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Remarks on Ramanujan function $A_q(z)$

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Abstract

In this short notes we will derive an inequality for scaled q^{-1}-Hermite orthogonal polynomials of Ismail and Masson, an inequality for scaled Stieltjes–Wigert, two inequalities for Ramanujan function and two definite integrals for Ramanujan function.

Keywords: q-Orthogonal polynomials; q-Airy function; Ramanujan's entire function; q^{-1}-Hermite polynomials; Stieltjes–Wigert polynomials; Plancherel–Rotach asymptotics; Scaling; Inequalities; Definite integrals

1. Introduction

Ramanujan function $A_q(z)$, which is also called q-Airy function in the literature, appears repeatedly in Ramanujan's work starting from the Rogers–Ramanujan identities, where $A_q(-1)$ and $A_q(-q)$ are expressed as infinite products [1], to properties of and conjectures about its zeros [2,3,5,8]. It is called q-Airy function because it appears repeatedly in the Plancherel–Rotach type asymptotics [6,9,10] of q-orthogonal polynomials, just like classical Airy function in the classical Plancherel–Rotach asymptotics of classical orthogonal polynomials [7,12]. In our joint work [10], we derived Plancherel–Rotach asymptotic expansions for the q^{-1}-Hermite of Ismail and Masson, q-Laguerre and Stieltjes–Wigert polynomials using a discrete analogue of Laplace's method. We found that when certain variables are above some critical values, the main terms in the asymptotics in the bulk contain Ramanujan function $A_q(z)$, when the variables are below these critical values, however, the main terms in the asymptotics expansion in the bulk involve theta functions.

In this paper we further investigate the properties of Ramanujan function $A_q(z)$. In Section 2 we introduce the notations and prove inequalities on Ismail–Masson polynomials $\{h_n(x|q)\}_{n=0}^{\infty}$ and Stieltjes–Wigert polynomials $\{S_n(x;q)\}_{n=0}^{\infty}$. In Section 3, we derive two inequalities for Ramanujan function $A_q(z)$. We use the asymptotic formulas in [10] to prove two definite integrals of $A_q(z)$ in Section 4.

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2. Preliminaries

In this section and the next section we will tacitly assume that all the log and power functions are taken as their principle branches, unless it is stated otherwise. As in our papers [9,10], we will follow the usual notations from q-series [3,4,7]

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{k=0}^{n}\bigl(1-aq^k\bigr), \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}. \tag{1}$$

Though out this paper, we shall always assume that

$$0 < q < 1, \quad t > 0, \tag{2}$$

hence $n = \infty$ is allowed in the above definitions. Then,

$$0 < \frac{(q;q)_n}{(q;q)_{n-k}} \leqslant 1 \tag{3}$$

and

$$0 < \begin{bmatrix} n \\ k \end{bmatrix}_q \leqslant \frac{1}{(q;q)_k} \tag{4}$$

for $k = 0, 1, \ldots, n$.

We will use the q-binomial theorem [3,4,7],

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k \tag{5}$$

and the following limiting cases, also known as Euler's formulas,

$$(z;q)_\infty = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q;q)_k}(-z)^k, \qquad \frac{1}{(z;q)_\infty} = \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k}. \tag{6}$$

Ramanujan function $A_q(z)$ is defined as [7,11]

$$A_q(z) := \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k}(-z)^k. \tag{7}$$

2.1. *Ismail–Masson polynomials* $\{h_n(x|q)\}_{n=0}^{\infty}$

Ismail–Masson polynomials $\{h_n(x|q)\}_{n=0}^{\infty}$ are defined as [7]

$$h_n(\sinh\xi|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)}(-1)^k e^{(n-2k)\xi}. \tag{8}$$

Ismail–Masson polynomials satisfy

$$\int_{-\infty}^{\infty} h_m(x|q)h_n(x|q)w_{IM}(x|q)\,dx = q^{-n(n+1)/2}(q;q)_n\delta_{m,n} \tag{9}$$

for $n, m = 0, 1, \ldots,$ where

$$w_{IM}(x|q) = \sqrt{\frac{-2q^{1/4}}{\pi \log q}}\,\exp\biggl\{\frac{2}{\log q}\bigl[\log\bigl(x+\sqrt{x^2+1}\bigr)\bigr]^2\biggr\}. \tag{10}$$

It is clear that their orthonormal polynomials are defined as

$$\tilde{h}_n(x|q) = \frac{q^{n(n+1)/4}}{\sqrt{(q;q)_n}} h_n(x|q). \tag{11}$$

Let

$$\sinh\xi_n := \frac{q^{-nt}u - q^{nt}u^{-1}}{2}, \tag{12}$$

and assume that

$$u \in \mathbb{C}\backslash\{0\}. \tag{13}$$

It is easy to see that

$$w_{IM}(\sinh\xi_n|q) = w_{IM}(\sinh u|q)u^{-4nt}q^{2n^2t^2}. \tag{14}$$

It is also clear from (8) and (12) that

$$h_n(\sinh\xi_n|q) = u^n q^{-n^2t}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q q^{k^2}\left(-\frac{q^{n(2t-1)}}{u^2}\right)^k. \tag{15}$$

Thus

$$\left|h_n(\sinh\xi_n|q)\right| \leqslant \frac{|u|^n}{q^{n^2t}}\sum_{k=0}^{n}\frac{q^{k^2}}{(q;q)_k}\left(\frac{q^{n(2t-1)}}{|u|^2}\right)^k \leqslant \frac{|u|^n}{q^{n^2t}}\sum_{k=0}^{\infty}\frac{q^{k^2}}{(q;q)_k}\left(\frac{q^{n(2t-1)}}{|u|^2}\right)^k,$$

or

$$\left|h_n(\sinh\xi_n|q)\right| \leqslant \frac{|u|^n}{q^{n^2t}}A_q\left(-\frac{q^{n(2t-1)}}{|u|^2}\right). \tag{16}$$

2.2. *Stieltjes–Wigert polynomials* $\{S_n(x;q)\}_{n=0}^{\infty}$

Stieltjes–Wigert polynomials $\{S_n(x;q)\}_{n=0}^{\infty}$ are defined as [7]

$$S_n(x;q) = \sum_{k=0}^{n}\frac{q^{k^2}(-x)^k}{(q;q)_k(q;q)_{n-k}}. \tag{17}$$

They are orthogonal respect to the weight function

$$w_{SW}(x;q) = \sqrt{\frac{-1}{2\pi\log q}}\exp\left\{\frac{1}{2\log q}\left[\log\left(\frac{x}{\sqrt{q}}\right)\right]^2\right\}, \tag{18}$$

with

$$\int_0^\infty S_n(x;q)S_m(x;q)w_{SW}(x;q)\,dx = \frac{q^{-n}}{(q;q)_n}\delta_{m,n} \tag{19}$$

for $n, m = 0, 1, \ldots.$ The orthonormal Stieltjes–Wigert polynomials with positive leading coefficients are

$$\tilde{s}_n(x;q) = (-1)^n\sqrt{q^n(q;q)_n}\,S_n(x;q). \tag{20}$$

In the case of the Stieltjes–Wigert polynomials the appropriate scaling is

$$x_n(t,u) = q^{-nt}u. \tag{21}$$

A calculation gives

$$w_{SW}\left(q^{-nt}u;q\right) = w_{SW}(u;q)u^{-nt}q^{(n^2t^2+nt)/2}. \tag{22}$$

Set $x = x_n(t,u)$ in (17) then replace k by $n-k$ to see that

$$S_n\left(x_n(t,u);q\right) = u^n q^{n^2(1-t)}\sum_{k=0}^{n}\frac{q^{k^2}(-\frac{q^{n(t-2)}}{u})^k}{(q;q)_k(q;q)_{n-k}}. \tag{23}$$

Thus

$$\left|S_n\big(x_n(t,u);q\big)(q;q)_n\right| \leqslant \frac{|u|^n}{q^{n^2(t-1)}}\sum_{k=0}^{n}\frac{q^{k^2}}{(q;q)_k}\left(\frac{q^{n(t-2)}}{|u|}\right)^k \leqslant \frac{|u|^n}{q^{n^2(t-1)}}\sum_{k=0}^{\infty}\frac{q^{k^2}}{(q;q)_k}\left(\frac{q^{n(t-2)}}{|u|}\right)^k$$

or

$$\left|S_n\big(x_n(t,u);q\big)\right| \leqslant \frac{|u|^n A_q(-\frac{q^{n(t-2)}}{|u|})}{(q;q)_\infty q^{n^2(t-1)}}. \tag{24}$$

3. Some inequalities for $A_q(z)$

It is clear that

$$n \geqslant \frac{1-q^n}{1-q} \geqslant nq^{n-1} \tag{25}$$

for $n \in \mathbb{N}$, then,

$$\left|\frac{(1-q)^k}{(q;q)_k}q^{k^2}(-z)^k\right| \leqslant \frac{(q|z|)^k}{k!} \tag{26}$$

for $k = 0, 1, \ldots$ and for any complex number z. Applying Lebesgue's dominated convergent theorem we have

$$\lim_{q\to 1} A_q\big((1-q)z\big) = e^{-z} \tag{27}$$

for any $z \in \mathbb{C}$, hence $A_q(z)$ is really one of many q-analogues of the exponential function. From (26) we also have obtained the inequality

$$\left|A_q\big((1-q)z\big)\right| \leqslant e^{q|z|} \tag{28}$$

or

$$\left|A_q(z)\right| \leqslant e^{q|z|/(1-q)} \tag{29}$$

for any complex number z. For any nonzero complex number z, then

$$\left|A_q(z)\right| \leqslant \sum_{k=0}^{\infty}\frac{q^k}{(q;q)_k}\big(q^{k-1}|z|\big)^k. \tag{30}$$

For $k = 0, 1, \ldots,$ the terms $q^{k(k-1)}|z|^k$ are bounded by

$$\left(\frac{|z|}{\sqrt{q}}\right)^{1/2}\exp\left\{-\frac{\log^2|z|}{4\log q}\right\}. \tag{31}$$

Thus

$$\left|A_q(z)\right| \leqslant \left(\frac{|z|}{\sqrt{q}}\right)^{1/2}\exp\left\{-\frac{\log^2|z|}{4\log q}\right\}\sum_{k=0}^{\infty}\frac{q^k}{(q;q)_k} \leqslant \frac{(\frac{|z|}{\sqrt{q}})^{1/2}\exp\{-\frac{\log^2|z|}{4\log q}\}}{(q;q)_\infty}$$

or we have

$$\left|A_q(z)\right| \leqslant \frac{(\frac{|z|}{\sqrt{q}})^{1/2}\exp\{-\frac{\log^2|z|}{4\log q}\}}{(q;q)_\infty} \tag{32}$$

for any nonzero complex number z.

Theorem 3.1. *Assume that $A_q(z)$ is Ramanujan function defined in* (6)*, then, for any complex number z*

$$\left|A_q(z)\right| \leqslant e^{q|z|/(1-q)}, \tag{33}$$

and

$$\left|A_q(z)\right| \leqslant \frac{(\frac{|z|}{\sqrt{q}})^{1/2}\exp\{-\frac{\log^2|z|}{4\log q}\}}{(q;q)_\infty}, \tag{34}$$

for any complex number $z \neq 0$.

Remark 3.2. The trivial inequality (25) can be used to show that a basic hypergeometric series converges to its hypergeometric series counter-part under suitable scaling and conditions. Also, using (34) the formulas (16) and (24) could be recast into other forms.

4. Definite integrals for $A_q(z)$

Put $t = \frac{1}{2}$ in formula (64) of [10], we have

$$\sqrt{\frac{(q;q)_n w_H(\sinh\xi_n|q)}{q^{n/2}w_H(\sinh u|q)}}\tilde{h}_n(\sinh\xi_n|q) = A_q\left(u^{-2}\right) + r_{IM} \tag{35}$$

with

$$|r_{IM}| \leqslant \frac{4(-q^3;q)_\infty A_q(-|u|^{-2})}{(q;q)_\infty^2}\left(q^{n/2} + q^{n^2/4}|u|^{-2\lfloor n/2\rfloor-2}\right). \tag{36}$$

Theorem 4.1. *Assuming that* $A_q(z)$ *and* $w_{IM}(x|q)$ *are defined as in* (7) *and* (10). *Then,*

$$\int_0^\infty A_q^2\left(u^{-2}\right)w_{IM}(u|q)\,du = 2(q;q)_\infty. \tag{37}$$

Proof. For the orthonormal Ismail–Masson polynomials $\tilde{h}_n(x|q)$ satisfy

$$\int_{-\infty}^\infty \left\{\tilde{h}_n(x|q)\right\}^2 w_{IM}(x|q)\,dx = 1. \tag{38}$$

Assume $u > 0$ and make the change of variable

$$x = \sinh\xi_n = \frac{q^{-n/2}u - q^{n/2}u^{-1}}{2} \tag{39}$$

in (38), we have

$$\int_0^\infty \left\{\tilde{h}_n(\sinh\xi_n|q)\right\}^2 w_{IM}(\sinh\xi_n|q)\left(q^{-n/2} + q^{n/2}u^{-2}\right)du = 2 \tag{40}$$

or

$$\int_0^\infty \left\{\sqrt{\frac{(q;q)_n w_{IM}(\sinh\xi_n|q)}{q^{n/2}w_{IM}(\sinh u|q)}}\,\tilde{h}_n(\sinh\xi_n|q)\right\}^2\left(1 + q^n u^{-2}\right)w_{IM}(\sinh u|q)\,du = 2(q;q)_n. \tag{41}$$

Thus we have

$$\lim_{n\to\infty}\int_0^\infty \left\{\sqrt{\frac{(q;q)_n w_{IM}(\sinh\xi_n|q)}{q^{n/2}w_{IM}(\sinh u|q)}}\,\tilde{h}_n(\sinh\xi_n|q)\right\}^2\left(1 + q^n u^{-2}\right)w_{IM}(\sinh u|q)\,du = 2(q;q)_\infty. \tag{42}$$

From (11), (14) and (16) we have

$$\left\{\sqrt{\frac{(q;q)_n w_{IM}(\sinh\xi_n|q)}{q^{n/2} w_{IM}(\sinh u|q)}}\,\left|\tilde{h}_n(\sinh\xi_n|q)\right|\right\}^2 \leqslant A_q^2\left(-u^{-2}\right), \tag{43}$$

and from (34) and (10) we know that

$$u^{-2} w_{IM}(\sinh u|q) A_q^2\left(-u^{-2}\right) \tag{44}$$

is bounded for $0 < u \leqslant 1$. From (33) we know that

$$A_q^2\left(-u^{-2}\right) \tag{45}$$

is bounded for $u \geqslant 1$. Lebesgue dominated convergence theorem allows us to take limit inside the integral. We use (35) to get (37). □

Take $t = 2$ in the formula (69) of [10],

$$\sqrt{\frac{q^{-n} w_{SW}(q^{-2n}u;q)}{(q;q)_n w_{SW}(u;q)}}\tilde{s}_n\left(q^{-2n}u;q\right) = \frac{\{A_q(u^{-1}) + r_{SW}(n)\}}{(q;q)_\infty} \tag{46}$$

with

$$\left|r_{SW}(n)\right| \leqslant \frac{2(-q^3;q)_\infty A_q(-|u|^{-1})}{(q;q)_\infty}\left\{q^{n/2} + \frac{q^{n^2/4}}{|u|^{1+\lfloor n/2\rfloor}}\right\}. \tag{47}$$

Theorem 4.2. *Assuming that $A_q(z)$, and $w_{SW}(x;q)$ are defined as in* (7) *and* (18)*. Then we have*

$$\int_0^\infty A_q^2\left(u^{-1}\right) w_{SW}(u;q)\,du = (q;q)_\infty. \tag{48}$$

Proof. From the orthogonality of Stieltjes–Wigert polynomials, we know that the orthonormal polynomials $\tilde{s}_N(x;q)$ satisfy

$$\int_0^\infty \tilde{s}_n^2(x;q) w_{SW}(x;q)\,dx = 1. \tag{49}$$

Let us make a change of variable

$$x = q^{-2n}u \tag{50}$$

in (49) with $u > 0$, then,

$$\int_0^\infty \left[\sqrt{\frac{q^{-2n} w_{SW}(q^{-2n}u;q)}{(q;q)_n w_{SW}(u;q)}}\,\tilde{s}_n\left(q^{-2n}u;q\right)\right]^2 w_{SW}(u;q)\,du = \frac{1}{(q;q)_n}, \tag{51}$$

therefore,

$$\lim_{n\to\infty}\int_0^\infty \left[\sqrt{\frac{q^{-2n} w_{SW}(q^{-2n}u;q)}{(q;q)_n w_{SW}(u;q)}}\,\tilde{s}_n\left(q^{-2n}u;q\right)\right]^2 w_{SW}(u;q)\,du = \frac{1}{(q;q)_\infty}. \tag{52}$$

From (20), (22) and (24) we have

$$\left\{\sqrt{\frac{q^{-2n} w_{SW}(q^{-2n}u;q)}{(q;q)_n w_{SW}(u;q)}}\,\left|\tilde{s}_n\left(q^{-2n}u;q\right)\right|\right\}^2 \leqslant \frac{A_q^2(-u^{-1})}{(q;q)_\infty^2},$$

and by (14) and (34)

$$A_q^2\left(-u^{-1}\right)w_{SW}(u;q)$$

is bounded for $0 < u \leqslant 1$, by (33),

$$A_q^2\left(-u^{-2}\right)$$

is bounded for $u \geqslant 1$. By Lebesgue dominated convergence theorem we interchange the orders of limit and integration, then apply (46) to get (48). □

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