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Elliptic inequalities with lower order terms and L^1 data in Orlicz spaces

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Abstract

We prove an existence result for solutions of nonlinear elliptic unilateral problems having natural growth terms and L^1 data in Orlicz–Sobolev spaces.

Keywords: Unilateral problems; Orlicz–Sobolev spaces; Truncations

1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geqslant 2$, with the segment property and let $f \in L^1(\Omega)$. Consider the following nonlinear Dirichlet problem:

$$A(u) + g(x, u, \nabla u) = f, \tag{1.1}$$

where $A(u) = -\operatorname{div}(a(x, u, \nabla u))$ is a Leray–Lions operator defined on $D(A) \subset W_0^1 L_M(\Omega)$, with M is an N-function and $g(x, s, \zeta)$ is a nonlinearity having the same sign of s and satisfying the following natural growth condition:

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$$\left|g(x,s,\zeta)\right| \leqslant b\big(|s|\big)\big(c(x)+M\big(|\zeta|\big)\big).$$

In the variational case (i.e., where $f \in W^{-1}E_{\overline{M}}(\Omega)$), it is well known that Gossez and Mustonen solved the following obstacle problem in the case where $g(x,u,\nabla u) \equiv g(x,u)$:

$$\begin{cases} u \in K_\phi, \\ \langle A(u), u-v\rangle + \int_\Omega g(x,u)(u-v)\,dx \leqslant \langle f, u-v\rangle, \quad \forall v \in K_\phi \cap L^\infty(\Omega), \end{cases} \tag{1.2}$$

where K_ϕ is a convex subset in $W_0^1L_M(\Omega)$ given by

$$K_\phi = \left\{v \in W_0^1L_M(\Omega)\colon\ v \geqslant \phi \text{ a.e. in } \Omega\right\},$$

where ϕ is a measurable function satisfying some regularity condition. Contributions in this direction include, for equations, [3,7,11].

In the general case where f belongs to $L^1(\Omega)$, many results have been obtained in this case, see, for example, [2] if $g \equiv g(x,u,\nabla u)$ satisfying further the following coercivity condition:

$$\left|g(x,s,\zeta)\right| \geqslant \beta|\zeta|^p \quad \text{for } |s| \geqslant \lambda. \tag{1.3}$$

Recently, the condition (1.3) is removed by the authors in [6].

It is our purpose in this paper to prove an existence theorem for unilateral problems corresponding to (1.1) without assuming the Δ_2 condition on the N-function M. So that, we generalize all previous works [4–6,8,12,13].

As examples of problems to which the present result can be applied (see also Remark 3.2), we give:

$$-\operatorname{div}\big(\exp\big(|\nabla u|\big)\nabla u\big) + u\exp(-u)\exp\big(|\nabla u|\big)|\nabla u|^2 = f,$$
$$-\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\log^\alpha\big(1+|\nabla u|\big)\big) + u\left|\cos(u)\right||\nabla u|^p\log^\alpha\big(1+|\nabla u|\big) = f$$

with $f \in L^1(\Omega)$, $p \geqslant 1$ and $\alpha > 0$.

2. Preliminaries

2.1. Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e., M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$.

Equivalently, M admits the representation: $M(t) = \int_0^t a(s)\,ds$, where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t)$ tends to ∞ as $t \to \infty$.

The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \int_0^t \bar{a}(s)\,ds$, where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\bar{a}(t) = \sup\{s\colon\ a(s) \leqslant t\}$ (see [1]).

The N-function is said to satisfy the Δ_2 condition, if for some $k > 0$,

$$M(2t) \leqslant kM(t) \quad \forall t \geqslant 0, \tag{2.1}$$

when (2.1) holds only for $t \geqslant$ some $t_0 > 0$ then M is said to satisfy the Δ_2 condition near infinity.

We will extend these N-functions into even functions on all $\mathbb{R}$.

Let P and Q be two N-functions. $P \ll Q$ means that P grows essentially less rapidly than Q, i.e., for each $\epsilon > 0$, $\frac{P(t)}{Q(\epsilon t)} \to 0$ as $t \to \infty$.

This is the case if and only if $\lim_{t\to\infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0$.

2.2. Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $K_M(\Omega)$ (respectively the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that:

$$\int_\Omega M\big(u(x)\big)\,dx < +\infty \qquad \left(\text{respectively } \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx < +\infty \text{ for some } \lambda > 0\right).$$

$L_M(\Omega)$ is Banach space under the norm

$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0\colon \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx \leqslant 1\right\}$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The equality $E_M(\Omega) = L_M(\Omega)$ holds if only if M satisfies Δ_2 condition, for all t or for t large according to whether Ω has infinite measure or not. The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega uv\,dx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|.\|_{\overline{M},\Omega}$. The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2 condition, for all t or for t large, according to whether Ω has infinite measure or not.

2.3. We now turn to the Orlicz–Sobolev space, $W^1L_M(\Omega)$ (respectively $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (respectively $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\leqslant 1} \big\|D^\alpha u\big\|_M.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $D(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $D(\Omega)$ in $W^1L_M(\Omega)$. We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$

$$\int_\Omega M\left(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\right)dx \to 0 \quad \text{for all } |\alpha| \leqslant 1.$$

This implies convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

If M satisfies Δ_2 condition on $\mathbb{R}^+$, then modular convergence coincides with norm convergence.

2.4. Let $W^{-1}L_{\overline{M}}(\Omega)$ (respectively $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order $\leqslant 1$ of functions in $L_{\overline{M}}$ (respectively $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property then the space $D(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (cf. [9,10]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined.

2.5. We recall some lemmas introduced in [3] (see also [11]) which we will be used in this paper.

Lemma 2.1. *Let $F:\mathbb{R}\to\mathbb{R}$ be uniformly Lipschitzian, with $F(0)=0$. Let M be an N-function and let $u\in W^1L_M(\Omega)$ (respectively $W^1E_M(\Omega)$). Then $F(u)\in W^1L_M(\Omega)$ (respectively $W^1E_M(\Omega)$). Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i}F(u)=\begin{cases}F'(u)\frac{\partial}{\partial x_i}u & \text{a.e. in } \{x\in\Omega\colon\ u(x)\notin D\},\\ 0 & \text{a.e. in } \{x\in\Omega\colon\ u(x)\notin D\}.\end{cases}$$

Lemma 2.2. *Let $F:\mathbb{R}\to\mathbb{R}$ be uniformly Lipschitzian, with $F(0)=0$. We suppose that the set of discontinuity points of F' is finite. Let M be an N-function, then the mapping $F:W^1L_M(\Omega)\to W^1L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\Pi L_M,\Pi E_{\overline{M}})$.*

2.6. We give now the following lemma which concerns operators of the Nemytskii type in Orlicz spaces (see [3]).

Lemma 2.3. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. Let M, P and Q be N-functions such that $Q\ll P$, and let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function such that for a.e. $x\in\Omega$ and for all $s\in\mathbb{R}$,*

$$\bigl|f(x,s)\bigr|\leqslant c(x)+k_1P^{-1}M\bigl(k_2|s|\bigr),$$

where k_1,k_2 are real constants and $c(x)\in E_Q(\Omega)$. Then the Nemytskii operator N_f defined by $N_f(u)(x)=f(x,u(x))$ is strongly continuous from

$$\mathcal{P}\left(E_M(\Omega),\frac{1}{k_2}\right)=\left\{u\in L_M(\Omega)\colon\ d\bigl(u,E_M(\Omega)\bigr)<\frac{1}{k_2}\right\}$$

into $E_Q(\Omega)$.

3. The main result

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N\geqslant 2$, with the segment property. Let

$$K_\psi=\bigl\{v\in W_0^1L_M(\Omega)\colon\ v\geqslant\psi \text{ a.e. in } \Omega\bigr\},$$

where $\psi:\Omega\to\overline{\mathbb{R}}$ is a given measurable function. Let M and P be two N-functions such that $P\ll M$. Let $A(u)=-\operatorname{div}(a(x,u,\nabla u))$ be a Leray–Lions operator defined on $D(A)\subset W_0^1L_M(\Omega)$ into $W^{-1}L_{\overline{M}}(\Omega)$ where $a:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function satisfying for a.e. $x\in\Omega$ and for all $\zeta,\zeta'\in\mathbb{R}^N$ ($\zeta\neq\zeta'$) and all $s\in\mathbb{R}$,

$$\bigl|a(x,s,\zeta)\bigr|\leqslant h(x)+k_1\overline{P}^{-1}M\bigl(k_2|s|\bigr)+k_3\overline{M}^{-1}M\bigl(k_4|\zeta|\bigr),\tag{3.1}$$

$$\bigl(a(x,s,\zeta)-a(x,s,\zeta')\bigr)(\zeta-\zeta')>0,\tag{3.2}$$

$$a(x,s,\zeta)\bigl(\zeta-\nabla\bar{v}(x)\bigr)\geqslant\alpha M\bigl(|\zeta|\bigr)-d(x)\tag{3.3}$$

with $\bar{v}(x)\in K_\psi\cap L^\infty(\Omega)\cap W_0^1E_M(\Omega)$, $d\in L^1(\Omega)$, $\alpha,k_1,k_2,k_3,k_4>0$ and $h\in E_{\overline{M}}(\Omega)$.

Furthermore, let $g:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ be a Carathéodory function such that for a.e. $x\in\Omega$ and for all $s\in\mathbb{R}$ and all $\zeta\in\mathbb{R}^N$,

$$g(x,s,\zeta)s \geqslant 0, \tag{3.4}$$

$$\left|g(x,s,\zeta)\right| \leqslant b\big(|s|\big)\big(c(x)+M\big(|\zeta|\big)\big), \tag{3.5}$$

where $b:\mathbb{R}_+ \to \mathbb{R}$ is a continuous and nondecreasing function and $c(x)$ is a given nonnegative function in $L^1(\Omega)$.

Now, assume that

$$K_\psi \cap W_0^1 E_M(\Omega) \quad \text{is dense in } K_\psi \tag{3.6}$$

for the modular convergence in $W_0^1 L_M(\Omega)$. Finally, we assume that

$$f \in L^1(\Omega). \tag{3.7}$$

We define by $T_0^{1,M}(\Omega)$ as the set of measurable functions $u:\Omega \to \mathbb{R}$ such that $T_k(u) \in W_0^1 L_M(\Omega) \cap D(A)$, where $T_k(s) = \max(-k, \min(k,s))$, $\forall s \in \mathbb{R}$, $\forall k \geqslant 0$.

We shall prove the following existence theorem.

Theorem 3.1. *Assume that* (3.1)–(3.7) *hold true. Then there exists at least one solution of the following obstacle problem*:

$$\begin{cases} u \in T_0^{1,M}(\Omega), \\ u \geqslant \psi \text{ a.e. in } \Omega, \quad g(x,u,\nabla u) \in L^1(\Omega), \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \\ \qquad \leqslant \displaystyle\int_\Omega f T_k(u-v)\,dx, \quad \forall v \in K_\psi \cap L^\infty(\Omega),\ \forall k > 0. \end{cases} \tag{P_ψ}$$

Remark 3.1. If $\psi \in W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ or if there exists $\overline{\psi} \in K_\psi \cap L^\infty(\Omega) \cap W_0^1 E_M(\Omega)$ such that $\psi - \overline{\psi}$ is continuous then (3.6) is satisfied.

Note that if M satisfies the Δ_2 condition, then the density (3.6) is trivially satisfied.

Remark 3.2. Let $m:\mathbb{R} \to \mathbb{R}$ be continuous, odd, strictly increasing from $-\infty$ to $+\infty$ and consider the Dirichlet problem

$$-\operatorname{div}\left(a(x,u)m\big(|\nabla u|\big)\frac{\nabla u}{|\nabla u|}\right) + g(u)m\big(|\nabla u|\big)|\nabla u| = f \quad \text{in } \Omega,$$

where $a(x,u)$ is a Carathéodory function such that $\alpha \leqslant a(x,u) \leqslant \beta$ and g is a continuous function satisfying $g(s)s \geqslant 0$. Then, the assumptions (3.1)–(3.5) of Theorem 3.1 hold true (see Remark 8 of [12]).

Proof of Theorem 3.1. *Step* 1: A priori estimates.

For the sake of simplicity, we assume that $d(x) = 0$. Let now λ such that $\lambda \geqslant \|\bar{v}\|_\infty$, $\gamma = (\frac{b(\lambda)}{2\alpha})^2$ and $\phi(s) = s\exp(\gamma s^2)$. It is well known that

$$\phi'(s) - \frac{b(\lambda)}{\alpha}\left|\phi(s)\right| \geqslant \frac{1}{2}, \quad \forall s \in \mathbb{R}. \tag{3.8}$$

Consider the approximate problems:

$$\begin{cases} u_n \in K_\psi \cap D(A), \\ \langle A(u_n), u_n - w\rangle + \int_\Omega g_n(x, u_n, \nabla u_n)(u_n - w)\,dx \leqslant \int_\Omega f_n(u_n - w)\,dx, \\ \quad \forall w \in K_\psi, \end{cases} \tag{3.9}$$

where $g_n(x, s, \zeta) = T_n(g(x, s, \zeta))$ and f_n is a sequence of smooth functions which converges strongly to f in $L^1(\Omega)$.

By Proposition 1 of [12], there exists at least one solution u_n of (3.9). By taking $v = u_n - \delta\phi(T_{2\|\bar{v}\|_\infty}(u_n - \bar{v}))$, as test function in (3.9), with $\delta = \exp(-4\gamma\|\bar{v}\|_\infty^2)$, we obtain

$$\begin{aligned} &\int_{\{|u_n-\bar{v}|<2\|\bar{v}\|_\infty\}} a(x, u_n, \nabla u_n)\nabla(u_n - \bar{v})\phi'\big(T_{2\|\bar{v}\|_\infty}(u_n - \bar{v})\big)\,dx \\ &\qquad + \int_\Omega g_n(x, u_n, \nabla u_n)\phi\big(T_{2\|\bar{v}\|_\infty}(u_n - \bar{v})\big)\,dx \\ &\quad \leqslant \int_\Omega f_n\phi\big(T_{2\|\bar{v}\|_\infty}(u_n - \bar{v})\big)\,dx \end{aligned}$$

which gives, since $g_n(x, u_n, \nabla u_n)\phi(T_{2\|\bar{v}\|_\infty}(u_n - \bar{v})) \geqslant 0$ on the set $\{x \in \Omega\colon |u_n| \geqslant \|\bar{v}\|_\infty\}$,

$$\begin{aligned} &\int_{\{|u_n-\bar{v}|<2\|\bar{v}\|_\infty\}} a(x, u_n, \nabla u_n)\nabla(u_n - \bar{v})\phi'\big(T_{2\|\bar{v}\|_\infty}(u_n - \bar{v})\big)\,dx \\ &\qquad + \int_{\{|u_n|<\|\bar{v}\|_\infty\}} g_n(x, u_n, \nabla u_n)\phi\big(T_{2\|\bar{v}\|_\infty}(u_n - \bar{v})\big)\,dx \\ &\quad \leqslant \int_\Omega f_n\phi\big(T_{2\|\bar{v}\|_\infty}(u_n - \bar{v})\big)\,dx. \end{aligned}$$

Thanks to (3.5), one easily obtains

$$\begin{aligned} &\int_{\{|u_n-\bar{v}|<2\|\bar{v}\|_\infty\}} a(x, u_n, \nabla u_n)\nabla(u_n - \bar{v})\phi'\big(T_{2\|\bar{v}\|_\infty}(u_n - \bar{v})\big)\,dx \\ &\quad \leqslant \int_{\{|u_n|<\|\bar{v}\|_\infty\}} b\big(\|\bar{v}\|_\infty\big)\big|\phi\big(T_{2\|\bar{v}\|_\infty}(u_n - \bar{v})\big)\big|\big(c(x) + M\big(|\zeta|\big)\big)\,dx + C \end{aligned}$$

which implies

$$\int_{\{|u_n-\bar{v}|<2\|\bar{v}\|_\infty\}} \alpha M\big(|\nabla u_n|\big)\big[\phi'\big(T_{2\|\bar{v}\|_\infty}(u_n - \bar{v})\big) - b\big(\|\bar{v}\|_\infty\big)\phi\big(T_{2\|\bar{v}\|_\infty}(u_n - \bar{v})\big)\big]\,dx \leqslant C$$

and by using (3.9), one easily has

$$\int_{\{|u_n-\bar{v}|<2\|\bar{v}\|_\infty\}} M\big(|\nabla u_n|\big)\,dx \leqslant C, \quad \forall n,$$

consequently

$$\int\limits_{\{|u_n|<\|\bar{v}\|_\infty\}} M\big(|\nabla u_n|\big)\,dx \leqslant C, \quad \forall n. \tag{3.10}$$

On the other hand, the choice of $w = u_n - T_k(u_n - v)$ as test function in (3.9) with $v \in K_\psi$, yields

$$\begin{cases} \big\langle A(u_n), T_k(u_n - v)\big\rangle + \int\limits_\Omega g_n(x, u_n, \nabla u_n) T_k(u_n - v)\,dx \leqslant \int\limits_\Omega f_n T_k(u_n - v)\,dx, \\ \forall v \in K_\psi,\ \forall k > 0. \end{cases} \tag{P_n}$$

Take now, $v = \bar{v}$ as test function in (P_n), we obtain for every $k > 0$,

$$\int\limits_{\{|u_n-\bar{v}|<k\}} a(x, u_n, \nabla u_n)\nabla(u_n - \bar{v})\,dx + \int\limits_{\{|u_n|<\|\bar{v}\|_\infty\}} g_n(x, u_n, \nabla u_n) T_k(u_n - \bar{v})\,dx$$
$$\leqslant \int\limits_\Omega f_n T_k(u_n - \bar{v})\,dx.$$

Consequently from (3.5) and (3.10), one easily has

$$\int\limits_{\{|u_n-\bar{v}|\leqslant k\}} a(x, u_n, \nabla u_n)\nabla(u_n - \bar{v})\,dx \leqslant Ck. \tag{3.11}$$

Thus by using (3.3) (with $d(x) = 0$) we obtain

$$\alpha \int\limits_{\{|u_n-\bar{v}|\leqslant k\}} M\big(|\nabla u_n|\big)\,dx \leqslant Ck.$$

Finally, we have for any $h > 0$,

$$\int\limits_{\{|u_n|\leqslant h\}} M\big(|\nabla u_n|\big)\,dx \leqslant \int\limits_{\{|u_n-\bar{v}|\leqslant h+\|\bar{v}\|_\infty\}} M\big(|\nabla u_n|\big)\,dx \leqslant C\big(h + \|\bar{v}\|_\infty\big) \tag{3.12}$$

which shows that

$$\int\limits_\Omega M\big(\big|\nabla T_h(u_n)\big|\big)\,dx \leqslant C\big(h + \|\bar{v}\|_\infty\big), \tag{3.13}$$

thanks to Lemma 5.7 of [9] there exist two positive constants c_1 and c_2 such that

$$\int\limits_\Omega M(v)\,dx \leqslant c_1 \int\limits_\Omega M\big(c_2|\nabla v|\big)\,dx, \quad \forall v \in W_0^1 L_M(\Omega). \tag{3.14}$$

Choosing, now $v = \frac{|T_h(u_n)|}{c_2}$ in (3.14) and using (3.13), we get

$$\int\limits_\Omega M\left(\frac{|T_h(u_n)|}{c_2}\right)dx \leqslant c_3\big(h + \|\bar{v}\|_\infty\big)$$

which implies that

$$\operatorname{meas}\{|u_n| > h\} \leqslant \frac{c_3(h + \|\bar{v}\|_\infty)}{M(\frac{h}{c_2})}, \quad \forall n,\ \forall k \geqslant \|\bar{v}\|_\infty.$$

We have for any $\delta > 0$

$$\begin{aligned}\operatorname{meas}\{|u_n - u_m| > \delta\} \leqslant{}& \operatorname{meas}\{|u_n| > h\} + \operatorname{meas}\{|u_m| > h\}\\ &+ \operatorname{meas}\{|T_h(u_n) - T_h(u_m)| > \delta\}\end{aligned}$$

which gives

$$\operatorname{meas}\{|u_n - u_m| > \delta\} \leqslant \frac{2c_3(h + \|\bar{v}\|_\infty)}{M(\frac{h}{c_2})} + \operatorname{meas}\{|T_h(u_n) - T_h(u_m)| > \delta\}. \tag{3.15}$$

Thanks to (3.13), we deduce that $(T_h(u_n))$ is bounded in $W_0^1 L_M(\Omega)$ and then we can assume that $(T_h(u_n))$ is a Cauchy sequence in measure in Ω.

Let $\epsilon > 0$, then by (3.15) and the fact that $\frac{t}{M(\frac{t}{c_2})} \to 0$ as $t \to \infty$, there exists $h(\epsilon) > 0$ such that

$$\operatorname{meas}\{|u_n - u_m| > \delta\} \leqslant \epsilon \quad \text{for all } n, m \geqslant n_0(h(\epsilon), \delta).$$

This proves that (u_n) is a Cauchy sequence in measure and then converges almost everywhere to some measurable function u. Finally, we deduce from (3.13) and Lemma 4.4 of [9], that

$$T_h(u_n) \to T_h(u) \quad \text{weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}), \quad \text{strongly in } E_M(\Omega). \tag{3.16}$$

Let us show now, that $(a(x, T_h(u_n), \nabla T_h(u_n))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$. Let $\varphi \in (E_M(\Omega))^N$, then by using (3.2), one easily has for every $k > 0$,

$$\begin{aligned}&\int\limits_{\{|u_n - \bar{v}| \leqslant k\}} a(x, u_n, \nabla u_n)(k_4\varphi) - \nabla\bar{v}\, dx\\ &\quad\leqslant \int\limits_{\{|u_n - \bar{v}| \leqslant k\}} a(x, u_n, \nabla u_n)(\nabla u_n - \nabla\bar{v})\, dx - \int\limits_{\{|u_n - \bar{v}| \leqslant k\}} a(x, u_n, k_4\varphi)(\nabla u_n - k_4\varphi)\, dx\end{aligned}$$

which gives by (3.11)

$$\int\limits_{\{|u_n - \bar{v}| \leqslant k\}} a(x, u_n, \nabla u_n)(k_4\varphi) - \nabla\bar{v}\, dx \leqslant Ck - \int\limits_{\{|u_n - \bar{v}| \leqslant k\}} a(x, u_n, k_4\varphi)(\nabla u_n - k_4\varphi)\, dx.$$

Since φ is arbitrary in $(E_M(\Omega))^N$, we choose $\eta = k_4\varphi - \nabla\bar{v}$ in the last inequality with $\|\eta\|_{(L_M(\Omega))^N} = 1$ and we find

$$\int\limits_{\{|u_n - \bar{v}| \leqslant k\}} a(x, u_n, \nabla u_n)\eta\, dx \leqslant Ck - \int\limits_{\{|u_n - \bar{v}| \leqslant k\}} a(x, u_n, \eta + \bar{v})(\nabla u_n - \eta - \nabla\bar{v})\, dx$$

which implies by using (3.1), that

$$\int\limits_{\{|u_n - \bar{v}| \leqslant k\}} a(x, u_n, \nabla u_n)\eta\, dx \leqslant C_{k,\bar{v}},$$

where $C_{k,\bar{v}}$ is a constant which depends on k and $\bar{v}$ but not on n.

Consequently by using the dual norm, one has $|a(x,u_n,\nabla u_n)|\chi_{\{|u_n-\bar{v}|\leqslant k\}}$ is bounded in $(L_{\overline{M}}(\Omega))^N$.

On the other hand, we have

$$\int_\Omega a\big(x,T_h(u_n),\nabla T_h(u_n)\big)\eta\,dx \leqslant \int_\Omega \big|a(x,u_n,\nabla u_n)\big|\chi_{\{|u_n-\bar{v})|\leqslant h+\|\bar{v}\|_\infty\}}\eta\,dx$$

which gives by Hölder inequality

$$\int_\Omega a\big(x,T_h(u_n),\nabla T_h(u_n)\big)\eta\,dx \leqslant 2\big\|a(x,u_n,\nabla u_n)\chi_{\{|u_n-\bar{v}|\leqslant h+\|\phi\|_\infty\}}\big\|_{(L_{\overline{M}}(\Omega))^N},$$

where we have used the fact that $\|\eta\|_{(L_M(\Omega))^N}=1$. So that $a(x,T_h(u_n),\nabla T_h(u_n))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$.

Step 2: Convergence of truncations.

Thanks to the assumption (3.6), there exists a sequence $w_j\in K_\psi(\Omega)\cap W^1E_M(\Omega)$ which converges to $T_k(u)$ for the modular convergence in $W_0^1L_M(\Omega)$.

Consider now the function θ_m, $m>0$ defined by

$$\theta_m(t)=1-\big|T_m\big(u_n-T_m(u_n)\big)\big|.$$

Let $v_{n,m,j}=u_n-\eta\theta_m(u_n-\bar{v})\phi(z_n^j)$, with $\eta=\exp(-4\gamma k^2)$, $z_n=T_k(u_n)-T_k(w_j)$ and $m>k+\|\bar{v}\|_\infty$, with $k\geqslant\|\bar{v}\|_\infty$. The use of $v_{n,m,j}$ as test function in (P_n) gives, for all $h>0$,

$$\begin{aligned}&\big\langle A(u_n),T_h\big(\eta\theta_m(u_n-\bar{v})\phi\big(z_n^j\big)\big)\big\rangle+\int_\Omega g_n(x,u_n,\nabla u_n)T_h\big(\eta\theta_m(u_n-\bar{v})\phi\big(z_n^j\big)\big)\,dx\\&\quad\leqslant\int_\Omega f_nT_h\big(\eta\theta_m(u_n-\bar{v})\phi\big(z_n^j\big)\big)\,dx,\end{aligned}$$

and by taking $h>2k$ we obtain

$$\begin{aligned}&\big\langle A(u_n),\theta_m(u_n-\bar{v})\phi\big(z_n^j\big)\big\rangle+\int_\Omega g_n(x,u_n,\nabla u_n)\theta_m(u_n-\bar{v})\phi\big(z_n^j\big)\,dx\\&\quad\leqslant\int_\Omega f_n\theta_m(u_n-\bar{v})\phi\big(z_n^j\big)\,dx\end{aligned}$$

which gives

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\big[\nabla T_k(u_n)-\nabla T_k(wj)\big]\theta_m(u_n-\bar{v})\phi'\big(z_n^j\big)\,dx\\&\qquad+\int_\Omega a(x,u_n,\nabla u_n)(\nabla u_n-\nabla\bar{v})\theta_m'(u_n-\bar{v})\phi\big(z_n^j\big)\,dx\\&\qquad+\int_\Omega g_n(x,u_n,\nabla u_n)\theta_m(u_n-\bar{v})\phi\big(z_n^j\big)\,dx\\&\quad\leqslant\int_\Omega f_n\theta_m(u_n-\bar{v})\phi\big(z_n^j\big)\,dx.\end{aligned}\tag{3.17}$$

Denote by $\epsilon^1(n,j), \epsilon^2(n,j), \ldots$ various sequences of real numbers which converge to zero when n and j tend to infinity in this order. Since $g_n(x,u_n,\nabla u_n)\theta_m(u_n)\phi(z_n^j) \geqslant 0$ on the subset $\{x \in \Omega\colon |u_n(x)| > k\}$, we deduce from (3.17) that

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\big[\nabla T_k(u_n) - \nabla T_k(w_j)\big]\theta_m(u_n-\bar v)\phi'\big(z_n^j\big)\,dx \\
&\qquad + \int_\Omega a(x,u_n,\nabla u_n)\nabla u_n - \nabla \bar v \theta_m'(u_n-\bar v)\phi\big(z_n^j\big)\,dx \\
&\qquad + \int_{\{|u_n|\leqslant k\}} g_n(x,u_n,\nabla u_n)\theta_m(u_n-\bar v)\phi\big(z_n^j\big)\,dx \\
&\quad \leqslant \int_\Omega f_n\theta_m(u_n-\bar v)\phi\big(z_n^j\big)\,dx = \epsilon^1(n,j).
\end{aligned} \tag{3.18}$$

For the first term of the left-hand side of the last inequality, we have

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\big[\nabla T_k(u_n) - \nabla T_k(w_j)\big]\theta_m(u_n-\bar v)\phi'\big(z_n^j\big)\,dx \\
&\quad = \int_{\{|u_n|\leqslant k\}} a(x,u_n,\nabla u_n)\big[\nabla T_k(u_n) - \nabla T_k(w_j)\big]\theta_m(u_n-\bar v)\phi'\big(z_n^j\big)\,dx \\
&\qquad - \int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(w_j)\theta_m(u_n-\bar v)\phi'\big(z_n^j\big)\,dx,
\end{aligned}$$

by using the fact that $\theta_m(u_n-\bar v) = 0$ on the set $\{x \in \Omega\colon |u_n| > 2m\}$ and $\theta_m(u_n-\bar v) = 1$ on the set $\{x \in \Omega\colon |u_n| \leqslant k\}$, since $m > k + \|\bar v\|_\infty$, we obtain

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\big[\nabla T_k(u_n) - \nabla T_k(w_j)\big]\theta_m(u_n-\bar v)\phi'\big(z_n^j\big)\,dx \\
&\quad = \int_{\{|u_n|\leqslant k\}} a(x,u_n,\nabla u_n)\big[\nabla T_k(u_n) - \nabla T_k(w_j)\big]\phi'\big(z_n^j\big)\,dx \\
&\qquad - \int_{\{2m\geqslant|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(w_j)\theta_m(u_n-\bar v)\phi'\big(z_n^j\big)\,dx.
\end{aligned}$$

The second term of the right-hand side of the last equality reads as

$$\begin{aligned}
&-\int_{\{|u_n|>k\}} a\big(x,T_{2m}(u_n),\nabla T_{2m}(u_n)\big)\nabla T_k(w_j)\theta_m(u_n-\bar v)\phi'\big(z_n^j\big)\,dx \\
&\quad = -\int_{\{|u|>k\}} h_{2m}\nabla T_k(u)\theta_m(u-\bar v)\,dx + \epsilon^2(n,j).
\end{aligned}$$

Since $\nabla T_k(u) = 0$ on $\{|u| > k\}$, we deduce that

$$-\int_{\{2m \geqslant |u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(w_j) \theta_m(u_n - \bar{v}) \phi'(z_n)\, dx = \epsilon(n, j),$$

where we have used the fact that

$$a\big(x, T_{2m}(u_n), \nabla T_{2m}(u_n)\big) \to h_{2m} \quad \text{weakly in } \big(L_{\overline{M}}(\Omega)\big)^N \text{ for } \sigma\big(\Pi L_{\overline{M}}(\Omega), \Pi E_M(\Omega)\big).$$

Denote now by $\chi_{j,s}$ and χ_s respectively the characteristic functions of the sets $\Omega_s^j = \{x \in \Omega\colon |\nabla T_k(w_j)| \leqslant s\}$ and $\Omega_s = \{x \in \Omega\colon |\nabla T_k(u)| \leqslant s\}$. We have

$$\begin{aligned}
&\int_{\{|u_n| \leqslant k\}} a(x, u_n, \nabla u_n)\big[\nabla T_k(u_n) - \nabla T_k(w_j)\big]\phi'\big(z_n^j\big)\, dx \\
&\quad = \int_\Omega \big[a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(w_j)\chi_{j,s}\big)\big] \\
&\qquad \times \big[\nabla T_k(u_n) - \nabla T_k(u)\chi_{j,s}\big]\phi'\big(z_n^j\big)\, dx \\
&\qquad + \int_\Omega a\big(x, T_k(u_n), \nabla T_k(w_j)\chi_{j,s}\big)\big[\nabla T_k(u_n) - \nabla T_k(w_j)\chi_{j,s}\big]\phi'\big(z_n^j\big)\, dx \\
&\qquad - \int_\Omega a\big(x, T_k(u_n), \nabla T_k(u_n)\big)\nabla T_k(w_j)\chi_{\Omega\setminus\Omega_s^j}\phi'\big(z_n^j\big)\, dx.
\end{aligned} \tag{3.19}$$

The second term of the right-hand side of (3.19) tends $\int_{\Omega\setminus\Omega_s} a(x, T_k(u), 0)\nabla T_k(u)\, dx$, as n and j tend to infinity. Indeed, since

$$a\big(x, T_k(u_n), \nabla T_k(w_j)\chi_{j,s}\big)\phi'\big(z_n^j\big) \to a\big(x, T_k(u), \nabla T_k(w_j)\chi_{j,s}\big) \quad \text{strongly in } \big(E_{\overline{M}}(\Omega)\big)^N$$

by Lemma 2.3 and

$$\nabla T_k(u_n) \rightharpoonup \nabla T_k(u) \quad \text{weakly in } \big(L_M(\Omega)\big)^N \text{ for } \sigma\big(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega)\big).$$

For what concerns the third term, one can remark that

$$\begin{aligned}
&\int_\Omega a\big(x, T_k(u_n), \nabla T_k(w_j)\chi_{j,s}\big)\big[\nabla T_k(u_n) - \nabla T_k(w_j)\chi_{j,s}\big]\phi'\big(z_n^j\big)\, dx \\
&\quad = \int_\Omega a\big(x, T_k(u), \nabla T_k(w_j)\chi_{j,s}\big)\big[\nabla T_k(u) - \nabla T_k(w_j)\chi_{j,s}\big]\, dx \\
&\quad = \int_{\Omega\setminus\Omega_s} a\big(x, T_k(u), 0\big)\nabla T_k(u)\, dx + \epsilon^3(n, j),
\end{aligned}$$

where we have used the fact

$$a\big(x, T_k(u), \nabla T_k(w_j)\chi_{j,s}\big) \to a\big(x, T_k(u), \nabla T_k(u)\chi_s\big) \quad \text{strongly in } \big(E_{\overline{M}}(\Omega)\big)^N$$

and

$$T_k(w_j) \to T_k(u) \quad \text{for the modular convergence in } W_0^1 L_M(\Omega).$$

The third term of (3.19) tends to $-\int_\Omega h_k \nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\,dx$ as $n, j \to \infty$ since

$$a\big(x, T_k(u_n), \nabla T_k(u_n)\big) \rightharpoonup h_k \quad \text{weakly for } \sigma\big(\Pi L_{\overline{M}}(\Omega), \Pi E_M(\Omega)\big)$$

while $\nabla T_k(w_j)\chi_{\Omega\setminus\Omega_s^j} \in E_M(\Omega)$ and $\nabla T_k(w_j)\chi_{\Omega\setminus\Omega_s^j} \to \nabla T_k(u)\chi_{\Omega\setminus\Omega_s}$ as j tends to infinity.

Consequently, from (3.19) we have

$$\begin{aligned}
&\int_\Omega a(x, u_n, \nabla u_n)\big[\nabla T_k(u_n) - \nabla T_k(w_j)\big]\phi'\big(z_n^j\big)\theta_m(u_n)\,dx \\
&\quad = \int_\Omega \big[a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(w_j)\chi_{j,s}\big)\big] \\
&\qquad \times \big[\nabla T_k(u_n) - \nabla T_k(w_j)\chi_{j,s}\big]\phi'\big(z_n^j\big)\,dx \\
&\qquad - \int_\Omega h_k \nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\,dx + \int_{\Omega\setminus\Omega_s} a\big(x, T_k(u), 0\big)\nabla T_k(u)\,dx + \epsilon^3(n, j).
\end{aligned} \tag{3.20}$$

On the other hand,

$$\begin{aligned}
&\left|\int_\Omega a(x, u_n, \nabla u_n)(\nabla u_n - \nabla \bar{v})\theta_m'(u_n - \bar{v})\phi\big(z_n^j\big)\,dx\right| \\
&\quad \leqslant \frac{2\phi(2k)}{m} \int_{\{m \leqslant |u_n - \bar{v}| \leqslant 2m\}} a(x, u_n, \nabla u_n)(\nabla u_n - \nabla \bar{v})\,dx
\end{aligned}$$

and by using $u_n - T_m(u_n - \bar{v} - T_m(u_n - \bar{v}))$ as test function in (3.8), we obtain

$$\left|\int_\Omega a(x, u_n, \nabla u_n)(\nabla u_n - \nabla \bar{v})\theta_m'(u_n - \bar{v})\phi(z_n)\,dx\right| \leqslant 2\phi(2k) \int_{\{|u_n - \bar{v}| \geqslant m\}} |f_n|\,dx. \tag{3.21}$$

If we denote by $K_{n,m,j}$ the third term of the left-hand side of (3.19), one has by using the fact that

$$0 \leqslant \theta_m(u_n - \bar{v}) \leqslant 1, \tag{3.22}$$

$$\begin{aligned}
|K_{n,m,j}| &\leqslant \int_{\{|u_n| \leqslant k\}} b(k)\big(c(x) + M\big(|\nabla u_n|\big)\big)\big|\phi\big(z_n^j\big)\big|\,dx \\
&\leqslant b(k)\int_\Omega c(x)\big|\phi\big(z_n^j\big)\big|\,dx + \frac{b(k)}{\alpha}\int_\Omega a\big(x, T_k(u_n), \nabla T_k(u_n)\big)\nabla T_k(u_n)\big|\phi\big(z_n^j\big)\big|\,dx \\
&\leqslant \epsilon^4(n, j) + \frac{b(k)}{\alpha}\int_\Omega \big[a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(w_j)\chi_{j,s}\big)\big] \\
&\qquad \times \big[\nabla T_k(u_n) - \nabla T_k(w_j)\chi_{j,s}\big]\big|\phi\big(z_n^j\big)\big|\,dx,
\end{aligned} \tag{3.23}$$

indeed, we have

$$\int_\Omega a\big(x, T_k(u_n), \nabla T_k(u_n)\big)\nabla T_k(u_n)\big|\phi\big(z_n^j\big)\big|\,dx$$
$$= \int_\Omega \big[a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(w_j)\chi_{j,s}\big)\big]$$
$$\times \big[\nabla T_k(u_n) - \nabla T_k(w_j)\chi_{j,s}\big]\big|\phi\big(z_n^j\big)\big|\,dx$$
$$+ \int_\Omega a\big(x, T_k(u_n), \nabla T_k(u_n)\big)\nabla T_k(w_j)\chi_{j,s}\big|\phi\big(z_n^j\big)\big|\,dx$$
$$+ \int_\Omega a\big(x, T_k(u_n), \nabla T_k(w_j)\chi_{j,s}\big)\big[\nabla T_k(u_n) - \nabla T_k(w_j)\chi_{j,s}\big]\big|\phi\big(z_n^j\big)\big|\,dx. \tag{3.24}$$

It is easy to see that the second term of the right-hand side of the last equality can be reads as

$$\int_\Omega a\big(x, T_k(u_n), \nabla T_k(u_n)\big)\nabla T_k(w_j)\chi_{j,s}\big|\phi\big(z_n^j\big)\big|\,dx$$
$$= \int_\Omega h_k \nabla T_k(w_j)\chi_{j,s}\big|\phi\big(T_k(u) - T_k(w_j)\big)\big| + \epsilon^j(n) = \epsilon^5(n, j),$$

where $\epsilon^j(n)$ is a sequence which converges to 0 as $n \to \infty$ for j fixed.

For the third term of the right-hand side of (3.24), it is easily seen that

$$\int_\Omega a\big(x, T_k(u_n), \nabla T_k(w_j)\chi_{j,s}\big)\big[\nabla T_k(u_n) - \nabla T_k(w_j)\chi_{j,s}\big]\big|\phi\big(z_n^j\big)\big|\,dx = \epsilon^6(n, j).$$

Combining (3.20), (3.21) and (3.23) we obtain

$$\int_\Omega \big[a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(w_j)\chi_{j,s}\big)\big]$$
$$\times \big[\nabla T_k(u_n) - \nabla T_k(w_j)\chi_{j,s}\big]\left(\phi'\big(z_n^j\big) - \frac{b(k)}{\alpha}\big|\phi\big(z_n^j\big)\big|\right)dx$$
$$\leqslant \epsilon^7(n, j) + \int_\Omega h_k \nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\,dx + \phi(2k)\int_{\{|u_n-\bar{v}|\geqslant m\}} |f_n|\,dx$$
$$+ \int_{\Omega\setminus\Omega_s} \big|a\big(x, T_k(u), 0\big)\big|\big|\nabla T_k(u)\big|\,dx$$

which implies, by using (3.8)

$$\int_\Omega \big[a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(w_j)\chi_{j,s}\big)\big]\big[\nabla T_k(u_n) - \nabla T_k(w_j)\chi_{j,s}\big]$$
$$\leqslant 2\epsilon^7(n, j) + 2\int_\Omega h_k \nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\,dx + 4\phi(2k)\int_{\{|u_n-\bar{v}|\geqslant m\}} |f_n|\,dx$$
$$+ \int_{\Omega\setminus\Omega_s} 2\big|a\big(x, T_k(u), 0\big)\big|\big|\nabla T_k(u)\big|\,dx. \tag{3.25}$$

Remark now that

$$\begin{aligned}
&\int_\Omega \big[a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(u)\chi_s\big)\big]\big[\nabla T_k(u_n) - \nabla T_k(u)\chi_s\big]\,dx \\
&\quad = \int_\Omega \big[a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(w_j)\chi_{j,s}\big)\big] \\
&\qquad \times \big[\nabla T_k(u_n) - \nabla T_k(w_j)\chi_{j,s}\big]\,dx \\
&\qquad + \int_\Omega a\big(x, T_k(u_n), \nabla T_k(u_n)\big)\big[\nabla T_k(w_j)\chi_{j,s} - \nabla T_k(u)\chi_s\big]\,dx \\
&\qquad - \int_\Omega a\big(x, T_k(u_n), \nabla T_k(u)\chi_s\big)\big[\nabla T_k(u_n) - \nabla T_k(u)\chi_s\big]\,dx \\
&\qquad + \int_\Omega a\big(x, T_k(u_n), \nabla T_k(w_j)\chi_{j,s}\big)\big[\nabla T_k(u_n) - \nabla T_k(w_j)\chi_{j,s}\big]\,dx. \qquad (3.26)
\end{aligned}$$

We argue as above to show that

$$\int_\Omega a\big(x, T_k(u_n), \nabla T_k(u_n)\big)\big[\nabla T_k(w_j)\chi_{j,s} - \nabla T_k(u)\chi_s\big]\,dx = \epsilon^8(n, j),$$

$$\begin{aligned}
&-\int_\Omega a\big(x, T_k(u_n), \nabla T_k(u)\chi_s\big)\big[\nabla T_k(u_n) - \nabla T_k(u)\chi_s\big]\,dx \\
&\quad = -\int_{\Omega\setminus\Omega_s} a\big(x, T_k(u), 0\big)\nabla T_k(u)\,dx + \epsilon^9(n, j)
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega a\big(x, T_k(u_n), \nabla T_k(w_j)\chi_{j,s}\big)\big[\nabla T_k(u_n) - \nabla T_k(w_j)\chi_{j,s}\big]\,dx \\
&\quad = \int_{\Omega\setminus\Omega_s} a\big(x, T_k(u), 0\big)\nabla T_k(u)\,dx + \epsilon^{10}(n, j).
\end{aligned}$$

Consequently, one has

$$\begin{aligned}
&\int_\Omega \big[a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(u)\chi_s\big)\big]\big[\nabla T_k(u_n) - \nabla T_k(u)\chi_s\big]\,dx \\
&\quad = \int_\Omega \big[a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(w_j)\chi_{j,s}\big)\big] \\
&\qquad \times \big[\nabla T_k(u_n) - \nabla T_k(w_j)\chi_{j,s}\big]\,dx \\
&\qquad + \int_\Omega a\big(x, T_k(u_n), \nabla T_k(u_n)\big)\big[\nabla T_k(w_j)\chi_{j,s} - \nabla T_k(u)\chi_s\big]\,dx + \epsilon^{11}(n, j). \qquad (3.27)
\end{aligned}$$

Let now $r \leqslant s$, then

$$\int_{\Omega_r} \big[a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(u)\big)\big]\big[\nabla T_k(u_n) - \nabla T_k(u)\big]\,dx$$
$$\leqslant \int_{\Omega} \big[a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(u)\chi_s\big)\big]\big[\nabla T_k(u_n) - \nabla T_k(u)\chi_s\big]\,dx,$$

hence, from (3.27) and (3.25)

$$\int_{\Omega_r} \big[a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(u)\big)\big]\big[\nabla T_k(u_n) - \nabla T_k(u)\big]\,dx$$
$$\leqslant 2\epsilon^7(n, j) + \epsilon^{11}(n, j) + 2\int_{\Omega} h_k \nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\,dx + 4\phi(2k) \int_{\{|u_n|\geqslant m\}} |f_n|\,dx$$
$$+ \int_{\Omega\setminus\Omega_s} 2\big|a\big(x, T_k(u), 0\big)\big|\big|\nabla T_k(u)\big|\,dx.$$

By letting respectively n, j, m and s to infinity, one easily has

$$\lim_{n\to\infty} \int_{\Omega_r} \big[a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(u)\big)\big]\big[\nabla T_k(u_n) - \nabla T_k(u)\big]\,dx = 0$$

and then as in [4]

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega. \tag{3.28}$$

On the other hand, we have from (3.25) and (3.27)

$$\int_{\Omega} a\big(x, T_k(u_n), \nabla T_k(u_n)\big)\nabla T_k(u_n)\,dx$$
$$\leqslant \int_{\Omega} a\big(x, T_k(u_n), \nabla T_k(u_n)\big)\nabla T_k(u)\chi_s\,dx$$
$$+ \int_{\Omega} a\big(x, T_k(u_n), \nabla T_k(u)\chi_s\big)\big[\nabla T_k(u_n) - \nabla T_k(u)\chi_s\big]\,dx + \epsilon^{11}(n, j)$$
$$+ 2\int_{\Omega} h_k \nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\,dx + 2\int_{\Omega\setminus\Omega_s} \big|a\big(x, T_k(u), 0\big)\nabla T_k(u)\big|\,dx$$
$$+ 4\phi(2k) \int_{\{|u_n|\geqslant m\}} |f_n|\,dx,$$

by passing to the limit sup on n, one has

$$\limsup_{n\to+\infty} \int_{\Omega} a\big(x, T_k(u_n), \nabla T_k(u_n)\big)\nabla T_k(u_n)\,dx$$
$$\leqslant \limsup_{n\to+\infty} \int_{\Omega} a\big(x, T_k(u_n), \nabla T_k(u_n)\big)\nabla T_k(u)\chi_s\,dx$$

$$
\begin{aligned}
&+\limsup_{n\to+\infty}\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u)\chi_s\big)\big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\big]\,dx\\
&+\limsup_{n\to+\infty}\epsilon^{11}(n,j)+2\int_\Omega h_k\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\,dx\\
&+2\int_{\Omega\setminus\Omega_s}\big|a\big(x,T_k(u),0\big)\nabla T_k(u)\big|\,dx+4\phi(2k)\int_{\{|u|\geqslant m\}}|f|\,dx. \qquad (3.29)
\end{aligned}
$$

The second term of the last inequality tends to $\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\chi_s\,dx$ since

$$
\begin{aligned}
&a\big(x,T_k(u_n),\nabla T_k(u_n)\big)\rightharpoonup a\big(x,T_k(u),\nabla T_k(u)\big)\\
&\quad\text{weakly in } \big(L_{\overline{M}}(\Omega)\big)^N \text{ for } \sigma\big(\Pi L_{\overline{M}}(\Omega),\Pi E_M(\Omega)\big),
\end{aligned}
$$

while $\nabla T_k(u)\chi_s\in E_M(\Omega)$.

The third term of inequality (3.29) tends to $\int_\Omega a(x,T_k(u),0)\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\,dx$ since

$$
a\big(x,T_k(u_n),\nabla T_k(u)\chi_s\big)\to a\big(x,T_k(u),\nabla T_k(u)\chi_s\big)\quad\text{strongly in } \big(E_{\overline{M}}(\Omega)\big)^N,
$$

by Lemma 2.3 while $\nabla T_k(u_n)$ tends weakly to $\nabla T_k(u)$. Consequently, we get, by letting j to infinity

$$
\begin{aligned}
&\limsup_{n\to+\infty}\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u_n)\big)\nabla T_k(u_n)\,dx\\
&\quad\leqslant\int_\Omega a\big(x,T_k(u),\nabla T_k(u)\big)\nabla T_k(u)\chi_s\,dx+2\int_\Omega h_k\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\,dx\\
&\qquad+4\phi(2k)\int_{\{|u|\geqslant m\}}|f|\,dx+3\int_\Omega\big|a\big(x,T_k(u),0\big)\nabla T_k(u)\big|\chi_{\Omega\setminus\Omega_s}\,dx.
\end{aligned}
$$

By using the fact that $a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)$, $|a(x,T_k(u),0)\nabla T_k(u)|$ and $h_k\nabla T_k(u)$ belong to $L^1(\Omega)$ and by letting $s\to\infty$, we get since $\operatorname{meas}(\Omega\setminus\Omega_s)\to 0$,

$$
\begin{aligned}
&\limsup_{n\to+\infty}\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u_n)\big)\nabla T_k(u_n)\,dx\\
&\quad\leqslant\int_\Omega a\big(x,T_k(u),\nabla T_k(u)\big)\nabla T_k(u)\,dx+4\phi(2k)\int_{\{|u|\geqslant m\}}|f|\,dx
\end{aligned}
$$

and by letting $m\to\infty$, we obtain

$$
\limsup_{n\to+\infty}\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u_n)\big)\nabla T_k(u_n)\,dx\leqslant\int_\Omega a\big(x,T_k(u),\nabla T_k(u)\big)\nabla T_k(u)\,dx
$$

which gives by Fatou's lemma

$$
\lim_{n\to+\infty}\int_\Omega a\big(x,T_k(u_n),\nabla T_k(u_n)\big)\nabla T_k(u_n)\,dx=\int_\Omega a\big(x,T_k(u),\nabla T_k(u)\big)\nabla T_k(u)\,dx. \qquad (3.30)
$$

Step 3: Passage to the limit.

Let now $v \in K_\psi \cap L^\infty(\Omega)$, then there exists a sequence $v_j \in K_\psi \cap W^1E_M(\Omega)$ such that $v_j \to v$ in $W_0^1L_M(\Omega)$ for the modular convergence.

By using $T_h(v_j)$, $h \geqslant \|v\|_\infty$, as test function in (P_n), one has

$$\int_\Omega a\big(x, T_H(u_n), \nabla T_H u_n\big)\nabla T_k\big(u_n - T_h(v_j)\big)\,dx + \int_\Omega g_n(x, u_n, \nabla u_n)T_k\big(u_n - T_h(v_j)\big)\,dx$$
$$\leqslant \int_\Omega f_n T_k\big(u_n - T_h(v_j)\big)\,dx, \tag{3.31}$$

where $H = k + h$.

On the one hand,

$$\int_\Omega a\big(x, T_H(u_n), \nabla T_H(u_n)\nabla T_k\big(u_n - T_h(v_j)\big)\big)\,dx$$
$$= \int_\Omega a\big(x, T_H(u_n), \nabla T_H(u_n) - a\big(x, T_H(u_n), \nabla T_h(v_j)\chi_{j,s}\big)\big)\nabla T_k\big(u_n - T_h(v_j)\chi_{j,s}\big)\,dx$$
$$+ \int_\Omega a\big(x, T_H(u_n), \nabla T_h(v_j)\chi_{j,s}\big)\nabla T_k\big(u_n - T_h(v_j)\chi_{j,s}\big)\,dx$$
$$+ \int_{\{|\nabla T_h(v_j)| \geqslant s\}} a\big(x, T_H(u_n), \nabla T_H(u_n)\big)\nabla T_h(v_j)\,dx,$$

by using Fatou's lemma and the fact that $\nabla T_h(v_j) \in (E_M(\Omega))^N$,

$$a\big(x, T_H(u_n), \nabla T_H(u_n)\big) \to a\big(x, T_H(u), \nabla T_H(u)\big) \quad \text{for } \sigma(\Pi L_{\overline{M}}, \Pi E_M)$$

and

$$a\big(x, T_H(u_n), \nabla T_h(v_j)\chi_{j,s}\big) \to a\big(x, T_H(u), \nabla T_h(v_j)\chi_{j,s}\big) \quad \text{strongly in } \big(E_{\overline{M}}(\Omega)\big)^N,$$

we obtain as n and $s \to \infty$,

$$\liminf_{n\to+\infty} \int_\Omega a(x, u_n, \nabla u_n)\nabla T_k\big(u_n - T_h(v_j)\big)\,dx$$
$$\geqslant \int_\Omega a(x, u, \nabla u)\nabla T_k\big(u - T_h(v_j)\big)\,dx. \tag{3.32}$$

About the second term of (3.31), one can write

$$\int_\Omega g_n(x, u_n, \nabla u_n)T_k\big(u_n - T_h(v_j)\big)\,dx$$
$$= \int_{\{|u_n| < h\}} g_n(x, u_n, \nabla u_n)T_k\big(u_n - T_h(v_j)\big)\,dx$$
$$+ \int_{\{|u_n| > h\}} g_n(x, u_n, \nabla u_n)T_k\big(u_n - T_h(v_j)\big)\,dx$$

and consequently by using Fatou's lemma in the first term of the last inequality and the convergence (3.30) in the second

$$\limsup_{n\to+\infty}\int_\Omega g_n(x,u_n,\nabla u_n)T_k\big(u_n-T_h(v_j)\big)\,dx \geqslant \int_\Omega g(x,u,\nabla u)T_k\big(u-T_h(v_j)\big)\,dx. \quad (3.33)$$

Combining (3.31)–(3.33) to obtain finally

$$\int_\Omega a\big(x,T_H(u),\nabla T_H(u)\big)\nabla T_k\big(u-T_h(v_j)\big)\,dx + \int_\Omega g(x,u,\nabla u)T_k\big(u-T_h(v_j)\big)\,dx \leqslant \int_\Omega fT_k\big(u-T_h(v_j)\big)\,dx$$

in which we can pass to the limit in j thanks to the modular convergence of v_j, to obtain

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \leqslant \int_\Omega fT_k(u-v)\,dx,$$

where we have used the fact that $T_h(v)=v$ since $h\geqslant\|v\|_\infty$. This completes the proof. □

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