J. Math. Anal. Appl. 356 (2009) 769–792

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Riesz extremal measures on the sphere for axis-supported external fields

J.S. Brauchart [a,1], P.D. Dragnev [b], E.B. Saff [a,*,2]

[a] Center for Constructive Approximation, Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA
[b] Department of Mathematical Sciences, Indiana-Purdue University, Fort Wayne, IN 46805, USA

ARTICLE INFO

Article history:
Received 29 January 2009
Available online 26 March 2009
Submitted by D. Khavinson

Keywords:
Balayage
Equilibrium measures
Extremal measures
Minimum energy
Riesz kernel
Weighted energy

ABSTRACT

We investigate the minimal Riesz s-energy problem for positive measures on the d-dimensional unit sphere $\mathbb{S}^d$ in the presence of an external field induced by a point charge, and more generally by a line charge. The model interaction is that of Riesz potentials $|\mathbf{x}-\mathbf{y}|^{-s}$ with $d-2 \leqslant s < d$. For a given axis-supported external field, the support and the density of the corresponding extremal measure on $\mathbb{S}^d$ is determined. The special case $s = d-2$ yields interesting phenomena, which we investigate in detail. A weak* asymptotic analysis is provided as $s \to (d-2)^+$.

1. Introduction and results

1.1. Potential-theoretical preliminaries

Let $\mathbb{S}^d := \{\mathbf{x} \in \mathbb{R}^{d+1}\colon |\mathbf{x}| = 1\}$ be the unit sphere in $\mathbb{R}^{d+1}$, where $|\cdot|$ denotes the Euclidean norm, and let $\sigma = \sigma_d$ be the unit Lebesgue surface measure on $\mathbb{S}^d$. Recall that, using cylindrical coordinates

$$\mathbf{x} = \big(\sqrt{1-u^2}\,\bar{\mathbf{x}}, u\big), \quad -1 \leqslant u \leqslant 1,\ \bar{\mathbf{x}} \in \mathbb{S}^{d-1}, \tag{1}$$

we can write the decomposition

$$\mathrm{d}\sigma_d(\mathbf{x}) = \frac{\omega_{d-1}}{\omega_d}\big(1-u^2\big)^{d/2-1}\,\mathrm{d}u\,\mathrm{d}\sigma_{d-1}(\bar{\mathbf{x}}). \tag{2}$$

Here ω_d is the surface area of $\mathbb{S}^d$, and the ratio of these areas can be evaluated as

$$\frac{\omega_d}{\omega_{d-1}} = \int_{-1}^{1}\big(1-u^2\big)^{d/2-1}\,\mathrm{d}u = \frac{\sqrt{\pi}\,\Gamma(d/2)}{\Gamma((d+1)/2)} = 2^{d-1}\frac{[\Gamma(d/2)]^2}{\Gamma(d)}. \tag{3}$$

Given a compact set $E \subset \mathbb{S}^d$, consider the class $\mathcal{M}(E)$ of unit positive Borel measures supported on E. For $0 < s < d$, the *Riesz s-potential* and *Riesz s-energy* of a measure $\mu \in \mathcal{M}(E)$ are given, respectively, by

* Corresponding author. Fax: +1 615 343 0215.
E-mail addresses: johann.brauchart@vanderbilt.edu (J.S. Brauchart), dragnevp@ipfw.edu (P.D. Dragnev), edward.b.saff@vanderbilt.edu (E.B. Saff).

[1] The research of this author was supported, in part, by the "Scholar-in-Residence" program at IPFW, by the Austrian Science Foundation (FWF) under grant S9603-N13, and the Oberwolfach Leibniz Fellow Programme.
[2] The research of this author was supported, in part, by the U.S. National Science Foundation under grants DMS-0603828 and DMS-0808093.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.03.060

$$U_s^{\mu}(\mathbf{x}) := \int k_s(\mathbf{x},\mathbf{y})\,\mathrm{d}\mu(\mathbf{y}), \qquad \mathcal{I}_s(\mu) := \iint k_s(\mathbf{x},\mathbf{y})\,\mathrm{d}\mu(\mathbf{x})\,\mathrm{d}\mu(\mathbf{y}),$$

where $k_s(\mathbf{x},\mathbf{y}) := |\mathbf{x}-\mathbf{y}|^{-s}$ is the so-called *Riesz kernel* (for $s=0$ we use the logarithmic kernel $k_0(\mathbf{x},\mathbf{y}) := \log(1/|\mathbf{x}-\mathbf{y}|)$ instead). The *s-energy* of E is $W_s(E) := \inf\{\mathcal{I}_s(\mu)\colon\ \mu \in \mathcal{M}(E)\}$ and if $W_s(E)$ is finite, there is a unique measure $\mu_{E,s}$ achieving this minimal energy, which is called the *s-extremal measure* on E. The *s-capacity of E* is defined as $\mathrm{cap}_s(E) := 1/W_s(E)$ for $s>0$. (In the logarithmic case $s=0$ we define $\mathrm{cap}_0(E) := \exp\{-W_0(E)\}$, cf. (27).) A property is said to hold *quasi-everywhere* (q.e.) if the exceptional set has s-capacity zero. For more details see [13, Chapter II]. We remind the reader that the s-energy of $\mathbb{S}^d$ is given by

$$W_s(\mathbb{S}^d) = \frac{\Gamma(d)\Gamma((d-s)/2)}{2^s\Gamma(d/2)\Gamma(d-s/2)}, \quad 0<s<d. \tag{4}$$

The weighted s-energy $\mathcal{I}_Q(\mu)$ associated with a non-negative lower semi-continuous *external field* $Q : E \to [0,\infty]$ and its extremal value V_Q are given by

$$\mathcal{I}_Q(\mu) := \mathcal{I}_s(\mu) + 2\int Q(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x}), \qquad V_Q := \inf\{\mathcal{I}_Q(\mu)\colon\ \mu \in \mathcal{M}(E)\}.$$

A measure $\mu_Q \in \mathcal{M}(E)$ such that $\mathcal{I}_Q(\mu_Q) = V_Q$ is called an *extremal* (*or positive equilibrium*) *measure on E associated with* $Q(\mathbf{x})$. The measure μ_Q is characterized by the Gauss variational inequalities

$$U_s^{\mu_Q}(\mathbf{x}) + Q(\mathbf{x}) \geqslant F_Q \quad \text{q.e. on } E, \tag{5}$$

$$U_s^{\mu_Q}(\mathbf{x}) + Q(\mathbf{x}) \leqslant F_Q \quad \text{everywhere on } \mathrm{supp}(\mu_Q), \tag{6}$$

where

$$F_Q := V_Q - \int Q(\mathbf{x})\,\mathrm{d}\mu_Q(\mathbf{x}).$$

For simplicity, we suppressed in some of the above notation the dependence on s; that is, $\mathcal{I}_Q = \mathcal{I}_{Q,s}$, $\mu_Q = \mu_{Q,s}$, etc. We note that for suitable external fields (e.g. continuous on $E = \mathbb{S}^d$), the inequality in (5) holds everywhere, which implies that equality holds in (6).

The existence, uniqueness, and characterization-related questions concerning equilibrium potentials with external fields in the most general setting can be found in [22–24]. We remark that the logarithmic potential with external fields is treated in depth in [20].

When $Q \equiv 0$ and $\mathrm{cap}_s(E) > 0$, the extremal measure μ_Q is the same as the measure $\mu_E = \mu_{E,s}$.

In [5] *Riesz external fields*

$$Q_{\mathbf{a},q}(\mathbf{x}) := Q_{\mathbf{a},q,s}(\mathbf{x}) := q|\mathbf{x}-\mathbf{a}|^{-s} \quad \text{on } E = \mathbb{S}^d,\ d-2<s<d, \tag{7}$$

were considered, where $q>0$ and $\mathbf{a}$ is a fixed point on $\mathbb{S}^d$.[3] The motivation for that investigation was to obtain new separation results for minimal s-energy points on the sphere (cf. [3,11,12]). In the current work we extend that investigation to Riesz external fields $Q_{\mathbf{a},q}$ with $\mathbf{a} \notin \mathbb{S}^d$ and develop a technique for finding the extremal measure associated with more general axis-supported external fields.

1.2. Signed equilibrium

We note that for $d=2$ and $s=1$ it is a standard electrostatic problem to find the charge density (signed measure) on a charged, insulated, conducting sphere in the presence of a point charge q placed off the sphere (see [9, Chapter 2]). This motivates us to give the following definition (see [4]).

Definition 1. Given a compact subset $E \subset \mathbb{R}^p$ ($p \geqslant 3$) and an external field Q, we call a signed measure $\eta_{E,Q} = \eta_{E,Q,s}$ supported on E and of total charge $\eta_{E,Q}(E) = 1$ *a signed s-equilibrium on E associated with* Q if its weighted Riesz s-potential is constant on E, that is

$$U_s^{\eta_{E,Q}}(\mathbf{x}) + Q(\mathbf{x}) = F_{E,Q} \quad \text{for all } \mathbf{x} \in E. \tag{8}$$

The choice of the normalization $\eta_{E,Q}(E) = 1$ is just for convenience in the applications here. Lemma 2.1 below establishes that if a signed equilibrium $\eta_{E,Q}$ exists, then it is unique.

In [7] Fabrikant et al. give a derivation of certain signed Riesz equilibria on suitably parametrized surfaces in $\mathbb{R}^3$, including spherical caps when $Q(\mathbf{x}) \equiv 0$. We remark that the determination of signed equilibria is a substantially easier problem

[3] The case $d=1$, $s=0$, where $\mathbf{a}$ is a point on the unit circle was investigated in [14].

than that of finding non-negative extremal measures, which is the goal of this paper. However, the solution to the former problem is useful in solving the latter problem.

Our first result establishes existence of the signed s-equilibrium associated with the Riesz external field $Q_{\mathbf{a},q}$, $\mathbf{a} \notin \mathbb{S}^d$, defined in (7). We assume that $\mathbf{a}$ lies above the North Pole $\mathbf{p} := (\mathbf{0}, 1)$, that is $\mathbf{a} = (\mathbf{0}, R)$ and $R > 1$ (the case $R < 1$ is handled by inversion).

Throughout, ${}_2\mathrm{F}_1\binom{a,b}{c};z)$ and ${}_2\tilde{\mathrm{F}}_1\binom{a,b}{c};z)$ denote the Gauss hypergeometric function and its regularized form[4] with series expansions

$$ {}_2\mathrm{F}_1\left(\begin{matrix}a,b\\c\end{matrix};z\right) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \qquad {}_2\tilde{\mathrm{F}}_1\left(\begin{matrix}a,b\\c\end{matrix};z\right) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{\Gamma(n+c)}\frac{z^n}{n!}, \quad |z|<1, \tag{9} $$

where $(a)_0 := 1$ and $(a)_n := a(a+1)\cdots(a+n-1)$ for $n \geqslant 1$ is the Pochhammer symbol. The incomplete Beta function and the Beta function are defined as

$$ \mathrm{B}(x;\alpha,\beta) := \int_0^x v^{\alpha-1}(1-v)^{\beta-1}\,\mathrm{d}v, \qquad \mathrm{B}(\alpha,\beta) := \mathrm{B}(1;\alpha,\beta), \tag{10} $$

whereas the regularized incomplete Beta function is given by

$$ \mathrm{I}(x;a,b) := \mathrm{B}(x;a,b)/\mathrm{B}(a,b). \tag{11} $$

Theorem 2. *Let $0 < s < d$ and $R > 1$. The signed s-equilibrium $\eta_{\mathbf{a}} = \eta_{\mathbb{S}^d, Q_{\mathbf{a},q}, s}$ on $\mathbb{S}^d$ associated with the Riesz external field $Q_{\mathbf{a},q}$, $\mathbf{a} = R\mathbf{p}$, is given by*

$$ \mathrm{d}\eta_{\mathbf{a}}(\mathbf{x}) = \eta_{\mathbf{a}}'(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{x}), \quad \eta_{\mathbf{a}}'(\mathbf{x}) := 1 + \frac{q U_s^\sigma(\mathbf{a})}{W_s(\mathbb{S}^d)} - \frac{q(R^2-1)^{d-s}}{W_s(\mathbb{S}^d)|\mathbf{x}-\mathbf{a}|^{2d-s}}. \tag{12} $$

Furthermore, $U_s^\sigma(\mathbf{a}) = \int k_s(\mathbf{a},\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y})$ has the following representation:

$$ U_s^\sigma(\mathbf{a}) = (R+1)^{-s}{}_2\mathrm{F}_1\left(\begin{matrix}s/2, d/2\\d\end{matrix}; 4R/(R+1)^2\right). \tag{13} $$

We remark that in the Coulomb case $d = 2$ and $s = 1$, the representation (12) is well known from elementary physics (cf. [9, p. 61]).

The next result explicitly shows the relationship between q and R so that $\mu_{Q_{\mathbf{a},q}}$ coincides with the signed equilibrium and has as support the entire sphere.

Corollary 3. *Let $0 < s < d$, $R = |\mathbf{a}| > 1$. Then $\operatorname{supp}(\mu_{Q_{\mathbf{a},q}}) = \mathbb{S}^d$ if and only if*

$$ \frac{W_s(\mathbb{S}^d)}{q} \geqslant \frac{(R+1)^{d-s}}{(R-1)^d} - U_s^\sigma(\mathbf{a}) = \sum_{k=0}^{\infty}\left[1 - \frac{(s/2)_k}{(d)_k}\right]\frac{(d/2)_k}{k!}\frac{(4R)^k}{(R+1)^{s+2k}}. \tag{14} $$

In such a case, $\mu_{Q_{\mathbf{a},q}} = \eta_{\mathbf{a}}$.

Remark 4. Observe that the right-most part of (14) is a strictly decreasing function of R for $R > 1$. Thus, for any fixed charge q there is a critical R_q given by equality in (14), such that for $R \geqslant R_q$ the extremal support is the entire sphere.

1.3. The Newtonian case $s = d - 1$

The following example deals with the classical case of a Newtonian potential (relative to the manifold dimension). The example answers a question of A.A. Gonchar; namely, how far from the unit sphere should a unit point charge be placed so that the support of the extremal measure associated with the external field exerted by the charge be the entire sphere?

Example 5. Let $d \geqslant 2$, $s = d - 1$, $q = 1$ and $\mathbf{a} = (\mathbf{0}, R)$. Then $W_s(\mathbb{S}^d) = 1$ (cf. (4)) and from the mean-value property for harmonic functions we can write

$$ U_s^\sigma(\mathbf{a}) = 1/R^{d-1} \quad \text{for } R \geqslant 1. $$

Thus (14) in this case is equivalent to the inequality

[4] Which is well defined even for c a negative integer.

$$1 \geqslant (R+1)(R-1)^{-d} - R^{1-d} \quad \text{or} \quad 1 \geqslant (\rho+2)\rho^{-d} - (\rho+1)^{1-d},$$

where ρ measures the distance between the unit charge and the surface of the sphere. Equality holds if ρ is an algebraic number satisfying

$$P(d;\rho) := \big(\rho^d - 2 - \rho\big)(\rho+1)^{d-1} + \rho^d = 0,$$

or on expanding the polynomial $P(d;\rho)$,

$$\sum_{m=0}^{d-1} \binom{d-1}{m} \rho^{m+d} - \sum_{m=0}^{d-1} \left[\binom{d}{m} + \binom{d-1}{m} \right] \rho^m = 0.$$

The monic polynomial[5] $P(d;\rho)$ with integer coefficients has odd degree $2d-1$. Furthermore, $P(d;1) < 0$ and hence $P(d;\rho)$ has at least one positive root; but, by Descartes' Sign Rule, this is the only positive root. This simple root ρ_+ must be in the interval $(1,2]$, since $P(d;\rho) > 0$ for $\rho > 2$. Asymptotic analysis shows that

$$\rho_+ = 1 + (\log 3)/d + \mathcal{O}\big(1/d^2\big) \quad \text{as } d \to \infty.$$

Of particular interest is the case when $d = 2$. Then one easily computes that the distance between the point charge and the surface of the sphere is given precisely by the golden ratio $\rho_+ = (1+\sqrt{5})/2$. We note that the fact that the inequality $R - 1 \geqslant \rho_+$ implies $\operatorname{supp}(\mu_{Q_{\mathbf{a},1}}) = \mathbb{S}^2$ follows from an elementary physics argument.

1.4. The Mhaskar–Saff $\mathcal{F}_s$-functional and the extremal support

An important tool in our analysis is the Riesz analog of the *Mhaskar–Saff F-functional* from classical logarithmic potential in the plane (see [15] and [20, Chapter IV, p. 194]).

Definition 6. The *$\mathcal{F}_s$-functional* of a compact subset $K \subset \mathbb{S}^d$ of positive s-capacity is defined as

$$\mathcal{F}_s(K) := W_s(K) + \int Q(\mathbf{x})\, \mathrm{d}\mu_K(\mathbf{x}), \tag{15}$$

where $W_s(K)$ is the s-energy of K and μ_K is the s-extremal measure (without external field) on K.

Remark 7. We caution the reader that (15) is the negative of the F-functional defined in [15,20].

Remark 8. When $d-2 < s < d$, there is a remarkable relationship between the signed s-equilibrium and the $\mathcal{F}_s$-functional. Namely, if the signed s-equilibrium on a compact set K associated with Q exists, then $\mathcal{F}_s(K) = F_{K,Q}$, where $F_{K,Q}$ is the constant from (8). Indeed, if $\eta_{K,Q}$ exists, we integrate (8) with respect to μ_K and interchange the order of integration to obtain the asserted equality.

The following optimization property is the main motivation for introducing the $\mathcal{F}_s$-functional.

Theorem 9. *Let $d-2 \leqslant s < d$ with $s > 0$ and Q be an external field on $\mathbb{S}^d$. Then the $\mathcal{F}_s$-functional is minimized for $S_Q := \operatorname{supp}(\mu_Q)$.*

The next theorem provides sufficient conditions on a general external field Q that guarantee that the extremal support S_Q is a spherical zone or a spherical cap.

Theorem 10. *Let $d-2 \leqslant s < d$ with $s > 0$ and the external field $Q : \mathbb{S}^d \to [0,\infty]$ be rotationally invariant about the polar axis; that is, $Q(\mathbf{z}) = f(\xi)$, where ξ is the altitude of $\mathbf{z} = (\sqrt{1-\xi^2}\,\overline{\mathbf{z}}, \xi)$ (see (1)). Suppose that f is a convex function on $[-1,1]$. Then the support of the s-extremal measure μ_Q on $\mathbb{S}^d$ is a spherical zone; namely, there are numbers $-1 \leqslant t_1 \leqslant t_2 \leqslant 1$ such that*

$$\operatorname{supp}(\mu_Q) = \Sigma_{t_1,t_2} := \big\{ \big(\sqrt{1-u^2}\,\overline{\mathbf{x}}, u\big) \colon t_1 \leqslant u \leqslant t_2,\ \overline{\mathbf{x}} \in \mathbb{S}^{d-1} \big\}. \tag{16}$$

Moreover, if additionally f is increasing, then $t_1 = -1$ and the support of μ_Q is a spherical cap centered at the South Pole.

It is easy to see that the external field $Q_{\mathbf{a},q}(\mathbf{z}) = q|1 - 2R\xi + R^2|^{-s/2}$ is rotationally invariant about the polar axis and is an increasing and convex function of the altitude ξ of $\mathbf{z}$. Therefore, from Theorem 10 we conclude that the support of the extremal measure $\mu_{Q_{\mathbf{a},q}}$ on $\mathbb{S}^d$ is a spherical cap. In view of Theorem 9 we thus need only to minimize the $\mathcal{F}_s$-functional

[5] Properties of these polynomials will be investigated in a future publication.

over the collection of spherical caps centered at the South Pole in order to determine S_Q. For this purpose, in consideration of Remark 8, we first seek an explicit representation for the signed equilibria for these spherical caps.

Denote by Σ_t the spherical cap centered at the South Pole

$$\Sigma_t := \Sigma_{-1,t} \tag{17}$$

(cf. (16)), and let η_t be the signed s-equilibrium on Σ_t associated with $Q_{\mathbf{a},q}$. Using M. Riesz's approach to s-balayage as presented in [13, Chapter IV], we introduce the following s-balayage measures onto Σ_t:

$$\epsilon_t = \epsilon_{t,s} := \mathrm{Bal}_s(\delta_{\mathbf{a}}, \Sigma_t), \qquad \nu_t = \nu_{t,s} := \mathrm{Bal}_s(\sigma, \Sigma_t), \tag{18}$$

where $\delta_{\mathbf{a}}$ is the unit Dirac-delta measure at $\mathbf{a}$. Recall that given a measure ν and a compact set K (of the sphere $\mathbb{S}^d$), the balayage measure $\hat{\nu} := \mathrm{Bal}_s(\nu, K)$ preserves the Riesz s-potential of ν onto the set K and diminishes it elsewhere (on the sphere $\mathbb{S}^d$). We remark that in what follows an important role is played by the function

$$\Phi_s(t) := W_s\big(\mathbb{S}^d\big)\big(1 + q\|\epsilon_t\|\big)/\|\nu_t\|, \quad d-2 < s < d. \tag{19}$$

The next assertion is an immediate consequence of the definition of the balayage measures in (18). In Lemmas 24 and 25 below we present explicit formulas for their densities. Their norms are calculated in Lemmas 30 and 29, respectively. Below we combine these formulas to give an explicit form for the density of the signed equilibrium. The only statement requiring further proof is the formula for the weighted s-potential (22) when $\xi > t$. We shall do this in Section 5.

Theorem 11. *Let $d-2 < s < d$. The signed s-equilibrium η_t on the spherical cap $\Sigma_t \subset \mathbb{S}^d$ associated with $Q_{\mathbf{a},q}$ is given by*

$$\eta_t = \big[\Phi_s(t)/W_s\big(\mathbb{S}^d\big)\big]\nu_t - q\epsilon_t. \tag{20}$$

It is absolutely continuous in the sense that for $\mathbf{x} = (\sqrt{1-u^2}\bar{\mathbf{x}}, u) \in \Sigma_t$,

$$\mathrm{d}\eta_t(\mathbf{x}) = \eta_t'(u)\frac{\omega_{d-1}}{\omega_d}\big(1-u^2\big)^{d/2-1}\,\mathrm{d}u\,\mathrm{d}\sigma_{d-1}(\bar{\mathbf{x}}),$$

where (with $R = |\mathbf{a}|$ and $r = \sqrt{R^2 - 2Rt + 1}$)

$$\begin{aligned}\eta_t'(u) = {} & \frac{1}{W_s(\mathbb{S}^d)}\frac{\Gamma(d/2)}{\Gamma(d-s/2)}\left(\frac{1-t}{1-u}\right)^{d/2}\left(\frac{t-u}{1-t}\right)^{(s-d)/2}\left\{\Phi_s(t)\,{}_2\tilde{\mathrm{F}}_1\left(\begin{matrix}1, d/2\\ 1-(d-s)/2\end{matrix}; \frac{t-u}{1-u}\right)\right.\\ & \left. - \frac{q(R+1)^{d-s}}{r^d}\,{}_2\tilde{\mathrm{F}}_1\left(\begin{matrix}1, d/2\\ 1-(d-s)/2\end{matrix}; \frac{(R-1)^2}{r^2}\frac{t-u}{1-u}\right)\right\}.\end{aligned} \tag{21}$$

Furthermore, if $\mathbf{z} = (\sqrt{1-\xi^2}\bar{\mathbf{z}}, \xi) \in \mathbb{S}^d$, the weighted s-potential is given by

$$\begin{aligned}& U_s^{\eta_t}(\mathbf{z}) + Q_{\mathbf{a},q}(\mathbf{z}) = \Phi_s(t), \quad \mathbf{z} \in \Sigma_t,\\ & U_s^{\eta_t}(\mathbf{z}) + Q_{\mathbf{a},q}(\mathbf{z}) = \Phi_s(t) + q\frac{1}{\rho^s}\mathrm{I}\left(\frac{(R+1)^2}{r^2}\frac{\xi-t}{1+\xi}; \frac{d-s}{2}, \frac{s}{2}\right) - \Phi_s(t)\,\mathrm{I}\left(\frac{\xi-t}{1+\xi}; \frac{d-s}{2}, \frac{s}{2}\right), \quad \mathbf{z} \in \mathbb{S}^d \setminus \Sigma_t,\end{aligned} \tag{22}$$

where $\rho = \sqrt{R^2 - 2R\xi + 1}$ and $\mathrm{I}(x; a, b)$ is the regularized incomplete Beta function.

The corresponding statement for the case $s = d-2$ is given in Theorem 15.

Remark 12. According to Remark 8 we have from Theorem 11 that $\mathcal{F}_s(\Sigma_t) = \Phi_s(t)$. Concerning the minimization of this function, we derive the following result.

Theorem 13. *Let $d-2 < s < d$. For the external field $Q_{\mathbf{a},q}(\mathbf{x})$, $\mathbf{a} = (\mathbf{0}, R)$, $R > 1$, the function $\Phi_s(t)$ has precisely one global minimum $t_0 \in (-1, 1]$. This minimum is either the unique solution $t_0 \in (-1, 1)$ of the equation*

$$\Phi_s(t) = q(R+1)^{d-s}/\big(R^2 - 2Rt + 1\big)^{d/2},$$

or $t_0 = 1$ when such a solution does not exist. Moreover, $t_0 = \max\{t\colon\ \eta_t \geqslant 0\}$. The extremal measure $\mu_{Q_{\mathbf{a},q}}$ on $\mathbb{S}^d$ is given by η_{t_0} (see (20)), and $\mathrm{supp}(\mu_{Q_{\mathbf{a},q}}) = \Sigma_{t_0}$.

Note that, in view of formulas (46) and (47) for $\|\epsilon_t\|$ and $\|\nu_t\|$ given below, the equation in Theorem 13 can be written in terms of hypergeometric functions.

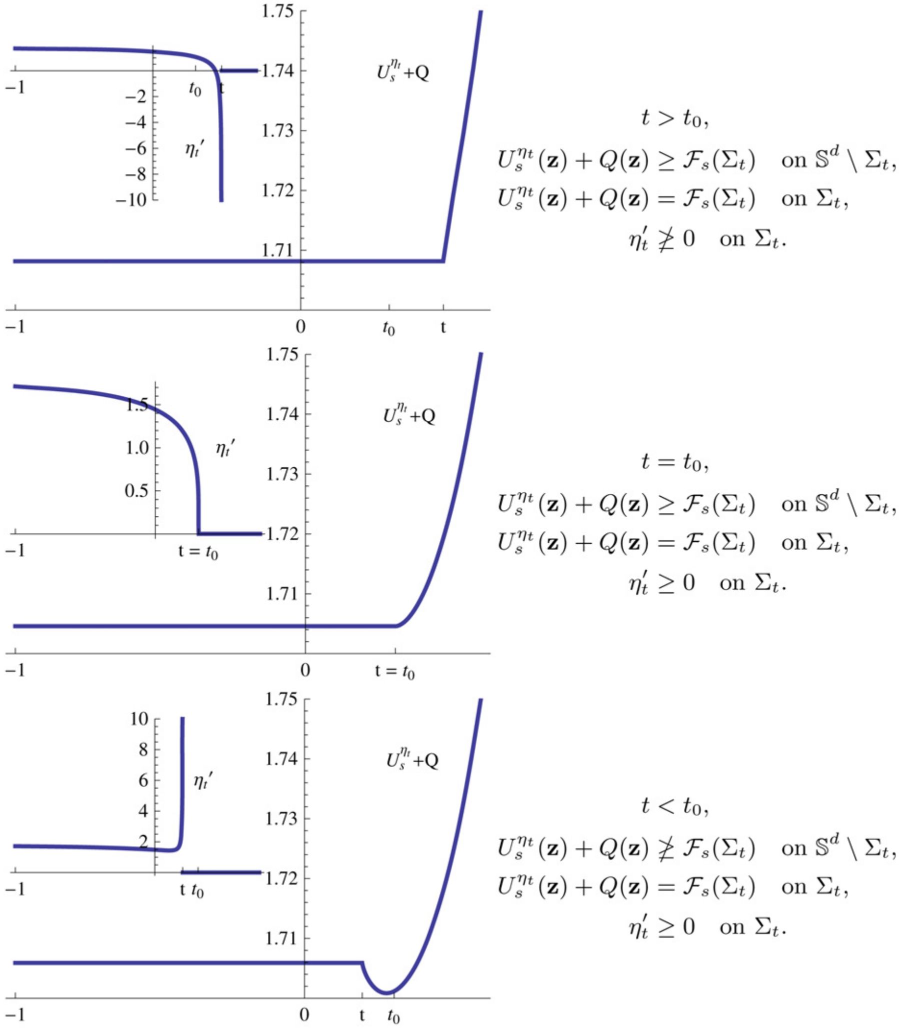

Fig. 1. The weighted s-potential of η_t for $t > t_0$, $t = t_0$, and $t < t_0$ versus altitude ξ of $\mathbf{z}$ for $d = 2$, $s = 1/2$, $q = 1$, and $R = 3/2$, cf. Theorems 11 and 13. Insets show the respective density η_t'.

Remark 14. The restriction on the parameter s arises in the process of applying the balayage method and the principle of domination. It is a topic for further investigation to extend the range of s for which the conclusion of Theorem 13 remains true.

Fig. 1 gives an overview of the qualitative behavior of the weighted s-potential of η_t on $\mathbb{S}^d$ associated with Q and its density with respect to $\sigma_d|_{\Sigma_t}$ for s in the range $d-2 < s < d$ and the choices $t < t_0$, $t = t_0$ and $t > t_0$. We remark that the tangent line to the graph of the weighted s-potential becomes vertical as $\xi \to t^+$ for $t \neq t_0$ and is horizontal for $t = t_0 < 1$ (cf. Remark 31).

1.5. *The exceptional case $s = d - 2$*

In this case M. Riesz's approach [13, Chapter IV] has to be modified. Somewhat surprisingly it turns out, as shown in Lemmas 33 and 36, that the s-balayage measures from (18)

$$\bar{\epsilon}_t := \epsilon_{t,d-2} = \mathrm{Bal}_{d-2}(\delta_{\mathbf{a}}, \Sigma_t), \qquad \bar{\nu}_t := \nu_{t,d-2} = \mathrm{Bal}_{d-2}(\sigma, \Sigma_t) \tag{23}$$

exist and both have a component that is uniformly distributed on the boundary of Σ_t. Moreover, unlike the case $d-2 < s < d$, the density for $\mu_{Q_{\mathbf{a},q}}$, where $s = d-2$, does not vanish on the boundary of its support. We introduce the measure

$$\beta_t(\mathbf{x}) := \delta_t(u) \cdot \sigma_{d-1}(\bar{\mathbf{x}}), \quad \mathbf{x} = \left(\sqrt{1-u^2}\,\bar{\mathbf{x}}, u\right).$$

Theorem 15. *Let $d \geqslant 3$. The signed s-equilibrium $\overline{\eta}_t$ on the spherical cap Σ_t associated with $\overline{Q}_{\mathbf{a},q}(\mathbf{x}) = q|\mathbf{x}-\mathbf{a}|^{2-d}$ is given by*

$$\overline{\eta}_t = \left[\overline{\Phi}_{d-2}(t)/W_{d-2}(\mathbb{S}^d)\right]\overline{\nu}_t - q\overline{\epsilon}_t, \quad \overline{\Phi}_{d-2}(t) := W_{d-2}(\mathbb{S}^d)\left(1+q\|\overline{\epsilon}_t\|\right)/\|\overline{\nu}_t\|,$$

where $\overline{\nu}_t$ and $\overline{\epsilon}_t$ are given in (23). More explicitly, for $\mathbf{x} = (\sqrt{1-u^2}\overline{\mathbf{x}}, u) \in \mathbb{S}^d$

$$\begin{aligned} \mathrm{d}\overline{\eta}_t(\mathbf{x}) &= \frac{1}{W_{d-2}(\mathbb{S}^d)}\left[\overline{\Phi}_{d-2}(t) - \frac{q(R^2-1)^2}{(R^2-2Ru+1)^{d/2+1}}\right]\mathrm{d}\sigma_d\Big|_{\Sigma_t}(\mathbf{x}) \\ &\quad + \frac{1-t}{2}\left(1-t^2\right)^{d/2-1}\left[\overline{\Phi}_{d-2}(t) - \frac{q(R+1)^2}{(R^2-2Rt+1)^{d/2}}\right]\mathrm{d}\beta_t(\mathbf{x}). \end{aligned} \tag{24}$$

Furthermore, for any fixed $t \in (-1,1)$, the following weak convergence holds:*

$$\nu_{t,s} \xrightarrow{*} \overline{\nu}_t, \qquad \epsilon_{t,s} \xrightarrow{*} \overline{\epsilon}_t, \quad \text{as } s \to (d-2)^+. \tag{25}$$

The function $\overline{\Phi}_{d-2}(t)$ has precisely one global minimum $t_0 \in (-1,1]$. This minimum is either the unique solution $t_0 \in (-1,1)$ of

$$\overline{\Phi}_{d-2}(t) = q(R+1)^2/\left(R^2-2Rt+1\right)^{d/2},$$

or $t_0 = 1$ when such a solution does not exist. Moreover, $t_0 = \max\{t\colon \overline{\eta}_t \geqslant 0\}$.

The extremal measure $\mu_{\overline{Q}_{\mathbf{a},q}}$ on $\mathbb{S}^d$ with $\operatorname{supp}(\mu_{\overline{Q}_{\mathbf{a},q}}) = \Sigma_{t_0}$ is given by

$$\mathrm{d}\mu_{\overline{Q}_{\mathbf{a},q}}(\mathbf{x}) = \mathrm{d}\overline{\eta}_{t_0}(\mathbf{x}) = \frac{\overline{\Phi}_{d-2}(t_0)}{W_{d-2}(\mathbb{S}^d)}\left[1 - \frac{(R-1)^2(R^2-2Rt_0+1)^{d/2}}{(R^2-2Ru+1)^{d/2+1}}\right]\mathrm{d}\sigma_d\Big|_{\Sigma_{t_0}}(\mathbf{x}). \tag{26}$$

In Lemmas 33 and 36 we give the s-potentials of the balayage measures $\overline{\nu}_t$ and $\overline{\epsilon}_t$ from which the weighted s-potential of $\overline{\eta}_t$ at every $\mathbf{z} \in \mathbb{S}^d$ can be easily obtained.

Remark 16. As can be seen from (24), depending on the sign of the coefficient of β_t, the signed equilibrium $\overline{\eta}_t$ has positive or negative charge on $\partial\Sigma_t$ unless $t = t_0$, in which case the charge on the boundary disappears (see Fig. 2).

Next, we describe the results when $d = 2$ and $s = 0$. The external field in this case is $\overline{Q}(\mathbf{x}) = \overline{Q}_{\mathbf{a},q}(\mathbf{x}) = q\log(1/|\mathbf{x}-\mathbf{a}|)$. The balayage process for logarithmic kernels preserves the mass of the measures, but changes the potentials by a constant. Hence, $\|\overline{\nu}_{t,0}\| = \|\overline{\epsilon}_{t,0}\| = 1$, and thus $\overline{\Phi}_{d-2}(t) = 1+q$. However, the Mhaskar–Saff functional $\mathcal{F}_0(\Sigma_t)$ from (15) is no longer equal to $\overline{\Phi}_{d-2}(t)$ (cf. Remark 12 and Lemma 40). The logarithmic energy satisfies

$$W_0(K) = \lim_{s\to 0^+} \mathrm{d}W_s(K)/\mathrm{d}s\big|_{s=0}. \tag{27}$$

For $K = \mathbb{S}^2$ we have $W_0(\mathbb{S}^2) = 1/2 - \log 2 < 0$. Since Theorem 10 can be extended to $s = 0$ if $d = 2$, we deduce that $S_{\overline{Q}} := \operatorname{supp}(\mu_{\overline{Q}})$ will be a spherical cap Σ_{t_0}. Direct calculations show that the Mhaskar–Saff functional $\mathcal{F}_0$ for spherical caps is still minimized for $S_{\overline{Q}}$. Fig. 2 shows the qualitative behavior for the weighted potential in the logarithmic case. (Note, that for $t \neq t_0$ the tangent line to the graph of the weighted logarithmic potential at $\xi \to t^+$ is **not** vertical like in the case $d-2 < s < d$ (cf. Fig. 1), but it becomes horizontal if $t = t_0 < 1$.)

Theorem 17. *Let $d = 2$ and $s = 0$. The signed s-equilibrium $\overline{\eta}_{t,0}$ on the spherical cap Σ_t associated with $\overline{Q}_{\mathbf{a},q}(\mathbf{x}) = q\log(1/|\mathbf{x}-\mathbf{a}|)$ is given by*

$$\overline{\eta}_{t,0} = (1+q)\overline{\nu}_{t,0} - q\overline{\epsilon}_{t,0},$$

where $\overline{\nu}_{t,0} = \mathrm{Bal}_0(\sigma_2, \Sigma_t)$ and $\overline{\epsilon}_{t,0} = \mathrm{Bal}_0(\delta_{\mathbf{a}}, \Sigma_t)$. For $\mathbf{x} = (\sqrt{1-u^2}\overline{\mathbf{x}}, u) \in \Sigma_t$

$$\mathrm{d}\overline{\eta}_{t,0}(\mathbf{x}) = \left[1+q - \frac{q(R^2-1)^2}{(R^2-2Ru+1)^2}\right]\mathrm{d}\sigma_2\Big|_{\Sigma_t}(\mathbf{x}) + \frac{1-t}{2}\left[1+q - \frac{q(R+1)^2}{R^2-2Rt+1}\right]\mathrm{d}\beta_t(\mathbf{x}).$$

For $\mathbf{z} = (\sqrt{1-\xi^2}\overline{\mathbf{z}}, \xi) \in \mathbb{S}^1$ the weighted logarithmic potential of $\overline{\eta}_{t,0}$ satisfies

$$\begin{aligned} &U_0^{\overline{\eta}_{t,0}}(\mathbf{z}) + \overline{Q}_{\mathbf{a},q}(\mathbf{z}) = \mathcal{F}_0(\Sigma_t), \quad \mathbf{z} \in \Sigma_t, \\ &U_0^{\overline{\eta}_{t,0}}(\mathbf{z}) + \overline{Q}_{\mathbf{a},q}(\mathbf{z}) = \mathcal{F}_0(\Sigma_t) + \frac{1}{2}\log\frac{1+t}{1+\xi} + \frac{q}{2}\log\frac{R^2-2Rt+1}{R^2-2R\xi+1}, \quad \mathbf{z} \in \mathbb{S}^2 \setminus \Sigma_t, \end{aligned}$$

where $\mathcal{F}_0(\Sigma_t)$ is given below in Lemma 40.

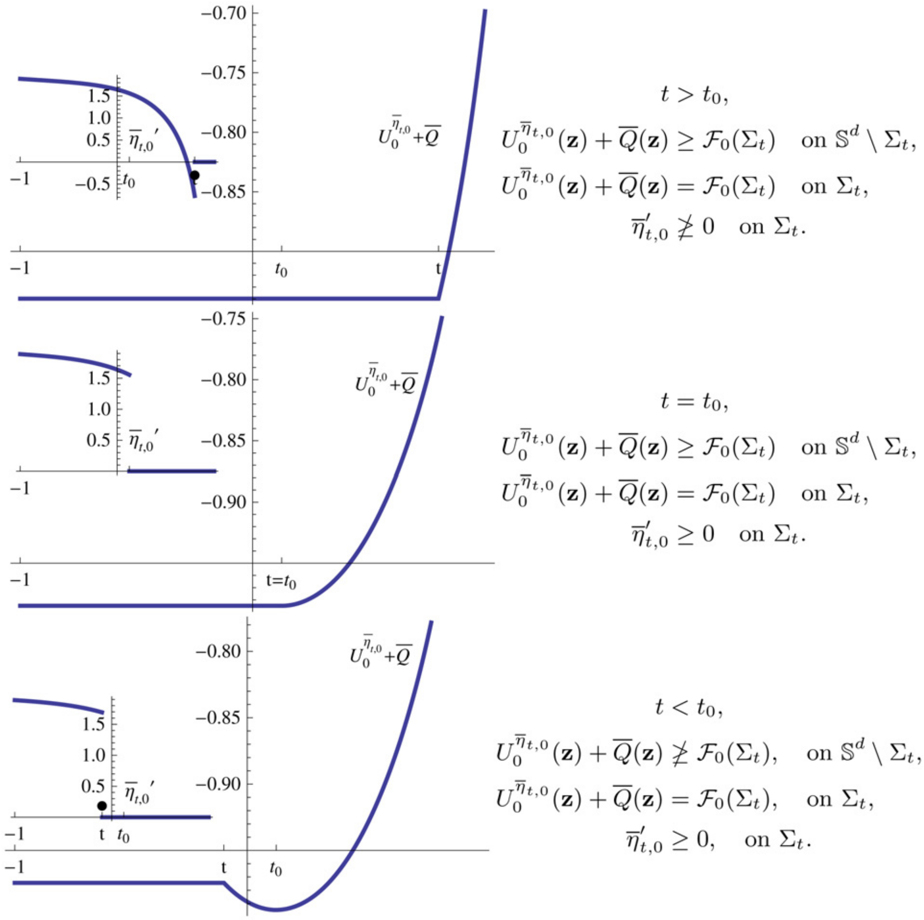

Fig. 2. The weighted logarithmic potential of $\overline{\eta}_{t,0}$ for $t > t_0$, $t = t_0$, and $t < t_0$ versus altitude ξ of $\mathbf{z}$ for $d = 2$, $s = 0$, $q = 1$, and $R = 2$, cf. Theorem 17. Insets show the respective density $\overline{\eta}_{t,0}'$. The dot indicates the component on the boundary of Σ_t.

The Mhaskar–Saff functional $\mathcal{F}_0$ is minimized for Σ_{t_0}, where

$$t_0 = \min\left\{1, \frac{(R+1)^2}{2R(1+q)} - 1\right\} = \min\left\{1, 1 - \frac{4qR - (R-1)^2}{2R(1+q)}\right\}.$$

Moreover, $t_0 = \max\{t\colon \overline{\eta}_{t,0} \geqslant 0\}$.

The logarithmic extremal measure $\mu_{\overline{Q}_{\mathbf{a},q}}$ on $\mathbb{S}^2$ with $\operatorname{supp}(\mu_{\overline{Q}_{\mathbf{a},q}}) = \Sigma_{t_0}$ *is*

$$\mathrm{d}\mu_{\overline{Q}_{\mathbf{a},q}}(\mathbf{x}) = \mathrm{d}\overline{\eta}_{t_0,0}(\mathbf{x}) = \left[1 + q - \frac{q(R^2-1)^2}{(R^2 - 2Ru + 1)^2}\right] \mathrm{d}\sigma_2\bigg|_{\Sigma_{t_0}}(\mathbf{x}). \tag{28}$$

Remark 18. In general, the density $\overline{\eta}_{t_0,0}'(u)$ in (28) does not vanish on the boundary of Σ_{t_0}. In fact, if $t_0 \in (-1, 1)$, then

$$\lim_{u \to t_0} \overline{\eta}_{t_0,0}'(u) = \left[(1+q)/q\right]\left[4qR - (R-1)^2\right]/(R+1)^2 > 0.$$

1.6. Axis-supported external fields

It is well known that the balayage of a measure can be represented as a superposition of balayages of Dirac-delta measures. Using this, we extend our results to external fields that are axis-supported s-potentials.

Definition 19. We call an external field Q *positive-axis supported*, if

$$Q(\mathbf{x}) = \int |\mathbf{x} - R\mathbf{p}|^{-s}\, \mathrm{d}\lambda(R), \quad \mathbf{x} \in \mathbb{S}^d, \tag{29}$$

for some finite positive measure λ supported on a compact subset of $(0, \infty)$.

Remark 20. Since $\mathrm{Bal}_s(\delta_{(1/R)\mathbf{p}}, \mathbb{S}^d) = R^s \mathrm{Bal}_s(\delta_{R\mathbf{p}}, \mathbb{S}^d)$, we can restrict ourselves to measures λ with support in $[1,\infty)$. It is possible to generalize the setting to fields supported on both the negative and positive polar axis as well. This generalization shall be reserved for a later occasion.

We begin with a result, which generalizes Theorem 2 and Corollary 3 to axis-supported external fields Q.

Theorem 21. *Let $0 < s < d$ and Q be as in* (29) *with* $\mathrm{supp}(\lambda) \subset [1,\infty)$. *Then*

$$\mathrm{d}\tilde{\eta}_\lambda(\mathbf{x}) = \frac{1}{W_s(\mathbb{S}^d)} \left\{ \mathcal{F}_s(\mathbb{S}^d) - \int (R^2-1)^{d-s} (R^2 - 2Ru + 1)^{s/2-d} \, \mathrm{d}\lambda(R) \right\} \mathrm{d}\sigma(\mathbf{x}).$$

Moreover, $\mathrm{supp}(\mu_Q) = \mathbb{S}^d$ *(that is $\mu_Q = \tilde{\eta}_\lambda$) if and only if*

$$\mathcal{F}_s(\mathbb{S}^d) \geqslant \int (R+1)^{d-s}(R-1)^{-d} \, \mathrm{d}\lambda(R).$$

The next assertion deals with the signed equilibrium measure $\tilde{\eta}_t$ on spherical caps $\Sigma_t \subset \mathbb{S}^d$ associated with the axis-supported external field Q.

Theorem 22. *Let $d-2 < s < d$ and Q be as in* (29) *with* $\mathrm{supp}(\lambda) \subset [1,\infty)$. *The signed s-equilibrium $\tilde{\eta}_t$ on the spherical cap Σ_t associated with Q is given by*

$$\tilde{\eta}_t = \left[\tilde{\Phi}_s(t)/W_s(\mathbb{S}^d)\right]\nu_t - \tilde{\epsilon}_t, \quad \tilde{\Phi}_s(t) := W_s(\mathbb{S}^d)\left(1 + \|\tilde{\epsilon}_t\|\right)/\|\nu_t\|,$$

where ν_t is defined in (18) *and* $\tilde{\epsilon}_t := \mathrm{Bal}_s(\lambda, \Sigma_t) = \int \mathrm{Bal}_s(\delta_{R\mathbf{p}}, \Sigma_t)\, \mathrm{d}\lambda(R)$. *For $\mathbf{x} = (\sqrt{1-u^2}\bar{\mathbf{x}}, u) \in \Sigma_t$ the signed s-equilibrium $\tilde{\eta}_t$ can be written as*

$$\mathrm{d}\tilde{\eta}_t(\mathbf{x}) = \tilde{\eta}_t'(u,R) \frac{\omega_{d-1}}{\omega_d} (1-u^2)^{d/2-1} \, \mathrm{d}u \, \mathrm{d}\sigma_{d-1}(\bar{\mathbf{x}}),$$

where

$$\begin{aligned} \tilde{\eta}_t'(u,R) = \frac{1}{W_s(\mathbb{S}^d)} \frac{\Gamma(d/2)}{\Gamma(d-s/2)} \left(\frac{1-t}{1-u}\right)^{d/2} \left(\frac{t-u}{1-t}\right)^{(s-d)/2} \Bigg\{ & \tilde{\Phi}_s(t) {}_2\tilde{\mathrm{F}}_1\left(\begin{matrix} 1, d/2 \\ 1-(d-s)/2 \end{matrix}; \frac{t-u}{1-u}\right) \\ & - \int \frac{(R+1)^{d-s}}{(R^2-2Rt+1)^{d/2}} {}_2\tilde{\mathrm{F}}_1\left(\begin{matrix} 1, d/2 \\ 1-(d-s)/2 \end{matrix}; \frac{(R-1)^2}{R^2-2Rt+1} \frac{t-u}{1-u}\right) \mathrm{d}\lambda(R) \Bigg\}. \end{aligned}$$

Furthermore, the function $\tilde{\Phi}_s(t)$ has precisely one global minimum in $(-1,1]$. This minimum is either the unique solution $t_\lambda \in (-1,1)$ of the equation

$$\tilde{\Phi}_s(t) = \int (R+1)^{d-s} (R^2 - 2Rt + 1)^{-d/2} \, \mathrm{d}\lambda(R),$$

or $t_\lambda = 1$ when such a solution does not exist. Moreover, $t_\lambda = \max\{t\colon\ \tilde{\eta}_t \geqslant 0\}$, $\mu_Q = \tilde{\eta}_{t_\lambda}$, and $\mathrm{supp}(\mu_Q) = \Sigma_{t_\lambda}$, where μ_Q is the extremal measure on $\mathbb{S}^d$.

Theorem 22 can be also extended to the case $s = d-2$ when $d \geqslant 3$ and also to the logarithmic case $s = 0$ for $d = 2$. For details, we refer the reader to [2].

The remainder of this paper is structured as follows. In Section 2 we show the uniqueness of the signed equilibrium and prove Theorem 2 and Corollary 3. In Section 3 a suitable Kelvin transformation of points and measures is considered and explicit formulas for the densities of the measures in (20) are found in Lemmas 24 and 25. Furthermore, the norms of these measures are computed. The proofs of Theorems 9, 10, and 13 are given in Section 4. The weighted s-potential of the signed equilibrium is given in Section 5. Section 6 considers the special case $s = d-2$ and the proofs of Theorems 15 and 17 are provided. Finally, in Section 7 we prove the generalization of the results to axis-supported external fields.

2. Signed equilibrium associated with an external field

First, we consider some preliminaries on the Kelvin transformation (spherical inversion) of points and measures. Inversion in a sphere is a basic technique in electrostatics (method of electrical images, cf. Jackson [9]) and in general in potential theory (cf. Kellog [10] and Landkof [13]). Kelvin transformation (of a function) is linear, preserves harmonicity (in the classical case), and preserves positivity. We shall make use of this method and of balayage to conveniently infer representations of the signed equilibrium associated with an external field from known results.

2.1. The Kelvin transformation

Let us denote by K_R the Kelvin transformation (stereographic projection) with center $\mathbf{a}=(\mathbf{0},R)$ and radius $\sqrt{R^2-1}$, that is for any point $\mathbf{x}\in\mathbb{R}^{d+1}$ the image $\mathbf{x}^*:=\mathrm{K}_R(\mathbf{x})$ lies on a ray stemming from $\mathbf{a}$, and passing through $\mathbf{x}$ such that

$$|\mathbf{x}-\mathbf{a}|\cdot\left|\mathbf{x}^*-\mathbf{a}\right|=R^2-1. \tag{30}$$

Thus, the transformation of the distance is given by the formula

$$\left|\mathbf{x}^*-\mathbf{y}^*\right|=\left(R^2-1\right)\frac{|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}-\mathbf{a}||\mathbf{y}-\mathbf{a}|},\quad \mathbf{x},\mathbf{y}\in\mathbb{S}^d. \tag{31}$$

It is easy to see that $\mathrm{K}_R(\mathbb{S}^d)=\mathbb{S}^d$, where K_R sends the spherical cap $A_R:=\{(\sqrt{1-u^2}\bar{\mathbf{x}},u)\colon\ 1/R\leqslant u\leqslant 1,\ \bar{\mathbf{x}}\in\mathbb{S}^{d-1}\}$ to $B_R:=\{(\sqrt{1-u^2}\bar{\mathbf{x}},u)\colon\ -1\leqslant u\leqslant 1/R,\ \bar{\mathbf{x}}\in\mathbb{S}^{d-1}\}$ and vice versa, with the points on the boundary being fixed. In particular, the North Pole $\mathbf{p}=(\mathbf{0},1)$ goes to the South Pole $\mathbf{q}:=(\mathbf{0},-1)$. The image of $\mathbf{x}=(\sqrt{1-u^2}\bar{\mathbf{x}},u)$ is $\mathbf{x}^*=(\sqrt{1-(u^*)^2}\bar{\mathbf{x}},u^*)$, where

$$1+u^*=\frac{(R+1)^2}{R^2-2Ru+1}(1-u). \tag{32}$$

The last equation is derived from the similar triangles proportion

$$\left|\mathbf{x}^*-\mathbf{q}\right|/|\mathbf{q}-\mathbf{a}|=|\mathbf{x}-\mathbf{p}|/|\mathbf{x}-\mathbf{a}|$$

and the formulas $|\mathbf{x}^*-\mathbf{q}|^2=2(1+u^*)$, $|\mathbf{x}-\mathbf{p}|^2=2(1-u)$, $|\mathbf{q}-\mathbf{a}|=R+1$, and $|\mathbf{x}-\mathbf{a}|^2=R^2-2Ru+1$. Finally, we point out that

$$\left|\mathbf{x}^*-\mathbf{a}\right|^{-d}\mathrm{d}\sigma\left(\mathbf{x}^*\right)=|\mathbf{x}-\mathbf{a}|^{-d}\,\mathrm{d}\sigma(\mathbf{x}), \tag{33}$$

which can be easily seen from the relation $(\mathbf{x}^*-\mathbf{a})/|\mathbf{x}^*-\mathbf{a}|=(\mathbf{x}-\mathbf{a})/|\mathbf{x}-\mathbf{a}|$.

Next, we recall that given a measure λ with no point mass at $\mathbf{a}$, its Kelvin transformation (associated with a fixed s) $\lambda^*=\mathcal{K}_{R,s}(\lambda)$ is a measure defined by

$$\mathrm{d}\lambda^*\left(\mathbf{x}^*\right):=\left(R^2-1\right)^{s/2}|\mathbf{x}-\mathbf{a}|^{-s}\,\mathrm{d}\lambda(\mathbf{x}). \tag{34}$$

The s-potentials of the two measures are related as follows (e.g. [5, Eq. (5.1)])

$$U_s^{\lambda^*}\left(\mathbf{x}^*\right)=\int\frac{\mathrm{d}\lambda^*(\mathbf{y}^*)}{|\mathbf{x}^*-\mathbf{y}^*|^s}=\int\frac{|\mathbf{x}-\mathbf{a}|^s\,\mathrm{d}\lambda(\mathbf{y})}{(R^2-1)^{s/2}|\mathbf{x}-\mathbf{y}|^s}=\frac{|\mathbf{x}-\mathbf{a}|^s}{(R^2-1)^{s/2}}U_s^{\lambda}(\mathbf{x}). \tag{35}$$

Note that the Kelvin transformation has the duality property $\mathcal{K}_{R,s}(\lambda^*(\mathbf{x}^*))=\lambda(\mathbf{x})$.

2.2. Signed equilibrium

We first establish the uniqueness of the signed equilibrium, provided it exists.

Lemma 23. *Let $0\leqslant s<d$. If a signed equilibrium $\eta_{E,Q}$ exists, then it is unique.*

Proof. The lemma follows from the positivity of the s-energy of signed measures. Indeed, suppose η_1 and η_2 are two signed equilibria on E associated with Q. Then

$$U_s^{\eta_1}(\mathbf{x})+Q(\mathbf{x})=F_1,\qquad U_s^{\eta_2}(\mathbf{x})+Q(\mathbf{x})=F_2\quad\text{for all }\mathbf{x}\in E.$$

Subtracting the two equations and integrating with respect to $\eta_1-\eta_2$ we obtain

$$\mathcal{I}_s(\eta_1-\eta_2)=\int\left[U_s^{\eta_1}(\mathbf{x})-U_s^{\eta_2}(\mathbf{x})\right]\mathrm{d}(\eta_1-\eta_2)(\mathbf{x})=0,$$

and from [13, Theorem 1.15] we conclude that $\eta_1=\eta_2$ (see also [8, Section 5]). When $d=2$ and $s=0$ instead of [13, Theorem 1.15] we could use [21, Theorem 4.1] to prove the assertion of the lemma. When $d>2$ and $s=0$ we could use [18, p. 6]. Note that $\eta_1-\eta_2$ is the difference of two signed measures with total charge 1. □

We are now in a position to find the signed equilibrium for the external field $Q_{\mathbf{a},q}$ defined by a point charge q at $\mathbf{a}$ (see (7)).

Proof of Theorem 2. We apply the Kelvin transformation (30) to the s-potential

$$U_s^{\epsilon_{\mathbf{a}}}(\mathbf{z}) = \int\limits_{\mathbb{S}^d} \frac{\mathrm{d}\epsilon_{\mathbf{a}}(\mathbf{x})}{|\mathbf{z}-\mathbf{x}|^s}, \quad \mathrm{d}\epsilon_{\mathbf{a}}(\mathbf{x}) := \frac{(R^2-1)^{d-s}}{W_s(\mathbb{S}^d)|\mathbf{x}-\mathbf{a}|^{2d-s}}\,\mathrm{d}\sigma(\mathbf{x}), \qquad \sigma = \sigma_d.$$

From (31) and (33) (recall that $\mathrm{K}_R(\mathbb{S}^d) = \mathbb{S}^d$) we obtain

$$U_s^{\epsilon_{\mathbf{a}}}(\mathbf{z}) = |\mathbf{z}-\mathbf{a}|^{-s} \int\limits_{\mathbb{S}^d} \frac{1}{W_s(\mathbb{S}^d)|\mathbf{z}^*-\mathbf{x}^*|^s}\,\mathrm{d}\sigma\big(\mathbf{x}^*\big) = \frac{1}{|\mathbf{z}-\mathbf{a}|^s},$$

where we used that $U_s^{\sigma}(\mathbf{z}^*) = W_s(\mathbb{S}^d)$ for all $\mathbf{z}^* \in \mathbb{S}^d$. Hence, $\epsilon_{\mathbf{a}} = \epsilon_1$ (see (18)). For $\eta_{\mathbf{a}}$ defined in (12), we therefore derive

$$U_s^{\eta_{\mathbf{a}}}(\mathbf{z}) + Q_{\mathbf{a},q}(\mathbf{z}) = W_s\big(\mathbb{S}^d\big) + qU_s^{\sigma}(\mathbf{a}) \quad \text{for all } \mathbf{z} \in \mathbb{S}^d.$$

In addition, one similarly finds

$$\int\limits_{\mathbb{S}^d} \big(R^2-1\big)^{d-s}|\mathbf{x}-\mathbf{a}|^{s-2d}\,\mathrm{d}\sigma(\mathbf{x}) = \int\limits_{\mathbb{S}^d} \big|\mathbf{x}^*-\mathbf{a}\big|^{-s}\,\mathrm{d}\sigma\big(\mathbf{x}^*\big) = U_s^{\sigma}(\mathbf{a}),$$

and consequently $\eta_{\mathbf{a}}(\mathbb{S}^d) = 1$. Therefore, $\eta_{\mathbf{a}}$ is the required signed equilibrium.

Finally, to derive (13), using (2) and (3), we evaluate

$$U_s^{\sigma}(\mathbf{a}) = \int\limits_{\mathbb{S}^d} \frac{\mathrm{d}\sigma_d(\mathbf{x})}{|\mathbf{x}-\mathbf{a}|^s} = \frac{\omega_{d-1}}{\omega_d}\int\limits_{-1}^{1} \big(1-u^2\big)^{d/2-1}\big(R^2-2Ru+1\big)^{-s/2}\,\mathrm{d}u = (R+1)^{-s}{}_2\mathrm{F}_1\left(\begin{matrix} s/2, d/2 \\ d \end{matrix}; 4R/(R+1)^2\right).$$

In the last step we used the standard substitution $2v = 1+u$ and the integral representation of the hypergeometric function [1, Eq. 15.3.1]. □

The proof of Corollary 3 is an easy consequence of the uniqueness of the extremal measure associated with an external field.

Proof of Corollary 3. The (strictly decreasing) density $\eta'_{\mathbf{a}}(\mathbf{x})$ in (12) attains its minimum value at the North Pole $\mathbf{p}$. So, non-negativity there implies that $\eta_{\mathbf{a}} > 0$ everywhere else on $\mathbb{S}^d$, in which case it coincides with the extremal measure on $\mathbb{S}^d$. On the other hand, if $\operatorname{supp}(\mu_{Q_{\mathbf{a},q}}) = \mathbb{S}^d$, then the variational inequalities (5) and (6) yield $\mu_{Q_{\mathbf{a},q}} = \eta_{\mathbf{a}}$; and $\eta'_{\mathbf{a}}(\mathbf{x})$ is again non-negative at $\mathbf{p}$. What remains to show is that the inequality in (14) is equivalent to $\eta'_{\mathbf{a}}(\mathbf{x}) \geqslant 0$, which can be seen by using $|\mathbf{p}-\mathbf{a}| = R-1$. Finally, using the series expansion of (13) and

$$(R+1)^d/(R-1)^d = \big\{1-\big[4R/(R+1)^2\big]\big\}^{-d/2} = \sum_{k=0}^{\infty} \frac{(d/2)_k}{k!}\frac{(4R)^k}{(R+1)^{2k}},$$

we derive the second part of (14). □

3. The s-balayage measures ν_t and ϵ_t

In this section we show that for s in the range $d-2 < s < d$, the measures ν_t and ϵ_t are absolutely continuous with respect to the normalized area surface measure σ_d (restricted to the spherical cap Σ_t) and we find their densities.

3.1. The balayage measures

We now focus on the two balayage measures in (18). The second one, ν_t, has already been found in [5, Eqs. (3.19) and (4.6)].[6] It is absolutely continuous in the following sense:

$$\mathrm{d}\nu_t(\mathbf{x}) = \big(1+J_t(\mathbf{x})\big)\frac{\omega_{d-1}}{\omega_d}\big(1-u^2\big)^{d/2-1}\,\mathrm{d}u\,\mathrm{d}\sigma_{d-1}(\bar{\mathbf{x}}), \quad \mathbf{x} \in \Sigma_t, \tag{36}$$

where Σ_t is the spherical cap centered at the South Pole (see (17)) and

$$J_t(\mathbf{x}) := \frac{\sin(\pi(d-s)/2)}{\pi}\frac{(1-t)^{d-s/2}}{(t-u)^{(d-s)/2}}\int\limits_0^1 \frac{v^{d/2-1}(1-v)^{(d-s)/2}}{1-u-(1-t)v}\,\mathrm{d}v.$$

[6] Here, we use normalized surface area measure.

It is convenient to obtain a closed form for $J_t(\mathbf{x})$ in terms of (regularized) hypergeometric functions (cf. (9)). By [1, Eq. 15.3.1]

$$J_t(\mathbf{x}) := \frac{\Gamma(d/2)(d-s)/2}{\Gamma(1-(d-s)/2)}\left(\frac{1-t}{1-u}\right)^{d/2}\left(\frac{t-u}{1-t}\right)^{(s-d)/2} {}_2\tilde{\mathrm{F}}_1\left(\begin{matrix}1, d/2\\ 1+d-s/2\end{matrix}; \frac{1-t}{1-u}\right).$$

The application of [1, Eq. 15.3.6] yields an expansion near $u=t$,

$$J_t(\mathbf{x}) = -1 + \frac{\Gamma(d/2)}{\Gamma(d-s/2)}\left(\frac{1-t}{1-u}\right)^{d/2}\left(\frac{t-u}{1-t}\right)^{(s-d)/2} {}_2\tilde{\mathrm{F}}_1\left(\begin{matrix}1, d/2\\ 1-(d-s)/2\end{matrix}; \frac{t-u}{1-u}\right).$$

Substituting the last relation into (36) and simplifying we get the following lemma.

Lemma 24. *Let $d-2<s<d$. The measure $\nu_t = \mathrm{Bal}_s(\sigma, \Sigma_t)$ is given by*

$$\mathrm{d}\nu_t(\mathbf{x}) = \nu_t'(u)\frac{\omega_{d-1}}{\omega_d}\left(1-u^2\right)^{d/2-1}\,\mathrm{d}u\,\mathrm{d}\sigma_{d-1}(\bar{\mathbf{x}}), \quad \mathbf{x}\in\Sigma_t, \tag{37}$$

where the density $\nu_t'(u)$ is given by

$$\nu_t'(u) := \frac{\Gamma(d/2)}{\Gamma(d-s/2)}\left(\frac{1-t}{1-u}\right)^{d/2}\left(\frac{t-u}{1-t}\right)^{(s-d)/2} {}_2\tilde{\mathrm{F}}_1\left(\begin{matrix}1, d/2\\ 1-(d-s)/2\end{matrix}; \frac{t-u}{1-u}\right). \tag{38}$$

To determine the s-balayage ϵ_t, we recall the formulas for the Kelvin transformation of measures and the relation of the corresponding potentials (see (34) and (35)). Let λ^* be the extremal measure on $\Sigma_t^* := \mathrm{K}_R(\Sigma_t)$, normalized so that its potential $U_s^{\lambda^*}(\mathbf{x}^*) = 1$ on Σ_t^*. Then, using (30) and (35) we derive just as in [5, Section 3, Eq. (3.7)] that

$$\epsilon_t(\mathbf{x}) = \left(R^2-1\right)^{-s/2}\mathcal{K}_{R,s}\left(\lambda^*\left(\mathbf{x}^*\right)\right). \tag{39}$$

Since the image Σ_t^* of Σ_t is also a spherical cap, this time centered at the North Pole, we can utilize a formula similar to (37) for its extremal measure. If $\Sigma_t = \{\mathbf{x}\colon\ -1 \leqslant u \leqslant t\}$, then $\Sigma_t^* = \{\mathbf{x}\colon\ 1 \geqslant u^* \geqslant t^*\}$, where u^* and t^* are related to u and t by (32). If we set $\nu_t^* := \mathrm{Bal}(\sigma, \Sigma_t^*)$, then $\lambda^* = \nu_t^*/W_s(\mathbb{S}^d)$; hence we get

$$\mathrm{d}\lambda^*\left(\mathbf{x}^*\right) = \left(\lambda^*\right)'\left(u^*\right)\frac{\omega_{d-1}}{\omega_d}\left[1-\left(u^*\right)^2\right]^{d/2-1}\,\mathrm{d}u^*\,\mathrm{d}\sigma_{d-1}\left(\bar{\mathbf{x}}^*\right), \tag{40}$$

where the density is given by

$$\left(\lambda^*\right)'\left(u^*\right) := \frac{\Gamma(d/2)/W_s(\mathbb{S}^d)}{\Gamma(d-s/2)}\left(\frac{1+t^*}{1+u^*}\right)^{d/2}\left(\frac{u^*-t^*}{1+t^*}\right)^{(s-d)/2} {}_2\tilde{\mathrm{F}}_1\left(\begin{matrix}1, d/2\\ 1-(d-s)/2\end{matrix}; \frac{u^*-t^*}{1+u^*}\right).$$

(We remark that the last formula (up to a normalization constant) for the special case $d=2$ was first derived by Fabrikant et al. [7].) From (32) we get

$$\frac{1+u^*}{1+t^*} = \frac{R^2-2Rt+1}{R^2-2Ru+1}\cdot\frac{1-u}{1-t}, \tag{41}$$

from which it follows that

$$\left[1-\left(u^*\right)^2\right]^{d/2-1}\mathrm{d}u^* = \left(\frac{R^2-1}{R^2-2Ru+1}\right)^d\left(1-u^2\right)^{d/2-1}\,\mathrm{d}u. \tag{42}$$

Substituting (41) and (42) in (40) and using (39) and (34) we obtain

Lemma 25. *Let $d-2<s<d$. The measure $\epsilon_t = \mathrm{Bal}_s(\delta_{\mathbf{a}}, \Sigma_t)$ is given by*

$$\mathrm{d}\epsilon_t(\mathbf{x}) = \epsilon_t'(u)\frac{\omega_{d-1}}{\omega_d}\left(1-u^2\right)^{d/2-1}\,\mathrm{d}u\,\mathrm{d}\sigma_{d-1}(\bar{\mathbf{x}}), \quad \mathbf{x}\in\Sigma_t, \tag{43}$$

and setting $r^2 := R^2-2Rt+1$, the density is given by

$$\epsilon_t'(u) := \frac{1}{W_s(\mathbb{S}^d)}\frac{\Gamma(d/2)}{\Gamma(d-s/2)}\frac{(R+1)^{d-s}}{r^d}\left(\frac{1-t}{1-u}\right)^{d/2}\left(\frac{t-u}{1-t}\right)^{(s-d)/2} {}_2\tilde{\mathrm{F}}_1\left(\begin{matrix}1, d/2\\ 1-(d-s)/2\end{matrix}; \frac{(R-1)^2}{r^2}\frac{t-u}{1-u}\right). \tag{44}$$

3.2. Positivity of the signed equilibrium of a spherical cap

The following lemma establishes a condition for positivity of the signed equilibrium

$$\mathrm{d}\eta_t(\mathbf{x}) = \eta_t'(u)\frac{\omega_{d-1}}{\omega_d}\left(1-u^2\right)^{d/2-1}\,\mathrm{d}u\,\mathrm{d}\sigma_{d-1}(\bar{\mathbf{x}}).$$

Lemma 26. *Let $d-2<s<d$. If for some $\gamma>0$ we have $\eta_t'(u)\geqslant 0$ for $u\in(t-\gamma,t)$, then*

$$\Phi_s(t)\geqslant q(R+1)^{d-s}/r^d, \qquad r^2=R^2-2Rt+1, \tag{45}$$

and, consequently, $\eta_t'(u)>0$ for all $-1\leqslant u<t<1$.

Proof. Using (21) and the non-negativity hypothesis for $\eta_t'(u)$, we get

$$\lim_{u\to t^-}\left[(t-u)^{(d-s)/2}\eta_t'(u)\right]=\frac{\Gamma(d/2)(1-t)^{(d-s)/2}\{\Phi_s(t)-q(R+1)^{d-s}/r^d\}}{W_s(\mathbb{S}^d)\Gamma(d-s/2)\Gamma(1-(d-s)/2)}>0.$$

In particular, the expression in braces is non-negative for $d-2<s<d$.

For $R\neq 1$ we have $(R-1)^2<r^2$. Thus, the first hypergeometric function in (21) is strictly larger than the second one for all $-1\leqslant u<t$ and $d-2<s<d$. Hence, using $\Phi_s(t)\geqslant q(R+1)^{d-s}/r^d$, we have

$$\eta_t'(u)>\frac{1}{W_s(\mathbb{S}^d)}\frac{\Gamma(d/2)}{\Gamma(d-s/2)}\left(\frac{1-t}{1-u}\right)^{d/2}\left(\frac{t-u}{1-t}\right)^{(s-d)/2}{}_2\tilde{\mathrm{F}}_1\left(\begin{matrix}1,d/2\\1-(d-s)/2\end{matrix};\frac{t-u}{1-u}\right)\left\{\Phi_s(t)-\frac{q(R+1)^{d-s}}{r^d}\right\}\geqslant 0,$$

which shows that $\eta_t'(u)>0$ for all $-1\leqslant u<t$. □

Remark 27. We note that in the limit $R\to 1$ relation (45) becomes the same as in [5, Eq. (5.9)]. It also follows from the proof of Lemma 26 that the sign of the difference $\Phi_s(t)-q(R+1)^{d-s}/r^d$ is determined by the sign of $\eta_t'(u)$ near the boundary of the spherical cap Σ_t, that is for u near t^-, and vice versa.

Remark 28. Equality in relation (45) yields $\lim_{u\to t^-}\eta_t'(u)=0$. This follows from (21) and series expansion (9).

3.3. Norms of the measures in (20)

Lemma 29. *Let $d-2<s<d$. Then*

$$\|\epsilon_t\|=\frac{2^{1-d}\Gamma(d)}{\Gamma(d-s/2)\Gamma(s/2)}\frac{(R+1)^{d-s}}{W_s(\mathbb{S}^d)}\int_{-1}^{t}\frac{(1+u)^{s/2-1}(1-u)^{d-s/2-1}}{(R^2-2Ru+1)^{d/2}}\,\mathrm{d}u. \tag{46}$$

Proof. Substitution of (43) and (44) into $\|\epsilon_t\|=\frac{\omega_{d-1}}{\omega_d}\int_{-1}^t\epsilon_t'(u)(1-u^2)^{d/2-1}\,\mathrm{d}u$ and further simplifications (see [2, Appendix] for details) yield

$$\|\epsilon_t\|=2^{(d-s)/2-1}\frac{\Gamma(d/2)}{\Gamma(d-s/2)}\frac{\Gamma(d/2)}{\Gamma(s/2)}\frac{\omega_{d-1}}{\omega_d}\frac{(R+1)^{d-s}}{W_s(\mathbb{S}^d)r^d}(1-t)^{d/2}(1+t)^{s/2}$$
$$\times(1-xy)^{-d/2}\int_0^1 v^{s/2-1}(1-xv)^{d-s/2-1}\left(1-\frac{x(1-y)}{1-xy}v\right)^{-d/2}\mathrm{d}v,$$

where $x=(1+t)/2$ and $y=(R-1)^2/r^2$. Substituting

$$1-xy=\frac{(R+1)^2}{r^2}\frac{1-t}{2}, \qquad \frac{x(1-y)}{1-xy}=\frac{4R}{(R+1)^2}\frac{1+t}{2},$$

and (3) we get the Euler-type integral of an Appell function [6, Eq. 5.8(5)]

$$\|\epsilon_t\|=\frac{2^{-s/2}\Gamma(d)}{\Gamma(d-s/2)\Gamma(s/2)}\frac{1}{W_s(\mathbb{S}^d)}(R+1)^{-s}(1+t)^{s/2}\int_0^1 u^{s/2-1}\left(1-\frac{1+t}{2}u\right)^{d-s/2-1}\left(1-\frac{4R}{(R+1)^2}\frac{1+t}{2}u\right)^{-d/2}\mathrm{d}u.$$

A change of variables $1+v=(1+t)u$ yields (46). □

Lemma 30. *Let $d-2<s<d$. Then*

$$\|\nu_t\| = \frac{2^{1-d}\Gamma(d)}{\Gamma(d-s/2)\Gamma(s/2)} \int_{-1}^{t} (1+u)^{s/2-1}(1-u)^{d-s/2-1}\,\mathrm{d}u = 1 - \mathrm{I}\big((1-t)/2; d-s/2, s/2\big). \tag{47}$$

Proof. We proceed as in the proof of Lemma 29. In fact, the densities ϵ_t' and ν_t' differ by a multiplicative factor $(R+1)^{d-s}/[W_s(\mathbb{S}^d)r^d]$ and a factor $(R-1)^2/r^2$ in the argument of the hypergeometric function. From (37) and (38)

$$\|\nu_t\| = \frac{\Gamma(d)}{\Gamma(d-s/2)\Gamma(s/2)}\left(\frac{1+t}{2}\right)^{s/2} \int_0^1 v^{s/2-1}\left(1-\frac{1+t}{2}v\right)^{d-s/2-1}\mathrm{d}v.$$

A change of variable $1+u=(1+t)v$ yields the first part of (47).

A manipulation of the integral (extending the integral over the complete interval $[-1,1]$ and using the standard substitution $2v=1-u$) yields the second part of (47). □

4. The extremal support and measure: Proofs of Theorems 9, 10, and 13

Our first proof deals with the minimization property of S_Q.

Proof of Theorem 9. Let K be any compact subset of $\mathbb{S}^d$ with positive s-capacity. For the considered range of the parameter s, we have that the potential of the extremal measure $\mu_K=\mu_{K,s}$ satisfies the following (in)equalities

$$U_s^{\mu_K}(\mathbf{x}) = W_s(K) \quad \text{q.e. on } K, \qquad U_s^{\mu_K}(\mathbf{x}) \leqslant W_s(K) \quad \text{on } \mathbb{S}^d. \tag{48}$$

This follows trivially from the general theory (see [13, Chapter II]) for $d-1\leqslant s<d$, with the inequality holding on the entire space $\mathbb{R}^{d+1}$. To derive (48) for the extended range, we observe that for $K=\mathbb{S}^d$ this is obvious ($\mu_K=\sigma_d$). If $\mathbb{S}^d\setminus K$ is non-empty, there is a spherical cap Σ that contains K. The s-potential of μ_Σ equals $W_s(\Sigma)$ everywhere on Σ (see [5]), so the measure $\nu:=[W_s(K)/W_s(\Sigma)]\mu_K$ has a potential that equals $W_s(K)$ on Σ. Since $U_s^{\mu_K}(\mathbf{x})\leqslant W_s(K)$ on $\operatorname{supp}(\mu_K)$ (see [13, p. 136(b)]), we could derive the inequality in (48) by comparing the potentials of μ_K and ν and applying the restricted version of the Principle of Domination as given in [5, Lemma 5.1] (for $s=d-2$ we adapt the argument in Lemma 5.1 using [13, Theorem 1.27]). Since $U_s^{\mu_K}(\mathbf{x})\geqslant W_s(K)$ q.e. on K (see [13, p. 136(a)]), we conclude the equality in (48) as well.

Clearly, $\mathcal{F}_s(S_Q)=F_Q$ (see (5) and (6)). We now show that for any compact set $K\subset\mathbb{S}^d$ with positive s-capacity we have $\mathcal{F}_s(K)\geqslant\mathcal{F}_s(S_Q)$. Indeed, let us integrate (5) with respect to μ_K. Since μ_K has finite energy, the inequality holds also μ_K-a.e. and, using the inequality in (48), we conclude that

$$W_s(K) \geqslant \int U_s^{\mu_K}(\mathbf{x})\,\mathrm{d}\mu_Q(\mathbf{x}) = \int U_s^{\mu_Q}(\mathbf{x})\,\mathrm{d}\mu_K(\mathbf{x}) \geqslant F_Q - \int Q(\mathbf{x})\,\mathrm{d}\mu_K(\mathbf{x}),$$

which proves our claim. □

Next, we prove sufficient conditions on Q, that guarantee that the extremal support is a spherical zone (cap).

Proof of Theorem 10. Convexity of $f(\xi)$ implies that $Q(\mathbf{z})$ is continuous and the existence and uniqueness of the extremal measure μ_Q follows from standard potential-theoretical arguments (see [22,23]). The rotational invariance of $Q(\mathbf{z})$ implies that the extremal support is also rotationally invariant. Hence, there is a compact set $A\subset[-1,1]$ and an integrable function $g:A\to\mathbb{R}^+$, such that the extremal measure and its support are given by

$$\mathrm{d}\mu_Q(\mathbf{x}) = g(u)\,\mathrm{d}u\,\mathrm{d}\sigma_{d-1}(\bar{\mathbf{x}}), \qquad \operatorname{supp}(\mu_Q) = \big\{\big(\sqrt{1-u^2}\,\bar{\mathbf{x}}, u\big):\ u\in A,\ \bar{\mathbf{x}}\in\mathbb{S}^{d-1}\big\}.$$

We show that A is connected. For this purpose we adapt the argument given in [15]. Suppose A is not connected. Then there is an interval $[\alpha,\beta]\subset(-1,1)$, such that $[\alpha,\beta]\cap A=\{\alpha,\beta\}$. Let $A^-:=A\cap[-1,\alpha]$ and $A^+:=A\cap[\beta,1]$. For

$$\mathbf{x} = \big(\sqrt{1-u^2}\,\bar{\mathbf{x}}, u\big), \quad u\in A^-\cup A^+,\ \bar{\mathbf{x}}\in\mathbb{S}^{d-1}, \qquad \mathbf{z} = \big(\sqrt{1-\xi^2}\,\bar{\mathbf{z}}, \xi\big), \quad \xi\in(\alpha,\beta),\ \bar{\mathbf{z}}\in\mathbb{S}^{d-1},$$

we represent the weighted s-potential of μ_Q as follows:

$$U_s^{\mu_Q}(\mathbf{z}) + Q(\mathbf{z}) = \int_A g(u)\left(\int_{\mathbb{S}^{d-1}} \frac{\mathrm{d}\sigma_{d-1}(\bar{\mathbf{x}})}{|\mathbf{z}-\mathbf{x}|^s}\right)\mathrm{d}u + Q(\mathbf{z}) =: \int_{A^-} g(u)\kappa(u,\xi)\,\mathrm{d}u + \int_{A^+} g(u)\kappa(u,\xi)\,\mathrm{d}u + f(\xi),$$

where $\kappa(u,\xi)$ has been evaluated in [5, Section 4] for the case $\xi>u$ ($u\in A^-$) to be

$$\kappa(u,\xi) := \int\limits_{\mathbb{S}^{d-1}} \frac{\mathrm{d}\sigma_{d-1}(\bar{\mathbf{x}})}{|\mathbf{z}-\mathbf{x}|^s} \tag{49}$$

$$= (1-u)^{-s/2}(1+\xi)^{-s/2}{}_2\mathrm{F}_1\left(\begin{matrix} s/2, 1-(d-s)/2 \\ d/2 \end{matrix}; \frac{1+u}{1-u}\frac{1-\xi}{1+\xi}\right) \tag{50}$$

$$= \sum_{k=0}^{\infty} \frac{(s/2)_k(1-(d-s)/2)_k(1+u)^k}{(d/2)_k k!(1-u)^{k+s/2}} \frac{(1-\xi)^k}{(1+\xi)^{k+s/2}}. \tag{51}$$

By symmetry we derive that when $\xi < u$ ($u \in A^+$)

$$\kappa(u,\xi) = \sum_{k=0}^{\infty} \frac{(s/2)_k(1-(d-s)/2)_k(1-u)^k}{(d/2)_k k!(1+u)^{k+s/2}} \frac{(1+\xi)^k}{(1-\xi)^{k+s/2}}. \tag{52}$$

It is easy to verify that the functions

$$(1-\xi)^k/(1+\xi)^{k+s/2}, \qquad (1+\xi)^k/(1-\xi)^{k+s/2}, \quad k=0,1,2,\ldots,$$

are strictly convex for $\xi \in (-1,1)$. Hence, from (51) and (52) we derive that $\kappa(u,\xi)$ is a convex function in ξ on (α,β) for any fixed $u \in A^- \cup A^+$. Therefore, using the convexity of $f(\xi)$ we deduce that the weighted s-potential is strictly convex on $[\alpha,\beta]$. This clearly contradicts the inequalities (5) and (6), which proves (16).

Now suppose that, in addition, $f(\xi)$ is also increasing. If $t_1 > -1$, for $u \in [t_1,t_2]$ and $\xi \in (-1,t_1)$, the kernel is calculated using (52), in which case we easily obtain that $\partial\kappa(u,\xi)/\partial\xi > 0$. This yields that the weighted s-potential is strictly increasing on $[-1,t_1]$, which contradicts (5) and (6) similarly. □

Proof of Theorem 13. The external field is given by

$$Q_{\mathbf{a},q}(\mathbf{z}) = q/|\mathbf{a}-\mathbf{z}|^s = q\left|R^2 - 2R\xi + 1\right|^{-s/2} =: f(\xi), \quad \mathbf{z} = \left(\sqrt{1-\xi^2}\bar{\mathbf{z}}, \xi\right) \in \mathbb{S}^d,$$

where $f'(\xi) > 0$ and $f''(\xi) > 0$ for $\xi \in [-1,1]$. By Theorem 10, $\operatorname{supp}(\mu_{Q_{\mathbf{a},q}})$ is a spherical cap centered at the South Pole. So, by Theorem 9 we have to minimize the $\mathcal{F}_s$-functional over all such caps. Recall that (see (19) and Remark 12)

$$\mathcal{F}_s(\Sigma_t) = \Phi_s(t) = W_s(\mathbb{S}^d)\left(1+q\|\epsilon_t\|\right)/\|\nu_t\|.$$

Applying the Quotient Rule and using (46), (47), and the Fundamental Theorem of Calculus, we get (note that $\|\nu_t\| > 0$ for $t > -1$ and $\|\nu_t\|' > 0$ for $-1 < t < 1$)

$$\frac{\mathrm{d}\Phi_s}{\mathrm{d}t} = \frac{q\|\epsilon_t\|'\|\nu_t\| - (1+q\|\epsilon_t\|)\|\nu_t\|'}{\|\nu_t\|^2/W_s(\mathbb{S}^d)} = -\frac{\|\nu_t\|'}{\|\nu_t\|}\left[\Phi_s(t) - qW_s\left(\mathbb{S}^d\right)\frac{\|\epsilon_t\|'}{\|\nu_t\|'}\right]$$
$$= -\frac{\|\nu_t\|'}{\|\nu_t\|}\left[\Phi_s(t) - \frac{q(R+1)^{d-s}}{r^d}\right] =: -\frac{\|\nu_t\|'}{\|\nu_t\|}\Delta(t), \tag{53}$$

where $r = r(t) = \sqrt{R^2-2Rt+1}$. Observe, that $\Delta(t) \to \infty$ as $t \to -1$. Hence, there is a largest $t_0 \in (-1,1]$ such that $\Delta(t) > 0$ on $(-1,t_0)$. If $t_0 = 1$, then $\Phi_s(t)$ is strictly decreasing on $(-1,1)$ and attains its minimum at $t=1$. (We note that $\Delta(1) \geqslant 0$ is equivalent to the condition in Corollary 3.) If $t_0 < 1$, then by continuity $\Delta(t_0) = 0$. Clearly, $\Phi_s'(t) < 0$ on $(-1,t_0)$ and $\Phi_s'(t_0) = 0$. Suppose, $\Phi_s'(\tau) = 0$ for some $\tau \in (-1,1)$. Then $\Delta(\tau) = 0$. Applying the product rule we get

$$\frac{\mathrm{d}^2\Phi_s}{\mathrm{d}t^2}(\tau) = -\frac{\|\nu_t\|'}{\|\nu_t\|}\left[\Phi_s'(t) - \frac{dq(R+1)^{d-s}R}{r^{d+2}}\right]\Bigg|_{t=\tau} = \frac{\|\nu_t\|'}{\|\nu_t\|}\frac{dq(R+1)^{d-s}R}{r^{d+2}}\Bigg|_{t=\tau} > 0.$$

Hence, any zero of Φ_s' is a minimum of Φ_s. Since Φ_s is twice continuously differentiable on $(-1,1)$ (see Lemmas 29 and 30), the latter observation implies that Φ_s has only one local minimum in $(-1,1)$, namely t_0, which has to be also a global minimum. Observe, that $\Phi_s'(t) < 0$ for $t \in (-1,t_0)$ and $\Phi_s'(t) > 0$ for $t \in (t_0,1)$. From (53) we conclude that $\Delta(t) > 0$ on $(-1,t_0)$ and $\Delta(t) < 0$ on $(t_0,1)$. This shows that $\Phi_s(t)$ has precisely one global minimum in $(-1,1]$, which is either the unique solution $t_0 \in (-1,1)$ of the equation $\Delta(t) = 0$ if it exists, or $t_0 = 1$. Moreover, $\Delta(t) \geqslant 0$ if and only if $t \leqslant t_0$. By Lemma 26 and Remark 27 we have $t_0 = \max\{t\colon\ \eta_t \geqslant 0\}$. Clearly, $S_{Q_{\mathbf{a},q}} = \Sigma_{t_0}$, from the minimization property. Since the signed equilibrium for Σ_{t_0} is a positive measure, by the uniqueness of the extremal measure we derive that $\mu_{Q_{\mathbf{a},q}} = \eta_{t_0}$. □

5. The weighted s-potential of η_t on $\mathbb{S}^d \setminus \Sigma_t$: Alternative proof of Theorem 13

In this section we complete the proof of Theorem 11, namely formula (22) on $\mathbb{S}^d \setminus \Sigma_t$. For $\mathbf{z} = (\sqrt{1-\xi^2}\bar{\mathbf{z}}, \xi)$ with $\xi > t$ the s-potential of η_t is given by

$$U_s^{\eta_t}(\mathbf{z}) = \int \frac{\mathrm{d}\eta_t(\mathbf{x})}{|\mathbf{z}-\mathbf{x}|^s} = \frac{\omega_{d-1}}{\omega_d} \int_{-1}^{t} \kappa(u,\xi)\eta_t'(u)\big(1-u^2\big)^{d/2-1}\,\mathrm{d}u,$$

where $\kappa(u,\xi)$ is given in (50). Using appropriately chosen constants C and c_t the densities of ϵ_t and ν_t in (20) both can be written as (cf. Lemmas 24 and 25)

$$\gamma_t'(u) = C\left(\frac{1-t}{1-u}\right)^{d/2}\left(\frac{t-u}{1-t}\right)^{(s-d)/2} \sum_{n=0}^{\infty} \frac{(d/2)_n}{\Gamma(n+1-(d-s)/2)}\left(c_t^2\frac{t-u}{1-u}\right)^n.$$

Hence, it is sufficient to study the s-potential of $\mathrm{d}\gamma_t = \gamma_t'\,\mathrm{d}\sigma_d|_{\Sigma_t}$.

Using the series representation (51) of $\kappa(u,\xi)$ and integrating term-wise we get

$$U_s^{\gamma_t}(\mathbf{z}) = \frac{C\omega_{d-1}/\omega_d}{(1+\xi)^{s/2}} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(s/2)_m(1-(d-s)/2)_m(d/2)_n}{m!(d/2)_m\Gamma(n+1-(d-s)/2)}\left[\frac{1-\xi}{1+\xi}\right]^m c_t^{2n}\mathcal{H}_{m,n}(t;u),$$

where $\mathcal{H}_{m,n}(t;u)$ is the integral (the second step follows from [16, Eq. 2.2.6(9)])

$$\begin{aligned}\mathcal{H}_{m,n}(t;u) &= (1-t)^{d-s/2}\int_{-1}^{t} \frac{(t-u)^{n-(d-s)/2}(1+u)^{m+d/2-1}}{(1-u)^{m+n+1+s/2}}\,\mathrm{d}u \\ &= \frac{\Gamma(m+d/2)\Gamma(n+1-(d-s)/2)}{\Gamma(m+n+1+s/2)}\frac{(1-t)^{d-s/2}(1+t)^{m+n+s/2}}{(1-t)^{m+d/2}(1+t)^{n+1-(d-s)/2}}.\end{aligned}$$

Putting everything together, we arrive at

$$\begin{aligned}U_s^{\gamma_t}(\mathbf{z}) &= 2^{d-s-1}C\frac{\omega_{d-1}}{\omega_d}\frac{\Gamma(d/2)}{\Gamma(1+s/2)}\left(\frac{1-t}{2}\right)^{(d-s)/2}\left(\frac{1+t}{1+\xi}\right)^{s/2} \\ &\quad\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(s/2)_m(1)_n(1-(d-s)/2)_m(d/2)_n}{(1+s/2)_{m+n}m!n!}\left(\frac{1-\xi}{1+\xi}\frac{1+t}{1-t}\right)^m\left(c_t^2\frac{1+t}{2}\right)^n.\end{aligned}$$

The double sum in the last expression is, in fact, the series expansion of the generalized F_3-hypergeometric function (cf. [17, Eq. 7.2.4(3)])

$$\mathrm{F}_3\left(\begin{matrix}a,a',b,b'\\c\end{matrix};w,z\right) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}m!n!}w^m z^n, \quad |w|,|z|<1.$$

Moreover, the F_3-function in question is of the form [17, Eq. 7.2.4(76)]

$$\mathrm{F}_3\left(\begin{matrix}a,c-a,b,c-b\\c\end{matrix};w,z\right) = (1-z)^{a+b-c}{}_2\mathrm{F}_1\left(\begin{matrix}a,b\\c\end{matrix};w+z-wz\right).$$

Let $r=\sqrt{R^2-2Rt+1}$ and $\rho = |\mathbf{z}-\mathbf{a}| = \sqrt{R^2-2R\xi+1}$. For $c_t=1$, $C=\Gamma(d/2)/\Gamma(d-s/2)$, and using (3), we have

$$U_s^{\nu_t}(\mathbf{z}) = W_s\big(\mathbb{S}^d\big)A_{s,d}\left(\frac{1+t}{1+\xi}\right)^{s/2}{}_2\mathrm{F}_1\left(\begin{matrix}s/2,1-(d-s)/2\\1+s/2\end{matrix};\frac{1+t}{1+\xi}\right).$$

For $c_t^2 = (R-1)^2/r^2$ and $C = (1/W_s(\mathbb{S}^d))\Gamma(d/2)/\Gamma(d-s/2)(R+1)^{d-s}/r^d$, we get

$$U_s^{\epsilon_t}(\mathbf{z}) = A_{s,d}\frac{1}{r^s}\left(\frac{1+t}{1+\xi}\right)^{s/2}{}_2\mathrm{F}_1\left(\begin{matrix}s/2,1-(d-s)/2\\1+s/2\end{matrix};\frac{\rho^2}{r^2}\frac{1+t}{1+\xi}\right).$$

The normalization constant $A_{s,d}$ is given by

$$A_{s,d} := \frac{\Gamma(d/2)}{\Gamma((d-s)/2)\Gamma(1+s/2)} = 1\Big/{}_2\mathrm{F}_1\left(\begin{matrix}s/2,1-(d-s)/2\\1+s/2\end{matrix};1\right).$$

(The last relation holds by [1, Eq. 15.1.20].) Note that the hypergeometric functions above can be expressed in terms of (incomplete) Beta functions (see (10)). Thus

$$U_s^{\nu_t}(\mathbf{z}) = W_s(\mathbb{S}^d)\,\mathrm{I}\left(\frac{1+t}{1+\xi}; \frac{s}{2}, \frac{d-s}{2}\right), \qquad U_s^{\epsilon_t}(\mathbf{z}) = \frac{1}{\rho^s}\,\mathrm{I}\left(\frac{\rho^2}{r^2}\frac{1+t}{1+\xi}; \frac{s}{2}, \frac{d-s}{2}\right), \tag{54}$$

which are valid for $\mathbf{z} \in \mathbb{S}^d \setminus \Sigma_t$. Hence, using (20), we obtain

$$U_s^{\eta_t}(\mathbf{z}) = \Phi_s(t)\,\mathrm{I}\left(\frac{1+t}{1+\xi}; \frac{s}{2}, \frac{d-s}{2}\right) - \frac{q}{\rho^s}\,\mathrm{I}\left(\frac{\rho^2}{r^2}\frac{1+t}{1+\xi}; \frac{s}{2}, \frac{d-s}{2}\right).$$

Application of the functional equation $\mathrm{I}(x; a, b) = 1 - \mathrm{I}(1-x; b, a)$ gives (22).

Next, we provide an alternative proof of Theorem 13. The (series) expansion

$$\mathrm{I}(z; a, b) = \left[\Gamma(a+b)/\Gamma(b)\right] z^a (1-z)^b {}_2\tilde{\mathrm{F}}_1\left(\begin{matrix}1, a+b\\ a+1\end{matrix}; z\right),$$

applied to (22) yields for $\xi > t > -1$

$$U_s^{\eta_t}(\mathbf{z}) + Q(\mathbf{z}) = \Phi_s(t) + \frac{\Gamma(d/2)}{\Gamma(s/2)}\left(\frac{\xi - t}{1+\xi}\right)^{(d-s)/2}\left(\frac{1+t}{1+\xi}\right)^{s/2}$$
$$\times \sum_{n=0}^{\infty} \frac{(d/2)_n}{\Gamma(n+1+(d-s)/2)}\left(\frac{\xi-t}{1+\xi}\right)^n \left\{\frac{q(R+1)^{d-s}}{r^d}\left[\frac{R^2+2R+1}{R^2-2Rt+1}\right]^n - \Phi_s(t)\right\}.$$

If $q(R+1)^{d-s}/r^d \geqslant \Phi_s(t)$, then the above infinite series is a positive function for $1 \geqslant \xi > t$. An immediate consequence in such a case is the inequality

$$U_s^{\eta_t}(\mathbf{z}) + Q(\mathbf{z}) > \Phi_s(t), \quad \mathbf{z} \in \mathbb{S}^d \setminus \Sigma_t.$$

In particular, the last relation holds when $t = t_0$ is a solution of $q(R+1)^{d-s}/r^d = \Phi_s(t)$. But then from Lemma 26 we have that the signed equilibrium is a positive measure. Since it satisfies the Gauss variational (in)equalities (5) and (6), it is the extremal measure μ_Q on $\mathbb{S}^d$. Easily, we derive that $t_0 = \max\{t\colon\ \eta_t \geqslant 0\}$.

Remark 31. An interesting observation is that for $t = t_0$ we could factor $(\xi - t)/(1+\xi)$ (to get $[(\xi-t)/(1+\xi)]^{1+(d-s)/2}$) and using product rule, it follows that

$$\partial\left\{U_s^{\eta_t}(\mathbf{z}) + Q(\mathbf{z})\right\}/\partial\xi|_{\xi\to t^+} = 0.$$

It can be also shown that for $q(R+1)^{d-s}/r^d \neq \Phi_s(t)$ one has

$$\frac{\partial}{\partial\xi}\left\{U_s^{\eta_t}(\mathbf{z}) + Q(\mathbf{z})\right\} = \frac{\Gamma(d/2)}{\Gamma((d-s)/2)\Gamma(s/2)}\left\{\left[q(R+1)^{d-s}/r^d\right] - \Phi_s(t)\right\}(1+t)^{(s-d)/2}(\xi-t)^{(d-s)/2-1} + \mathcal{O}\left((\xi-t)^{(d-s)/2}\right)$$

as $\xi \to t^+$. Thus, the partial derivative with respect to ξ of the weighted s-potential of the signed s-equilibrium η_t is singular at the boundary of Σ_t when approaching it from the "outside" if t is not a solution of the equilibrium condition. The sign of this partial derivative is determined by the difference in curly braces, see Fig. 1.

6. The exceptional case $s = d - 2$: Proof of Theorems 15 and 17

The proof of Theorem 15 will be split into several lemmas. We first find the s-balayage of a point charge $\mathbf{y} = (\sqrt{1-v^2}\bar{\mathbf{y}}, v) \in \mathbb{S}^d \setminus \Sigma_t$ onto Σ_t. Set

$$\epsilon_{\mathbf{y}} = \epsilon_{\mathbf{y},t,d-2} := \mathrm{Bal}_{d-2}(\delta_{\mathbf{y}}, \Sigma_t).$$

To determine $\epsilon_{\mathbf{y}}$ we proceed as in [5, Section 3] (see also [13, Chapter IV]). We apply an inversion (stereographical projection) with center $\mathbf{y}$ and radius $\sqrt{2}$. The image of $\mathbb{S}^d$ is a hyperplane passing through the origin. The image of Σ_t is a hyperdisc of radius $\tau = \sqrt{1-t^2}/(v-t)$. The $(d-2)$-extremal measure on this d-dimensional hyperdisc is the normalized (unit) uniform surface measure on its boundary $\mathrm{d}\lambda^*(\mathbf{x}^*) = \tau^{d-1}\,\mathrm{d}\sigma_{d-1}((\mathbf{x}^* - \mathbf{b}^*)/\tau)$, where $\mathbf{b}^*$ is the center of this hyperdisc. The potential of λ^* is found to be

$$U_{d-2}^{\lambda^*}(\mathbf{x}^*) = \tau \int_{\mathbb{S}^{d-1}} \frac{\mathrm{d}\sigma_{d-1}((\mathbf{x}^* - \mathbf{b}^*)/\tau)}{|(\mathbf{z}^* - \mathbf{b}^*)/\tau - (\mathbf{x}^* - \mathbf{b}^*)/\tau|^{d-2}} = \tau W_{d-2}(\mathbb{S}^{d-1}) = \tau.$$

Using the Kelvin transformation of this measure as given in Section 2.1 (cf. (34) and (35) with $R^2 - 1 = 2$), we compute that

$$\mathrm{d}\epsilon_{\mathbf{y}}(\mathbf{x}) = 2(v-t)\left(1-t^2\right)^{d/2-1}|\mathbf{x}-\mathbf{y}|^{-d}\,\mathrm{d}\sigma_{d-1}(\bar{\mathbf{x}}), \quad \mathbf{x} \in \partial\Sigma_t.$$

The corresponding balayage for $d-2<s<d$ was found in [5, Eq. (3.12)]:

$$\mathrm{d}\epsilon_{\mathbf{y},s}(\mathbf{x}) = \frac{2\sin(\pi(d-s)/2)}{\pi}\left(\frac{v-t}{t-u}\right)^{(d-s)/2}\left(1-u^2\right)^{d/2-1}\frac{\mathrm{d}u\,\mathrm{d}\sigma_{d-1}(\bar{\mathbf{x}})}{|\mathbf{x}-\mathbf{y}|^d}, \quad \mathbf{x}\in\Sigma_t.$$

The following lemma establishes the relationship between $\epsilon_{\mathbf{y},s}$ and $\epsilon_{\mathbf{y}}$.

Lemma 32. *Let $d\geqslant 3$. Let $\mathrm{d}\gamma_s := \frac{\sin(\pi(d-s)/2)}{\pi(t-u)^{(d-s)/2}}\,\mathrm{d}u$, $-1\leqslant u\leqslant t$. Then $\|\gamma_s\|\to 1$ and $\gamma_s \xrightarrow{*} \delta_t$, as $s\to(d-2)^+$. Consequently, $\epsilon_{\mathbf{y},s}\xrightarrow{*}\epsilon_{\mathbf{y}}$, as $s\to(d-2)^+$.*

Proof. We compute

$$\|\gamma_s\| = \int_{-1}^{t}\frac{\sin(\pi(d-s)/2)}{\pi(t-u)^{(d-s)/2}}\,\mathrm{d}u = \frac{\sin(\pi(1-(d-s)/2))}{\pi(1-(d-s)/2)}(1+t)^{1-(d-s)/2}.$$

Clearly, $\|\gamma_s\|\leqslant 2$ and $\|\gamma_s\|\to 1$ as $s\to(d-2)^+$. Let f be a continuous function on $[-1,t]$. Then what we have to prove is that

$$\lim_{s\to(d-2)^+}\int_{-1}^{t}\frac{\sin(\pi(d-s)/2)}{\pi(t-u)^{(d-s)/2}}f(u)\,\mathrm{d}u = f(t).$$

By $\|\gamma_s\|\to 1$ as $s\to(d-2)^+$, this is equivalent to

$$\lim_{s\to(d-2)^+}\int_{-1}^{t}\frac{\sin(\pi(d-s)/2)}{\pi(t-u)^{(d-s)/2}}\big[f(u)-f(t)\big]\,\mathrm{d}u = 0.$$

Suppose now that $f(\mathbf{x})$, where $\mathbf{x}=(\sqrt{1-u^2}\bar{\mathbf{x}},u)$, is a continuous function on $\mathbb{S}^d$. Then as $s\to(d-2)^+$ we have

$$\begin{aligned}\lim\int_{\Sigma_t} f\,\mathrm{d}\epsilon_{\mathbf{y},s} &= \lim\int_{-1}^{t}\left(\int_{\mathbb{S}^{d-1}} f(\mathbf{x})\frac{\mathrm{d}\sigma_{d-1}(\bar{\mathbf{x}})}{|\mathbf{x}-\mathbf{y}|^d}\right)2(v-t)^{(d-s)/2}\left(1-u^2\right)^{d/2-1}\mathrm{d}\gamma_s(u)\\ &= 2(v-t)\left(1-t^2\right)^{d/2-1}\left(\int_{\mathbb{S}^{d-1}} f(\mathbf{x})\frac{\mathrm{d}\sigma_{d-1}(\bar{\mathbf{x}})}{|\mathbf{x}-\mathbf{y}|^d}\right)\Bigg|_{u=t} = \int_{\Sigma_t} f\,\mathrm{d}\epsilon_{\mathbf{y}},\end{aligned}$$

which completes the proof of the lemma. □

Next, we determine the balayage measures in (23). We shall use that β_t, which is the unit charge uniformly distributed on the boundary of Σ_t, has $(d-2)$-potential

$$U_{d-2}^{\beta_t}(\mathbf{z}) = \int_{\mathbb{S}^{d-1},\,u=t}\frac{\mathrm{d}\sigma_{d-1}(\bar{\mathbf{x}})}{|\mathbf{z}-\mathbf{x}|^{d-2}} = \begin{cases}(1-t)^{1-d/2}(1+\xi)^{1-d/2} & \text{if } \xi\geqslant t,\\ (1+t)^{1-d/2}(1-\xi)^{1-d/2} & \text{if } \xi<t,\end{cases} \tag{55}$$

where $\mathbf{z}=(\sqrt{1-\xi^2}\bar{\mathbf{z}},u)\in\mathbb{S}^d$. This follows from (51) and (52).

Lemma 33. *Let $d\geqslant 3$. The measure $\bar{\nu}_t=\mathrm{Bal}_{d-2}(\sigma_d,\Sigma_t)$ is given by*

$$\mathrm{d}\bar{\nu}_t(\mathbf{x}) = \mathrm{d}\sigma_d|_{\Sigma_t}(\mathbf{x}) + W_{d-2}\big(\mathbb{S}^d\big)\frac{1-t}{2}\left(1-t^2\right)^{d/2-1}\mathrm{d}\delta_t(u)\,\mathrm{d}\sigma_{d-1}(\bar{\mathbf{x}}). \tag{56}$$

The $(d-2)$-potential of $\bar{\nu}_t$ is given by

$$U_{d-2}^{\bar{\nu}_t}(\mathbf{z}) = W_{d-2}\big(\mathbb{S}^d\big), \quad \mathbf{z}\in\Sigma_t, \tag{57}$$

$$U_{d-2}^{\bar{\nu}_t}(\mathbf{z}) = W_{d-2}\big(\mathbb{S}^d\big)(1+t)^{d/2-1}(1+\xi)^{1-d/2} < W_{d-2}\big(\mathbb{S}^d\big), \quad \mathbf{z}\in\mathbb{S}^d\setminus\Sigma_t. \tag{58}$$

Remark 34. It is interesting that the $(d-2)$-potential of $\bar{\nu}_t$ can be expressed using the potential of β_t (cf. (55))

$$U_{d-2}^{\bar{\nu}_t}(\mathbf{z}) = W_{d-2}\big(\mathbb{S}^d\big)\left(1-t^2\right)^{d/2-1}U_{d-2}^{\beta_t}(\mathbf{z}), \quad \mathbf{z}\in\mathbb{S}^d\setminus\Sigma_t.$$

Remark 35. In the proof of Lemma 33 and Lemma 36 below we shall obtain the balayage measures constructively. Alternatively, one could get this from the potential (in)equalities (57), (58) and (61), (62).

Proof of Lemma 33. It is well known that

$$\mathrm{Bal}_{d-2}(\sigma_d, \Sigma_t) = \sigma_d|_{\Sigma_t} + \mathrm{Bal}_{d-2}(\sigma_d|_{\mathbb{S}^d\setminus\Sigma_t}, \Sigma_t). \tag{59}$$

By the principle of superposition we have for $\mathbf{x} \in \partial\Sigma_t$

$$\mathrm{Bal}_{d-2}(\sigma_d|_{\mathbb{S}^d\setminus\Sigma_t}, \Sigma_t) = \int\limits_{\mathbb{S}^d\setminus\Sigma_t} \epsilon_{\mathbf{y}}(\mathbf{x})\,\mathrm{d}\sigma_d(\mathbf{y}) = 2\frac{\omega_{d-1}}{\omega_d}\left(1-t^2\right)^{d/2-1}\left(\int\limits_t^1 \left(1-v^2\right)^{d/2-1}(v-t)\int\limits_{\mathbb{S}^{d-1}}\frac{\mathrm{d}\sigma_{d-1}(\bar{\mathbf{y}})}{|\mathbf{x}-\mathbf{y}|^d}\,\mathrm{d}v\right)\sigma_{d-1}(\bar{\mathbf{x}}).$$

The inner integral can be computed using (52) with $s = d$

$$\int\limits_{\mathbb{S}^{d-1}} |\mathbf{x}-\mathbf{y}|^{-d}\,\mathrm{d}\sigma_{d-1}(\bar{\mathbf{y}}) = 1/\left[2(v-t)(1+v)^{d/2-1}(1-t)^{d/2-1}\right]. \tag{60}$$

Hence,

$$\begin{aligned}\mathrm{Bal}_{d-2}(\sigma_d|_{\mathbb{S}^d\setminus\Sigma_t}, \Sigma_t) &= \frac{\omega_{d-1}}{\omega_d}(1+t)^{d/2-1}\left(\int\limits_t^1 (1-v)^{d/2-1}\,\mathrm{d}v\right)\sigma_{d-1}(\bar{\mathbf{x}}) \\ &= \frac{2}{d}\frac{\omega_{d-1}}{\omega_d}(1+t)^{d/2-1}(1-t)^{d/2}\sigma_{d-1}(\bar{\mathbf{x}}) =: q_{\bar{\nu}_t}\sigma_{d-1}(\bar{\mathbf{x}}), \quad \mathbf{x}\in\partial\Sigma_t.\end{aligned}$$

Using $W_{d-2}(\mathbb{S}^d) = (4/d)(\omega_{d-1}/\omega_d)$ and (59) we derive (56).

Eq. (57) holds because of the balayage properties. Using (55) we have

$$U_{d-2}^{\bar{\nu}_t}(\mathbf{z}) = \int\limits_{\Sigma_t} |\mathbf{z}-\mathbf{x}|^{2-d}\,\mathrm{d}\sigma_d(\mathbf{x}) + q_{\bar{\nu}_t}U_{d-2}^{\beta_t}(\mathbf{z}) = W_{d-2}\left(\mathbb{S}^d\right)\frac{1+t}{2}\frac{(1+t)^{d/2-1}}{(1+\xi)^{d/2-1}} + W_{d-2}\left(\mathbb{S}^d\right)\frac{1-t}{2}\frac{(1+t)^{d/2-1}}{(1+\xi)^{d/2-1}},$$

from which follows (58). □

Lemma 36. *Let $d \geqslant 3$. The measure $\bar{\epsilon}_t = \mathrm{Bal}_{d-2}(\delta_{\mathbf{a}}, \Sigma_t)$ is given by*

$$\mathrm{d}\bar{\epsilon}_t(\mathbf{x}) = \bar{\epsilon}_t'(u)\,\mathrm{d}\sigma_d|_{\Sigma_t}(\mathbf{x}) + q_{\bar{\epsilon}_t}\,\mathrm{d}\delta_t(u)\,\mathrm{d}\sigma_{d-1}(\bar{\mathbf{x}}),$$

where the density $\bar{\epsilon}_t'(u)$ and the constant $q_{\bar{\epsilon}_t}$ are given by

$$\bar{\epsilon}_t'(u) := \frac{(R^2-1)^2/W_{d-2}(\mathbb{S}^d)}{(R^2-2Ru+1)^{d/2+1}}, \qquad q_{\bar{\epsilon}_t} = \frac{1-t}{2}\frac{(R+1)^2}{r^d}\left(1-t^2\right)^{d/2-1}.$$

The $(d-2)$-potential of $\bar{\epsilon}_t$ is given by

$$U_{d-2}^{\bar{\epsilon}_t}(\mathbf{z}) = |\mathbf{z}-\mathbf{a}|^{2-d} = U_{d-2}^{\delta_{\mathbf{a}}}(\mathbf{z}), \quad \mathbf{z}\in\Sigma_t, \tag{61}$$

$$U_{d-2}^{\bar{\epsilon}_t}(\mathbf{z}) = r^{2-d}(1+t)^{d/2-1}(1+\xi)^{1-d/2} < U_{d-2}^{\delta_{\mathbf{a}}}(\mathbf{z}), \quad \mathbf{z}\in\mathbb{S}^d\setminus\Sigma_t. \tag{62}$$

Proof. As in the proof of Theorem 2 we evaluate

$$\bar{\epsilon}_{\mathbf{a}} := \mathrm{Bal}_{d-2}\left(\delta_{\mathbf{a}}, \mathbb{S}^d\right), \qquad \mathrm{d}\bar{\epsilon}_{\mathbf{a}}(\mathbf{x}) = \bar{\epsilon}_t'(u)\,\mathrm{d}\sigma_d(\mathbf{x}).$$

Using balayage in steps and (59) we get

$$\mathrm{Bal}_{d-2}(\delta_{\mathbf{a}}, \Sigma_t) = \bar{\epsilon}_{\mathbf{a}}|_{\Sigma_t} + \mathrm{Bal}_{d-2}(\bar{\epsilon}_{\mathbf{a}}|_{\mathbb{S}^d\setminus\Sigma_t}, \Sigma_t).$$

By the principle of superposition we have for $\mathbf{x}\in\partial\Sigma_t$

$$\mathrm{Bal}_{d-2}(\bar{\epsilon}_{\mathbf{a}}|_{\mathbb{S}^d\setminus\Sigma_t}, \Sigma_t) = \int\limits_{\mathbb{S}^d\setminus\Sigma_t} \bar{\epsilon}_t'(v)\epsilon_{\mathbf{y}}(\mathbf{x})\,\mathrm{d}\sigma_d(\mathbf{y}) = \frac{2}{d}\frac{\omega_{d-1}}{\omega_d}\frac{(R^2-1)^2}{W_{d-2}(\mathbb{S}^d)}(1+t)^{d/2-1}\frac{(1-t)^{d/2}}{(R^2-2Rt+1)^{d/2}}\sigma_{d-1}(\bar{\mathbf{x}}) = q_{\bar{\epsilon}_t}\sigma_{d-1}(\bar{\mathbf{x}}),$$

where we applied (60) and used the change of variable $w = (R-1)^2/(1-v) + 2R$.

Similar computations with the substitution $w = (R+1)^2/(1+u) - 2R$ (see also (55)) lead to (62). That is, for $\mathbf{z} \in \mathbb{S}^d \setminus \Sigma_t$ one has

$$U_{d-2}^{\bar{\epsilon}_t}(\mathbf{z}) = \int\limits_{\Sigma_t} \frac{\bar{\epsilon}_t'(u)\,\mathrm{d}\sigma_d(\mathbf{x})}{|\mathbf{z}-\mathbf{x}|^{d-2}} + q_{\bar{\epsilon}_t} U_{d-2}^{\beta_t}(\mathbf{z}) = \frac{2}{d}\frac{\omega_{d-1}}{\omega_d}\frac{(R-1)^2}{W_{d-2}(\mathbb{S}^d) r^d}\frac{(1+t)^{d/2}}{(1+\xi)^{d/2-1}} + \frac{1-t}{2}\frac{(R+1)^2}{r^d}\frac{(1+t)^{d/2-1}}{(1+\xi)^{d/2-1}}$$
$$= \frac{1}{r^{d-2}}\frac{(1+t)^{d/2-1}}{(1+\xi)^{d/2-1}}.$$

As in the proof of Lemma 33 the balayage properties imply Eq. (61). □

The weak* convergence in (25) is shown next.

Lemma 37. *Let $t \in (-1,1)$ be fixed. Then*

$$\nu_{t,s} \xrightarrow{*} \bar{\nu}_t, \qquad \epsilon_{t,s} \xrightarrow{*} \bar{\epsilon}_t, \quad \text{as } s \to (d-2)^+.$$

Proof. The result follows easily from the following representation

$$\mathrm{Bal}_s(\mu, \Sigma_t)(\mathbf{x}) = \mu|_{\Sigma_t}(\mathbf{x}) + \int\limits_{\mathbb{S}^d \setminus \Sigma_t} \epsilon_{\mathbf{y},s}(\mathbf{x})\,\mathrm{d}\mu(\mathbf{y}), \quad \mu \in \mathcal{M}(\mathbb{S}^d),$$

and the weak* convergence $\epsilon_{\mathbf{y},s} \xrightarrow{*} \epsilon_{\mathbf{y}}$ as $s \to (d-2)^+$. □

The norms $\|\bar{\nu}_t\|$ and $\|\bar{\epsilon}_t\|$ are obtained from Lemmas 29 and 30 by taking the limit $s \to (d-2)^+$, which is justified by weak* convergence shown in Lemma 32.

Lemma 38. *Let $d \geqslant 3$. Then*

$$\|\bar{\epsilon}_t\| = \frac{d-2}{4}(R+1)^2 \int\limits_{-1}^{t} \frac{(1+u)^{d/2-2}(1-u)^{d/2}}{(R^2-2Ru+1)^{d/2}}\,\mathrm{d}u, \qquad \|\bar{\nu}_t\| = \frac{d-2}{4} W_{d-2}(\mathbb{S}^d) \int\limits_{-1}^{t} (1+u)^{d/2-2}(1-u)^{d/2}\,\mathrm{d}u.$$

Completion of the proof of Theorem 15. Proceeding as in the proof of Theorem 13, but using now ($r = r(t) = \sqrt{R^2-2Rt+1}$)

$$\overline{\Phi}_{d-2}'(t) = -\|\bar{\nu}_t\|'/\|\bar{\nu}_t\|\big[\overline{\Phi}_{d-2}(t) - q(R+1)^2/r^d\big] =: -\|\bar{\nu}_t\|'/\|\bar{\nu}_t\|\Delta(t),$$

it follows that the global minimum of $\overline{\Phi}_{d-2}$ is either the unique solution $t_0 \in (-1,1)$ of the equation $\Delta(t) = 0$, or $t_0 = 1$. In particular, $\Delta(t) \geqslant 0$ if and only if $t \leqslant t_0$.

The explicit form (24) follows from Lemmas 33 and 36. If $\bar{\eta}_t \geqslant 0$ then $\Delta(t) \geqslant 0$, so $t \leqslant t_0$. On the other hand, it is easy to see that if $t = t_0$, then $\bar{\eta}_{t_0}$ given in (26) is $\geqslant 0$ because of $(R-1)^2 < R^2 - 2Rt_0 + 1 < R^2 - 2Ru + 1$. Therefore, we have that $t_0 = \max\{t\colon \bar{\eta}_t \geqslant 0\}$, $\mu_{\overline{Q}_{\mathbf{a},q}} = \bar{\eta}_{t_0}$, and $\mathrm{supp}(\mu_{\overline{Q}_{\mathbf{a},q}}) = \Sigma_{t_0}$. □

The proof of Theorem 17 is also split into several lemmas. We must check that Theorem 10 also holds in the case $d = 2$ and $s = 0$. Then we can make use of the fact that the support $S_{\overline{Q}_{\mathbf{a},q}}$ of the extremal measure on $\mathbb{S}^2$ is a spherical cap.

Adaption of the proof of Theorem 10 for $d = 2$ and $s = 0$. The kernel

$$\kappa_0(u,\xi) := \int\limits_{\mathbb{S}^1} \log\frac{1}{|\mathbf{z}-\mathbf{x}|}\,\mathrm{d}\sigma_1(\bar{\mathbf{x}}) = -\frac{1}{2}\frac{1}{\pi}\int\limits_{-1}^{1} \frac{\log(2-2u\xi-2\sqrt{1-u^2}\sqrt{1-\xi^2}\tau)}{\sqrt{1-\tau^2}}\,\mathrm{d}\tau$$
$$= -\frac{1}{2}\log\big(1-u\xi+|\xi-u|\big) = \begin{cases} -\frac{1}{2}\log(1+\xi) - \frac{1}{2}\log(1-u), & \xi \geqslant u, \\ -\frac{1}{2}\log(1-\xi) - \frac{1}{2}\log(1+u), & \xi \leqslant u, \end{cases} \tag{63}$$

replaces $\kappa(u,\xi)$ in (49). (For the computation we used [20, Lemma 1.15].) It is easy to verify that the kernel $\kappa_0(u,\xi)$ is strictly convex for $\xi \in (-1,1)$ for any fixed $u \in (-1,1)$. Hence, we may use the arguments of the proof of Theorem 10 appropriately adapted for $d = 2$ and $s = 0$. □

It should be emphasized that in the logarithmic case balayage preserves mass, and that the logarithmic potentials of a measure and its logarithmic balayage onto a compact set K differ by a constant on K.

Lemma 39. *Let $d = 2$ and $s = 0$. The measure $\bar{\nu}_{t,0} = \mathrm{Bal}_0(\sigma_2, \Sigma_t)$ is given by*

$$d\bar{\nu}_{t,0}(\mathbf{x}) = d\sigma_2|_{\Sigma_t}(\mathbf{x}) + \frac{1-t}{2}\, d\delta_t(u)\, d\sigma_1(\bar{\mathbf{x}}) \tag{64}$$

and $\|\bar{\nu}_{t,0}(\mathbf{x})\| = 1$. The logarithmic potential of $\bar{\nu}_{t,0}$ is given by

$$U_0^{\bar{\nu}_{t,0}}(\mathbf{z}) = \frac{1+t}{4} - \frac{\log 2}{2} - \frac{1}{2}\log(1+t) = W_0(\Sigma_t), \quad \mathbf{z} \in \Sigma_t, \tag{65}$$
$$U_0^{\bar{\nu}_{t,0}}(\mathbf{z}) = \frac{1+t}{4} - \frac{\log 2}{2} - \frac{1}{2}\log(1+\xi), \quad \mathbf{z} \in \mathbb{S}^d \setminus \Sigma_t.$$

The measure $\bar{\nu}_{t,0}$ is the logarithmic extremal measure on Σ_t.

Proof. Using (63) we show that (64) satisfies the balayage properties. For $\mathbf{z} \in \Sigma_t$

$$U_0^{\bar{\nu}_{t,0}}(\mathbf{z}) = U_0^{\sigma_2}(\mathbf{z}) - U_0^{\sigma_2|_{\mathbb{S}^2\setminus\Sigma_t}}(\mathbf{z}) + \frac{1-t}{2} U_0^{\sigma_1|_{u=t}}(\mathbf{z}) = \frac{1+t}{4} - \frac{\log 2}{2} - \frac{1}{2}\log(1+t) = W_0(\Sigma_t).$$

For $\mathbf{z} \in \mathbb{S}^2 \setminus \Sigma_t$

$$U_0^{\bar{\nu}_{t,0}}(\mathbf{z}) = \frac{\omega_1}{\omega_2}\int_{-1}^{t}\left(\int_{\mathbb{S}^1} \log\frac{1}{|\mathbf{z}-\mathbf{x}|}\, d\sigma_1(\bar{\mathbf{x}})\right) du + \frac{1-t}{2}\int_{\mathbb{S}^1} \log\frac{1}{|\mathbf{z}-\mathbf{x}|}\bigg|_{u=t} d\sigma_1(\bar{\mathbf{x}})$$
$$= \frac{1+t}{4} - \frac{\log 2}{2} - \frac{1}{2}\log(1+\xi) = W_0(\Sigma_t) + \frac{1}{2}\log\frac{1+t}{1+\xi} < W_0(\Sigma_t).$$

Since it can be easily verified that $\|\bar{\nu}_{t,0}\| = 1$, it follows that $\bar{\nu}_{t,0}$ is a probability measure on Σ_t with constant logarithmic potential on Σ_t. By uniqueness of the logarithmic extremal measure μ_{Σ_t} on Σ_t one has $\mu_{\Sigma_t} = \bar{\nu}_{t,0}$. □

Lemma 40. *Let $d = 2$ and $s = 0$. Then the Mhaskar–Saff functional $\mathcal{F}_0$ for spherical caps Σ_t is given by*

$$\mathcal{F}_0(\Sigma_t) = (1+q)\frac{1+t}{4} + q\frac{(R-1)^2\log(R^2-2Rt+1)}{8R} - \frac{1}{2}\log(1+t) - \frac{\log 2}{2} - q\frac{(R+1)^2\log(R+1)^2}{8R}. \tag{66}$$

It has precisely one global minimum $t_0 \in (-1, 1]$. This minimum is given by

$$t_0 = \min\{1, (R^2 - 2Rq + 1)/[2R(1+q)]\}.$$

Proof. By Lemma 39 and $|\mathbf{x}-\mathbf{a}|^2 = R^2 - 2Ru + 1$ we obtain (with $\mu_{\Sigma_t,0} = \bar{\nu}_{t,0}$)

$$\int \overline{Q}_{\mathbf{a},q}\, d\mu_{\Sigma_t,0} = q\int_{\Sigma_t} \log\frac{1}{|\mathbf{x}-\mathbf{a}|}\, d\sigma_2(\mathbf{x}) + q\frac{1-t}{2}\int_{\mathbb{S}^1} \log\frac{1}{|\mathbf{x}-\mathbf{a}|}\bigg|_{u=t} d\sigma_1(\bar{\mathbf{x}})$$
$$= q\frac{1+t}{4} - q\frac{(R+1)^2\log(R+1)^2}{8R} + q\frac{(R-1)^2\log(R^2-2Rt+1)}{8R}.$$

Substitution of the last expression and $W_0(\Sigma_t)$ from (65) into

$$\mathcal{F}_0(t) := \mathcal{F}_0(\Sigma_t) = W_0(\Sigma_t) + \int \overline{Q}_{\mathbf{a},q}\, d\mu_{\Sigma_t,0},$$

yields (66). Observe, that $\mathcal{F}_0(t) \to \infty$ as $t \to -1$. Furthermore,

$$\mathcal{F}_0'(t) = \frac{R(1+q)(1-t)}{2(1+t)(R^2-2Rt+1)}\left[1 + t - \frac{(R+1)^2}{2R(1+q)}\right].$$

If $-1 < t < 1$, then the sign of $\mathcal{F}_0'(t)$ is given by the sign of the linear function in the brackets, which is negative at $t = -1$. If $(R+1)^2 \geqslant 4R(1+q)$, then $\mathcal{F}_0'(t) < 0$ everywhere on $(-1, 1)$, and $\mathcal{F}_0(\Sigma_t)$ is strictly monotonically decreasing on $(-1, 1)$ and has a global minimum at $t = 1$. Otherwise, if $(R+1)^2 < 4R(1+q)$, then $\mathcal{F}_0'(t)$ has exactly one zero $t_0 := (R^2 - 2Rq + 1)/[2R(1+q)]$ on $(-1, 1)$, and is negative on $(-1, t_0)$ and positive on $(t_0, 1)$. Clearly, $\mathcal{F}_0(t)$ achieves global minimum on $(-1, 1]$ at t_0. This completes the proof. □

Lemma 41. *Let $d=2$ and $s=0$. The measure $\bar{\epsilon}_{t,0}=\mathrm{Bal}_0(\delta_{\mathbf{a}},\Sigma_t)$ is given by*

$$\mathrm{d}\bar{\epsilon}_{t,0}(\mathbf{x})=\frac{(R^2-1)^2}{(R^2-2Ru+1)^2}\,\mathrm{d}\sigma_2\bigg|_{\Sigma_t}(\mathbf{x})+\frac{1-t}{2}\frac{(R+1)^2}{R^2-2Rt+1}\,\mathrm{d}\delta_t(u)\,\mathrm{d}\sigma_1(\bar{\mathbf{x}})$$

and $\|\bar{\epsilon}_{t,0}\|=1$. The logarithmic potential of $\bar{\epsilon}_{t,0}$ is given by

$$U_0^{\bar{\epsilon}_{t,0}}(\mathbf{z})=U_0^{\delta_{\mathbf{a}}}(\mathbf{z})+\frac{1}{2}\log\frac{R^2-2Rt+1}{2(1+t)}+\frac{(R+1)^2}{8R}\log\frac{(R+1)^2}{R^2-2Rt+1},\quad \mathbf{z}\in\Sigma_t,$$
$$U_{d-2}^{\bar{\epsilon}_{t,0}}(\mathbf{z})=U_0^{\delta_{\mathbf{a}}}(\mathbf{z})+\frac{1}{2}\log\frac{R^2-2R\xi+1}{2(1+\xi)}+\frac{(R+1)^2}{8R}\log\frac{(R+1)^2}{R^2-2Rt+1},\quad \mathbf{z}\in\mathbb{S}^d\setminus\Sigma_t.$$

Proof. Let $\mathbf{z}\in\Sigma_t$. We write

$$U_0^{\bar{\epsilon}_{t,0}}(\mathbf{z})=\frac{\omega_1}{\omega_2}\left(\int_{-1}^{\xi}+\int_{\xi}^{t}\right)\frac{(R^2-1)^2}{(R^2-2Ru+1)^2}\left(\int_{\mathbb{S}^1}\log\frac{1}{|\mathbf{z}-\mathbf{x}|}\,\mathrm{d}\sigma_1(\bar{\mathbf{x}})\right)\mathrm{d}u+\frac{1-t}{2}\frac{(R+1)^2}{R^2-2Rt+1}\int_{\mathbb{S}^1}\log\frac{1}{|\mathbf{z}-\mathbf{x}|}\bigg|_{u=t}\,\mathrm{d}\sigma_1(\bar{\mathbf{x}}).$$

Using relation (63) we arrive at

$$U_0^{\bar{\epsilon}_{t,0}}(\mathbf{z})=-\frac{1}{2}\log\left(R^2-2R\xi+1\right)+C(R;t),$$

where

$$C(R,t):=\frac{1}{2}\log\frac{R^2-2Rt+1}{2(1+t)}+\frac{(R+1)^2}{8R}\log\frac{(R+1)^2}{R^2-2Rt+1}.$$

Let $\mathbf{z}\in\mathbb{S}^2\setminus\Sigma_t$. Then

$$U_0^{\bar{\epsilon}_{t,0}}(\mathbf{z})=\frac{\omega_1}{\omega_2}\int_{-1}^{t}\frac{(R^2-1)^2}{(R^2-2Ru+1)^2}\left(\int_{\mathbb{S}^1}\log\frac{1}{|\mathbf{z}-\mathbf{x}|}\,\mathrm{d}\sigma_1(\bar{\mathbf{x}})\right)\mathrm{d}u+\frac{1-t}{2}\frac{(R+1)^2}{R^2-2Rt+1}\int_{\mathbb{S}^1}\log\frac{1}{|\mathbf{z}-\mathbf{x}|}\bigg|_{u=t}\,\mathrm{d}\sigma_1(\bar{\mathbf{x}}).$$

Using relation (63) and evaluating the integral one gets after some simplifications

$$U_0^{\bar{\epsilon}_{t,0}}(\mathbf{z})=-\frac{1}{2}\log\left[2(1+\xi)\right]+\frac{(R+1)^2}{8R}\log\frac{(R+1)^2}{R^2-2Rt+1},$$

which yields the representation outside of Σ_t. For $\mathbf{z}\in\mathbb{S}^2\setminus\Sigma_t$

$$U_0^{\bar{\epsilon}_{t,0}}(\mathbf{z})=U_0^{\delta_{\mathbf{a}}}(\mathbf{z})+C(R,t)+\frac{1}{2}\log\left[\frac{R^2-2R\xi+1}{R^2-2Rt+1}\frac{1+t}{1+\xi}\right]<U_0^{\delta_{\mathbf{a}}}(\mathbf{z})+C(R,t),$$

since the logarithmic term is negative for $\xi>t$. Hence, $\bar{\epsilon}_{t,0}$ has the properties of a logarithmic balayage measure. Finally, it can be easily verified that $\|\bar{\epsilon}_{t,0}\|=1$ (for details cf. [2]). This completes the proof. □

Proof of Theorem 17. Lemmas 39 and 41 imply that $\bar{\eta}_{t,0}=(1+q)\bar{\nu}_{t,0}-q\bar{\epsilon}_{t,0}$ is, indeed, the logarithmic signed equilibrium on Σ_t associated with $\overline{Q}_{\mathbf{a},q}$ as can be seen from its weighted logarithmic potential given in the theorem. Using $r=\sqrt{R^2-2Rt+1}$ and $\rho=\sqrt{R^2-2Ru+1}$, we can write

$$\mathrm{d}\bar{\eta}_{t,0}(\mathbf{x})=\left[1+q-\frac{q(R^2-1)^2}{\rho^4}\right]\mathrm{d}\sigma_2\bigg|_{\Sigma_t}(\mathbf{x})+\frac{1-t}{2}\left[1+q-\frac{q(R+1)^2}{r^2}\right]\mathrm{d}\beta_t(\mathbf{x}),$$

where $\mathbf{x}\in\Sigma_t$. If $\bar{\eta}_{t,0}\geqslant 0$, then $1+q-q(R+1)^2/(R^2-2Rt+1)\geqslant 0$, so $t\leqslant t_0$. On the other hand, it is easy to see that if $t=t_0$, then $\bar{\eta}_{t_0,0}$ given in (28) is $\geqslant 0$ because $\rho\leqslant\rho$ and $(R-1)^2<R^2-2Ru+1$. Therefore, we have that $t_0=\max\{t\colon\ \bar{\eta}_{t_0,0}\geqslant 0\}$, $\mu_{\overline{Q}_{\mathbf{a},q}}=\bar{\eta}_{t_0,0}$, and $\mathrm{supp}(\mu_{\overline{Q}_{\mathbf{a},q}})=\Sigma_{t_0}$. □

7. Axis-supported Riesz external fields

In this section we shall prove Theorems 21 and 22.

Proof of Theorem 21. Direct calculation shows that

$$U_s^{\bar{\eta}_\lambda}(\mathbf{z})=\frac{\mathcal{F}_s(\mathbb{S}^d)}{W_s(\mathbb{S}^d)}U_s^{\sigma_d}(\mathbf{z})-\int\left(\int_{\mathbb{S}^d}\frac{(R^2-1)^{d-s}\,\mathrm{d}\sigma_d(\mathbf{x})}{|\mathbf{z}-\mathbf{x}|^s|\mathbf{x}-\mathbf{a}|^{2d-s}}\right)\mathrm{d}\lambda(R)=\frac{\mathcal{F}_s(\mathbb{S}^d)}{W_s(\mathbb{S}^d)}W_s\left(\mathbb{S}^d\right)-\int\frac{\mathrm{d}\lambda(R)}{|\mathbf{z}-R\mathbf{p}|^s}=\mathcal{F}_s\left(\mathbb{S}^d\right)-Q(\mathbf{z}),$$

where we used the Kelvin transformation for points (cf. proof of Theorem 2).

The second part follows from the uniqueness of the s-extremal measure on $\mathbb{S}^d$ associated with Q (Lemma 23 and, in particular, [18,19]) and the fact that the density is minimal at the North Pole. □

Proof of Theorem 22. By construction $\tilde{\eta}_t$ is of total charge one. From

$$U_s^{\tilde{\epsilon}_t}(\mathbf{x}) = \int U_s^{\mathrm{Bal}_s(\delta_{R\mathbf{p}},\Sigma_t)}(\mathbf{x})\,\mathrm{d}\lambda(R) = \int |\mathbf{x} - R\mathbf{p}|^{-s}\,\mathrm{d}\lambda(R) = Q(\mathbf{x}), \quad \mathbf{x} \in \Sigma_t,$$

and $U_s^{\nu_t}(\mathbf{z}) = W_s(\mathbb{S}^d)$ on Σ_t, we get $U_s^{\tilde{\eta}_t}(\mathbf{z}) = \tilde{\Phi}_s(t) = \mathcal{F}_s(\Sigma_t)$ on Σ_t (Remark 8).

By definition of ν_t, $\tilde{\epsilon}_t$, and $\mathrm{Bal}_s(\delta_{R\mathbf{p}}, \Sigma_t) = \varepsilon_{t,R}$ we can write

$$\tilde{\eta}_t = \frac{\tilde{\Phi}_s(t)}{W_s(\mathbb{S}^d)}\frac{1}{\|\lambda\|}\int \nu_t\,\mathrm{d}\lambda(R) - \int \epsilon_{t,R}\,\mathrm{d}\lambda(R) = \int \left[\frac{\tilde{\Phi}_s(t)}{W_s(\mathbb{S}^d)}\frac{1}{\|\lambda\|}\nu_t - \epsilon_{t,R}\right]\mathrm{d}\lambda(R),$$

where subscript R indicates the dependence on the parameter R. Thus

$$\mathrm{d}\tilde{\eta}_t(\mathbf{x}) = \left[\int \tilde{\eta}_t''(u,R)\,\mathrm{d}\lambda(R)\right]\frac{\omega_{d-1}}{\omega_d}\left(1-u^2\right)^{d/2-1}\mathrm{d}u\,\mathrm{d}\sigma_{d-1}(\bar{\mathbf{x}}), \quad \mathbf{x} \in \Sigma_t,$$

where, when using Lemmas 24 and 25 and letting $y = (t-u)/(1-u)$, we have

$$\tilde{\eta}_t''(u,R) = \frac{1}{W_s(\mathbb{S}^d)}\frac{1}{\|\lambda\|}\frac{\Gamma(d/2)}{\Gamma(d-s/2)}\left(\frac{1-t}{1-u}\right)^{d/2}\left(\frac{t-u}{1-t}\right)^{(s-d)/2} \times \left\{\tilde{\Phi}_s(t)\,{}_2\tilde{\mathrm{F}}_1\left(\begin{matrix}1, d/2\\ 1-(d-s)/2\end{matrix}; y\right) - \frac{\|\lambda\|(R+1)^{d-s}}{r^d}\,{}_2\tilde{\mathrm{F}}_1\left(\begin{matrix}1, d/2\\ 1-(d-s)/2\end{matrix}; \frac{(R-1)^2}{r^2}y\right)\right\}.$$

We claim that the density (the integral in square brackets) is either positive for all $u \in [-1,t]$, or is positive on some interval $[-1,t_c)$ and negative on $(t_c,t]$. It suffices to consider the function $h(u)$ obtained by integrating the above expression in braces against $\mathrm{d}\lambda(R)$. Using series expansions we get

$$h(u) = \sum_{k=0}^{\infty}\frac{(d/2)_k y^k}{\Gamma(k+1-(d-s)/2)}\left\{\int\left[\tilde{\Phi}_s(t) - \frac{\|\lambda\|(R+1)^{d-s}}{r^d}\left(\frac{R-1}{r}\right)^{2k}\right]\mathrm{d}\lambda(R)\right\}.$$

The coefficients in braces form an increasing sequence with positive limit as $k \to \infty$. Hence, either all coefficients are positive, or the first n are negative and then all others are positive. So, for $y \in A_t := [0, (1+t)/2]$ we obtain

$$g(y) = \sum_{k=0}^{\infty}\frac{a_k}{k!}y^k, \quad a_k < 0 \text{ for } k < n \text{ and } a_k \geqslant 0 \text{ for } k \geqslant n.$$

We have that $g^{(n)}(y) > 0$ on A_t, so $g^{(n-1)}(y)$ is strictly increasing on A_t. Since $g^{(n-1)}(0) = a_{n-1} < 0$, there is a γ_{n-1} in A_t such that $g^{(n-1)}(y)$ is negative on $[0,\gamma_{n-1})$ and positive on $(\gamma_{n-1},(1+t)/2]$. Indeed, if such a γ_{n-1} does not exist, we get a contradiction, because $g^{(n-1)}(y)$ will be negative on A_t, which would imply that $g^{(n-2)}(y)$ is decreasing and negative on A_t, and so on. This argument yields $g(y) < 0$ on A_t, which is impossible because the total charge of $\tilde{\eta}_t$ is one.

By iteration one can show a sequence $\gamma_0 > \gamma_1 > \cdots > \gamma_{n-1}$ such that $g^{(m)}(y)$ is negative on $[0,\gamma_m)$ and positive on $(\gamma_m,(1+t)/2]$ for every $m = 0,1,\dots,n-1$. This establishes our claim ($t_c = \gamma_0$).

We now can complete the proof of the theorem as follows. If $\tilde{\eta}_1$ is not a positive measure, then there is a t_1 such that the density of $\tilde{\eta}_1$ is positive on $[-1,t_1)$ and negative on $(t_1,1]$. Then the signed equilibrium for Σ_{t_1} is given by

$$\tilde{\eta}_{t_1} = \tilde{\eta}_1^+ - \mathrm{Bal}_s\left(\tilde{\eta}_1^-,\Sigma_{t_1}\right) - \left(\left\|\tilde{\eta}_1^-\right\| - \left\|\mathrm{Bal}_s\left(\tilde{\eta}_1^-,\Sigma_{t_1}\right)\right\|\right)\nu_{t_1}/\|\nu_{t_1}\|.$$

If it is still not a positive measure, then there exists a t_2 such that $\tilde{\eta}_{t_1}$ has positive density on $[-1,t_2)$ and negative one on $(t_2,t_1]$. Continuing the argument we derive a decreasing sequence $\{t_k\}$ with the property that $\tilde{\eta}_{t_k}$ is positive on $[-1,t_{k+1})$ and negative on $(t_{k+1},t_k]$. The limit of this sequence is the number t_λ defined in Theorem 22. Thus, $t_\lambda = \max\{t\colon\ \tilde{\eta}_t \geqslant 0\}$, $\mu_Q = \tilde{\eta}_{t_\lambda}$, and $\mathrm{supp}(\mu_Q) = \Sigma_{t_\lambda}$.

The Mhaskar–Saff functional $\mathcal{F}_s$ is minimized for Σ_{t_λ}. Since $\mathcal{F}_s(\Sigma_t) = \tilde{\Phi}_s(t)$ (cf. Remark 8 and beginning of this proof), we will show similar as in the proof of Theorem 13 above that t_λ is, in fact, the unique solution in $(-1,1]$ of the relation

$$\Delta(t) := \tilde{\Phi}_s(t) - \int (R+1)^{d-s}\left(R^2 - 2Rt + 1\right)^{-d/2}\mathrm{d}\lambda(R) = 0, \tag{67}$$

or $t_\lambda = 1$ when such a solution does not exist.

Using Quotient Rule and $\|\tilde{\varepsilon}_t\|' = \mathrm{d}\|\tilde{\varepsilon}_t\|/\mathrm{d}t = \int \|\varepsilon_{t,R}\|'\,\mathrm{d}\lambda(R)$, we obtain

$$\tilde{\Phi}_s'(t) = -\|\nu_t\|'/\|\nu_t\|\Delta(t).$$

Observe that $\Delta(t) \to \infty$ as $t \to -1^+$. Hence, by the above relation, $\tilde{\Phi}_s(t)$ is strictly monotonically decreasing on $(-1, t')$ for some maximal $t' \in (-1, 1]$ (cf. (47)). If $t' = 1$, then $t_\lambda = 1$. Otherwise, $t' < 1$ and $\tilde{\Phi}_s(t') = 0$ meaning that t' is a solution of (67). Arguing as in the proof of Theorem 13 we have that every solution $t_0 \in (-1, 1)$ of (67) is actually a local minimum of $\tilde{\Phi}_s(t)$ because of $\tilde{\Phi}_s''(t_0) > 0$. We conclude that $\tilde{\Phi}_s(t)$ can have at most one minimum in $(-1, 1)$. Consequently $t_\lambda = t'$. We also infer that $\Delta(t) > 0$ on $(-1, t_\lambda)$ and $\Delta(t) < 0$ on $(t_\lambda, 1]$. □

Acknowledgments

The authors are grateful to Robert Scherrer for pointing out connections with elementary electromagnetic theory.

An important part of this research was done at the Mathematisches Forschungsinstitut Oberwolfach during the first author's stay within the Oberwolfach Leibniz Fellows Programme (OWLF) from September 28 to December 18, 2008. The first author is in particular grateful that the OWLF and the Leibniz-Gemeinschaft supporting this program made it possible to invite his collaborators to visit MFO. We would like to thank the MFO for the excellent working conditions.

References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover, 1970.
[2] J.S. Brauchart, P.D. Dragnev, E.B. Saff, Minimal Riesz energy on the sphere for axis-supported external fields, arXiv:0902.1558v1 [math-ph].
[3] P.D. Dragnev, On the separation of logarithmic points on the sphere, in: C.K. Chui, L.L. Schumaker, J. Stöckler (Eds.), Approximation Theory X: Abstract and Classical Analysis, 2002, pp. 137–144.
[4] P.D. Dragnev, On an energy problem with Riesz external field, Oberwolfach Rep. 4 (2) (2007) 1042–1044.
[5] P.D. Dragnev, E.B. Saff, Riesz spherical potentials with external fields and minimal energy points separation, Potential Anal. 26 (2007) 139–162.
[6] A. Erdélyi, et al., Higher Transcendental Functions, vol. 1, McGraw–Hill, New York, 1953.
[7] V.I. Fabrikant, et al., On the generalized potential problem for a surface of revolution, Proc. Amer. Math. Soc. 90 (1) (1984) 47–56.
[8] M. Goetz, On the Riesz energy of measures, J. Approx. Theory 122 (1) (2003) 62–78.
[9] J.D. Jackson, Classical Electrodynamics, 3rd ed., Wiley, New York, 1999.
[10] O.D. Kellogg, Foundations of Potential Theory, Grundlehren Math. Wiss., vol. 31, Springer-Verlag, Berlin, 1967.
[11] A.B.J. Kuijlaars, E.B. Saff, Asymptotics for minimal discrete energy on the sphere, Trans. Amer. Math. Soc. 350 (2) (1998) 523–538.
[12] A.B.J. Kuijlaars, E.B. Saff, X. Sun, On a separation of minimal Riesz energy points on spheres in Euclidean spaces, J. Comput. Appl. Math. 199 (1) (2007) 172–180.
[13] N.S. Landkof, Foundations of Modern Potential Theory, Grundlehren Math. Wiss., Springer-Verlag, Berlin, 1972.
[14] M. Lachance, E.B. Saff, R.S. Varga, Inequalities for polynomials with a prescribed zero, Math. Z. 168 (2) (1979) 105–116.
[15] H.N. Mhaskar, E.B. Saff, Where does the sup norm of a weighted polynomial live? A generalization of incomplete polynomials, Constr. Approx. 1 (1) (1985) 71–91.
[16] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, Integrals and Series, vol. 1, Gordon & Breach Science Publishers, New York, 1988.
[17] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, Integrals and Series. More Special Functions, vol. 3, Gordon & Breach Science Publishers, New York, 1988.
[18] M. Riesz, Intégrales de Riemann–Liouville et potentiels, Acta Lit. Sci. Szeged 9 (1938) 1–42.
[19] M. Riesz, Rectification au travail "Intégrales de Riemann–Liouville et potentiels", Acta Lit. Sci. Szeged 9 (1939) 116–118.
[20] E.B. Saff, V. Totik, Logarithmic Potentials with External Fields, Springer-Verlag, New York, 1997.
[21] P. Simeonov, A weighted energy problem for a class of admissible weights, Houston J. Math. 31 (4) (2005) 1245–1260.
[22] N.V. Zorii, Equilibrium potentials with external fields, Ukrain. Mat. Zh. 55 (9) (2003) 1423–1444.
[23] N.V. Zorii, Equilibrium problems for potentials with external fields, Ukrain. Mat. Zh. 55 (10) (2003) 1588–1618.
[24] N.V. Zorii, Theory of potential with respect to consistent kernels: Theorem on completeness and sequences of potentials, Ukrain. Mat. Zh. 56 (11) (2004) 1796–1812.