Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 326 (2007) 1334–1345
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Demiclosedness principle and convergence theorems for k-strictly asymptotically pseudocontractive maps

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Received 30 April 2005
Available online 2 May 2006
Submitted by S. Heikkilä

Abstract

Let E be a real q-uniformly smooth Banach space which is also uniformly convex (for example, L_p or ℓ_p spaces, $1 < p < \infty$), and K a nonempty closed convex (not necessarily bounded) subset of E. Let $T : K \to K$ be a k-strictly asymptotically pseudocontractive map with a nonempty fixed-point set. It is proved that $(I - T)$ is demiclosed at 0. Furthermore, weak and strong convergence of an averaging iteration method to a fixed point of T are proved.

Keywords: Fixed points; k-Strictly asymptotically pseudocontractive maps; Mann iteration; Ishikawa iteration; q-Uniformly smooth Banach spaces

1. Introduction

Let E be an arbitrary real Banach space and let J_q $(q > 1)$ denote the generalized duality mapping from E into 2^{E^*} given by

$$J_q(x) = \left\{ f \in E^*\colon \langle x, f \rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1} \right\},$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.12.052

where E^* denotes the dual space of E and $\langle .,. \rangle$ denotes the generalized duality pairing. In particular, J_2 is called the normalized duality mapping and it is usually denoted by J. It is well known (see, for example, [18]) that $J_q(x) = \|x\|^{q-2}J(x)$ if $x \neq 0$, and that if E^* is strictly convex then J_q is single-valued. In the sequel we shall denote single-valued generalized duality mapping by j_q.

Let K be a nonempty subset of E. A mapping $T : K \to K$ is called *k-strictly asymptotically pseudocontractive*, with sequence $\{k_n\} \subseteq [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ (see, e.g., [6,9,10]) if for all $x, y \in K$, there exist $j(x-y) \in J(x-y)$ and a constant $k \in [0,1)$ such that

$$\left\langle T^n x - T^n y, j(x-y)\right\rangle \leqslant \frac{1}{2}(1+k_n)\|x-y\|^2 - \frac{1}{2}(1-k)\left\|x - T^n x - \left(y - T^n y\right)\right\|^2, \quad (1)$$

for all $n \in \mathbb{N}$. If I denotes the identity operator, then (1) can be written in the form

$$\left\langle \left(I - T^n\right)x - \left(I - T^n\right)y, j(x-y)\right\rangle \geqslant \frac{1}{2}(1-k)\left\|\left(I - T^n\right)x - \left(I - T^n\right)y\right\|^2 - \frac{1}{2}(k_n - 1)\|x-y\|^2. \quad (2)$$

The class of k-strictly asymptotically pseudocontractive maps was first introduced in Hilbert spaces by Qihou [10]. In Hilbert spaces, j is the identity and it is shown by one of the authors [9] that (1) (and hence (2)) is equivalent to the inequality

$$\left\|T^n x - T^n y\right\|^2 \leqslant k_n\|x-y\|^2 + k\left\|\left(I - T^n\right)x - \left(I - T^n\right)y\right\|^2, \quad (3)$$

which is the inequality considered by Qihou [10].

A mapping T with domain $D(T)$ and range $R(T)$ in E is called *strictly pseudocontractive* in the terminology of Browder and Petryshyn [1] if there exists $\lambda > 0$ such that

$$\left\langle Tx - Ty, j(x-y)\right\rangle \leqslant \|x-y\|^2 - \lambda\left\|x - y - (Tx - Ty)\right\|^2, \quad (4)$$

for all $x, y \in D(T)$ and for all $j(x-y) \in J(x-y)$. Without loss of generality we may assume $\lambda \in (0,1)$. If I denotes the identity operator, then (4) can be written in the form

$$\left\langle (I-T)x - (I-T)y, j(x-y)\right\rangle \geqslant \lambda\left\|(I-T)x - (I-T)y\right\|^2. \quad (5)$$

In Hilbert spaces H, (4) (and hence (5)) is equivalent to the inequality

$$\|Tx - Ty\|^2 \leqslant \|x-y\|^2 + k\left\|(I-T)x - (I-T)y\right\|^2, \quad k = (1-2\lambda) < 1, \quad (6)$$

and we can assume also that $k \geqslant 0$, so that $k \in [0,1)$.

The class of strictly pseudocontractive mappings has been studied by several authors (see, for example, [1,3,8,11,12]). It is shown in [8] that a strictly pseudocontractive map is L-Lipschitzian (i.e., $\|Tx - Ty\| \leqslant L\|x-y\|$ for all $x, y \in D(T)$ and for some $L > 0$). The class of k-asymptotically pseudocontractive maps and the class of strictly pseudocontractive maps are independent (see our examples at the end of this paper).

T is said to be *uniformly L-Lipschitzian*, if there exists a constant $L > 0$, such that

$$\left\|T^n x - T^n y\right\| \leqslant L\|x-y\|$$

for all $x, y \in K$ and $n \in \mathbb{N}$, and is said to be *demiclosed at a point* p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx = p$. Furthermore, T is said to be *demicompact* if whenever $\{x_n\}$ is a bounded sequence in $D(T)$ such that $\{x_n - Tx_n\}$ converges strongly, then $\{x_n\}$ has a subsequence which converges strongly.

In [10] Qihou proved the following:

Theorem 1.1. [10, p. 1836] *Let H be a real Hilbert space and K a nonempty closed convex and bounded subset of H. Let $T : K \to K$ be a completely continuous uniformly L-Lipschitzian k-strictly asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ be a real sequence satisfying the condition*

(i) $0 < \epsilon \leqslant \alpha_n \leqslant 1 - k - \epsilon$, *for all $n \geqslant 1$ and for some $\epsilon > 0$. Then the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in K$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geqslant 1, \tag{7}$$

converges strongly to a fixed point of T.

The iteration scheme in Theorem 1.1 was introduced by Schu [13,14] and has been used by several authors (see, for example, [4,6,7,9,10,15]).

In [6] one of the authors extended Theorem 1.1 from Hilbert spaces to much more general real q-uniformly smooth Banach spaces, $1 < q < \infty$.

Let E be a real q-uniformly smooth Banach space which is also uniformly convex, K a nonempty closed convex (not necessarily bounded) subset of E, and $T : K \to K$ a k-strictly asymptotically pseudocontractive map with a nonempty fixed-point set.

It is our purpose in this paper to first prove that $(I - T)$ is demiclosed at 0. We then prove weak and strong convergence theorems for the iterative approximation of fixed points of T using the modified Mann iteration processes. Our class of Banach spaces includes the L_p, ℓ_p spaces, $1 < p < \infty$, and the Sobolev spaces W_m^p, $1 < p < \infty$. Our main convergence theorem does not require the assumption that T be completely continuous.

2. Preliminaries

In the sequel, we shall need the following: Let E be a real Banach space. The *modulus of smoothness* of E is the function

$$\rho_E : [0,\infty) \to [0,\infty)$$

defined by

$$\rho_E(\tau) = \sup\left\{\frac{1}{2}\big(\|x+y\| + \|x-y\|\big) - 1:\ \|x\| \leqslant 1,\ \|y\| \leqslant \tau\right\}.$$

E is *uniformly smooth* if and only if $\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0$.

Let $q > 1$. E is said to be *q-uniformly smooth* (or to have a modulus of smoothness of power type $q > 1$) if there exists a constant $c > 0$ such that $\rho_E(\tau) \leqslant c\tau^q$. Hilbert spaces, L_p (or ℓ_p) spaces, $1 < p < \infty$, and the Sobolev spaces, W_m^p, $1 < p < \infty$, are q-uniformly smooth. Hilbert spaces are 2-uniformly smooth while

$$L_p(\text{or } \ell_p) \text{ or } W_m^p \text{ is } \quad \begin{cases} p\text{-uniformly smooth} & \text{if } 1 < p \leqslant 2, \\ 2\text{-uniformly smooth} & \text{if } p \geqslant 2. \end{cases}$$

Theorem 2.1. [18, p. 1130] *Let $q > 1$ and let E be a real Banach space. Then the following are equivalent*:

(1) *E is q-uniformly smooth.*
(2) *There exists a constant $c_q > 0$ such that for all $x, y \in E$*

$$\|x+y\|^q \leqslant \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q. \tag{8}$$

(3) *There exists a constant d_q such that for all $x, y \in E$, and $t \in [0,1]$*

$$\big\|(1-t)x+ty\big\|^q \geqslant (1-t)\|x\|^q + t\|y\|^q - \omega_q(t)d_q\|x-y\|^q, \tag{9}$$

where $\omega_q(t) = t^q(1-t) + t(1-t)^q$.

Furthermore, it is proved in [17] (see Remark 5, p. 208) that if E is q-uniformly smooth ($q > 1$), then for all $x, y \in E$ there exists a constant $L_* > 0$ such that

$$\big\|j_q(x) - j_q(y)\big\| \leqslant L_*\|x-y\|^{q-1}. \tag{10}$$

E is said to have a *Fréchet differentiable norm* if for all $x \in U = \{x \in E\colon \|x\| = 1\}$

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists and is attained uniformly in $y \in U$. In this case there exists an increasing function $b : [0,\infty) \to [0,\infty)$ with $\lim_{t\to 0^+} b(t) = 0$ such that

$$\frac{1}{2}\|x\|^2 + \langle h, j(x)\rangle \leqslant \frac{1}{2}\|x+h\|^2 \leqslant \frac{1}{2}\|x\|^2 + \langle h, j(x)\rangle + b\big(\|h\|\big), \quad \forall x, h \in E. \tag{11}$$

Lemma 2.1. [5, Lemma 2.1, p. 29] *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space E, and let $T : C \to E$ be a nonexpansive mapping. Let $\{x_n\}$ be a sequence in C such that $\{x_n\}$ converges weakly to some point x. Then there exists an increasing continuous function $h : [0,\infty) \to [0,\infty)$ with $h(0) = 0$ depending on the diameter of C such that*

$$h\big(\|x - Tx\|\big) \leqslant \liminf_{n\to\infty} \|x_n - Tx_n\|.$$

Lemma 2.2. [9] *Let E be a real Banach space, K a nonempty subset of E and $T : K \to K$ a k-strictly asymptotically pseudocontractive mapping. Then T is uniformly L-Lipschitzian.*

Remark 2.1. Since $\|T^n x - T^n y\| \leqslant L\|x-y\|$, we have $\|x - T^n x - (y - T^n y)\| \leqslant (1+L)\|x-y\|$. Hence

$$\begin{aligned}
&\big\langle \big(I-T^n\big)x - \big(I-T^n\big)y, j_q(x-y)\big\rangle \\
&\quad = \|x-y\|^{q-2}\big\langle \big(I-T^n\big)x - \big(I-T^n\big)y, j(x-y)\big\rangle \\
&\quad \geqslant \|x-y\|^{q-2}\bigg\{\frac{1}{2}(1-k)\big\|\big(I-T^n\big)x - \big(I-T^n\big)y\big\|^2 \\
&\qquad - \frac{1}{2}(k_n - 1)\|x-y\|^2\bigg\} \quad \text{(using (2))} \\
&\quad \geqslant \frac{1}{2}(1-k)(1+L)^{-(q-2)}\big\|\big(I-T^n\big)x - \big(I-T^n\big)y\big\|^q - \frac{1}{2}(k_n-1)\|x-y\|^q.
\end{aligned} \tag{12}$$

In what follows, L is the uniformly Lipschitzian constant of T, c_q, d_q and L_* are the constants appearing in (8), (9) and (10), respectively.

3. Main results

We prove the following:

Lemma 3.1. *Let E be a real q-uniformly smooth Banach space and K a nonempty convex subset of E. Let $T : K \to K$ be a k-strictly asymptotically pseudocontractive map and let $\{\alpha_n\}$ be a real sequence in* $[0, 1]$. *Define $T_n : K \to K$ by*

$$T_n x := (1 - \alpha_n)x + \alpha_n T^n x, \quad x \in K.$$

Then for all $x, y \in K$ we have

$$\begin{aligned}\|T_n x - T_n y\|^q &\leqslant \left[1 + \frac{q}{2}\alpha_n(k_n - 1)\right]\|x - y\|^q \\ &\quad - \alpha_n\left[\frac{q}{2}(1 - k)(1 + L)^{-(q-2)} - c_q\alpha_n^{q-1}\right]\left\|\left(I - T^n\right)x - \left(I - T^n\right)y\right\|^q. \end{aligned} \tag{13}$$

Proof. Using (8) we obtain

$$\begin{aligned}\|T_n x - T_n y\|^q &= \left\|x - y - \alpha_n\left[\left(I - T^n\right)x - \left(I - T^n\right)y\right]\right\|^q \\ &\leqslant \|x - y\|^q - q\alpha_n\left\langle\left(I - T^n\right)x - \left(I - T^n\right)y, j_q(x - y)\right\rangle \\ &\quad + c_q\alpha_n^q\left\|\left(I - T^n\right)x - \left(I - T^n\right)y\right\|^q \\ &\leqslant \|x - y\|^q - q\alpha_n\left\{\frac{1}{2}(1 - k)(1 + L)^{-(q-2)}\left\|\left(I - T^n\right)x - \left(I - T^n\right)y\right\|^q \right. \\ &\quad \left. - \frac{1}{2}(k_n - 1)\|x - y\|^q\right\} + c_q\alpha_n^q\left\|\left(I - T^n\right)x - \left(I - T^n\right)y\right\|^q \\ &= \left[1 + \frac{1}{2}q\alpha_n(k_n - 1)\right]\|x - y\|^q \\ &\quad - \alpha_n\left[\frac{q}{2}(1 - k)(1 + L)^{-(q-2)} - c_q\alpha_n^{q-1}\right]\left\|\left(I - T^n\right)x - \left(I - T^n\right)y\right\|^q,\end{aligned}$$

completing the proof of Lemma 3.1. $\square$

Remark 3.1. Let $\gamma = \min\{1, [\frac{q}{2}(1 - k)(1 + L)^{-(q-2)}/c_q]^{1/(q-1)}\}$, and choose any $\alpha \in (0, \gamma)$. Set $\alpha_n = \alpha$, $\forall n \geqslant 1$, in Lemma 3.1. Then we obtain $T_{\alpha,n} : K \to K$ defined for all $x \in K$ by

$$T_{\alpha,n}x = (1 - \alpha)x + \alpha T^n x.$$

Observe that $\|T_{\alpha,n}x - T_{\alpha,n}y\|^q \leqslant [1 + \frac{q}{2}\alpha(k_n - 1)]\|x - y\|^q$. Thus

$$\|T_{\alpha,n}x - T_{\alpha,n}y\| \leqslant \left[1 + \frac{q}{2}\alpha(k_n - 1)\right]^{1/q}\|x - y\|, \quad \forall x, y \in K. \tag{14}$$

Theorem 3.1. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a nonempty closed convex subset of E and $T : K \to K$ a k-strictly asymptotically pseudocontractive mapping with a nonempty fixed point set. Then $(I - T)$ is demiclosed at zero.*

Proof. Let $\{x_n\}$ be a sequence in K which converges weakly to p and $\{x_n - Tx_n\}$ converges strongly to 0. We prove that $(I - T)(p) = 0$. Let $x^* \in F(T)$. Then there exists $R > 0$ such that $\|x_n - x^*\| \leqslant R$, $\forall n \geqslant 1$. Let $\bar{B}_R = \{x \in E\colon \|x - x^*\| \leqslant R\}$, and let $C = K \cap \bar{B}_R$. Then C is nonempty closed convex and bounded, and $\{x_n\} \subseteq C$. Let α and $T_{\alpha,n}$ be as in Remark 3.1. Then $\|T_{\alpha,n}x - T_{\alpha,n}y\| \leqslant [1 + \frac{q}{2}\alpha(k_n - 1)]^{1/q}\|x - y\| = a_n\|x - y\|$, $\forall x, y \in K$, where $a_n = [1 + \frac{q}{2}\alpha(k_n - 1)]^{1/q}$. Define $G_{\alpha,m} : C \to E$ by

$$G_{\alpha,m}x = \frac{1}{a_m}T_{\alpha,m}x, \quad m \geqslant 1.$$

Then $G_{\alpha,m}$ is nonexpansive and it follows from Lemma 2.1 that there exists an increasing continuous function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ depending on the diameter of C such that

$$h\big(\|p - G_{\alpha,m}p\|\big) \leqslant \liminf_{n\to\infty} \|x_n - G_{\alpha,m}x_n\|. \tag{15}$$

Observe that

$$\begin{aligned} \|x_n - G_{\alpha,m}x_n\| &= \left\|x_n - \frac{1}{a_m}T_{\alpha,m}x_n\right\| \\ &\leqslant \|x_n - T_{\alpha,m}x_n\| + \left(1 - \frac{1}{a_m}\right)\big[a_m\|x_n - x^*\| + \|x^*\|\big] \\ &\leqslant \|x_n - T_{\alpha,m}x_n\| + \left(1 - \frac{1}{a_m}\right)\big[a_m R + \|x^*\|\big]. \end{aligned} \tag{16}$$

Observe that

$$\|x_n - T_{\alpha,m}x_n\| = \alpha\big\|x_n - T^m x_n\big\| \leqslant \sum_{j=1}^{m}\big\|T^{j-1}x_n - T^j x_n\big\| \leqslant Lm\|x_n - Tx_n\| \to 0$$

as $n \to \infty$.

Thus it follows from (16) that

$$\limsup_{n\to\infty} \|x_n - G_{\alpha,m}x_n\| \leqslant \left(1 - \frac{1}{a_m}\right)\big[a_m R + \|x^*\|\big],$$

so that (15) implies that

$$h\big(\|p - G_{\alpha,m}p\|\big) \leqslant \left(1 - \frac{1}{a_m}\right)\big[a_m R + \|x^*\|\big]. \tag{17}$$

Observe that

$$\begin{aligned} \|p - G_{\alpha,m}p\| &\geqslant \|p - T_{\alpha,m}p\| - \left(1 - \frac{1}{a_m}\right)\|T_{\alpha,m}p\| \\ &\geqslant \|p - T_{\alpha,m}p\| - \left(1 - \frac{1}{a_m}\right)\big[a_m R + \|x^*\|\big], \end{aligned}$$

so that

$$\begin{aligned} \|p - T_{\alpha,m}p\| &\leqslant \|p - G_{\alpha,m}p\| + \left(1 - \frac{1}{a_m}\right)\big[a_m R + \|x^*\|\big] \\ &\leqslant h^{-1}\left(\left(1 - \frac{1}{a_m}\right)\big[a_m R + \|x^*\|\big]\right) + \left(1 - \frac{1}{a_m}\right)\big[a_m R + \|x^*\|\big] \to 0 \end{aligned}$$

as $m \to \infty$.

Since T is continuous, we have $(I - T)(p) = 0$, completing the proof of Theorem 3.1. $\square$

Lemma 3.2. *Let E be a real q-uniformly smooth Banach space and let K be a nonempty convex subset of E. Let $T : K \to K$ be a k-strictly asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subseteq [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and let $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence satisfying the conditions*:

(i) $0 \leqslant \alpha_n \leqslant 1$, $n \geqslant 1$;
(ii) $0 < a \leqslant \alpha_n^{q-1} \leqslant b < \frac{q(1-k)}{2c_q}(1 + L)^{-(q-2)}$, $n \geqslant 1$.

Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geqslant 1.$$

Then

(a) $\lim_{n\to\infty} \|x_n - x^*\|$ *exists for all* $x^* \in F(T)$;
(b) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. Let $x^* \in F(T)$ and set $x = x_n$, $y = x^*$ and $\delta_n = \frac{q}{2}\alpha(k_n - 1)$ in Lemma 3.1. Then

$$\begin{aligned}
&\|x_{n+1} - x^*\|^q \\
&\quad \leqslant [1 + \delta_n]\|x_n - x^*\|^q - \alpha_n\left[\frac{q}{2}(1 - k)(1 + L)^{-(q-2)} - c_q\alpha_n^{q-1}\right]\left\|x_n - T^n x_n\right\|^q && (18)\\
&\quad \leqslant [1 + \delta_n]\|x_n - x^*\|^q \quad \text{(using condition (ii))}. && (19)
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, it follows that $\sum_{n=1}^{\infty}\delta_n < \infty$, and hence (19) implies that $\{\|x_n - x^*\|\}$ is bounded. Let $\|x_n - x^*\| \leqslant M$, $\forall n \geqslant 1$. Then (19) implies that

$$\|x_{n+1} - x^*\|^q \leqslant \|x_n - x^*\|^q + M^q\delta_n,$$

so that it follows from [9], [16, Lemma 1.1] that $\lim_{n\to\infty} \|x_n - x^*\|$ exists, completing the proof of (a). Using (18) we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^q \leqslant{}& \|x_n - x^*\|^q - \alpha_n\left[\frac{q}{2}(1 - k)(1 + L)^{-(q-2)} - c_q\alpha_n^{q-1}\right]\left\|x_n - T^n x_n\right\|^q \\
&+ M^q\delta_n.
\end{aligned}$$

Hence $a^{1/(q-1)}[\frac{q}{2}(1-k)(1+L)^{-(q-2)} - c_q b]\sum_{j=1}^{n}\|x_j - T^j x_j\|^q \leqslant \|x_1 - x^*\|^q + M^q\sum_{j=1}^{n}\delta_j \leqslant \|x_1 - x^*\|^q + M^q\sum_{j=1}^{\infty}\delta_j < \infty$. Thus $\sum_{j=1}^{\infty}\|x_j - T^j x_j\| < \infty$, so that $\lim_{n\to\infty}\|x_n - T^n x_n\| = 0$. Since

$$\begin{aligned}
\|x_n - Tx_n\| &\leqslant \left\|x_n - T^n x_n\right\| + \left\|T^n x_n - Tx_n\right\| \leqslant \left\|x_n - T^n x_n\right\| + L\left\|T^{n-1}x_n - x_n\right\| \\
&\leqslant \left\|x_n - T^n x_n\right\| + L\left\|T^{n-1}x_n - T^{n-1}x_{n-1}\right\| + L\left\|T^{n-1}x_{n-1} - x_n\right\| \\
&\leqslant \left\|x_n - T^n x_n\right\| + L^2\|x_n - x_{n-1}\| + L\left\|T^{n-1}x_{n-1} - x_{n-1}\right\| + L\|x_{n-1} - x_n\| \\
&\leqslant \left\|x_n - T^n x_n\right\| + L(2 + L)\left\|x_{n-1} - T^{n-1}x_{n-1}\right\|,
\end{aligned}$$

we have $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. $\square$

Corollary 3.1. *Let E be a real q-uniformly smooth Banach space and K a nonempty closed convex subset of E. Let T, $\{\alpha_n\}$ and $\{x_n\}$ be as in Lemma 3.2. If $\{x_n\}$ clusters strongly at some point p, then $p \in F(T)$ and $\{x_n\}$ converges strongly to p.*

Proof. $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ which converges strongly to $p \in K$. Since T is continuous at p, then $Tx_{n_j} \to Tp$ as $j \to \infty$. Hence $\lim_{j\to\infty} \|x_{n_j} - Tx_{n_j}\| = \|p - Tp\| = 0$, so that $p \in F(T)$. From Lemma 3.2, $\lim_{n\to\infty} \|x_n - x^*\|$ exists, and since $\lim_{j\to\infty} \|x_{n_j} - p\| = 0$, we must have $\lim_{n\to\infty} \|x_n - p\| = 0$, completing the proof of Corollary 3.1. □

Remark 3.2. In view of Corollary 3.1, we can conclude that if K is also closed in Lemma 3.2, then either $\{x_n\}$ converges strongly to a fixed point of T or else $\{x_n\}$ has no subsequence which converges strongly. In particular, if T is in addition completely continuous, or demicompact, then $\{x_n\}$ converges strongly to a fixed point of T.

Lemma 3.3. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a nonempty convex subset of E and let $T : K \to K$ be a k-strictly asymptotically pseudocontractive map with a sequence $\{k_n\} \subseteq [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and let $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{x_n\}$ be as in Lemma* 3.2. *Then for all $p_1, p_2 \in F(T)$*

$$\lim_{n\to\infty} \left\|tx_n + (1-t)p_1 - p_2\right\|$$

exists for all $t \in [0,1]$.

Proof. Let $\sigma_n(t) := \|tx_n + (1-t)p_1 - p_2\|$. Then $\lim_{n\to\infty} \sigma_n(0) = \|p_1 - p_2\|$, and from Lemma 3.2 $\lim_{n\to\infty} \sigma_n(1) = \lim_{n\to\infty} \|x_n - p_2\|$ exists. It now remains to prove the lemma for $t \in (0,1)$. Let T_n be as in Lemma 3.1, then

$$\|T_n x - T_n y\| \leqslant [1+\delta_n]^{1/q} \|x - y\| = a_n \|x - y\|, \quad \forall x, y \in K,$$

where $a_n = (1+\delta_n)^{1/q}$. Since $\sum_{n=1}^{\infty} \delta_n < \infty$, then $\prod_{n=1}^{\infty} a_n < \infty$.

Set

$$S_{n,m} := T_{n+m-1}T_{n+m-2}\cdots T_n, \quad m \geqslant 1.$$

Then

$$\|S_{n,m}x - S_{n,m}y\| \leqslant \left(\prod_{j=n}^{n+m-1} a_j\right)\|x - y\|, \quad \forall x, y \in K;$$

$$S_{n,m}x_n = x_{n+m} \quad \text{and} \quad S_{n,m}p = p, \quad \forall p \in F(T).$$

Set $b_{n,m} := \|S_{n,m}(tx_n + (1-t)p_1) - tS_{n,m}x_n - (1-t)S_{n,m}p_1\|$; $D := (\prod_{j=1}^{\infty} a_j)^2 \|x_1 - p_1\|$. Let δ denote the *modulus of convexity* of E. We prove that

$$\frac{D}{2}\delta\left(\frac{4}{D}b_{n,m}\right) \leqslant \left(\prod_{j=n}^{n+m-1} a_j\right)\|x_n - p_1\| - \|x_{n+m} - p_1\|. \tag{20}$$

If $\|x_n - p_1\| = 0$ for some n_0, then $x_n = p_1$, $\forall n \geqslant n_0$, so that clearly (20) holds and in fact $\{x_n\}$ converges strongly to $p_1 \in F(T)$. Thus we may assume $\|x_n - p_1\| > 0$, $\forall n \geqslant 1$. It is well known (see, for example, [2, p. 108]) that

$$\left\|tx + (1-t)y\right\| \leqslant 1 - 2\min\bigl\{t, (1-t)\bigr\}\delta\bigl(\|x-y\|\bigr) \leqslant 1 - 2t(1-t)\delta\bigl(\|x-y\|\bigr) \tag{21}$$

for all $t \in [0,1]$ and for all $x, y \in E$ such that $\|x\| \leqslant 1$, $\|y\| \leqslant 1$. Set

$$w_{n,m} := \frac{S_{n,m}p_1 - S_{n,m}(tx_n + (1-t)p_1)}{t(\prod_{j=n}^{n+m-1} a_j)\|x_n - p_1\|}, \qquad z_{n,m} := \frac{S_{n,m}(tx_n + (1-t)p_1) - S_{n,m}x_n}{(1-t)(\prod_{j=n}^{n+m-1} a_j)\|x_n - p_1\|}.$$

Then $\|w_{n,m}\| \leqslant 1$ and $\|z_{n,m}\| \leqslant 1$ so that it follows from (21) that

$$2t(1-t)\delta\big(\|w_{n,m}-z_{n,m}\|\big) \leqslant 1-\big\|tw_{n,m}+(1-t)z_{n,m}\big\|. \tag{22}$$

Observe that

$$\|w_{n,m}-z_{n,m}\| = \frac{b_{n,m}}{t(1-t)(\prod_{j=n}^{n+m-1} a_j)\|x_n-p_1\|} \quad \text{and}$$

$$\big\|tw_{n,m}+(1-t)z_{n,m}\big\| = \frac{\|S_{n,m}x_n - S_{n,m}p_1\|}{(\prod_{j=n}^{n+m-1} a_j)\|x_n-p_1\|},$$

so that it follows from (22) that

$$\begin{aligned} &2t(1-t)\left(\prod_{j=n}^{n+m-1} a_j\right)\|x_n-p_1\|\delta\left(\frac{b_{n,m}}{t(1-t)(\prod_{j=n}^{n+m-1} a_j)\|x_n-p_1\|}\right) \\ &\quad \leqslant \left(\prod_{j=n}^{n+m-1} a_j\right)\|x_n-p_1\| - \|S_{n,m}x_n - S_{n,m}p_1\|. \end{aligned} \tag{23}$$

Observe that

$$\begin{aligned} &t(1-t)\left(\prod_{j=n}^{n+m-1} a_j\right)\|x_n-p_1\| \leqslant \frac{1}{4}\left(\prod_{j=1}^{\infty} a_j\right)^2\|x_1-p_1\| = \frac{D}{4} \\ &\quad \left(\text{since } t(1-t) \leqslant \frac{1}{4},\ \forall t \in [0,1]\right). \end{aligned}$$

Since E is uniformly convex, then $\delta(s)/s$ is nondecreasing and hence it follows from (23) that

$$\begin{aligned} \frac{D}{2}\delta\left(\frac{4}{D}b_{n,m}\right) &\leqslant \left(\prod_{j=n}^{n+m-1} a_j\right)\|x_n-p_1\| - \|S_{n,m}x_n - S_{n,m}p_1\| \\ &= \left(\prod_{j=n}^{n+m-1} a_j\right)\|x_n-p_1\| - \|x_{n+m}-p_1\|, \end{aligned}$$

establishing (20). From Lemma 3.2 $\lim_{n\to\infty}\|x_n-p_1\|$ exists and hence $\lim_{n\to\infty}\|x_n-p_1\| = \lim_{n\to\infty}\|x_{n+m}-p_1\|$. Since $\delta(0)=0$ and $\lim_{n\to\infty}\prod_{j=n}^{\infty} a_j = 1$, then the continuity of δ yields $\lim_{n\to\infty} b_{n,m} = 0$ uniformly for all $m \geqslant 1$. Observe that

$$\begin{aligned} &\sigma_{n+m}(t) \\ &\quad \leqslant \big\|tx_{n+m}+(1-t)p_1-p_2+\big(S_{n,m}\big(tx_n+(1-t)p_1\big)-tS_{n,m}x_n-(1-t)S_{n,m}p_1\big)\big\| \\ &\qquad + \big\|-\big(S_{n,m}\big(tx_n+(1-t)p_1\big)-tS_{n,m}x_n-(1-t)S_{n,m}p_1\big)\big\| \\ &\quad = \big\|S_{n,m}\big(tx_n+(1-t)p_1\big)-p_2\big\|+b_{n,m} = \big\|S_{n,m}\big(tx_n+(1-t)p_1\big)-S_{n,m}p_2\big\|+b_{n,m} \\ &\quad \leqslant \left(\prod_{j=n}^{n+m-1} a_j\right)\big\|tx_n+(1-t)p_1-p_2\big\|+b_{n,m} = \left(\prod_{j=n}^{n+m-1} a_j\right)\sigma_n(t)+b_{n,m}. \end{aligned}$$

Hence $\limsup_{n\to\infty}\sigma_n(t) \leqslant \liminf_{n\to\infty}\sigma_n(t)$, completing the proof of Lemma 3.3. □

Lemma 3.4. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a nonempty convex subset of E and let $T : K \to K$ be a k-strictly asymptotically pseudocontractive map with a sequence $\{k_n\} \subseteq [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, and let $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{x_n\}$ be as in Lemma 3.2. Then for all $p_1, p_2 \in F(T)$, $\lim_{n\to\infty}\langle x_n, j(p_1 - p_2)\rangle$ exists. Furthermore, if $\omega_w(x_n)$ denotes the set of weak subsequential limits of $\{x_n\}$, then $\langle p - q, j(p_1 - p_2)\rangle = 0$, $\forall p_1, p_2 \in F(T)$, and $\forall p, q \in \omega_w(x_n)$.*

Proof. Since E is both uniformly convex and uniformly smooth, it has a Fréchet differentiable norm. Set $x = p_1 - p_2$ and $h = t(x_n - p_1)$ in (8) to obtain

$$\begin{aligned}
&\frac{1}{2}\|p_1 - p_2\|^2 + t\bigl\langle x_n - p_1, j(p_1 - p_2)\bigr\rangle \\
&\quad\leqslant \frac{1}{2}\bigl\|tx_n + (1-t)p_1 - p_2\bigr\|^2 \\
&\quad\leqslant \frac{1}{2}\|p_1 - p_2\|^2 + t\bigl\langle x_n - p_1, j(p_1 - p_2)\bigr\rangle + b\bigl(t\|x_n - p_1\|\bigr).
\end{aligned}$$

Since b is increasing and $\|x_n - p_1\| \leqslant M$, $\forall n \geqslant 1$ and for some $M > 0$, then

$$\begin{aligned}
&\frac{1}{2}\|p_1 - p_2\|^2 + t\bigl\langle x_n - p_1, j(p_1 - p_2)\bigr\rangle \\
&\quad\leqslant \frac{1}{2}\bigl\|tx_n + (1-t)p_1 - p_2\bigr\|^2 \\
&\quad\leqslant \frac{1}{2}\|p_1 - p_2\|^2 + t\bigl\langle x_n - p_1, j(p_1 - p_2)\bigr\rangle + b(tM).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{2}\|p_1 - p_2\|^2 + t\limsup_{n\to\infty}\bigl\langle x_n - p_1, j(p_1 - p_2)\bigr\rangle \\
&\quad\leqslant \frac{1}{2}\lim_{n\to\infty}\bigl\|tx_n + (1-t)p_1 - p_2\bigr\|^2 \\
&\quad\leqslant \frac{1}{2}\|p_1 - p_2\|^2 + t\liminf_{n\to\infty}\bigl\langle x_n - p_1, j(p_1 - p_2)\bigr\rangle + b(tM).
\end{aligned}$$

Hence $\limsup_{n\to\infty}\langle x_n, j(p_1 - p_2)\rangle \leqslant \liminf_{n\to\infty}\langle x_n, j(p_1 - p_2)\rangle + b(tM)/t$. Since $\lim_{t\to 0^+} b(t)/t = 0$, then $\lim_{n\to\infty}\langle x_n, j(p_1 - p_2)\rangle$ exists. Since $\lim_{n\to\infty}\langle x_n, j(p_1 - p_2)\rangle = \langle p, j(p_1 - p_2)\rangle$, $\forall p \in \omega_w(x_n)$, we have $\langle p - q, j(p_1 - p_2)\rangle = 0$, $\forall p_1, p_2 \in F(T)$ and $\forall p, q \in \omega_w(x_n)$. □

Theorem 3.2. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a nonempty closed convex subset of E and let $T : K \to K$ be a k-strictly asymptotically pseudocontractive map with a sequence $\{k_n\} \subseteq [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, and let $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{x_n\}$ be as in Lemma 3.2. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Since $\{x_n\}$ is bounded, it has a weakly convergent subsequence $\{x_{n_j}\}_{j=1}^{\infty}$. Suppose $\{x_{n_j}\}$ converges weakly to p. Then $p \in K$ because K is weakly closed. Since $\lim_{j\to\infty}\|x_{n_j} - Tx_{n_j}\| = 0$ and $(I - T)$ is demiclosed at zero, we must have $p - Tp = 0$, so that $p \in F(T)$. If $\{x_{m_k}\}$ is another subsequence of $\{x_n\}$ which converges weakly to some q. Then as for p, we must have $q \in F(T)$, and it follows from Lemma 3.4 that $p = q$. Hence $\omega_w(x_n)$ is singleton, so that $\{x_n\}$ converges weakly to a fixed point of T. $\square$

We now show with the following examples that the class of k-strictly asymptotically pseudocontractive maps and the class of strictly pseudocontractive maps are independent.

Example 1. Let R denote the reals with the usual norm. Define $T : R \to R$ by $Tx = -2x$. Observe that

$$\bigl|x - Tx - (y - Ty)\bigr| = 9|x - y|^2,$$

so that

$$\bigl\langle x - Tx - (y - Ty), x - y\bigr\rangle = 3|x - y|^2 = \frac{1}{3}\bigl|x - Tx - (y - Ty)\bigr|^2.$$

Thus, T is strictly pseudocontractive.

For n even $(n > 1)$

$$\bigl\langle T^n x - T^n y, x - y\bigr\rangle = 2^n|x - y|^2 > 2|x - y|^2.$$

Since $\lim_{n\to\infty} k_n = 1$, there exists N such that $k_n < 2$, $\forall n \geqslant N$. Thus we have

$$\begin{aligned}\bigl\langle T^n x - T^n y, x - y\bigr\rangle &> 2|x - y|^2 > k_n|x - y|^2\\ &\geqslant k_n|x - y|^2 - \lambda\bigl|\bigl(I - T^n\bigr)x - \bigl(I - T^n\bigr)y\bigr|^2, \quad \forall\lambda \in (0, 1).\end{aligned}$$

Thus, T is not asymptotically strictly pseudocontractive.

Example 2. Let $X = \ell_2 = \{\bar{x} = \{x_i\}_{i=1}^{\infty}\colon\ x_i \in C,\ \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$, and let $\bar{B} = \{\bar{x} \in \ell_2\colon \|x\| \leqslant 1\}$. Define $T : \bar{B} \to \ell_2$ by

$$T\bar{x} = \bigl(0, x_1^2, a_2x_2, a_3x_2, \ldots\bigr),$$

where $\{a_j\}_{j=1}^{\infty}$ is a real sequence satisfying: $a_2 > 0$, $0 < a_j < 1$, $j \neq 2$, and $\prod_{j=2}^{\infty} a_j = 1/2$. Then

$$\begin{aligned}\bigl\|T^n\bar{x} - T^n\bar{y}\bigr\|^2 &\leqslant 2\Biggl(\prod_{j=2}^{n} a_j\Biggr)\|\bar{x} - \bar{y}\|^2\\ &\leqslant 2\Biggl(\prod_{j=2}^{n} a_j\Biggr)\|\bar{x} - \bar{y}\|^2 + \lambda\bigl\|\bigl(I - T^n\bigr)\bar{x} - \bigl(I - T^n\bigr)\bar{y}\bigr\|^2\end{aligned}$$

for all $\lambda \in (0, 1)$, $n \geqslant 2$ and $\bar{x}, \bar{y} \in X$. Since $\lim_{n\to\infty} 2(\prod_{j=2}^{n} a_j) = 1$, it follows that T is asymptotically strictly pseudocontractive.

Choose $\bar{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, \ldots)$, $\bar{y} = (0, 0, 0, \ldots)$ and $a_2 = 3$, then

$$\begin{aligned}\langle T\bar{x} - T\bar{y}, \bar{x} - \bar{y}\rangle &= \biggl\langle\biggl(0, \frac{1}{9}, 1, \frac{a_3}{3}, 0, 0, 0, \ldots\biggr), \biggl(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0, \ldots\biggr)\biggr\rangle = \frac{10}{27} > \frac{1}{3}\\ &= \|\bar{x} - \bar{y}\|^2 \geqslant \|\bar{x} - \bar{y}\|^2 - \lambda\bigl\|(I - T)\bar{x} - (I - T)\bar{y}\bigr\|^2.\end{aligned}$$

Hence T is *not* strictly pseudocontractive.

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