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Truncations of extremal quasiconformal mappings and their applications ☆

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Abstract

In this paper we give a necessary and sufficient condition to decide whether the Teichmüller equivalency class $[\alpha]$ of a truncation α induced by a uniquely extremal Beltrami differential is a Strebel point in T. We also obtain a necessary and sufficient condition of the unique extremality of α. Using the properties of truncations we provide a method to construct Hamilton sequences. We also get a sufficient condition for the extremality of $f(z,t)$ to be equivalent to that of $F(w,t)$. The corresponding results in the infinitesimal case are obtained, too.

Keywords: Unique extremality; Strebel points; Hamilton sequences; Truncations; Quasiconformal deformations

1. Introduction

Let R be a hyperbolic Riemann surface covered by the unit disk $\Delta = \{z \in C \mid |z| < 1\}$. We adopt the concepts and notations in [2] in this paper. Denote by $QC(R)$ all the quasiconformal mappings f from R onto $f(R)$. Two mappings f and g are equivalent if there is a conformal mapping c from $f(R)$ onto $g(R)$ and a homotopy through quasiconformal mappings h_t mapping R onto $g(R)$ such that $h_0 = c \circ f$, $h_1 = g$ and $h_t(p) = c \circ f(p) = g(p)$ for every p in the ideal boundary of R. We denote the equivalency class of a quasiconformal mapping f in $QC(R)$ by

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$[f]$ or $[\mu]$, where μ is the Beltrami coefficient of f. The Teichmüller space T is defined by the set of Teichmüller equivalence classes $[f]$ of $f \in QC(R)$. Denote by $L^\infty(R)$ the Banach space of Beltrami differentials $\mu = \mu(z)\,\overline{dz}/dz$ on R with finite L^∞-norm and denote by $M(R)$ the open unit ball in $L^\infty(R)$. Thus T can be represented as the space of equivalent classes of Beltrami differentials μ in $M(R)$. We say that u and v in $M(R)$ are Teichmüller equivalent if they induce quasiconformal mappings on R whose lifts to Δ have extensions to the closure of Δ with the same boundary values. If k is a constant and $|\mu| \equiv k$ a.e., then we say that μ has a constant absolute value.

Write $K(\mu) = (1 + \|\mu\|_\infty)/(1 - \|\mu\|_\infty)$. Let $K([\mu])$ be the infimum of $K(\nu)$ over all the Beltrami differentials ν in $[\mu]$. We say that ν is extremal in $[\mu]$ if $K(\nu) = K([\mu])$. A quasiconformal mapping f is extremal if its Beltrami differential is extremal. Hamilton [6], Krushkal [9] and Reich–Strebel [21] gave some criteria to determine whether a Beltrami differential or a quasiconformal mapping is extremal. Let τ be a point of T, we say that $\mu \in \tau$ is uniquely extremal if $K(\nu) > K(\mu) = K(\tau)$ for every $\nu \in \tau$ such that $\nu \neq \mu$. One can refer to [2,12,17–19,25] for current researches about unique extremality.

$A(R)$ is the set of all the holomorphic functions φ in R with $\|\varphi\| = \iint_R |\varphi| < \infty$. Write $A_1(R) = \{\varphi \mid \varphi \in A(R),\ \|\varphi\| = 1\}$. Suppose that $\mu, \nu \in L^\infty(R)$. We say that μ and ν are infinitesimal equivalent if $\iint_R \mu\varphi = \iint_R \nu\varphi$ holds for every $\varphi \in A(R)$. Denote by $[\mu]^B$ the infinitesimal equivalent class of μ. Write $B = \{[\mu]^B \mid \mu \in L^\infty(R)\}$.

A sequence $\{\varphi_n\} \subset A_1(R)$ is a Hamilton sequence for μ if and only if the Hamilton–Krushkal condition (see [6,9,21], see also [4]) holds, namely,

$$\lim_{n\to\infty} \left| \iint_R \mu\varphi_n \right| = \|\mu\|_\infty. \tag{1.1}$$

Particularly, we say that $\{\varphi_n\}$ is degenerating if $\lim_{n\to\infty} \varphi_n = 0$ locally uniformly in R.

The boundary dilatation $H([\mu])$ of the Teichmüller equivalent class of $[\mu] \in T$ is the infimum of the quantity $H^*(\nu)$ over all elements $\nu \in [\mu]$, where

$$H^*(\nu) = \inf\bigl\{K(\eta|_{R-E}) \bigm| \text{for all } \eta \in [\nu] \text{ and compact subsets } E \subset R\bigr\}.$$

Obviously $H([\mu]) \leqslant K([\mu])$. If $H([\mu]) < K([\mu])$ then $[\mu]$ is called a Strebel point (see [10]) of the Teichmüller space T. By Strebel's frame mapping theorem (see [22]), every Strebel point $[\mu]$ can be represented by a unique Beltrami coefficient of the form $k\bar{\varphi}/|\varphi|$, where $k = (K - 1)/(K+1)$, $K = K([\mu])$ and $\varphi \in A_1(R)$. There does not exist a degenerating Hamilton sequence for the extremal representative of any Strebel point (see [22]). In [3], Earle and Li proved that the converse also holds. The set of all the Strebel points in T is open and dense (see [10], also [5]).

Suppose that $E \subset R$ is a compact subset with positive measure and r is a positive constant. Set

$$\alpha = \begin{cases} \mu(z), & \text{on } E, \\ \mu(z)/(1+r), & \text{on } R - E. \end{cases} \tag{1.2}$$

We call α a truncation (see [2]) of μ decided by r and E.

Truncations are usually used to solve some extremal problems (see [2,10,13,27,33]). For example, Bozin, Lakic, Markovic and Mateljevic [2] proved that the Teichmüller equivalence class $[\alpha]$ of each truncation α induced by a uniquely extremal μ with a constant absolute value is a Strebel point in T. Using this result, they proved that the unique extremality of μ in $[\mu]$ is equivalent to that in $[\mu]^B$. Hence it attracts much attention to study properties of truncations. In [33] Zhu and Chen proved that for a uniquely extremal μ either $[\alpha]$ is a Strebel point in T or α is uniquely extremal. The purpose of this paper is to discuss the extremality of truncations and its applications.

First, only basing on calculating the essential supremum of the truncation $\alpha(z)$ as z varies over E, that is, $\|\alpha|_E\|_\infty$, we will give a necessary and sufficient condition to determine whether $[\alpha]$ is a Strebel point in T. We also get a necessary and sufficient condition to decide if α is uniquely extremal for a uniquely extremal μ which unnecessarily has a constant absolute value.

Next, we obtain a sufficient condition for α to be extremal when μ is extremal but unnecessarily uniquely extremal.

Then, since Hamilton sequences play a vital role in studying extremality or unique extremality of quasiconformal mappings, it is of great interest to construct a Hamilton sequence for a given extremal quasiconformal mapping (see [7,8,11,15,21,23,24,30–32] for construction methods and their development). Using the properties of truncations we will give a method to construct a Hamilton sequence $\{\varphi_n\}$ for a uniquely extremal μ_f, where $\{\varphi_n\}$ is decided by a sequence of truncations.

Last, suppose that $F(w,t)$ is a family of quasiconformal deformations (see [1] for the definition) such that $\bar\partial F(w,t)$ has a uniform bound M, $f(z,t)$ $(f(z,0)=z)$ is the solution of Löwner-type differential equation $dw/dt = F(w,t)$ (see [16] for more properties of the solution). In [26], for a Beltrami coefficient with separable variables, Shen obtained a sufficient condition for the extremality of $F(w,t)$ to be equivalent to that of $f(z,t)$. By considering a class of truncated Beltrami coefficients we will get another sufficient condition. Using this result and the properties of truncations we will obtain a sufficient condition for $f(z,t)$ and $F(w,t)$ to be extremal simultaneously.

This paper is organized as follows. Section 1 gives introduction. Section 2 obtains some properties of truncations. Section 3 finds a method to construct Hamilton sequences. Section 4 studies the extremality of $f(z,t)$ and $F(w,t)$. Section 5 discusses corresponding properties of truncations in the infinitesimally extremal case.

2. Properties of truncations of extremal Beltrami differentials

Denote by χ_E the characteristic function of a set E. In [33] Zhu and Chen proved the following Theorem A.

Theorem A.

(1) *If $\mu \in M(\Delta)$ is extremal, and for every compact subset E of Δ and every $r>0$, $[\mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}]$ is a Strebel point in T, then μ is uniquely extremal.*
(2) *If $\mu \in M(\Delta)$ is uniquely extremal, then for every compact subset E of Δ and every $r>0$, either $[\mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}]$ is a Strebel point in T, or $\mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}$ is uniquely extremal.*

Now let us study how to judge the above two cases at (2) of Theorem A. At first, we give a sufficient condition for $[\mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}]$ to be a Strebel point in T in Theorem 2.1 and a sufficient condition for α to be extremal (uniquely extremal) in Theorem 2.2 (Corollary 2.1). Then, by these results we give a necessary and sufficient condition for $[\mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}]$ to be a Strebel point in T and a necessary and sufficient condition for $\mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}$ to be uniquely extremal (see Theorem 2.3). From now on, we always assume that $K([\mu]) > 1$ and $\|\mu\|_\infty = k$ with $0<k<1$.

Theorem 2.1. *Suppose $\mu \in M(R)$ is uniquely extremal. Let α be a truncation of μ decided by a compact subset E of R with positive measure and $r > 0$. If $\|\alpha|_E\|_\infty > k/(1+r)$, then the Teichmüller equivalence class $[\alpha]$ is a Strebel point in T.*

Proof. Suppose that $\mu \in M(R)$. Let α be a truncation of μ decided by E and $r > 0$, where E is a compact subset of R with positive measure. If $\|\alpha|_E\|_\infty > k/(1+r)$ then $\|\alpha\|_\infty > k/(1+r)$. Now we will prove that $[\alpha]$ is a Strebel point in T, that is, $H([\alpha]) < K([\alpha])$.

For convenience, set $s = k/(1+r)$. We have $H([\alpha]) \leqslant (1+s)/(1-s)$ since $\alpha \in [\alpha]$. Suppose the result of Theorem 2.1 does not hold, namely, $H([\alpha]) = K([\alpha])$. Thus there at least exists a Beltrami differential $\eta \in [\alpha]$, such that $\|\eta\|_\infty \leqslant s$. Assume that f^μ, f^α and f^η are quasiconformal mappings of Δ onto itself, which are normalized to fix three boundary points -1, 1, i, and whose Beltrami coefficients are the lifts of μ, α and η, respectively. Let $F \subset \Delta$ be the lift of a compact set $E \subset R$ and $g = f^\mu \circ (f^\alpha)^{-1}$. Then

$$|\mu_g| = \left|\frac{\mu_{f^\mu} - \mu_{f^\alpha}}{1 - \mu_{f^\mu}\overline{\mu_{f^\alpha}}}\right| = \begin{cases} 0, & \text{on } f^\alpha(F), \\ \frac{r|\mu|/(1+r)}{1-|\mu|^2/(1+r)}, & \text{on } \Delta - f^\alpha(F). \end{cases} \tag{2.1}$$

It is clear that $\|\mu_g\|_\infty \leqslant [rk/(1+r)]/[1-k^2/(1+r)]$. Since $\alpha, \eta \in [\alpha]$, we know that $f^\mu = g \circ f^\alpha$ and $g \circ f^\eta$ have the same boundary values. Furthermore,

$$\begin{aligned} K\bigl[g \circ f^\eta\bigr] &\leqslant K[g] \cdot K\bigl[f^\eta\bigr] \leqslant \frac{1 + rk/(1+r-k^2)}{1 - rk/(1+r-k^2)} \cdot K\bigl[f^\eta\bigr] \\ &= \frac{1+k}{1-k} \cdot \frac{1+r-k}{1+r+k} \cdot K\bigl[f^\eta\bigr] \leqslant \frac{1+k}{1-k} \cdot \frac{1+r-k}{1+r+k} \cdot \frac{1+r+k}{1+r-k} = K\bigl[f^\mu\bigr]. \end{aligned}$$

Thus $g \circ f^\eta = f^\mu$ by the fact that μ is uniquely extremal, that is, $f^\eta = g^{-1} \circ f^\mu = f^\alpha$. Hence $\eta = \alpha$. But this contradicts the inequality $\|\eta\|_\infty \leqslant s < \|\alpha\|_\infty$. Therefore $[\alpha]$ is a Strebel point in T, namely, there exists $s_r > s$ and a unit vector $\varphi_r \in A_1(R)$ such that $\alpha \in [s_r\overline{\varphi_r}/|\varphi_r|]$. □

Theorem 2.2. *Suppose that $\mu \in M(R)$ is extremal. Let α be a truncation of μ decided by a compact subset E of R with positive measure and $r > 0$. If $\|\alpha|_E\|_\infty \leqslant k/(1+r)$, then α is extremal, too.*

Proof. From the assumption that $\|\alpha|_E\|_\infty \leqslant k/(1+r)$, we know that $\|\mu|_{R-E}\|_\infty = k$. Then $\|\alpha\|_\infty = k/(1+r)$. Since μ is extremal, there exists a Hamilton sequence $\{\varphi_n\} \subset A_1(R)$ for μ, that is,

$$\lim_{n\to\infty}\left|\iint_R \mu\varphi_n\right| = \|\mu\|_\infty = k. \tag{2.2}$$

In the following we are going to show that $\{\varphi_n\}$ is also a Hamilton sequence for α. By the definition of α and (2.2) it follows that

$$\begin{aligned} \left|\iint_R \alpha\varphi_n\right| &= \left|\iint_{R-E} \frac{\mu}{1+r}\varphi_n + \iint_E \mu\varphi_n\right| = \left|\iint_R \frac{\mu}{1+r}\varphi_n + \iint_E \mu\varphi_n - \iint_E \frac{\mu}{1+r}\varphi_n\right| \\ &\geqslant \frac{1}{1+r}\left|\iint_R \mu\varphi_n\right| - \left|\iint_E \frac{r}{1+r}\mu\varphi_n\right| \geqslant \frac{1}{1+r}\left|\iint_R \mu\varphi_n\right| - \frac{r}{1+r}k\iint_E |\varphi_n|. \end{aligned}$$

From the assumption of Theorem 2.2 it is clear that $|\mu| \leqslant k/(1+r) < k$ when $z \in E$. The sequence $\{\varphi_n\}$ is an absolute maximal sequence of the functional $\sup_{\varphi\in A(R),\|\varphi\|\leqslant 1}|\iint_R \mu\varphi|$ since it is a Hamilton sequence for μ (see [21]). Then from the properties of an absolute maximal sequence (see [21]) we have $\iint_E |\varphi_n| \to 0$ as $n \to \infty$. Thus

$$\left|\iint\limits_R \alpha\varphi_n\right| \geqslant \frac{1}{1+r}\left|\iint\limits_R \mu\varphi_n\right| - \frac{r}{1+r}k\iint\limits_E |\varphi_n| \to \frac{k}{1+r}, \quad n \to \infty.$$

On the other hand, it is clear that $|\iint_R \alpha\varphi_n| \leqslant \|\alpha\|_\infty = k/(1+r)$. Hence $\{\varphi_n\}$ is also a Hamilton sequence for α. Then α is an extremal Beltrami differential. □

Corollary 2.1. *If μ is uniquely extremal, then α is also uniquely extremal under all the assumptions of Theorem* 2.2*.*

Proof. From Theorem 2.2 we know that α is extremal. Now we only need to prove that α is also uniquely extremal. Otherwise, there exists an extremal Beltrami differential $\eta \in [\alpha]$, $\eta \neq \alpha$. Thus $\|\eta\|_\infty = k/(1+r)$, $f^\mu \circ (f^\alpha)^{-1} \circ f^\eta$ has the same boundary values as that of f^μ, and $f^\mu \neq f^\mu \circ (f^\alpha)^{-1} \circ f^\eta$. Since μ is uniquely extremal, it follows that

$$\begin{aligned}\frac{1+k}{1-k} &= K[f^\mu] < K\left[f^\mu \circ \left(f^\alpha\right)^{-1} \circ f^\eta\right] \leqslant K\left[f^\mu \circ \left(f^\alpha\right)^{-1}\right] \cdot K[f^\eta]\\ &= \frac{1+k}{1-k}\cdot\frac{1+r-k}{1+r+k}\cdot K[f^\eta] = \frac{1+k}{1-k}\cdot\frac{1+r-k}{1+r+k}\cdot\frac{1+r+k}{1+r-k} = \frac{1+k}{1-k}.\end{aligned}$$

It is impossible. Hence α is also uniquely extremal. □

Theorem 2.3. *Suppose that μ is uniquely extremal. Let α be a truncation of μ decided by a compact subset E of R with positive measure and $r > 0$. Then the Teichmüller equivalence class $[\alpha]$ is a Strebel point in T if and only if $\|\alpha|_E\|_\infty > k/(1+r)$, and α is uniquely extremal if and only if $\|\alpha|_E\|_\infty \leqslant k/(1+r)$.*

Proof. If $\|\alpha|_E\|_\infty > k/(1+r)$, then $[\alpha]$ is a Strebel point in T by Theorem 2.1. Conversely, if $[\alpha]$ is a Strebel point in T, then $\|\alpha|_E\|_\infty > k/(1+r)$. Otherwise, the inequality $\|\alpha|_E\|_\infty \leqslant k/(1+r)$ holds. By Corollary 2.1 we have that α is uniquely extremal. However α itself cannot be an extremal representative of a Strebel point $[\alpha]$ with its extremal representative $s_r\bar{\varphi}/|\varphi|$ satisfying $s_r > k/(1+r)$, since $\|\alpha|_E\|_\infty \leqslant k/(1+r)$, a contradiction.

If $\|\alpha|_E\|_\infty \leqslant k/(1+r)$, then α is uniquely extremal by Corollary 2.1. Conversely, if α is uniquely extremal, then $\|\alpha|_E\|_\infty \leqslant k/(1+r)$. Otherwise, the inequality $\|\alpha|_E\|_\infty > k/(1+r)$ holds, then from Theorem 2.1, we see that $[\alpha]$ is a Strebel point in T. So its extremal representative has a constant absolute value. Thus α itself cannot be uniquely extremal, since α cannot have a constant absolute value from the fact that $|\alpha|_{R-E}| \leqslant k/(1+r)$, a contradiction. □

3. A method to construct a Hamilton sequence

In 1969, Hamilton [6] proved that there really exist Hamilton sequences for every extremal quasiconformal mapping in an abstract way. Krushkal [9] obtained similar results in the special case that Beltrami coefficients have a constant absolute value. In 1974, Reich and Strebel [21] proved that the quadratic differential sequence $\{\varphi_n\}$ is a Hamilton sequence, where $\{\varphi_n\}$ is determined by the extremal quasiconformal mapping of the polygon P_n with Δ as its interior and n

vertices on $\partial\Delta$ onto another polygon P_n'. Hayman [7] and Reich [7,15] used the putative method to construct Hamilton sequences for Teichmüller mappings. But this method is invalid for the affine mapping in the chimney domain (see [15]). So the scope should be confined properly when using it to construct a Hamilton sequence. Recently, the applicable scope of putative method was extended to some extent (see [8,30–32]). But this problem is still not solved completely.

At the same time, many other methods to construct Hamilton sequences were given (see [11, 23,24]). For example, Strebel [23] used point shift differential sequences (see [23] for the definition) to construct Hamilton sequences. Sun and Wu [28] extended the applicable scope of Strebel's result in [23] after proving that a degenerating point shift differential sequence is a common Hamilton sequence (see [21,29] for the definition). Using the fact that the set of all the Strebel points is dense in T (see [10], see also [5]), Li [11] showed that the Strebel differential sequence induced by Strebel points which converges at μ in T is a Hamilton sequence for μ.

In this section, we will apply our results about truncations to provide a method to construct a Hamilton sequence.

Lemma 3.1. *Let $a > 1$. Then two functions*

$$f(x) = \frac{1+x^2}{1-x^2} - \frac{1+x^2/a^2}{1-x^2/a^2} \quad \text{and} \quad g(x) = \frac{1}{1-x^2} - \frac{1/a}{1-x^2/a^2} \tag{3.1}$$

increase in $(0, 1)$.

Proof. When $x \in (0, 1)$, by direct calculation we have

$$f'(x) = \frac{4x(1-1/a^2)(1-x^4/a^2)}{(1-x^2)^2(1-x^2/a^2)^2} > 0$$

and

$$g'(x) \geqslant \frac{(2/a)(1-1/a)^3(1+1/a)^2 x}{(1-x^2)^2(1-x^2/a^2)^2} > 0.$$

Thus both $f(x)$ and $g(x)$ increase in $(0, 1)$. □

By the main inequality (see [21]) Bozin, Lakic, Markovic and Mateljevic proved the following Lemma A (see [2]).

Lemma A. *If there exists a unit vector $\varphi \in A_1(R)$ such that $[\mu] = [k\bar{\varphi}/|\varphi|]$ in T for some $k \in (0, 1)$, then*

$$\frac{1+k}{1-k} \leqslant \iint_R |\varphi| \frac{|1+\mu\varphi/|\varphi||^2}{1-|\mu|^2}. \tag{3.2}$$

Theorem 3.1. *If there exists a sequence of truncations*

$$\alpha_n = \begin{cases} \mu(z), & z \in E_n, \\ \mu(z)/(1+1/n), & z \in R - E_n, \end{cases} \tag{3.3}$$

satisfying all the assumptions of Theorem 2.1, where $\{E_n\}$ is a sequence of compact subsets of R with positive measure, then the Strebel differential sequence $\{\varphi_n\}$ induced by a Strebel-point sequence $\{[\alpha_n]\} \subset T$ is a Hamilton sequence for μ.

Proof. Under the assumptions of Theorem 3.1, it is clear that $[\alpha_n]$ is a Strebel point in T by Theorem 2.1, namely, there exists $s_n > s = k/(1+1/n)$ and a unit vector $\varphi_n \in A_1(R)$ such that $\alpha_n \in [s_n\overline{\varphi_n}/|\varphi_n|]$. Thus by Lemma A, it follows that

$$\frac{1+k/(1+1/n)}{1-k/(1+1/n)} < \frac{1+s_n}{1-s_n} \leqslant \iint\limits_R |\varphi_n| \frac{|1+\alpha_n\varphi_n/|\varphi_n||^2}{1-|\alpha_n|^2}. \tag{3.4}$$

Let $\mu_n = \mu/(1+1/n)$. From (3.4) we have

$$\begin{aligned}
&\frac{1+k/(1+1/n)}{1-k/(1+1/n)} \\
&\quad< \iint\limits_{R-E_n} |\varphi_n| \frac{|1+\mu_n\varphi_n/|\varphi_n||^2}{1-|\mu_n|^2} + \iint\limits_{E_n} |\varphi_n| \frac{|1+\mu\varphi_n/|\varphi_n||^2}{1-|\mu|^2} \\
&\quad= \iint\limits_{R} |\varphi_n| \frac{|1+\mu_n\varphi_n/|\varphi_n||^2}{1-|\mu_n|^2} + \iint\limits_{E_n} \left(\frac{|1+\mu\varphi_n/|\varphi_n||^2}{1-|\mu|^2} - \frac{|1+\mu_n\varphi_n/|\varphi_n||^2}{1-|\mu_n|^2}\right)|\varphi_n|.
\end{aligned}$$

Let

$$\begin{aligned}
A &= \iint\limits_{R} |\varphi_n| \frac{|1+\mu_n\varphi_n/|\varphi_n||^2}{1-|\mu_n|^2}, \\
B &= \iint\limits_{E_n} \left(\frac{|1+\mu\varphi_n/|\varphi_n||^2}{1-|\mu|^2} - \frac{|1+\mu_n\varphi_n/|\varphi_n||^2}{1-|\mu_n|^2}\right)|\varphi_n|.
\end{aligned}$$

Then

$$\begin{aligned}
B &= \iint\limits_{E_n} \left(\frac{1+|\mu|^2}{1-|\mu|^2} - \frac{1+|\mu|^2/(1+1/n)^2}{1-|\mu|^2/(1+1/n)^2}\right)|\varphi_n| \\
&\quad + 2\operatorname{Re}\iint\limits_{E_n} \left(\frac{1}{1-|\mu|^2} - \frac{1/(1+1/n)}{1-|\mu|^2/(1+1/n)^2}\right)\mu\varphi_n.
\end{aligned}$$

By Lemma 3.1 we get

$$\begin{aligned}
B &\leqslant \left[\left(\frac{1+k^2}{1-k^2} - \frac{1+k^2/(1+1/n)^2}{1-k^2/(1+1/n)^2}\right) + 2k\left(\frac{1}{1-k^2} - \frac{1/(1+1/n)}{1-k^2/(1+1/n)^2}\right)\right] \iint\limits_{E_n} |\varphi_n| \\
&\leqslant \frac{8k}{(1-k^2)^2}\bigl(1-1/(1+1/n)\bigr) \iint\limits_{E_n} |\varphi_n| \leqslant \frac{8k}{(1-k^2)^2}\frac{1}{n} \iint\limits_{E_n} |\varphi_n|,
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
A &\leqslant \frac{1}{1-k^2/(1+1/n)^2} \iint\limits_R \left[1 + \frac{|\mu|^2}{(1+1/n)^2} + 2\frac{\operatorname{Re}\mu\varphi_n}{1+1/n}\right] \\
&\leqslant \frac{1+k^2/(1+1/n)^2}{1-k^2/(1+1/n)^2} + \frac{2}{[1-k^2/(1+1/n)^2](1+1/n)} \operatorname{Re}\iint\limits_R \mu\varphi_n.
\end{aligned} \tag{3.6}$$

From (3.4)–(3.6) we obtain

$$\frac{2k/(1+1/n)}{1-k^2/(1+1/n)^2} = \frac{1+k/(1+1/n)}{1-k/(1+1/n)} - \frac{1+k^2/(1+1/n)^2}{1-k^2/(1+1/n)^2}$$
$$< \frac{2}{1+1/n} \cdot \frac{1}{1-k^2/(1+1/n)^2} \operatorname{Re} \iint_R \mu\varphi_n + \frac{8k}{(1-k^2)^2} \cdot \frac{1}{n} \iint_{E_n} |\varphi_n|.$$

Using the above inequality it follows that

$$k - \operatorname{Re} \iint_R \mu\varphi_n < \frac{1-k^2/(1+1/n)^2}{2/(1+1/n)} \frac{8k}{(1-k^2)^2} \frac{1}{n} \iint_{E_n} |\varphi_n| \leqslant \frac{8k}{(1-k^2)^2} \frac{1}{n} \iint_{E_n} |\varphi_n|.$$

Thus $k - \operatorname{Re} \iint_R \mu\varphi_n \to 0$ as $n \to \infty$, that is, $\{\varphi_n\}$ is a Hamilton sequence for μ. □

Remark 3.1. If there exists a compact subset E of R with positive measure such that $\|\mu|_E\|_\infty = k$, then α_n can be chose as $[\mu\chi_E + (1/(1+1/n))\mu\chi_{R-E}]$.

4. Extremality of $f(z,t)$ and $F(w,t)$

In this section, we will deal with quasiconformal solutions $w = f(z,t)$ $(f(z,0) = z)$ of the following Löwner-type differential equation

$$\frac{dw}{dt} = F(w,t) \tag{4.1}$$

in Δ.

Given a family of quasiconformal deformations $F(w,t)$ such that $\bar\partial F$ has a uniform bound M, Reich proved that the solution $f(z,t)$ of (4.1) with the initial condition $f(z,0) = z$, which is an e^{2Mt}-quasiconformal mapping, is unique (see [16]). If, additionally, $F(w,t)$ satisfies the normalized condition

$$\Re\big[\bar w F(w,t)\big] = 0, \qquad F(1,t) = F(-1,t) = F(i,t) = 0, \quad w \in \partial\Delta, \tag{4.2}$$

then $f(z,t)$ maps Δ onto itself with $f(-1,t) = -1$, $f(1,t) = 1$, $f(i,t) = i$. Reich and Chen [20] proved that F is an extremal quasiconformal deformation if and only if its $\bar\partial$-derivative satisfies the Hamilton–Krushkal condition. The maximal dilatation $K[f]$ of f can be estimated in terms of the essential supremum of $\bar\partial F$. It is of interest to find out whether minimizing the essential supremum of $\bar\partial F$ is equivalent to minimizing the maximal dilatation $K[f]$.

To answer this question, Shen proved the following counterexample Theorem B in [26] by considering the family of Beltrami coefficients $\alpha(z,t) = t\chi_{\Delta-E}(z)\mu(z) + t^2\chi_E\mu(z)$, where $\mu(z)$ is an extremal Beltrami coefficient in Δ which has a degenerating Hamilton sequence and a constant absolute value.

Theorem B. *There exists a family of quasiconformal deformations $F(w,t)$ on $\bar\Delta \times [0,T]$ such that the solution $f(z,t)$ of the system* (4.1) *and $F(w,t)$ themselves satisfy the following*

(1) *For $t \in [0,t_1]$, neither $f(z,t)$ nor $F(w,t)$ is extremal.*
(2) *For $t \in [t_1,t_2]$, $f(z,t)$ is not extremal while $F(w,t)$ is.*
(3) *For $t \in [t_2,T]$, both $f(z,t)$ and $F(w,t)$ are extremal.*

He also gave a sufficient condition for the extremality of $f(z,t)$ to be equivalent to that of $F(w,t)$.

Theorem C. *Let $f(z,t)$ be the solution of the system* (4.1). *If the Beltrami coefficient $\alpha(z,t)$ of $f(z,t)$ has the form*

$$\alpha(z,t) = k(t)\mu(z)$$

for some differentiable function $k(t)$ with $k(0)=0$ and $k'(t)>0$, then for each fixed $t>0$, $f(z,t)$ is extremal if and only if $F(w,t)$ is extremal.

The class of Beltrami coefficients studied in Theorem C were confined to have separable variables t and z. Next we will consider another family of Beltrami coefficients with the form of truncations defined by

$$\alpha(z,t) = t\chi_E(z)\mu(z) + t^2\chi_{\Delta-E}(z)\mu(z), \tag{4.3}$$

where E is a compact subset of Δ with positive measure. A new sufficient condition for the extremality of $f(z,t)$ to be equivalent to that of $F(w,t)$ will be given in the following Theorem 4.1.

Let $f(z,t)$ be the solution of (4.1). As did in [16], differentiating both sides of the equation

$$\frac{df(z,t)}{dt} = F\big(f(z,t),t\big) \tag{4.4}$$

partially with respect to z and $\bar{z}$ yields the relation

$$\bar{\partial}F\big(f(z,t),t\big) = \frac{\partial_t\mu(z,t)}{1-|\mu(z,t)|^2}\cdot\frac{f_z(z,t)}{\overline{f_z(z,t)}}, \tag{4.5}$$

where $\mu(z,t)$ is the Beltrami coefficient of $f(z,t)$. Denote by $v(w,t)$ the Beltrami coefficient of inverse mapping $f^{-1}(w,t)$. Then the relation (4.5) is equivalent to

$$\bar{\partial}F(w,t) = -\frac{\partial_t\mu(z,t)}{\mu(z,t)}\cdot\frac{v(w,t)}{1-|v(w,t)|^2}\quad \big(z=f^{-1}(w,t)\big) \tag{4.6}$$

when $\mu(z,t)\neq 0$.

Theorem 4.1. *Let $f(z,t)$ be the solution of the system* (4.1). *If the Beltrami coefficient $\alpha(z,t)$ of $f(z,t)$ has the form*

$$\alpha(z,t) = t\chi_E(z)\mu(z) + t^2\chi_{\Delta-E}(z)\mu(z),$$

and satisfies $\|\alpha(z,t)|_E\|_\infty < kt^2$, and $E\subset\Delta$ is a compact subset with positive measure, then for each fixed $t\in(0,1)$, $f(z,t)$ is extremal if and only if $F(w,t)$ is extremal.

Proof. Let $t\in(0,1)$ be fixed, and $E\subset\Delta$ be a compact subset which has positive measure and satisfies $\|\alpha(z,t)|_E\|_\infty < kt^2$. Assume that $f(z,t)$ with a Beltrami coefficient $\alpha(z,t)$ is the solution of the system (4.1).

Suppose that $f(z,t)$ is extremal. We are going to verify that $F(w,t)$ is also extremal. By (4.5) and (4.6) we have

$$\big|\bar{\partial}F\big(f(z,t),t\big)\big| = \begin{cases} \frac{|\mu(z)|}{1-t^2|\mu(z)|^2}, & z\in E,\\ \frac{2t|\mu(z)|}{1-t^4|\mu(z)|^2}, & z\in\Delta-E \end{cases} \tag{4.7}$$

and

$$\bar\partial F(w,t)=\begin{cases}-\frac{1}{t}\cdot\frac{v(w,t)}{1-|v(w,t)|^2}, & w\in f(.,t)(E),\\ -\frac{2}{t}\cdot\frac{v(w,t)}{1-|v(w,t)|^2}, & w\in\Delta-f(.,t)(E).\end{cases}\tag{4.8}$$

When $0<s<1$, the functions $s/(1-t^2s^2)$ and $2ts/(1-t^4s^2)$ increase monotonically with respect to s. Thus it follows that

$$\begin{aligned}\left\|\bar\partial F\big(f(z,t),t\big)|_E\right\|_\infty &< kt/\big(1-t^4k^2\big)<2kt/\big(1-t^4k^2\big)\\ &=\left\|\bar\partial F\big(f(z,t),t\big)\big|_{\Delta-E}\right\|_\infty\end{aligned}\tag{4.9}$$

by (4.7) since $\|\mu(z)|_E\|_\infty<kt$ and $\|\mu(z)|_{\Delta-E}\|_\infty=k$. Hence

$$\begin{aligned}\left\|\bar\partial F(w,t)|_{f(.,t)(E)}\right\|_\infty &< \left\|\bar\partial F(w,t)|_{\Delta-f(.,t)(E)}\right\|_\infty\\ &=\frac{2}{t}\left\|\frac{v(w,t)}{1-|v(w,t)|^2}\Big|_{\Delta-f(.,t)(E)}\right\|_\infty.\end{aligned}\tag{4.10}$$

If $f(z,t)$ is extremal, then $v(w,t)$ is extremal. Hence $v(w,t)/(1-|v(w,t)|^2)$ is extremal (see [21]). Therefore $-\frac{2}{t}\frac{v(w,t)}{1-|v(w,t)|^2}$ is extremal. So there exists a Hamilton sequence $\{\psi_n\}\subset A_1(\Delta)$ for $-\frac{2}{t}\frac{v(w,t)}{1-|v(w,t)|^2}$. By the relation $\|v(w,t)|_{f(.,t)(E)}\|_\infty=\|\alpha(z,t)|_E\|_\infty<kt^2=\|\alpha(z,t)\|_\infty=\|v(w,t)\|_\infty$, it follows that

$$\left\|\frac{v(w,t)}{1-|v(w,t)|^2}\Big|_{f(.,t)(E)}\right\|_\infty<\left\|\frac{v(w,t)}{1-|v(w,t)|^2}\right\|_\infty=\frac{kt^2}{1-k^4t^4}.$$

Thus $\{\psi_n\}$ can be assumed to be degenerating.

From (4.8) we get

$$\begin{aligned}\left|\iint_\Delta\bar\partial F(w,t)\psi_n\right| &=\left|\iint_{\Delta-f(.,t)(E)}\bar\partial F(w,t)\psi_n+\iint_{f(.,t)(E)}\bar\partial F(w,t)\psi_n\right|\\ &\geqslant\frac{2}{t}\left|\iint_\Delta\frac{v(w,t)}{1-|v(w,t)|^2}\psi_n\right|-\frac{1}{t}\left|\iint_{f(.,t)(E)}\frac{v(w,t)}{1-|v(w,t)|^2}\psi_n\right|.\end{aligned}\tag{4.11}$$

Since E is a compact subset of Δ and $\{\psi_n\}$ is degenerating, we obtain

$$\left|\iint_{f(.,t)(E)}\frac{v(w,t)}{1-|v(w,t)|^2}\psi_n\right|\to 0,\quad n\to\infty.\tag{4.12}$$

By (4.10), (4.11) and (4.12) we have

$$\left|\iint_\Delta\bar\partial F(w,t)\psi_n\right|\to\frac{2}{t}\left\|\frac{v(w,t)}{1-|v(w,t)|^2}\right\|_\infty=\left\|\bar\partial F(w,t)\right\|_\infty,\quad n\to\infty.$$

Thus $\{\psi_n\}$ is also a Hamilton sequence of $\bar\partial F(w,t)$. So $F(w,t)$ is extremal by the Hamilton–Krushkal condition.

Conversely, suppose that $F(w,t)$ is extremal. We are going to prove that $f(z,t)$ is also extremal under the condition that $\|\alpha(z,t)|_E\|<kt^2$.

$F(w,t)$ is extremal, that is, $\bar\partial F(w,t)$ satisfies the Hamilton–Krushkal condition, namely, there exists a Hamilton sequence $\{\psi_n\} \subset A_1(\Delta)$ for $\bar\partial F(w,t)$. We can assume $\{\psi_n\}$ is degenerating since the relation (4.10) holds.

By (4.8) we obtain

$$\begin{aligned}\left|\iint_\Delta \frac{v(w,t)}{1-|v(w,t)|^2}\psi_n\right| &= \frac{t}{2}\left|\iint_\Delta -\frac{2}{t}\frac{v(w,t)}{1-|v(w,t)|^2}\psi_n\right| \\ &= \frac{t}{2}\left\{\left|\iint_\Delta \bar\partial F(w,t)\psi_n + \iint_{f(.,t)(E)} -\frac{1}{t}\frac{v(w,t)}{1-|v(w,t)|^2}\psi_n\right|\right\} \\ &\geqslant \frac{t}{2}\left|\iint_\Delta \bar\partial F(w,t)\psi_n\right| - \frac{t}{2}\left|\iint_{f(.,t)(E)} \bar\partial F(w,t)\psi_n\right|. \end{aligned} \tag{4.13}$$

Since $\{\psi_n\}$ is degenerating, by (4.9), (4.10) and (4.13) we have

$$\left|\iint_\Delta \frac{v(w,t)}{1-|v(w,t)|^2}\psi_n\right| \to \left\|\frac{v(w,t)}{1-|v(w,t)|^2}\right\|_\infty.$$

So $\{\psi_n\}$ is a Hamilton sequence for $\frac{v(w,t)}{1-|v(w,t)|^2}$, that is, $\frac{v(w,t)}{1-|v(w,t)|^2}$ satisfies Hamilton–Krushkal condition. Thus $v(w,t)$ is extremal. Therefore $\alpha(z,t)$ is extremal, namely, $f(z,t)$ is extremal. □

Corollary 4.1. *Suppose that μ is extremal. Then both $f(z,t)$ and $F(w,t)$ are extremal under the assumptions of Theorem* 4.1.

Proof. Let $t \in (0,1)$ be fixed. Suppose that $E \subset \Delta$ is a compact subset with positive measure and satisfying $\|\alpha(z,t)|_E\|_\infty < kt^2$.

Write $r = (1-t)/t$. Set

$$\beta(z,r) = \chi_E(z)\mu(z) + \frac{1}{1+r}\chi_{\Delta - E}(z)\mu(z).$$

Then $\beta(z,r)$ is a truncation of μ decided by E and r, $\alpha(z,t) = t\beta(z,r)$, and $\|\beta(z,r)|_E\|_\infty < k/(1+r)$.

By Theorem 2.2, $\beta(z,r)$ is extremal when μ is extremal. It is easy to show that $\alpha(z,t)$ is also extremal, that is, $f(z,t)$ is extremal when $\|\alpha(z,t)|_E\|_\infty < kt^2$. By Theorem 4.1, $F(w,t)$ is extremal, too. □

Remark 4.1. Suppose that μ is uniquely extremal and $\|\alpha(z,t)|_E\|_\infty > kt^2$. Using the same method as that in Theorem 2.1, one can verify that $[\alpha(z,t)]$ is a Strebel point in T.

5. Corresponding properties of the infinitesimally extremal case

Define $\|[\mu]^B\| = \inf\{\|\nu\|_\infty \mid \nu \in [\mu]^B\}$. If $\nu \in [\mu]^B$ and $\|\nu\|_\infty = \|[\mu]^B\|$, then we say that ν is infinitesimally extremal. Moreover, if $\|\eta\|_\infty > \|\mu\|$ for each $\eta \in [\mu]^B$ with $\eta \neq \nu$, then ν is infinitesimally uniquely extremal.

Similarly to the definition of the boundary dilatation of a Teichmüller equivalence class, we define the boundary seminorm $b([\mu]^B)$ of an infinitesimal equivalence class $[\mu]^B$ by

$$b\big([\mu]^B\big) = \inf\big\{\|\nu|_{R-E}\|_\infty \;\big|\; \nu \in [\mu]^B,\ E \subset R \text{ is compact with positive measure}\big\}.$$

Clearly $b([\mu]^B) \leqslant \|[\mu]^B\|$. If $b([\mu]^B) < \|[\mu]^B\|$, then we say that $[\mu]^B$ is an infinitesimal Strebel point in B.

Zhu and Chen [33] also proved

Theorem D.

(1) *If $\mu \in L^\infty(\Delta)$ is infinitesimally extremal, and for every compact subset E of Δ and every $r > 0$, $[\mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}]$ is an infinitesimal Strebel point, then μ is infinitesimally uniquely extremal.*

(2) *If $\mu \in L^\infty(\Delta)$ is infinitesimally uniquely extremal, then for every compact subset E of Δ and every $r > 0$, either $[\mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}]$ is an infinitesimally Strebel point, or $\mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}$ is infinitesimally uniquely extremal.*

In this section, corresponding to Sections 2 and 3, we will give some properties in the infinitesimally extremal case.

Suppose that $\mu \in L^\infty(R)$. If $\|\mu\|_\infty = M \geqslant 1$, then we turn to consider $v = \mu/(1+M)$. Then $v \in M(R)$ and the infinitesimal extremality of v is equivalent to that of μ. Thus we only consider the case that $\mu \in M(R)$ in the following.

Theorem 5.1. *Suppose that $\mu \in M(R)$ is infinitesimally uniquely extremal. Let α be a truncation of μ decided by a compact subset E of R with positive measure and $r > 0$. If $\|\alpha|_E\|_\infty > k/(1+r)$, then the infinitesimal equivalence class $[\alpha]^B$ is an infinitesimal Strebel point in B.*

Proof. Suppose that $\mu \in M(R)$. Let α be a truncation of μ decided by a compact subset E of R with positive measure and $r > 0$. If α satisfies that $\|\alpha|_E\|_\infty > k/(1+r)$, then $\|\alpha\|_\infty > k/(1+r)$. Now we will prove that $[\alpha]^B$ is an infinitesimal Strebel point in B, that is, $b([\alpha]^B) < \|[\alpha]^B\|$.

For convenience, set $s = k/(1+r)$. We have $b([\alpha]^B) \leqslant s$ since $\alpha \in [\alpha]^B$. Assume that the result does not hold, namely, $b([\alpha]^B) = \|[\alpha]^B\|$. Thus there at least exists a Beltrami coefficient $\eta \in [\alpha]^B$ such that $\|\eta\|_\infty \leqslant s$. Hence both μ and $\eta + \mu - \alpha$ belong to $[\mu]^B$, and

$$\mu - \alpha = \begin{cases} 0, & z \in E, \\ \frac{r\mu}{1+r}, & z \in R - E. \end{cases} \tag{5.1}$$

So $\|\mu - \alpha\|_\infty \leqslant rk/(1+r)$. Since

$$\|\eta + \mu - \alpha\|_\infty \leqslant \|\mu - \alpha\|_\infty + \|\eta\|_\infty \leqslant rk/(1+r) + k/(1+r) = k = \|\mu\|_\infty$$

and μ is infinitesimally uniquely extremal, it follows that $\eta + \mu - \alpha = \mu$. Hence $\eta = \alpha$. But this contradicts the inequality $\|\eta\|_\infty \leqslant s < \|\alpha\|_\infty$. Therefore $b([\alpha]^B) < \|[\alpha]^B\|$. Hence $[\alpha]^B$ is an infinitesimal Strebel point in B, namely, there exists $s_r > s$ and a unit vector $\varphi_r \in A_1(R)$ such that $\alpha \in [s_r\overline{\varphi_r}/|\varphi_r|]^B$. We call φ_r the infinitesimal Strebel differential induced by α. $\square$

Theorem 5.2. *Suppose that $\mu \in M(R)$ is infinitesimally extremal, and a truncation α of μ is decided by a compact subset E of R with positive measure and $r > 0$. If $\|\alpha|_E\|_\infty \leqslant k/(1+r)$, then α is infinitesimally extremal, too.*

Proof. By the extremal equivalence theorem (see [6,9,21]) (a Beltrami coefficient $\mu \in M(R)$ is extremal if and only if it is infinitesimally extremal), we know that μ is also extremal under the assumptions of Theorem 5.2. By Theorem 2.2 we have that the truncation α decided by E and r is extremal. Using the extremal equivalence theorem again we obtain that α is infinitesimally extremal, too. □

Corollary 5.1. *If μ is infinitesimally uniquely extremal, then α is also infinitesimally uniquely extremal under all the assumptions of Theorem* 5.2.

Proof. By the uniquely extremal equivalence theorem (see [2]) (a Beltrami coefficient μ is uniquely extremal if and only if it is infinitesimally uniquely extremal), it follows that μ is also uniquely extremal under the assumptions of Corollary 5.1. By Corollary 2.1 it is true that the truncation α decided by r and E is uniquely extremal. Using the uniquely extremal equivalence theorem again we obtain that α is infinitesimally uniquely extremal, too. □

Theorem 5.3. *Suppose that μ is infinitesimally uniquely extremal. Let α be a truncation decided by a compact subset E of R with positive measure and $r > 0$. Then the infinitesimally equivalent class $[\alpha]^B$ is a Strebel point in B if and only if $\|\alpha|_E\|_\infty > k/(1+r)$, α is infinitesimally uniquely extremal if and only if $\|\alpha|_E\|_\infty \leqslant k/(1+r)$.*

Proof. From Theorems 5.1, 5.2 and Corollary 5.1, Theorem 5.3 can be proved by the same method as that in Theorem 2.3. □

In [14], Reich proved the following infinitesimal main inequality.

Lemma B. *Suppose μ and ν are infinitesimally equivalent, for every $\varphi \in A_1(R)$ it is true that*

$$\iint_R |\varphi|\left(1 - |\mu|^2\right) \leqslant \iint_R |\varphi| \left|1 - \mu\frac{\varphi}{|\varphi|}\right|^2 \frac{\left|1 + \nu\frac{\varphi}{|\varphi|}\frac{1-\bar{\mu}\bar{\varphi}/|\varphi|}{1-\mu\varphi/|\varphi|}\right|^2}{1 - |\nu|^2}. \tag{5.2}$$

If there exists a unit vector $\varphi \in A_1(R)$ such that $[\mu]^B = [k\bar{\varphi}/|\varphi|]^B$ in B for some $k \in (0, 1)$, then

$$\frac{1+k}{1-k} \leqslant \iint_R |\varphi| \frac{|1 + \mu\varphi/|\varphi||^2}{1 - |\mu|^2}. \tag{5.3}$$

Theorem 5.4. *If there exists a sequence of truncations*

$$\alpha_n = \begin{cases} \mu(z), & z \in E_n, \\ \mu(z)/(1 + 1/n), & z \in R - E_n, \end{cases} \tag{5.4}$$

satisfying all the assumptions of Theorem 5.1, *where $\{E_n\}$ is a sequence of compact subsets of R with positive measure, then the infinitesimal Strebel differential sequence $\{\varphi_n\}$ induced by infinitesimal Strebel-point sequence $\{[\alpha_n]^B\} \subset B$ is a Hamilton sequence for μ.*

Proof. Using Theorem 5.1 and Lemma B, Theorem 5.4 can be proved by the same method as that in Theorem 3.1. □

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