $\{\xi^*(x_1, x_2)\}$, $\{\eta^*(x_1, x_2)\}$ and a number $\lambda = \lambda_{\delta h} \in \{0, 1\}$, not all zero, such that

$$\begin{aligned} & (\psi^*(x_1, x_2), \xi^*(x_1, x_2), u^*(x_1 - \delta, x_2), \eta^*(x_1, x_2)) \\ & \in Q^*(u^*(x_1, x_2), (\tilde{u}(x_1 - \delta, x_2), \tilde{u}(x_1, x_2 - h), \tilde{u}(x_1, x_2), \tilde{u}(x_1, x_2 + h), \\ & \quad \tilde{u}(x_1 + \delta, x_2), x_1, x_2)) + \{0\} \times \{0\} \\ & \quad \times \{\psi^*(x_1 + \delta, x_2) + \xi^*(x_1, x_2 + h) + \eta^*(x_1, x_2 - h) - \lambda \partial g(\tilde{u}(x_1, x_2), x_1, x_2)\} \times \{0\}, \\ & \psi^*(0, x_2) = 0, \quad u^*(1, x_2) = 0, \quad x_2 = h, \dots, 1 - h, \\ & \eta^*(x_1, 0) = 0, \quad \xi^*(x_1, 1) = 0, \quad x_1 = \delta, \dots, 1 - \delta. \end{aligned} \quad (4.4)$$

The main problem in (4.4) is to express LAM Q^* in terms of F^* .

Theorem 4.1. Let $Q(., x)$ be a multivalued mapping such that the cone of tangent directions $K_{\text{gph } Q(., x)}(u_1, u_2, u_3, u_4, v)$, $(u_1, u_2, u_3, u_4, v) \in \text{gph } Q(., x)$ is a local tent. Then

$$K_{\text{gph } Q(., x)}\left(u_3, \frac{v + u_1 + \theta(u_2 + u_4) - 2(1 + \theta)u_3}{\delta^2}\right)$$

is a locally tent to $\text{gph } F(., x)$ and the following inclusions are equivalent:

$$\begin{aligned} (a) \quad & (\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4) \in K_{\text{gph } Q(., x)}(u_1, u_2, u_3, u_4, v), \\ (b) \quad & \left(\bar{u}_3, \frac{\bar{v} + \bar{u}_1 + \theta(\bar{u}_2 + \bar{u}_4) - 2(1 + \theta)\bar{u}_3}{\delta^2}\right) \\ & \in K_{\text{gph } F(., x)}\left(u_3, \frac{v + u_1 + \theta(u_2 + u_4) - 2(1 + \theta)u_3}{\delta^2}\right). \end{aligned}$$

Proof. By the definition of a local tent there exist functions $r_i(\bar{z})$, $i = 0, 1, 2, 3, 4$, $\bar{z} = (\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{v})$ such that $r_i(\bar{z})\|\bar{z}\|^{-1} \rightarrow 0$ as $\bar{z} \rightarrow 0$ and

$$\begin{aligned} v + \bar{v} + r_0(\bar{z}) & \in 2(1 + \theta)(u_3 + \bar{u}_3 + r_3(\bar{z})) - u_1 - \bar{u}_1 - r_1(\bar{z}) \\ & \quad - \theta(u_4 + u_2 + \bar{u}_4 + \bar{u}_2 + r_4(\bar{z}) + r_2(\bar{z})) + \delta^2 F(u_3 + \bar{u}_3 + r_3(\bar{z}), x) \end{aligned}$$

for sufficiently small $\bar{z} \in K$, where $K \subseteq \text{ri } K_{\text{gph } Q(., x)}(z)$ is a convex cone.

Transforming this inclusion, we get

$$\begin{aligned} & \frac{v - 2(1 + \theta)u_3 + u_1 + \theta(u_2 + u_4)}{\delta^2} + \frac{\bar{v} - 2(1 + \theta)\bar{u}_3 + \bar{u}_1 + \theta(\bar{u}_2 + \bar{u}_4)}{\delta^2} \\ & \quad + \frac{r_0(\bar{z}) - 2(1 + \theta)r_3(\bar{z}) + r_1(\bar{z}) + \theta(r_4(\bar{z}) + r_2(\bar{z}))}{\delta^2} \in F(u_3 + \bar{u}_3 + r_3(\bar{z}), x). \end{aligned}$$

From this relation it is clear that

$$\left(\bar{u}_3, \frac{\bar{v} + \bar{u}_1 + \theta(\bar{u}_2 + \bar{u}_4) - 2(1 + \theta)\bar{u}_3}{\delta^2} \right) \in K_{\text{gph } F(.,x)} \left(u_3, \frac{v + u_1 + \theta(u_2 + u_4) - 2(1 + \theta)u_3}{\delta^2} \right). \quad (4.5)$$

By going in the reverse direction, it is also not hard to see from (4.5) that

$$(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4) \in K_{\text{gph } Q(.,x)}(u_1, u_2, u_3, u_4, v). \quad (4.6)$$

Therefore (4.5) and (4.6) are equivalent. \square

In what follows the next theorem is very important.

Theorem 4.2. Assume that the mapping $Q(.,x)$ is such that the cones of tangent directions $K_{\text{gph } Q(.,x)}(u_1, u_2, u_3, u_4, v)$ determine a local tent. Then the following inclusions are equivalent under the conditions that $v^* + u_1^* = 0$, $u_2^* = u_4^* = -\theta v^*$:

- (a) $(u_1^*, u_2^*, u_3^*, u_4^*) \in Q^*(v^*, (u_1, u_2, u_3, u_4, v), x)$,
- (b) $\frac{u_3^* - 2(1 + \theta)v^*}{\delta^2} \in F^*\left(v^*, \left(u_3, \frac{v + u_1 + \theta(u_2 + u_4) - 2(1 + \theta)u_3}{\delta^2}\right), x\right)$.

Proof. Suppose that the condition (a) is fulfilled. On a definition of LAM it means that in the case of (4.6),

$$\langle \bar{u}_1, u_1^* \rangle + \langle \bar{u}_2, u_2^* \rangle + \langle \bar{u}_3, u_3^* \rangle + \langle \bar{u}_4, u_4^* \rangle - \langle \bar{v}, v^* \rangle \geq 0. \quad (4.7)$$

Let us rewrite the inequality (4.7) in the form:

$$\langle \bar{u}_3, \psi_3^* \rangle - \left\langle \frac{\bar{v} + \bar{u}_1 + \theta(\bar{u}_2 + \bar{u}_4) - 2(1 + \theta)\bar{u}_3}{\delta^2}, \psi^* \right\rangle \geq 0, \quad (4.8)$$

where it is taken into account that the inclusions (4.5) and (4.6) are equivalent. Here ψ_3^* and ψ^* are to be determined. Carrying out the necessary transformations in (4.8) and comparing it with (4.7) it is not hard to see that

$$\psi^* = v^*, \quad -\psi^* = u_1^*, \quad -\theta\psi^* = u_2^*, \quad -\theta\psi^* = u_4^*, \quad \delta^2\psi_3^* + 2(1 + \theta)\psi^* = u_3^*.$$

These equalities imply that $v^* + u_1^* = 0$, $u_2^* = u_4^* = -\theta v^*$ and $\psi_3^* = (u_3^* - 2(1 + \theta)v^*)/\delta^2$. Then from Theorem 4.1, we see the accuracy of the inclusion (b), i.e., (a) \Rightarrow (b). By analogy it can be shown that (b) \Rightarrow (a). This ends the proof of the theorem. \square

Remark 4.1. If the mapping $F(.,x)$ is a convex, then using Lemma 2.1, Theorem 4.2 can be proved by another way, namely by calculating the subdifferential $\partial_u M(u, v^*, x)$, $u = (u_1, u_2, u_3, u_4)$ and expressing it via subdifferential of support function of mapping (4.1).

Let us return to the inclusion in (4.5). By equivalence Theorem 4.2, this condition has the form

$$\frac{u^*(x_1 - \delta, x_2) - \psi^*(x_1 + \delta, x_2) - \xi^*(x_1, x_2 + h) - \eta^*(x_1, x_2 - h) - 2(1 + \theta)u^*(x_1, x_2)}{\delta^2} \in F^*\left(u^*(x_1, x_2), (\tilde{u}(x_1, x_2), \tilde{A}_1 \tilde{u}(x_1, x_2) + \tilde{A}_2 \tilde{u}(x_1, x_2)), x_1, x_2\right) - \lambda \partial g(\tilde{u}(x_1, x_2), x_1, x_2), \quad (4.9)$$

$$u^*(x_1, x_2) = -\psi^*(x_1, x_2), \quad \xi^*(x_1, x_2) = \eta^*(x_1, x_2) = -\theta u^*(x_1, x_2),$$

$$\theta = \frac{\delta^2}{h^2}, \quad x_1 = \delta, 2\delta, \dots, 1 - \delta, \quad x_2 = h, 2h, \dots, 1 - h. \quad (4.10)$$

Further, using (4.10) it is not hard to verify that left side of the inclusion (4.9) has the form

$$\begin{aligned} & \frac{1}{\delta^2} [u^*(x_1 - \delta, x_2) + u^*(x_1 + \delta, x_2) + \theta(u^*(x_1, x_2 + h) + u^*(x_1, x_2 - h)) \\ & \quad - 2(1 + \theta)u^*(x_1, x_2)] \\ & = \frac{u^*(x_1 + \delta, x_2) - 2u^*(x_1, x_2) + u^*(x_1 - \delta, x_2)}{\delta^2} \\ & \quad + \frac{u^*(x_1, x_2 + h) - 2u^*(x_1, x_2) + u^*(x_1, x_2 - h)}{h^2}. \end{aligned} \quad (4.11)$$

On the other hand, from the boundary conditions in (4.5) and from (4.10) we obtain

$$\begin{aligned} u^*(x_1, 0) = 0, \quad u^*(x_1, 1) = 0, \quad x_1 = \delta, \dots, 1 - \delta, \\ u^*(0, x_2) = 0, \quad u^*(1, x_2) = 0, \quad x_2 = h, 2h, \dots, 1 - h. \end{aligned} \quad (4.12)$$

Taking into account the relations (4.9) and (4.11), (4.12), we can formulate the following result for problem (P_A) .

Theorem 4.3. Suppose $g(., x)$ is a convex proper function and continuous at the points on some feasible solution $\{u^0(x)\}$, $x \in \bar{\omega}_{\delta h}$. Then for the optimality of the solution $\{\tilde{u}(x)\}$ in the convex problem (P_A) it is necessary that there exist a number $\lambda = \lambda_{\delta h} \in \{0, 1\}$ and grid functions $\{u^*(x)\}$, $x \in \bar{\omega}_{\delta h}$, simultaneously not equal to zero such as:

- (i) $\tilde{A}_1 u^*(x) + \tilde{A}_2 u^*(x) \in F^*(u^*(x), (\tilde{u}(x), \tilde{A}_1 \tilde{u}(x) + \tilde{A}_2 \tilde{u}(x)), x) - \lambda \partial g(\tilde{u}(x), x)$.
- (ii) $u^*(x_1, 0) = u^*(x_1, 1) = 0, \quad x_1 = \delta, \dots, 1 - \delta,$
 $u^*(0, x_2) = u^*(1, x_2), \quad x_2 = h, \dots, 1 - h.$

Under the condition (H2), these conditions are also sufficient for the optimality of $\{\tilde{u}(x)\}$, $x \in \bar{\omega}_{\delta h}$.

Remark 4.2. As in Theorem 3.1 the conditions (i), (ii) of Theorem 4.3 are necessary for optimality in the nonconvex case of the problem (P_A) under Hypothesis (H1).

Remark 4.3. Observe that for problem (P_C) with nonsquare region R the boundary condition (ii) of Theorem 4.3 for boundary points consist of the following: $u^*(x) = 0$, $x \in \gamma_{\delta h} \subset B$.

5. Sufficient conditions for optimality for differential inclusions of elliptic type

Using results in Section 4, we formulate a sufficient condition of optimality of the continuous problem (P_C) . Therefore, let us pass to the formal limit in condition (i) of Theorem 4.3 and in the boundary condition (see Remark 4.3) as $\delta, h \rightarrow 0$ and set $\lambda = 1$. Then we have

- (a) $\Delta u^*(x) \in F^*(u^*(x), (\tilde{u}(x), \Delta \tilde{u}(x)), x) - \partial g(\tilde{u}(x), x), \quad x = (x_1, x_2) \in R,$
- (b) $u^*(x) = 0, \quad x \in B.$

Along with this we get the following condition (c) ensuring that the LAM $F^*(., ., x)$ is non-empty for every fixed $x \in R$ (see Lemma 2.1 in Section 2):

$$(c) \quad \Delta \tilde{u}(x) \in F(\tilde{u}(x), u^*(x), x).$$

The arguments in Section 4 guarantee the sufficiency of the conditions (a)–(c) for optimality. It turns out that the following assertion is true.

Theorem 5.1. *Assume that a continuous function g is convex with respect to u , and $F(., ., x)$ is a convex mapping for all fixed x . Then for the optimality of the solution $\tilde{u}(\cdot)$ among all feasible solutions in convex problem (P_C) it is sufficient that there exist a classical solution $u^*(\cdot)$ such that the conditions (a)–(c) hold.*

Proof. By Lemma 2.1 of Section 2,

$$F^*(v^*, (u, v), x) = \partial_u M(u, v^*, x), \quad v \in F(u, v^*, x).$$

Then applying the Moreau–Rockafellar theorem [14,30,31] and the fact that $-\partial g(., x) = \partial(-g(., x))$ from condition (a) we obtain

$$\Delta u^*(x) \in \partial_u [M(\tilde{u}(x), u^*(x), x) - g(\tilde{u}(x), x)], \quad x \in R,$$

or

$$\begin{aligned} M(u(x), u^*(x), x) - M(\tilde{u}(x), u^*(x), x) - g(u(x), x) + g(\tilde{u}(x), x) \\ \leq \langle \Delta u^*(x), u(x) - \tilde{u}(x) \rangle. \end{aligned}$$

Now taking into account the condition (c) of Theorem 5.1, definition of a function M and by integrating both sides of the last inequality over the domain R , we get

$$\begin{aligned} \int_R [g(u(x), x) - g(\tilde{u}(x), x)] dx - \int_R \langle \Delta(u(x) - \tilde{u}(x)), u^*(x) \rangle dx \\ + \int_R \langle u(x) - \tilde{u}(x), \Delta u^*(x) \rangle dx \geq 0. \end{aligned} \quad (5.1)$$

On the other hand, by the familiar Green's theorem [24, 32] we have

$$\begin{aligned} \int_R [\langle u(x) - \tilde{u}(x), \Delta u^*(x) \rangle - \langle \Delta(u(x) - \tilde{u}(x)), u^*(x) \rangle] dx \\ = \int_B \left[\left\langle u(x) - \tilde{u}(x), \frac{\partial u^*(x)}{\partial n} \right\rangle - \left\langle \frac{\partial(u(x) - \tilde{u}(x))}{\partial n}, u^*(x) \right\rangle \right] ds, \end{aligned} \quad (5.2)$$

where ds is a symbolic arc length element and n is the other normal for a curve B .

Since $u(\cdot)$ and $\tilde{u}(\cdot)$ are feasible solutions, that is $u(x) = \tilde{u}(x) = \beta(x)$, $x \in B$ and the condition (b) of theorem is fulfilled the integral (5.2) is equal to zero. Therefore from inequality (5.1) it follows that

$$\int_R g(u(x), x) dx \geq \int_R g(\tilde{u}(x), x) dx$$

for arbitrarily feasible solutions $u(\cdot)$. The theorem is proved. \square

Corollary 5.1. *In addition to assumptions of Theorem 5.1 let $F(\cdot, x)$ be a closed mapping. Then the conditions (a), (c) of Theorem 5.1 can be rewritten as follows:*

- (i) $\Delta u^*(x) \in \partial_u M(\tilde{u}(x), u^*(x), x) - \partial g(\tilde{u}(x), x)$,
- (ii) $\Delta \tilde{u}(x) \in \partial_{v^*} M(\tilde{u}(x), u^*(x), x)$.

Proof. In fact, on one hand by Lemma 2.1 the following equality is correct:

$$F^*(v^*, (u, v), x) = \partial_u M(u, v^*, x), \quad v \in F(u, v^*, x).$$

On the other hand, by Lemma 2.2, we have

$$\partial_{v^*} M(u, v^*, x) = F(u, v^*, x).$$

Therefore (i), (ii) are equivalent to the conditions (a), (c) of Theorem 5.1. \square

Remark 5.1. It follows from the condition (ii) of Corollary 5.1 and the condition (c) of Theorem 5.1 that

$$\langle u^*(x), \Delta \tilde{u}(x) \rangle = M(\tilde{u}(x), u^*(x), x).$$

So, in particular, if $F(\cdot, x)$ is a quasisuperlinear mapping and $M(\cdot, v^*, x)$ is a convex proper function, then by Corollary 2.4 this equality can be written as follows:

$$\langle u^*(x), \Delta \tilde{u}(x) \rangle = \inf_{\Delta u^*(x) \in F^*(u^*(x), x)} \{ \langle u^*(x), \Delta \tilde{u}(x) \rangle - H(\Delta u^*(x), u^*(x), x) \}.$$

Theorem 5.2. *Let us consider the nonconvex problem (P_C) . Moreover, let $\tilde{u}(\cdot)$ be some feasible solution of this nonconvex problem and $u^*(\cdot)$ is a classical solution satisfying the following conditions:*

- (i) $\Delta u^*(x) + u^*(x) \in F^*(u^*(x), (\tilde{u}(x), \Delta \tilde{u}(x)), x)$,
- (ii) $g(u, x) - g(\tilde{u}(x), x) \geq \langle u^*(x), u - \tilde{u}(x) \rangle$ for all u ,
- (iii) $\langle u^*(x), \Delta \tilde{u}(x) \rangle = M(\tilde{u}(x), u^*(x), x)$,

where the LAM $F^*(\cdot, \cdot, x)$ is given by Definition 2.2. Consequently the feasible solution $\tilde{u}(\cdot)$ is optimal.

Proof. Taking into account Definition 2.2 it follows from the condition (i) of theorem that for all feasible solutions $u(\cdot)$ is valid the inequality

$$M(u(x), u^*(x), x) - M(\tilde{u}(x), u^*(x), x) \leq \langle \Delta u^*(x) + u^*(x), u(x) - \tilde{u}(x) \rangle, \quad x \in R.$$

Then using the condition (iii), we have from this inequality

$$\langle \Delta(u(x) - \tilde{u}(x)), u^*(x) \rangle \leq \langle \Delta u^*(x) + u^*(x), u(x) - \tilde{u}(x) \rangle. \quad (5.3)$$

Now, from the condition (ii) of theorem for arbitrarily feasible solution $u(\cdot)$ and from inequality (5.3) it is easy to see that

$$g(u(x), x) - g(\tilde{u}(x), x) - \langle \Delta(u(x) - \tilde{u}(x)), u^*(x) \rangle + \langle u(x) - \tilde{u}(x), \Delta u^*(x) \rangle \geq 0, \quad x \in R.$$

Then by integrating this inequality over the domain R , we see that the obtained inequality takes the form (5.1). Thus in view of (5.1) it is easy to show as in the proof of Theorem 5.1 that $\tilde{u}(\cdot)$ is optimal. The proof is complete. \square

Let us consider the following example:

$$\begin{aligned} \text{minimize} \quad & J(u(\cdot)) = \int \int_R g(u(x), x) dx, \\ \text{subject to} \quad & \Delta u(x) = Au(x) + Bw(x), \quad w(x) \in V, \end{aligned} \quad (5.4)$$

where A is $n \times n$ matrix, B is a rectangular $n \times r$ matrix $V \subset R^r$ is a closed convex set and g is continuously differentiable function on x . It is required to find a controlling parameter $w(x) \in V$ such that the feasible solution corresponding to it minimizes $J(u(\cdot))$.

Let introduce a convex mapping $F(u) = Au + BV$. By elementary calculations, it can be shown that

$$F^*(v^*, (u, v)) = \begin{cases} A^*v^*, & -B^*v^* \in K_V^*(w), \\ \emptyset, & -B^*v^* \notin K_V^*(w). \end{cases}$$

Here $v = Au + Bw$ and $K_V^*(w)$ is the cone dual to the cone of tangent directions $K_V(w)$ at a point $w \in V$. Then using Theorem 5.1, we get the relations

$$\begin{aligned} \Delta u^*(x) &= A^*u^*(x) - g'(\tilde{u}(x), x), \quad x \in R, \\ u^*(x) &= 0, \quad x \in B, \\ \langle B\tilde{u}(x), u^*(x) \rangle &= \inf_{w \in V} \langle Bu, u^*(x) \rangle. \end{aligned} \quad (5.5)$$

Thus we obtain the following result.

Theorem 5.3. *The feasible solution $\tilde{u}(\cdot)$ corresponding to the control $\tilde{w}(\cdot)$ minimizes $J(u(\cdot))$ in the problem (5.4) if there exists a classical solution $u^*(\cdot)$ satisfying the conditions (5.5).*

6. Multidimensional optimal control problem for elliptic differential inclusion

In this section we study the following problem (P_M) with elliptic operator L considered in Section 2:

$$\begin{aligned} (P_M) \quad \text{minimize} \quad & J(u(\cdot)) = \int g(u(x), x) dx, \\ \text{subject to} \quad & Lu(x) \in F(u(x), x), \\ & u(x) = \alpha(x), \quad x \in S. \end{aligned}$$

Theorem 6.1. *If g is a continuous function convex with respect to u , and $F(\cdot, x)$ is a convex closed mapping for every fixed $x \in G$. Then a solution $\tilde{u}(\cdot)$ minimizes the functional $J(u(\cdot))$ among all feasible solutions of the problem (P_M) if there exists a classical solution $u^*(\cdot)$ of the following boundary value problem:*

- (i) $L^*u^*(x) \in F^*(u^*(x), (\tilde{u}(x), L\tilde{u}(x)), x) - \partial g(\tilde{u}(x), x)u^*(x) = 0, x \in S,$
- (ii) $L\tilde{u}(x) \in F(\tilde{u}(x), u^*(x), x),$

where L^* is the operator adjoint to L .

Proof. By arguments analogous to those in the proof of the preceding Theorem 5.1 and the condition (i) it is easy to see that

$$\begin{aligned} M(u(x), u^*(x), x) - M(\tilde{u}(x), u^*(x), x) \\ \leq g(u(x), x) - g(\tilde{u}(x), x) + L^*u^*(x)(u(x) - \tilde{u}(x)), \end{aligned}$$

where due to (ii),

$$M(\tilde{u}(x), u^*(x), x) = u^*(x)L\tilde{u}(x).$$

And so

$$\begin{aligned} \int_G [g(u(x), x) - g(\tilde{u}(x), x)] dx \\ \geq \int_G u^*(x)L[u(x) - \tilde{u}(x)] dx - \int_G L^*u^*(x)[u(x) - \tilde{u}(x)] dx. \end{aligned} \quad (6.1)$$

Then using the boundary conditions of (i) and the fact that the functions $u(\cdot)$ are feasible solutions, i.e., $u(x) = \tilde{u}(x) = \alpha(x)$, $x \in S$, we get from the familiar Green's formula in the multidimensional case, that right-hand side of the inequality (6.1) is equal to zero. It means that $J(u(x)) \geq J(\tilde{u}(x))$ for all feasible solutions in problem (P_M) . The theorem is proved. \square

Remark 6.1. In addition to assumptions of Theorem 6.1, let $F(\cdot, x)$ is a closed mapping. Then the conditions (i), (ii) of Theorem 6.1 can be rewritten as follows (see Corollary 5.1):

- (i) $L^*u^*(x) \in \partial_u M(\tilde{u}(x), u^*(x), x) - \partial g(\tilde{u}(x), x)$,
- (ii) $L\tilde{u}(x) \in \partial_{v^*} M(\tilde{u}(x), u^*(x), x)$.

Replacing Laplace operator Δ with elliptic operator L and extending the proof of Theorem 5.2 to the problem (P_M) for nonconvex case it is not hard to get the following theorem.

Theorem 6.2. Suppose $\tilde{u}(\cdot)$ is some feasible solution of the nonconvex problem (P_M) and $u^*(\cdot)$ is a classical solution satisfying the following conditions:

- (i) $L^*u^*(x) + u^*(x) \in F^*(u^*(x), (\tilde{u}(x), L\tilde{u}(x)), x)$, $u^*(x) = 0$, $x \in S$,
- (ii) $\langle u^*(x), L\tilde{u}(x) \rangle = M(\tilde{u}(x), u^*(x), x)$,
- (iii) $g(u, x) - g(\tilde{u}(x), x) \geq \langle u^*(x), u - \tilde{u}(x) \rangle$ for all u ,

where the LAM $F^*(\cdot, \cdot, x)$ is given by Definition 2.2, then the feasible solution $\tilde{u}(\cdot)$ is optimal.

In conclusion we consider the possibility of passing to more general function spaces of solutions in the problems discussed above. It is known that for the theory of partial differential equations, the concept of generalized solution is important both from the theoretical and from the practical point of view [24,31]. The definition of such solutions associates with a given equation, a certain integral identity that uses, in turn, the class of generalized derivatives.

Therefore, on this path the most natural approach for elliptic differential inclusions is apparently the use of single-valued branches (selections) of a multi-valued mapping [14].

Thus suppose that we have the problem (P_M) with homogeneous boundary conditions and let $H^1(G)$ is the Hilbert space consisting of the elements $u(\cdot) \in L_2(G)$ having square-integrable generalized derivatives on G , where the inner product and norm are defined by the expressions, respectively

$$\langle u_1, u_2 \rangle_{H^1(G)} = \int_G (u_1 u_2 + u'_{1x} u'_{2x}) dx, \quad \|u\|_{H^1(G)} = \sqrt{\langle u, u \rangle_{H^1(G)}}.$$

By analogy with the classical theory of the Dirichlet problem for elliptic equation [24,32] we call a function $u(\cdot) \in H(G)$ a generalized solution of our problem if it satisfies the integral identity

$$\int_G (a_{ij} u'_{x_i} \eta'_{x_j} - b_i u'_{x_i} \eta - c u \eta) dx = - \int_G g \eta dx$$

for all $\eta(\cdot) \in \mathring{H}^1(G)$ (for a more detailed study see, for example, [24,32]). Here $g = g(u, x)$ is an arbitrary measurable selection of the multi-valued mapping $F(u, x)$. A generalized solution is defined analogously for the adjoint boundary value problem.

We now emphasize that for all the results obtained here have been used the formula of integration by parts and the Green and Gauss–Ostrogradskii formulae following from it. The latter can be used for getting the indicated classes of generalized solutions. Therefore, it is not difficult to verify the validity of all the assertions in this general case.

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