



# Invariance of quasi-arithmetic means with function weights

Justyna Jarczyk

Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, Szafrana 4A, PL-65-516 Zielona Góra, Poland

## ARTICLE INFO

### Article history:

Received 28 August 2008

Available online 24 November 2008

Submitted by D. O'Regan

### Keywords:

Mean

Functional equation

Invariant mean

Quasi-arithmetic mean

Function weight

## ABSTRACT

Let  $I \subset \mathbb{R}$  be a non-trivial interval and let  $\lambda, \mu, \nu: I^2 \rightarrow (0, 1)$ . We present some results concerning the following functional equation, generalizing the Matkowski–Sutô equation,

$$\begin{aligned} &\lambda(x, y)\varphi^{-1}(\mu(x, y)\varphi(x) + (1 - \mu(x, y))\varphi(y)) \\ &\quad + (1 - \lambda(x, y))\psi^{-1}(\nu(x, y)\psi(x) + (1 - \nu(x, y))\psi(y)) \\ &= \lambda(x, y)x + (1 - \lambda(x, y))y, \end{aligned}$$

where  $\varphi, \psi: I \rightarrow \mathbb{R}$  are continuous and strictly monotonic unknown functions.

© 2008 Elsevier Inc. All rights reserved.

## 1. Introduction

Given a continuous strictly monotonic function  $\alpha$ , defined on a real interval  $I$ , and  $\lambda: I^2 \rightarrow (0, 1)$  consider the quasi-arithmetic mean  $A_\lambda^\alpha: I^2 \rightarrow I$  generated by  $\alpha$  and weighted by  $\lambda$ :

$$A_\lambda^\alpha(x, y) = \alpha^{-1}(\lambda(x, y)\alpha(x) + (1 - \lambda(x, y))\alpha(y)). \quad (1)$$

Observe that any strict mean  $M: I^2 \rightarrow I$ , i.e. function satisfying

$$\min\{x, y\} < M(x, y) < \max\{x, y\}, \quad x, y \in I, \quad x \neq y,$$

can be written as  $A_\lambda^\alpha$  with some  $\lambda: I^2 \rightarrow (0, 1)$ ; it is enough to define  $\lambda$  by

$$\lambda(x, y) = \frac{\varphi(M(x, y)) - \varphi(y)}{\varphi(x) - \varphi(y)}.$$

In the present paper we study the functional equation

$$\begin{aligned} &\lambda(x, y)\varphi^{-1}(\mu(x, y)\varphi(x) + (1 - \mu(x, y))\varphi(y)) + (1 - \lambda(x, y))\psi^{-1}(\nu(x, y)\psi(x) + (1 - \nu(x, y))\psi(y)) \\ &= \lambda(x, y)x + (1 - \lambda(x, y))y, \end{aligned} \quad (2)$$

that is, equivalently,

$$\lambda(x, y)A_\mu^\varphi(x, y) + (1 - \lambda(x, y))A_\nu^\psi(x, y) = \lambda(x, y)x + (1 - \lambda(x, y))y. \quad (3)$$

In the whole paper  $I$  stands for a non-trivial interval and  $\lambda, \mu, \nu: I^2 \rightarrow (0, 1)$  are fixed functions.

In the case when  $\lambda$  is constant or, more generally,

E-mail address: j.jarczyk@wmie.uz.zgora.pl.

$$\lambda(A_\mu^\varphi(x, y), A_\nu^\psi(x, y)) = \lambda(x, y), \quad x, y \in I,$$

applying (1) we can rewrite (3) as

$$A_\lambda^{\text{id}}(A_\mu^\varphi(x, y), A_\nu^\psi(x, y)) = A_\lambda^{\text{id}}(x, y). \quad (4)$$

This is the equation of the invariance of  $A_\lambda^{\text{id}}$  with respect to the pair  $(A_\mu^\varphi, A_\nu^\psi)$ . In the case of constant weight functions  $\lambda, \mu, \nu$  Eq. (4) was investigated and solved by several authors under various regularity assumptions. In particular, there is a vast literature concerning the invariance of classical arithmetic mean (cf. [3,4,11]). The general solution, without any regularity assumption, was published in [5]. The invariance equation involving three quasi-arithmetic means with scalar weights was studied in [2,7]. The final answer with no additional regularity assumption has been recently obtained in [8], where a method elaborated by Z. Daróczy and Zs. Páles in [4] was adopted as well as results from [7] and [6].

Below we quote the main result of [8] (see [8, Theorem 1]). To shorten its formulation accept the following terminology. A pair  $(\varphi, \psi)$  of real functions defined on  $I$  is said to be *affine* if there exist  $a, c \in \mathbb{R} \setminus \{0\}$  and  $b, d \in \mathbb{R}$  such that

$$\varphi(x) = ax + b \quad \text{and} \quad \psi(x) = cx + d, \quad x \in I,$$

and, given a  $p \in \mathbb{R} \setminus \{0\}$ , a pair  $(\varphi, \psi)$  is said to be *p-exponential* if there exist  $a, c \in \mathbb{R} \setminus \{0\}$  and  $b, d \in \mathbb{R}$  such that

$$\varphi(x) = ae^{px} + b \quad \text{and} \quad \psi(x) = ce^{-px} + d, \quad x \in I.$$

**Theorem 1.** Assume that  $\lambda, \mu, \nu$  are constant. A pair  $(\varphi, \psi)$  of continuous strictly monotonic functions  $\varphi, \psi : I \rightarrow \mathbb{R}$  satisfies Eq. (2) if and only if two conditions are fulfilled:

- (i)  $\lambda = \frac{\nu}{1-\mu+\nu}$ ,
- (ii) either  $(\varphi, \psi)$  is affine, or  $\lambda = 1/2$  and  $(\varphi, \psi)$  is *p-exponential* with some  $p \in \mathbb{R} \setminus \{0\}$ .

In what follows by the diagonal we mean the set  $\{(x, y) \in I^2 : x = y\}$ .

**Lemma 1.** Assume that  $\lambda$  is differentiable in first variable on the diagonal and let  $\alpha : I \rightarrow \mathbb{R}$  be a differentiable function with non-vanishing derivative. Then

$$\partial_1 A_\lambda^\alpha(x, x) = \lambda(x, x), \quad x \in I.$$

If, in addition,  $\lambda$  is twice differentiable in first variable on the diagonal and  $\alpha$  is twice differentiable, then

$$\partial_1^2 A_\lambda^\alpha(x, x) = \lambda(x, x)(1 - \lambda(x, x)) \frac{\alpha''(x)}{\alpha'(x)} + 2\partial_1 \lambda(x, x), \quad x \in I.$$

**Proof.** Differentiating the functions in (1) with respect to the first variable on the diagonal we obtain

$$\alpha'(A_\lambda^\alpha(x, x)) \partial_1 A_\lambda^\alpha(x, x) = \lambda(x, x) \alpha'(x)$$

for every  $x \in I$ . Using the equality  $A_\lambda^\alpha(x, x) = x$  we obtain the first assertion. Now assume that  $\lambda$  is twice differentiable in first variable on the diagonal and fix any  $x \in I$ . Then, by (1), we have

$$\alpha'(A_\lambda^\alpha(y, x)) \partial_1 A_\lambda^\alpha(y, x) = \partial_1 \lambda(y, x)(\alpha(y) - \alpha(x)) + \lambda(y, x) \alpha'(y) \quad (5)$$

for  $y$ 's from a neighbourhood of  $x$ . Differentiating the functions appearing here with respect to the first variable, we get

$$\alpha''(A_\lambda^\alpha(x, x)) (\partial_1 A_\lambda^\alpha(x, x))^2 + \alpha'(A_\lambda^\alpha(x, x)) \partial_1^2 A_\lambda^\alpha(x, x) 2\partial_1 \lambda(x, x) \alpha'(x) + \lambda(x, x) \alpha''(x).$$

Making use of the first part of the lemma and the relation  $A_\lambda^\alpha(x, x) = x$  we come to the second assertion.  $\square$

In the next corollary we present a relationship between  $\lambda, \mu, \nu$  on the diagonal.

**Corollary 1.** Assume that  $\lambda, \mu, \nu$  are differentiable in first variable on the diagonal. If Eq. (2) is satisfied by a pair of differentiable functions with non-vanishing derivatives, then

$$\lambda(x, x) = \frac{\nu(x, x)}{1 - \mu(x, x) + \nu(x, x)}, \quad x \in I.$$

**Proof.** Differentiating the functions in (3) with respect to the first variable on the diagonal, we have

$$\partial_1 \lambda(x, x) (A_\mu^\varphi(x, x) - A_\nu^\psi(x, x)) + \lambda(x, x) \partial_1 A_\mu^\varphi(x, x) + (1 - \lambda(x, x)) \partial_1 A_\nu^\psi(x, x) = \lambda(x, x)$$

for every  $x \in I$ . Using the first part of Lemma 1 and the relation  $A_\lambda^\alpha(x, x) = x$  we obtain the claimed condition.  $\square$

## 2. Main result

We start with a result which is fundamental for next considerations.

**Theorem 2.** Assume that  $\lambda, \mu, \nu$  are three times differentiable in first variable on the diagonal. Then there exist twice differentiable functions  $a, b : I \rightarrow \mathbb{R}$  such that  $\Phi := \varphi''/\varphi'$  satisfies the Riccati equation

$$(\mu(x, x) - \nu(x, x))\Phi'(x) = \frac{1 - \mu(x, x) - \nu(x, x)}{1 - \nu(x, x)}\Phi(x)^2 + a(x)\Phi(x) + b(x) \quad (6)$$

whenever  $\varphi : I \rightarrow \mathbb{R}$  and  $\psi : I \rightarrow \mathbb{R}$  are three times differentiable functions with non-vanishing first derivatives and the pair  $(\varphi, \psi)$  satisfies Eq. (2).

To prove this theorem we need some lemmas.

**Lemma 2.** Let  $n \geq 2$  be an integer. Assume that  $\lambda, \mu, \nu$  are  $n$ -times differentiable in first variable on the diagonal. Then there exists an  $(n-1)$ -times differentiable function  $f : I \rightarrow \mathbb{R}$  such that

$$\frac{\psi''(x)}{\psi'(x)} = -\frac{\mu(x, x)}{1 - \nu(x, x)} \frac{\varphi''(x)}{\varphi'(x)} + f(x), \quad x \in I, \quad (7)$$

for all twice differentiable functions  $\varphi : I \rightarrow \mathbb{R}$  and  $\psi : I \rightarrow \mathbb{R}$ , with non-vanishing first derivatives and such that  $(\varphi, \psi)$  satisfies Eq. (2); the function  $f$  is given by

$$f(x) = 2\partial_1\lambda(x, x) \frac{(1 - \mu(x, x) + \nu(x, x))^2}{(1 - \mu(x, x))\nu(x, x)(1 - \nu(x, x))} - 2\frac{\partial_1\mu(x, x)}{(1 - \mu(x, x))(1 - \nu(x, x))} - 2\frac{\partial_1\nu(x, x)}{\nu(x, x)(1 - \nu(x, x))}.$$

**Proof.** Fix any  $x \in I$ . Then, by (3), we get

$$\partial_1\lambda(y, x)(A_\mu^\varphi(y, x) - A_\nu^\psi(y, x)) + \lambda(y, x)\partial_1A_\mu^\varphi(y, x) + (1 - \lambda(y, x))\partial_1A_\nu^\psi(y, x) = \partial_1\lambda(y, x)(y - x) + \lambda(y, x) \quad (8)$$

for  $y$ 's running through a neighbourhood of  $x$ . Differentiating the functions appearing here with respect to the first variable, we obtain

$$\begin{aligned} \partial_1^2\lambda(x, x)(A_\mu^\varphi(x, x) - A_\nu^\psi(x, x)) + 2\partial_1\lambda(x, x)(\partial_1A_\mu^\varphi(x, x) - \partial_1A_\nu^\psi(x, x)) \\ + \lambda(x, x)\partial_1^2A_\mu^\varphi(x, x) + (1 - \lambda(x, x))\partial_1^2A_\nu^\psi(x, x) = 2\partial_1\lambda(x, x). \end{aligned}$$

Using Lemma 1, Corollary 1, and the equality  $A_\lambda^\alpha(x, x) = x$  we obtain the desired condition.  $\square$

**Corollary 2.** Assume that  $\lambda, \mu, \nu$  are twice differentiable in first variable on the diagonal and

$$\mu(x, x) + \nu(x, x) = 1, \quad x \in I. \quad (9)$$

If Eq. (2) has a solution  $(\varphi, \psi)$ , where  $\varphi : I \rightarrow \mathbb{R}$  and  $\psi : I \rightarrow \mathbb{R}$  are twice differentiable functions with non-vanishing first derivatives, then

$$\partial_1\lambda(x, x) = 0 \quad (10)$$

or

$$\lambda(x, x) = \mu(x, x) = \nu(x, x) = \frac{1}{2} \quad (11)$$

for every  $x \in I$ .

**Proof.** Take any twice differentiable functions  $\varphi : I \rightarrow \mathbb{R}$  and  $\psi : I \rightarrow \mathbb{R}$  with non-vanishing first derivatives and such that  $(\varphi, \psi)$  satisfies (2). Then, by Lemma 2 and (9),

$$\frac{\psi''(x)}{\psi'(x)} = -\frac{\varphi''(x)}{\varphi'(x)} + \frac{8\partial_1\lambda(x, x)\nu(x, x) - 2\partial_1\mu(x, x) - 2\partial_1\nu(x, x)}{\nu(x, x)(1 - \nu(x, x))} \quad (12)$$

for every  $x \in I$ . Replacing the pairs  $(\mu, \nu)$  and  $(\varphi, \psi)$  by  $(\nu, \mu)$  and  $(\psi, \varphi)$ , respectively, and using Lemma 2 and (9) again we see that

$$\frac{\varphi''(x)}{\varphi'(x)} = -\frac{\psi''(x)}{\psi'(x)} + \frac{8\partial_1\lambda(x, x)\mu(x, x) - 2\partial_1\nu(x, x) - 2\partial_1\mu(x, x)}{\mu(x, x)(1 - \mu(x, x))} \quad (13)$$

for every  $x \in I$ . Consequently, (12), (13) and (9) once more, give

$$\partial_1 \lambda(x, x)(\mu(x, x) - \nu(x, x)) = 0, \quad x \in I.$$

Fix any  $x \in I$ . Then (10) holds or  $\mu(x, x) = \nu(x, x)$ . In the second case (9) and Corollary 1 yield condition (11).  $\square$

The next lemma provides the form of the third derivative of the function  $A_\lambda^\alpha$  on the diagonal.

**Lemma 3.** Assume that  $\lambda$  is three times differentiable in first variable on the diagonal and let  $\alpha : I \rightarrow \mathbb{R}$  be a three times differentiable function with non-vanishing first derivative. Then

$$\begin{aligned} \partial_1^3 A_\lambda^\alpha(x, x) &= \lambda(x, x)(1 - \lambda(x, x)^2) \left( \frac{\alpha''(x)}{\alpha'(x)} \right)' + \lambda(x, x)(1 - \lambda(x, x))(1 - 2\lambda(x, x)) \left( \frac{\alpha''(x)}{\alpha'(x)} \right)^2 \\ &\quad + 3(1 - 2\lambda(x, x)) \partial_1 \lambda(x, x) \frac{\alpha''(x)}{\alpha'(x)} + 3\partial_1^2 \lambda(x, x) \end{aligned}$$

for every  $x \in I$ .

**Proof.** Fix any  $x \in I$ . Then it follows from (5) that

$$\alpha''(A_\lambda^\alpha(y, x))(\partial_1 A_\lambda^\alpha(y, x))^2 + \alpha'(A_\lambda^\alpha(y, x))\partial_1^2 A_\lambda^\alpha(y, x) = \partial_1^2 \lambda(y, x)(\alpha(y) - \alpha(x)) + 2\partial_1 \lambda(y, x)\alpha'(y) + \lambda(y, x)\alpha''(x)$$

for  $y$ 's from a neighbourhood of  $x$ . Differentiating the functions appearing here with respect to the first variable we have

$$\begin{aligned} \alpha'''(A_\lambda^\alpha(x, x))(\partial_1 A_\lambda^\alpha(x, x))^3 + 3\alpha''(A_\lambda^\alpha(x, x))\partial_1 A_\lambda^\alpha(x, x)\partial_1^2 A_\lambda^\alpha(x, x) + \alpha'(A_\lambda^\alpha(x, x))\partial_1^3 A_\lambda^\alpha(x, x) \\ = 3\partial_1^2 \lambda(x, x)\alpha'(x) + 3\partial_1 \lambda(x, x)\alpha''(x) + \lambda(x, x)\alpha'''(x). \end{aligned}$$

By Lemma 1 and condition  $A_\lambda^\alpha(x, x) = x$  we obtain

$$\begin{aligned} \partial_1^3 A_\lambda^\alpha(x, x) &= \lambda(x, x)(1 - \lambda(x, x)^2) \frac{\alpha'''(x)}{\alpha'(x)} - 3\lambda(x, x)^2(1 - \lambda(x, x)) \left( \frac{\alpha''(x)}{\alpha'(x)} \right)^2 \\ &\quad + 3(1 - 2\lambda(x, x)) \partial_1 \lambda(x, x) \frac{\alpha''(x)}{\alpha'(x)} + 3\partial_1^2 \lambda(x, x). \end{aligned}$$

Now, using the identity

$$\frac{\alpha'''(x)}{\alpha'(x)} = \left( \frac{\alpha''(x)}{\alpha'(x)} \right)' + \left( \frac{\alpha''(x)}{\alpha'(x)} \right)^2,$$

we obtain the claimed formula.  $\square$

Now we are in position to give

**Proof of Theorem 2.** Fix any  $x \in I$ . Then, by (8), we get

$$\begin{aligned} \partial_1^2 \lambda(y, x)(A_\mu^\varphi(y, x) - A_\nu^\psi(y, x)) + 2\partial_1 \lambda(y, x)(\partial_1 A_\mu^\varphi(y, x) - \partial_1 A_\nu^\psi(y, x)) + \lambda(y, x)\partial_1^2 A_\mu^\varphi(y, x) + (1 - \lambda(y, x))\partial_1^2 A_\nu^\psi(y, x) \\ = \partial_1^2 \lambda(y, x)(y - x) + 2\partial_1 \lambda(y, x) \end{aligned}$$

for  $y$ 's from a neighbourhood of  $x$ . Differentiating the functions appearing here with respect to the first variable at  $x$ , that is calculating the third derivatives of the functions in (3) at the point  $(x, x)$ , we obtain

$$\begin{aligned} \partial_1^3 \lambda(x, x)(A_\mu^\varphi(x, x) - A_\nu^\psi(x, x)) + 3\partial_1^2 \lambda(x, x)(\partial_1 A_\mu^\varphi(x, x) - \partial_1 A_\nu^\psi(x, x)) + 3\partial_1 \lambda(x, x)(\partial_1^2 A_\mu^\varphi(x, x) - \partial_1^2 A_\nu^\psi(x, x)) \\ + \lambda(x, x)\partial_1^3 A_\mu^\varphi(x, x) + (1 - \lambda(x, x))\partial_1^3 A_\nu^\psi(x, x) = 3\partial_1^2 \lambda(x, x). \end{aligned}$$

Using Lemmas 1, 3 and the equality  $A_\lambda^\alpha(x, x) = x$ , we get

$$\begin{aligned}
& 3\partial_1^2\lambda(x, x)(\mu(x, x) - v(x, x)) + 3\partial_1\lambda(x, x)\left(\mu(x, x)(1 - \mu(x, x))\frac{\varphi''(x)}{\varphi'(x)} + 2\partial_1\mu(x, x)\right) \\
& + 3\partial_1\lambda(x, x)\left(-v(x, x)(1 - v(x, x))\frac{\psi''(x)}{\psi'(x)} - 2\partial_1v(x, x)\right) + \lambda(x, x)\mu(x, x)(1 - \mu(x, x))^2\left(\frac{\varphi''(x)}{\varphi'(x)}\right)' \\
& + \lambda(x, x)\mu(x, x)(1 - \mu(x, x))(1 - 2\mu(x, x))\left(\frac{\varphi''(x)}{\varphi'(x)}\right)^2 + 3\lambda(x, x)(1 - 2\mu(x, x))\partial_1\mu(x, x)\frac{\varphi''(x)}{\varphi'(x)} + 3\lambda(x, x)\partial_1^2\mu(x, x) \\
& + (1 - \lambda(x, x))v(x, x)(1 - v(x, x))^2\left(\frac{\psi''(x)}{\psi'(x)}\right)' + (1 - \lambda(x, x))v(x, x)(1 - v(x, x))(1 - 2v(x, x))\left(\frac{\psi''(x)}{\psi'(x)}\right)^2 \\
& + 3(1 - \lambda(x, x))(1 - 2v(x, x))\partial_1v(x, x)\frac{\psi''(x)}{\psi'(x)} + 3(1 - \lambda(x, x))\partial_1^2v(x, x) = 3\partial_1^2\lambda(x, x). \tag{14}
\end{aligned}$$

By formula (7) we eliminate the function  $\psi''/\psi'$  from this equality. Let  $\Phi = \varphi''/\varphi'$ . Then, collecting all the terms suitably, we infer that the coefficients staying by  $\Phi'(x)$  and  $\Phi(x)^2$  are equal

$$\lambda(x, x)\mu(x, x)(1 - \mu(x, x))^2 - (1 - \lambda(x, x))\mu(x, x)v(x, x)(1 + v(x, x))$$

and

$$\lambda(x, x)\mu(x, x)(1 - \mu(x, x))(1 - 2\mu(x, x)) + \frac{(1 - \lambda(x, x))\mu(x, x)^2v(x, x)(1 - 2v(x, x))}{1 - v(x, x)},$$

respectively. Now, multiplying the functions occurring on both sides of (14) by  $1 - \mu(x, x) + v(x, x)$  and dividing by  $\mu(x, x)(1 - \mu(x, x))v(x, x)$ , using Corollary 1, and denoting by  $a(x)$  the coefficient staying by  $\Phi(x)$  and by  $b(x)$  the free term, we come to the desired formula.  $\square$

### 3. Further results

It is well known that when solving the Riccati equation we come to some difficulties. In particular, there are no, in general, methods and formulas giving its solutions effectively. However, in the corollaries below we consider some special cases when Eq. (6) can be effectively solved. In particular, if

$$\mu(x, x) = v(x, x), \quad x \in I, \tag{15}$$

then (6) is equivalent to an algebraic quadratic equation. Then we obtain

**Corollary 3.** Assume that  $\lambda, \mu, v$  are three times differentiable in first variable on the diagonal and condition (15) holds. Then there exist twice differentiable functions  $a, b : I \rightarrow \mathbb{R}$  such that  $\varphi''/\varphi'$  satisfies the equation

$$\frac{1 - 2\mu(x, x)}{1 - \mu(x, x)}\Phi(x)^2 + a(x)\Phi(x) + b(x) = 0$$

for all three times differentiable functions  $\varphi : I \rightarrow \mathbb{R}$  and  $\psi : I \rightarrow \mathbb{R}$ , with non-vanishing first derivatives and such that  $(\varphi, \psi)$  satisfies Eq. (2).

In the case when condition (9) is satisfied, the Riccati equation (6) is a linear differential one, which is easy to solve.

**Corollary 4.** Assume that  $\lambda, \mu, v$  are three times differentiable in first variable on the diagonal and condition (9) holds. Then there exist twice differentiable functions  $a, b : I \rightarrow \mathbb{R}$  such that  $\varphi''/\varphi'$  satisfies the differential equation

$$(2\mu(x, x) - 1)\Phi'(x) = a(x)\Phi(x) + b(x)$$

for all three times differentiable functions  $\varphi : I \rightarrow \mathbb{R}$  and  $\psi : I \rightarrow \mathbb{R}$ , with non-vanishing first derivatives and such that  $(\varphi, \psi)$  satisfies Eq. (2).

If (9) holds and some additional condition is satisfied, then we can say even more. The next result follows immediately from Corollary 2 and formula (14).

**Corollary 5.** Assume that  $\lambda, \mu, v$  are three times differentiable in first variable on the diagonal, condition (9) holds, and

$$(2\partial_1\lambda(x, x) - \partial_1\mu(x, x) - \partial_1v(x, x))' = 2\partial_1^2\lambda(x, x) - \partial_1^2\mu(x, x) - \partial_1^2v(x, x), \quad x \in I. \tag{16}$$

If  $\varphi : I \rightarrow \mathbb{R}$  and  $\psi : I \rightarrow \mathbb{R}$  are three times differentiable functions, with non-vanishing first derivatives and such that  $(\varphi, \psi)$  satisfies Eq. (2), then equality (10) or

$$\frac{\varphi''(x)}{\varphi'(x)} = 8(\partial_1 \lambda(x, x) - \partial_1 \mu(x, x)) \quad \text{and} \quad \frac{\psi''(x)}{\psi'(x)} = 8(\partial_1 \lambda(x, x) - \partial_1 \nu(x, x))$$

hold for every  $x \in I$ .

Clearly condition (16) is satisfied if, for instance, the function  $2\partial_1 \lambda - \partial_1 \mu - \partial_1 \nu$  is constant in a neighbourhood of the diagonal.

**Example.** Given a function  $s: I \rightarrow (0, \infty)$  we come to an important class of weighted quasi-arithmetic means  $B_s^\alpha$  of the form

$$B_s^\alpha(x, y) = \alpha^{-1} \left( \frac{s(x)}{s(x) + s(y)} \alpha(x) + \frac{s(y)}{s(x) + s(y)} \alpha(y) \right),$$

known as the Bajraktarević means (cf. [1]). In paper [9] we study the invariance equation

$$A_{1/2}^{\text{id}} \circ (B_s^\varphi, B_s^\psi) = A_{1/2}^{\text{id}},$$

whereas in [10] the equations

$$B_r^{\text{id}} \circ (B_s^\varphi, B_t^\psi) = B_r^{\text{id}}$$

and

$$A_{1/2}^{\text{id}} \circ (B_s^\varphi, B_{1/s}^\psi) = A_{1/2}^{\text{id}}$$

are investigated. Observe that all these equations are of the form (4). The functions  $\lambda, \mu, \nu$  are defined by

$$\begin{aligned} \lambda(x, y) &= \frac{1}{2} \quad \text{and} \quad \mu(x, y) = \nu(x, y) = \frac{s(x)}{s(x) + s(y)}, \\ \lambda(x, y) &= \frac{r(x)}{r(x) + r(y)}, \quad \mu(x, y) = \frac{s(x)}{s(x) + s(y)}, \quad \nu(x, y) = \frac{t(x)}{t(x) + t(y)}, \end{aligned}$$

and

$$\lambda(x, y) = \frac{1}{2}, \quad \mu(x, y) = \frac{s(x)}{s(x) + s(y)}, \quad \nu(x, y) = \frac{s(y)}{s(x) + s(y)},$$

respectively. In all the cases  $\mu(x, x) = \nu(x, x) = 1/2$  for every  $x \in I$ , so conditions (9) and (15) are fulfilled. However, in the last case we have

$$\partial_1 \lambda(x, x) = 0, \quad \partial_1 \mu(x, x) = \frac{s'(x)}{4s(x)} \quad \text{and} \quad \partial_1 \nu(x, x) = -\frac{s'(x)}{4s(x)}, \quad x \in I,$$

that is also condition (16) holds.

Corollary 5 provides a rather surprising observation: in some cases the solution  $(\varphi, \psi)$  of Eq. (2) is completely determined in the whole  $I^2$  by the values of the weights  $\lambda, \mu, \nu$  counted arbitrarily close to the diagonal only.

In general, however, we are still far from determining all the solutions of Eq. (2). If we assume that a pair  $(\varphi, \psi)$  satisfies (2), then, by Theorem 2 or Corollaries 3 and 5, we know only the form of  $\varphi''/\varphi'$ . By twice integration we obtain the form of  $\varphi$  and then, by Lemma 2, also  $\psi$ . In particular, when we obtain a linear differential equation (cf. Corollary 5) we get at most four parameter families of functions  $\varphi$  and  $\psi$ . It is usually difficult to verify if the pair of functions, obtained in that way, really satisfies Eq. (2). Sometimes, however, it is quite easy; this is the case, for instance, in the situations described in the rest of the paper.

As an immediate consequence of Corollary 5 we obtain

**Corollary 6.** Assume that  $\lambda, \mu, \nu$  are three times differentiable in first variable on the diagonal, condition (9) holds,  $\partial_1 \lambda(x, x) \neq 0$  for every  $x \in I$ , and

$$\partial_1 \lambda(x, x) = \partial_1 \mu(x, x) = \partial_1 \nu(x, x), \quad x \in I. \quad (17)$$

If  $\varphi: I \rightarrow \mathbb{R}$  and  $\psi: I \rightarrow \mathbb{R}$  are three times differentiable functions, with non-vanishing first derivatives and such that  $(\varphi, \psi)$  satisfies Eq. (2), then the pair  $(\varphi, \psi)$  is affine.

As  $\lambda$  and  $-\lambda$  simultaneously generate the mean  $A_\alpha^\lambda$  we may assume below for simplicity that both  $\varphi$  and  $\psi$  are strictly increasing.

**Theorem 3.** Assume that  $\lambda, \mu, \nu$  are twice differentiable in first variable on the diagonal, condition (9) holds, and

$$\partial_1 \mu(x, x) + \partial_1 \nu(x, x) = 4\partial_1 \lambda(x, x)\mu(x, x), \quad x \in I,$$

or

$$\partial_1 \mu(x, x) + \partial_1 \nu(x, x) = 4\partial_1 \lambda(x, x)\nu(x, x), \quad x \in I.$$

Let  $\varphi : I \rightarrow \mathbb{R}$  and  $\psi : I \rightarrow \mathbb{R}$  be strictly increasing and twice differentiable functions such that  $(\varphi, \psi)$  satisfies Eq. (2). If  $\varphi$  and  $\psi$  are simultaneously convex or simultaneously concave, then the pair  $(\varphi, \psi)$  is affine.

**Proof.** At first assume additionally that  $\varphi'$  and  $\psi'$  do not vanish. By Lemma 2 and (9) (cf. formulas (13) and (12)) we infer that

$$\frac{\psi''(x)}{\psi'(x)} = -\frac{\varphi''(x)}{\varphi'(x)}, \quad x \in I.$$

Therefore, as  $\varphi'$  and  $\psi'$  are positive, the convexity assumptions implies that  $\varphi''(x) = \psi''(x) = 0$  for every  $x \in I$ , that is  $\varphi$  and  $\psi$  are affine.

For the proof in general case we show that  $\varphi'(x) \neq 0$  and  $\psi'(x) \neq 0$  for all  $x \in I$ . Denote by  $Z_\varphi$  and  $Z_\psi$  the sets of all zeros of  $\varphi'$  and  $\psi'$ , respectively. It is enough to show that these sets are empty. Since they are closed sets with empty interiors,  $\text{int } I \setminus (Z_\varphi \cup Z_\psi) \neq \emptyset$ . It can be represented as a union of a countable family of pairwise disjoint open intervals. Let  $I_0$  be one of them, and suppose that  $I_0 \neq I$ . Then at least one end of  $I_0$ , say  $x_0$ , belongs to  $\text{int } I$ . Clearly,  $x_0 \in Z_\varphi \cup Z_\psi$ , that is either  $\varphi'(x_0) = 0$ , or  $\psi'(x_0) = 0$ . On the other hand, applying the just proved case of the theorem to  $\varphi|_{I_0}$  and  $\psi|_{I_0}$ , and the continuity of  $\varphi'$  and  $\psi'$  at  $x_0$ , we infer that  $\varphi'(x_0) \neq 0$  and  $\psi'(x_0) \neq 0$ . This contradiction shows that  $I_0 = I$  and, consequently,  $Z_\varphi = Z_\psi = \emptyset$ .  $\square$

As an immediate consequence we obtain what follows.

**Corollary 7.** Assume that  $\lambda, \mu, \nu$  are twice differentiable in first variable on the diagonal,

$$\mu(x, x) = \nu(x, x) = \frac{1}{2}, \quad x \in I,$$

and condition (17) holds. Let  $\varphi : I \rightarrow \mathbb{R}$  and  $\psi : I \rightarrow \mathbb{R}$  be strictly increasing and twice differentiable functions such that  $(\varphi, \psi)$  satisfies Eq. (2). If  $\varphi$  and  $\psi$  are simultaneously convex or simultaneously concave, then the pair  $(\varphi, \psi)$  is affine.

Assuming additionally that all the weight functions are the same, we come to the equation

$$\begin{aligned} \lambda(x, y)\varphi^{-1}(\lambda(x, y)\varphi(x) + (1 - \lambda(x, y))\varphi(y)) + (1 - \lambda(x, y))\psi^{-1}(\lambda(x, y)\psi(x) + (1 - \lambda(x, y))\psi(y)) \\ = \lambda(x, y)x + (1 - \lambda(x, y))y. \end{aligned} \quad (18)$$

The final result follows directly from Theorem 3.

**Corollary 8.** Assume that  $\lambda$  is twice differentiable in first variable on the diagonal and such that

$$\lambda(x, x) = \frac{1}{2}, \quad x \in I.$$

Let  $\varphi : I \rightarrow \mathbb{R}$  and  $\psi : I \rightarrow \mathbb{R}$  be strictly increasing and twice differentiable functions, simultaneously convex or simultaneously concave. The pair  $(\varphi, \psi)$  satisfies Eq. (18) if and only if it is affine.

## Acknowledgment

The author is indebted to the referee for his/her valuable remarks improving the paper.

## References

- [1] M. Bajraktarević, Sur une équation fonctionnelle aux valeurs moyennes, Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske. Ser. II 13 (1958) 243–248.
- [2] P. Burai, A Matkowski–Sutô type equation, Publ. Math. Debrecen 70 (1–2) (2007) 233–247.
- [3] Z. Daróczy, Zs. Páles, On mens that are both quasi-arithmetic and conjugate arithmetic, Acta Math. Hungar. 90 (2001) 271–282.
- [4] Z. Daróczy, Zs. Páles, Gauss-composition of means and the solution of the Matkowski–Sutô problem, Publ. Math. Debrecen 61 (2002) 157–218.
- [5] Z. Daróczy, Zs. Páles, A Matkowski–Sutô-type problem for weight quasi-arithmetic means, Ann. Univ. Sci. Budapest. Sect. Comput. 22 (2003) 69–81.
- [6] A. Járjai, Regularity Properties of Functional Equations in Several Variables, Adv. Math., vol. 8, Springer, New York, 2005.
- [7] J. Jarczyk, J. Matkowski, Invariance in the class of weighted quasi-arithmetic means, Ann. Polon. Math. 88 (2006) 39–51.
- [8] J. Jarczyk, Invariance in the class of weighted quasi-arithmetic means with continuous generators, Publ. Math. Debrecen 71 (2007) 279–294.
- [9] J. Jarczyk, Invariance in a class of Bajraktarević means, submitted for publication.
- [10] J. Jarczyk, Some results concerning the invariance in a class of Bajraktarević means, submitted for publication.
- [11] J. Matkowski, Invariant and complementary quasi-arithmetic means, Aequationes Math. 57 (1999) 87–107.