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Uniqueness of meromorphic functions and differential polynomials sharing one value IM

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Abstract

In this paper, we study the uniqueness of meromorphic functions concerning differential polynomials, prove two theorems which generalize some results given by M.L. Fang and S.S. Bhoosnurmath and R.S. Dyavanal.

Keywords: Meromorphic functions; Sharing values; Differential polynomials

1. Introduction and results

Let $f(z)$ be a non-constant meromorphic function defined in the whole complex plane. It is assumed that the reader is familiar with the notations of the Nevanlinna theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $S(r,f)$ and so on, that can be found, for instance, in [1–3].

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let a be a finite complex number. We say that $f(z)$, $g(z)$ share the value a CM (counting multiplicities) if $f(z)$, $g(z)$ have the same a-points with the same multiplicities and we say that $f(z)$, $g(z)$ share the value a IM (ignoring multiplicities) if we do not consider the multiplicities. We denote by $N_{11}(r,\frac{1}{f-1})$ the counting function for common simple 1-points of $f(z)$ and $g(z)$ where multiplicity is not counted. $\overline{N}_L(r,\frac{1}{f^{(k)}-1})$ is the counting function for 1-points of both $f^{(k)}$ and $g^{(k)}$ about which $f^{(k)}$ has larger multiplicity than $g^{(k)}$, with multiplicity being not counted. For any constant a, we define

$$\Theta(a,f)=1-\varlimsup_{r\to\infty}\frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)}.$$

Let $f(z)$ be a non-constant meromorphic function. Let a be a finite complex number, and k be a positive integer, we denote by $N_{k)}(r,\frac{1}{f-a})$ (or $\overline{N}_{k)}(r,\frac{1}{f-a})$) the counting function for zeros of $f-a$ with multiplicity $\leqslant k$ (ignoring

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multiplicities), and by $N_{(k}(r, \frac{1}{f-a})$ (or $\overline{N}_{(k}(r, \frac{1}{f-a})$) the counting function for zeros of $f-a$ with multiplicity at least k (ignoring multiplicities). Set

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + \cdots + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

We further define

$$\delta_k(a, f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_k(r, \frac{1}{f-a})}{T(r, f)}.$$

Fang [4] proved the following result.

Theorem A. *Let $f(z)$ and $g(z)$ be two non-constant entire functions, and let n, k be two positive integers with $n > 2k+4$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c, c_1 and c_2 are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$, or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Theorem B. *Let $f(z)$ and $g(z)$ be two non-constant entire functions, let n, k be two positive integers with $n \geqslant 2k+8$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f(z) \equiv g(z)$.*

Recently, S.S. Bhoosnurmath and R.S. Dyavanal [5] extended Theorems A and B and proved the following theorem.

Theorem C. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let n, k be two positive integers with $n > 3k+8$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c, c_1 and c_2 are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$, or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Theorem D. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions satisfying $\Theta(\infty, f) > \frac{3}{n+1}$, and let n, k be two positive integers with $n \geqslant 3k+13$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f(z) \equiv g(z)$.*

In this paper, we generalize and improve Theorems C and D and obtain the following two theorems.

Theorem 1. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let n, k be two positive integers with $n > 6k+14$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 IM, then either $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c, c_1 and c_2 are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$, or $f = tg$ for a constant t such that $t^n = 1$.*

Theorem 2. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions satisfying $\Theta(\infty, f) > \frac{3}{n+1}$, and let n, k be two positive integers with $n \geqslant 6k+20$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 IM, then $f(z) \equiv g(z)$.*

2. Lemmas

For the proof of our results, we need the following lemmas.

Lemma 1. *(See [1].) Let $f(z)$ be a non-constant meromorphic function, $a_0, a_1, \ldots, a_n$ be finite complex numbers such that $a_n \neq 0$. Then*

$$T\left(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0\right) = nT(r, f) + S(r, f).$$

Lemma 2. *(See [1].) Let $f(z)$ be a non-constant meromorphic function, k be a positive integer, and let c be a non-zero finite complex number. Then*

$$\begin{aligned} T(r, f) &\leqslant \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - c}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f) \\ &\leqslant \overline{N}(r, f) + N_{k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - c}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f). \end{aligned}$$

Here $N_0(r, \frac{1}{f^{(k+1)}})$ is the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)} - c) \neq 0$.

Lemma 3. *(See [2].) Let $f(z)$ be a transcendental meromorphic function, and let $a_1(z)$, $a_2(z)$ be two meromorphic functions such that $T(r,a_i)=S(r,f)$, $i=1,2$. Then*

$$T(r,f)\leqslant \overline{N}(r,f)+N\left(r,\frac{1}{f-a_1}\right)+N\left(r,\frac{1}{f-a_1}\right)+S(r,f).$$

Lemma 4. *(See [6].) Let $f(z)$ be a non-constant entire function, and let $k\geqslant 2$ be a positive integer. If $f(z)f^{(k)}(z)\neq 0$, then $f=e^{az+b}$, where $a\neq 0$, b are constants.*

Lemma 5. *Let $f(z)$ and $g(z)$ be two meromorphic functions, and let k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share the value 1 IM and*

$$\Delta=(2k+3)\Theta(\infty,f)+(2k+4)\Theta(\infty,g)+2\Theta(0,f)+3\Theta(0,g)+\delta_{k+1}(0,f)+\delta_{k+1}(0,g)>4k+13 \quad (1)$$

then either $f^{(k)}g^{(k)}\equiv 1$ or $f\equiv g$.

Proof. Let

$$h(z)=\frac{f^{(k+2)}(z)}{f^{(k+1)}(z)}-2\frac{f^{(k+1)}(z)}{f^{(k)}(z)-1}-\frac{g^{(k+2)}(z)}{g^{(k+1)}(z)}+2\frac{g^{(k+1)}(z)}{g^{(k)}(z)-1}. \quad (2)$$

If z_0 is a common simple 1-point of $f^{(k)}$ and $g^{(k)}$, substituting their Taylor series at z_0 into (2), we see that z_0 is a zero of $h(z)$. Thus, we have

$$N_{11}\left(r,\frac{1}{f^{(k)}-1}\right)=N_{11}\left(r,\frac{1}{g^{(k)}-1}\right)\leqslant \overline{N}\left(r,\frac{1}{h}\right)\leqslant T(r,h)+O(1)\leqslant N(r,h)+S(r,f)+S(r,g). \quad (3)$$

By our assumptions, $h(z)$ have poles only at zeros of $f^{(k+1)}$ and $g^{(k+1)}$ and poles of f and g, and 1-points of $f^{(k)}$ whose multiplicities are not equal to the multiplicities of the corresponding 1-points of $g^{(k)}$.

Thus, we deduce from (2) that

$$N(r,h)\leqslant \overline{N}(r,f)+\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)$$
$$+N_0\left(r,\frac{1}{f^{(k+1)}}\right)+N_0\left(r,\frac{1}{g^{(k+1)}}\right). \quad (4)$$

Here $N_0(r,\frac{1}{f^{(k+1)}})$ has the same meaning as in Lemma 2. By Lemma 2, we have

$$T(r,f)\leqslant \overline{N}(r,f)+N_{k+1}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)-N_0\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f), \quad (5)$$

$$T(r,g)\leqslant \overline{N}(r,g)+N_{k+1}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)-N_0\left(r,\frac{1}{g^{(k+1)}}\right)+S(r,g). \quad (6)$$

Since $f^{(k)}$ and $g^{(k)}$ share the value 1 IM, we have

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)\\
&\quad\leqslant N_{11}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)+N\left(r,\frac{1}{f^{(k)}-1}\right)\\
&\quad\leqslant N_{11}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)+T\left(r,f^{(k)}\right)+O(1)\\
&\quad\leqslant N_{11}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)+m\left(r,f^{(k)}\right)+N\left(r,f^{(k)}\right)+O(1)
\end{aligned}$$

$$\leqslant N_{11}\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}_L\left(r, \frac{1}{g^{(k)}-1}\right) + m(r, f) + m\left(r, \frac{f^{(k)}}{f}\right) + N(r, f) + k\overline{N}(r, f) + O(1)$$
$$\leqslant N_{11}\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}_L\left(r, \frac{1}{g^{(k)}-1}\right) + T(r, f) + k\overline{N}(r, f) + S(r, f). \tag{7}$$

Noting that

$$\overline{N}\left(r, \frac{1}{f^{(k)}}\right) \leqslant \overline{N}\left(r, \frac{f}{f^{(k)}}\right) + \overline{N}\left(r, \frac{1}{f}\right) \leqslant T\left(r, \frac{f}{f^{(k)}}\right) + \overline{N}\left(r, \frac{1}{f}\right) \leqslant T\left(r, \frac{f^{(k)}}{f}\right) + \overline{N}\left(r, \frac{1}{f}\right) + S(r, f)$$
$$\leqslant N\left(r, \frac{f^{(k)}}{f}\right) + m\left(r, \frac{f^{(k)}}{f}\right) + \overline{N}\left(r, \frac{1}{f}\right) + S(r, f) \leqslant k\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + S(r, f), \tag{8}$$

$$\overline{N}_L\left(r, \frac{1}{f^{(k)}-1}\right) \leqslant N\left(r, \frac{1}{f^{(k)}-1}\right) - \overline{N}\left(r, \frac{1}{f^{(k)}-1}\right) \leqslant N\left(r, \frac{f^{(k)}}{f^{(k+1)}}\right)$$
$$\leqslant N\left(r, \frac{f^{(k+1)}}{f^{(k)}}\right) + S(r, f) \leqslant \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + S(r, f)$$

we have

$$\overline{N}_L\left(r, \frac{1}{f^{(k)}-1}\right) \leqslant (k+1)\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + S(r, f). \tag{9}$$

Similarly

$$\overline{N}_L\left(r, \frac{1}{g^{(k)}-1}\right) \leqslant (k+1)\overline{N}(r, g) + \overline{N}\left(r, \frac{1}{g}\right) + S(r, g). \tag{10}$$

We obtain from (3)–(10) that

$$T(r, g) \leqslant (2k+3)\overline{N}(r, f) + (2k+4)\overline{N}(r, g) + 2\overline{N}\left(r, \frac{1}{f}\right) + 3\overline{N}\left(r, \frac{1}{g}\right) + N_{k+1}\left(r, \frac{1}{f}\right) + N_{k+1}\left(r, \frac{1}{g}\right)$$
$$+ S(r, f) + S(r, g).$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r, f) \leqslant T(r, g)$ for $r \in I$. Hence

$$T(r, g) \leqslant \bigl\{\bigl[4k+13 - (2k+3)\Theta(\infty, f) - (2k+4)\Theta(\infty, g) - 2\Theta(0, f) - 3\Theta(0, g)$$
$$- \delta_{k+1}(0, f) - \delta_{k+1}(0, g)\bigr] + \varepsilon\bigr\}T(r, g) + S(r, g) \tag{11}$$

for $r \in I$ and $0 < \varepsilon < \Delta - (4k+13)$.

Thus we obtain from (1) and (11) that $T(r, g) \leqslant S(r, g)$ for $r \in I$, a contradiction.

Hence, we get $h(z) \equiv 0$; that is

$$\frac{f^{(k+2)}(z)}{f^{(k+1)}(z)} - 2\frac{f^{(k+1)}(z)}{f^{(k)}(z)-1} \equiv \frac{g^{(k+2)}(z)}{g^{(k+1)}(z)} - 2\frac{g^{(k+1)}(z)}{g^{(k)}(z)-1}.$$

By solving this equation, we obtain

$$\frac{1}{f^{(k)}(z)-1} \equiv \frac{bg^{(k)}(z)+a-b}{g^{(k)}(z)-1} \tag{12}$$

where a, b are two constants. Next, we consider three cases.

Case 1. $b \neq 0$ and $a = b$.

Subcase 1. $b=-1$. Then, we deduce from (12) that $f^{(k)}g^{(k)} \equiv 1$.

Subcase 2. $b \neq -1$. Then, we get from (12) that

$$\frac{1}{f^{(k)}(z)} \equiv \frac{bg^{(k)}(z)}{(1+b)g^{(k)}(z)-1}.$$

So

$$\overline{N}\left(r, \frac{1}{g^{(k)}(z)-(1/(1+b))}\right) \leqslant \overline{N}\left(r, \frac{1}{f^{(k)}}\right). \tag{13}$$

From (13) and (8), we get

$$\overline{N}\left(r, \frac{1}{g^{(k)}(z)-(1/(1+b))}\right) \leqslant k\overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + S(r,f).$$

By Lemma 2, we have

$$\begin{aligned}
T(r,g) &\leqslant \overline{N}(r,g) + N_{k+1}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g^{(k)}-(1/(1+b))}\right) - N_0\left(r, \frac{1}{g^{(k+1)}}\right) + S(r,g) \\
&\leqslant \overline{N}(r,g) + N_{k+1}\left(r, \frac{1}{g}\right) + k\overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + S(r,f) + S(r,g) \\
&\leqslant (2k+3)\overline{N}(r,f) + (2k+4)\overline{N}(r,g) + 2\overline{N}\left(r, \frac{1}{f}\right) + 3\overline{N}\left(r, \frac{1}{g}\right) + N_{k+1}\left(r, \frac{1}{f}\right) + N_{k+1}\left(r, \frac{1}{g}\right) \\
&\quad + S(r,f) + S(r,g).
\end{aligned}$$

That is $T(r,g) \leqslant (4k+14-\Delta)T(r,g) + S(r,g)$. Thus, we obtain that $T(r,g) \leqslant S(r,g)$ for $r \in I$, a contradiction.

Case 2. $b \neq 0$ and $a \neq b$.

Subcase 1. $b=-1$. Then we obtain from (12) that

$$f^{(k)}(z) \equiv \frac{a}{-g^{(k)}(z)+a+1}.$$

Therefore

$$\overline{N}\left(r, \frac{a}{-g^{(k)}(z)+a+1}\right) = \overline{N}\left(r, f^{(k)}(z)\right) = \overline{N}(r,f).$$

By Lemma 2, we have

$$\begin{aligned}
T(r,g) &\leqslant \overline{N}(r,g) + N_{k+1}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g^{(k)}-(a+1)}\right) + S(r,g) \\
&\leqslant \overline{N}(r,g) + N_{k+1}\left(r, \frac{1}{g}\right) + \overline{N}(r,f) + S(r,f) + S(r,g) \\
&\leqslant (2k+3)\overline{N}(r,f) + (2k+4)\overline{N}(r,g) + 2\overline{N}\left(r, \frac{1}{f}\right) + 3\overline{N}\left(r, \frac{1}{g}\right) + N_{k+1}\left(r, \frac{1}{f}\right) + N_{k+1}\left(r, \frac{1}{g}\right) \\
&\quad + S(r,f) + S(r,g).
\end{aligned}$$

Using the argument as in Case 1, we get a contradiction.

Subcase 2. $b \neq -1$. Then we get from (12) that

$$f^{(k)}(z) - \left(1 + \frac{1}{b}\right) \equiv \frac{-a}{b^2(g^{(k)}(z) + (a-b)/b)}.$$

Therefore

$$\overline{N}\left(r, \frac{1}{g^{(k)}(z)+(a-b)/b}\right) = \overline{N}\left(r, f^{(k)}(z) - \left(1+\frac{1}{b}\right)\right) = \overline{N}(r,f).$$

By Lemma 2, we have

$$\begin{aligned}
T(r,g) &\leqslant \overline{N}(r,g) + N_{k+1}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g^{(k)}+\frac{a-b}{b}}\right) + S(r,g)\\
&\leqslant \overline{N}(r,g) + N_{k+1}\left(r,\frac{1}{g}\right) + \overline{N}(r,f) + S(r,f) + S(r,g)\\
&\leqslant (2k+3)\overline{N}(r,f) + (2k+4)\overline{N}(r,g) + 2\overline{N}\left(r,\frac{1}{f}\right) + 3\overline{N}\left(r,\frac{1}{g}\right) + N_{k+1}\left(r,\frac{1}{f}\right) + N_{k+1}\left(r,\frac{1}{g}\right)\\
&\quad + S(r,f) + S(r,g).
\end{aligned}$$

Using the argument as in Case 1, we get a contradiction.

Case 3. $b = 0$.

From (12), we obtain

$$f = \frac{1}{a}g + P(z), \tag{14}$$

where $P(z)$ is a polynomial. If $P(z) \not\equiv 0$, then by Lemma 3, we have

$$\begin{aligned}
T(r,f) &\leqslant \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f-P}\right) + S(r,f)\\
&\leqslant \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right) + S(r,f).
\end{aligned} \tag{15}$$

From (14), we obtain $T(r,f) = T(r,g) + S(r,f)$.

Hence, substituting this into (15), we get

$$T(r,f) \leqslant \left\{3 - \left[\Theta(\infty,f) + \Theta(0,f) + \Theta(0,g)\right] + \varepsilon\right\}T(r,f) + S(r,f)$$

where

$$\begin{aligned}
0 < \varepsilon < 1 - \delta_{k+1}(0,f) + 1 - \delta_{k+1}(0,g) + (2k+2)\left[1-\Theta(\infty,f)\right] + (2k+4)\left[1-\Theta(\infty,g)\right] + 1 - \Theta(0,f)\\
+ 2\left[1-\Theta(0,g)\right].
\end{aligned}$$

Therefore $T(r,f) \leqslant \{4k + 14 - \Delta\}T(r,f) + S(r,f)$.

Hence, by (1), we deduce that $T(r,f) \leqslant S(r,f)$ for $r \in I$, a contradiction.

Therefore, we deduce that $P(z) \equiv 0$, that is

$$f = \frac{1}{a}g. \tag{16}$$

If $a \neq 1$, then $f^{(k)}$ and $g^{(k)}$ sharing the value 1 IM, we deduce from (16) that $g^{(k)} \neq 1$. That is $\overline{N}(r, \frac{1}{g^{(k)}-1}) = 0$.

Next, we can deduce a contradiction as in Case 2. Thus, we get that $a = 1$, that is $f \equiv g$. Thus the proof of Lemma 5 is completed. □

3. Proof of Theorem 1

Consider $F(z) = f^n(z)$ and $G(z) = g^n(z)$. We have

$$\Delta = (2k+3)\Theta(\infty,F) + (2k+4)\Theta(\infty,G) + 2\Theta(0,F) + 3\Theta(0,G) + \delta_{k+1}(0,F) + \delta_{k+1}(0,G).$$

Consider

$$\Theta(0,F) = 1 - \overline{\lim_{r\to\infty}}\frac{\overline{N}(r,\frac{1}{F})}{T(r,F)} = 1 - \overline{\lim_{r\to\infty}}\frac{\overline{N}(r,\frac{1}{f^n})}{nT(r,f)} \geqslant 1 - \overline{\lim_{r\to\infty}}\frac{T(r,f)}{nT(r,f)} \geqslant \frac{n-1}{n}. \tag{17}$$

Similarly

$$\Theta(0, G) \geqslant \frac{n-1}{n}, \tag{18}$$

$$\Theta(\infty, F) \geqslant \frac{n-1}{n}, \tag{19}$$

$$\Theta(\infty, G) \geqslant \frac{n-1}{n}. \tag{20}$$

Next, we have

$$\begin{aligned} \delta_{k+1}(0, F) &= 1 - \overline{\lim_{r\to\infty}} \frac{N_{k+1}(r, \frac{1}{F})}{T(r, F)} \geqslant 1 - \overline{\lim_{r\to\infty}} \frac{(k+1)\overline{N}(r, \frac{1}{F})}{T(r, F)} = 1 - \overline{\lim_{r\to\infty}} \frac{(k+1)\overline{N}(r, \frac{1}{f})}{nT(r, f)} \\ &\geqslant 1 - \frac{k+1}{n} = \frac{n-(k+1)}{n}. \end{aligned} \tag{21}$$

Similarly

$$\delta_{k+1}(0, G) \geqslant 1 - \frac{k+1}{n} = \frac{n-(k+1)}{n}. \tag{22}$$

From (17)–(22), we get

$$\Delta = (2k+3)\frac{n-1}{n} + (2k+4)\frac{n-1}{n} + 2\frac{n-1}{n} + 3\frac{n-1}{n} + \frac{n-(k+1)}{n} + \frac{n-(k+1)}{n}.$$

Since $n > 6k+14$, we get $\Delta > 4k+13$.

Considering $F^{(k)}(z) = [f^n(z)]^{(k)}$ and $G^{(k)}(z) = [g^n(z)]^{(k)}$ share the value 1 IM, then by Lemma 5, we deduce that either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.

Next, we consider two cases.

Case 1. $F^{(k)}G^{(k)} \equiv 1$; that is

$$\left[f^n(z)\right]^{(k)}\left[g^n(z)\right]^{(k)} \equiv 1. \tag{23}$$

We proved that

$$f \neq 0, \infty \quad \text{and} \quad g \neq 0, \infty. \tag{24}$$

Suppose that f has a zero z_0 of order p, then z_0 is a zero of $[f^n(z)]^{(k)}$ of order $(6k+k_1)p - k = 6pk + k_1p - k$ and z_0 is a pole of $[g^n(z)]^{(k)}$ of order $(6k+k_1)q + k = 6qk + k_1q + k$, where $k_1 > 14$. From (23), we get

$$6pk + k_1p - k = 6qk + k_1q + k,$$

i.e. $(6k+k_1)(p-q) = 2k$, which is impossible since p and q are integers and $k_1 > 14$.

Therefore $f \neq 0$ and $g \neq 0$. Similarly $f \neq \infty$ and $g \neq \infty$. From (23) and (24), we get

$$\left[f^n(z)\right]^{(k)} \neq 0 \quad \text{and} \quad \left[g^n(z)\right]^{(k)} \neq 0. \tag{25}$$

From (23)–(25) and Lemma 4, we get that $f(z) = c_1e^{cz}$ and $g(z) = c_2e^{-cz}$, where c, c_1 and c_2 are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$.

Case 2. $F \equiv G$; that is $f^n(z) = g^n(z)$. This implies $f \equiv tg$ for a constant t such that $t^n = 1$.

4. Proof of Theorem 2

Let $F(z)=f^n(f-1)$ and $G(z)=g^n(g-1)$. We have

$$\Delta=(2k+3)\Theta(\infty,F)+(2k+4)\Theta(\infty,G)+2\Theta(0,F)+3\Theta(0,G)+\delta_{k+1}(0,F)+\delta_{k+1}(0,G).$$

Using the argument as in the proof of Theorem 1, we get

$$\Theta(0,F)=1-\varlimsup_{r\to\infty}\frac{\overline{N}(r,\frac{1}{f^n(f-1)})}{(n+1)T(r,f)}=1-\varlimsup_{r\to\infty}\frac{\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{(f-1)})}{(n+1)T(r,f)}\geqslant 1-\varlimsup_{r\to\infty}\frac{2T(r,f)}{(n+1)T(r,f)}$$
$$\geqslant 1-\frac{2}{n+1}=\frac{n-1}{n+1}. \tag{26}$$

Similarly

$$\Theta(0,G)\geqslant\frac{n-1}{n+1}, \tag{27}$$
$$\Theta(\infty,F)\geqslant\frac{n}{n+1}, \tag{28}$$
$$\Theta(\infty,G)\geqslant\frac{n}{n+1}. \tag{29}$$

Next, we have

$$\delta_{k+1}(0,F)=1-\varlimsup_{r\to\infty}\frac{N_{k+1}(r,\frac{1}{F})}{T(r,F)}=1-\varlimsup_{r\to\infty}\frac{N_{k+1}(r,\frac{1}{f^n(f-1)})}{(n+1)T(r,f)}$$
$$\geqslant 1-\varlimsup_{r\to\infty}\frac{(k+2)T(r,f)}{(n+1)T(r,f)}\geqslant 1-\frac{k+2}{n+1}=\frac{n-(k+1)}{n+1}. \tag{30}$$

Similarly

$$\delta_{k+1}(0,G)\geqslant\frac{n-(k+1)}{n+1}. \tag{31}$$

From (26)–(31), we get

$$\Delta=(2k+3)\frac{n}{n+1}+(2k+4)\frac{n}{n+1}+2\frac{n-1}{n+1}+3\frac{n-1}{n+1}+\frac{n-(k+1)}{n+1}+\frac{n-(k+1)}{n+1}.$$

Since $n>6k+20$, we get $\Delta>4k+13$. Considering $F^{(k)}(z)$ and $G^{(k)}(z)$ share the value 1 IM, then by Lemma 5, we deduce that either $F^{(k)}G^{(k)}\equiv 1$ or $F\equiv G$.

Next, we consider two cases.

Case 1. $F^{(k)}G^{(k)}\equiv 1$, that is

$$\left[f^n(z)\left[f(z)\right]-1\right]^{(k)}\left[g^n(z)\left[g(z)-1\right]\right]^{(k)}\equiv 1. \tag{32}$$

Suppose that f has a zero z_0 of order p, then z_0 is a pole of g of order q. From (32), we get $np-k=nq+q+k$, i.e. $n(p-q)=q+2k$, which implies that $p\geqslant q+1$ and $q+2k\geqslant n$. Therefore $p\geqslant n-2k+1$.

Let z_1 be a zero of $f-1$ of order p_1, then z_1 is a zero of $[f^n(z)(f-1)]^{(k)}$ of order p_1-k and z_1 is a pole of g of order q_1. From (32), we get $p_1-k=nq_1+q_1+k$, i.e. $p_1=(n+1)q_1+2k$. Therefore $p_1\geqslant n+2k+1$.

Let z_2 be a zero of f' of order p_2 that is not a zero of $f(f-1)$, as above, we obtain from (32) $p_2-(k-1)=nq_2+q_2+k$.

Therefore $p_2\geqslant n+2k$.

Moreover, in the same manner as above, we have similar results for zeros of $[g^n(z)[g(z)-1]]^{(k)}$.

On the other hand, suppose that z_3 is a pole of f. From (32), we get that z_3 is the zero of $[g^n(z)[g(z)-1]]^{(k)}$. Thus

$$\begin{aligned}\overline{N}(r,f) &\leqslant \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+\overline{N}\left(r,\frac{1}{g'}\right)\\ &\leqslant \frac{1}{n-2k+1}N\left(r,\frac{1}{g}\right)+\frac{1}{n+2k+1}N\left(r,\frac{1}{g-1}\right)+\frac{1}{n+2k}N\left(r,\frac{1}{g'}\right).\end{aligned}$$

Since $n>6k+20$, we get

$$\begin{aligned}\overline{N}(r,f) &\leqslant \frac{1}{4k+21}N\left(r,\frac{1}{g}\right)+\frac{1}{8k+21}N\left(r,\frac{1}{g-1}\right)+\frac{1}{8k+20}N\left(r,\frac{1}{g'}\right)\\ &\leqslant \frac{1}{25}N\left(r,\frac{1}{g}\right)+\frac{1}{29}N\left(r,\frac{1}{g-1}\right)+\frac{1}{28}N\left(r,\frac{1}{g'}\right)\\ &\leqslant \left(\frac{1}{25}+\frac{1}{29}+\frac{1}{28}\right)T(r,g)+S(r,g)\leqslant 0.11T(r,g)+S(r,g).\end{aligned} \tag{33}$$

From Lemma 3 and (33), we obtain

$$\begin{aligned}T(r,f) &\leqslant \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}(r,f)+S(r,f)\\ &\leqslant \frac{1}{25}N\left(r,\frac{1}{f}\right)+\frac{1}{29}N\left(r,\frac{1}{f-1}\right)+0.11T(r,g)+S(r,f)\\ &\leqslant 0.07T(r,f)+0.11T(r,g)+S(r,f).\end{aligned} \tag{34}$$

Similarly, we have

$$T(r,g)\leqslant 0.07T(r,g)+0.11T(r,f)+S(r,f). \tag{35}$$

By (34) and (35), we get $T(r,f)+T(r,g)\leqslant 0.18(T(r,g)+T(r,f))+S(r,f)$, which is a contradiction.

Case 2. $F\equiv G$; that is

$$f^n(f-1)=g^n(g-1). \tag{36}$$

Suppose $f\not\equiv g$, then we consider two cases:

(i) Let $H=\frac{f}{g}$ be a constant. Then from (36) it follows that $H\neq 1$, $H^n\neq 1$, $H^{n+1}\neq 1$ and $g=\frac{1-H^n}{1-H^{n+1}}=$ constant, which leads to a contradiction.

(ii) Let $H=\frac{f}{g}$ be not a constant. Since $f\not\equiv g$, we have $H\not\equiv 1$ and hence we deduce that $g=\frac{1-H^n}{1-H^{n+1}}$ and $f=(\frac{1-H^n}{1-H^{n+1}})H=\frac{(1+H+H^2+\cdots+H^{n-1})H}{1+H+H^2+\cdots+H^n}$, where H is a non-constant meromorphic function. It follows that

$$T(r,f)=T(r,gH)=(n+1)T(r,H)+S(r,f).$$

On the other hand, by the second fundamental theorem, we deduce

$$\overline{N}(r,f)=\sum_{j=1}^{n}\overline{N}\left(r,\frac{1}{H-\alpha_j}\right)\geqslant (n-2)T(r,H)+S(r,f),$$

where $\alpha_j(\neq 1)$ $(j=1,2,\ldots,n)$ are distinct roots of the algebraic equation $H^{n+1}=1$.

We have

$$\begin{aligned}\Theta(\infty,f) &= 1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,f)}{T(r,f)}\leqslant 1-\overline{\lim_{r\to\infty}}\frac{(n-2)T(r,H)+S(r,f)}{T(r,f)}\\ &\leqslant 1-\overline{\lim_{r\to\infty}}\frac{(n-2)T(r,H)+S(r,f)}{(n+1)T(r,H)+S(r,f)}\leqslant 1-\frac{n-2}{n+1}=\frac{3}{n+1}\end{aligned}$$

which contradicts the assumption $\Theta(\infty,f)\geqslant\frac{3}{n+1}$.

Thus $f\equiv g$. This completes the proof of Theorem 2.

RETRACTED

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