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q-Difference equation and the Cauchy operator identities

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ABSTRACT

In this paper, we verify the Cauchy operator identities by a new method. And by using the Cauchy operator identities, we obtain a generating function for Rogers–Szegö polynomials. Applying the technique of parameter augmentation to two multiple generalizations of q-Chu–Vandermonde summation theorem given by Milne, we also obtain two multiple generalizations of the Kalnins–Miller transformation.

1. Introduction

In an attempt to find efficient q-shift operators to deal with basic hypergeometric series identities in the framework of the q-umbral calculus [1,2,10], Chen and Liu [7,8] introduced two q-exponential operators, Fang [9] introduced a new q-exponential operator, Chen and Gu [6] introduced a Cauchy operator for deriving identities from their special cases. In this paper, motivated by their work, we study some applications of the Cauchy operator for basic hypergeometric series.

Following [5] we will define the q-shifted factorial by

$$(a;q)_0=1, \qquad (a;q)_n=\prod_{k=0}^{n-1}\left(1-aq^k\right), \qquad (a;q)_\infty=\prod_{k=0}^{\infty}\left(1-aq^k\right),$$

where a is a complex variable. And for convenience, we always assume $0<q<1$ throughout the paper.

For a complex number α, we define

$$(a;q)_\alpha=(a;q)_\infty/\left(aq^\alpha;q\right)_\infty. \tag{1.1}$$

We also adopt the following compact notation

$$(a_1,a_2,\ldots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\ldots(a_m;q)_n, \quad n=0,1,2,\ldots,\infty.$$

In this paper, we will frequently use the following property

$$\left(aq^{1-n}/c;q\right)_\infty=(-a/c)^n q^{\binom{-n}{2}}(c/a;q)_n(aq/c;q)_\infty, \quad n=0,1,2,\ldots,\infty. \tag{1.2}$$

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The q-binomial coefficient and the q-binomial theorem are given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x| < 1, \tag{1.3}$$

respectively.

Recall that the q-difference operator is defined by

$$D_q\{f(a)\} = \frac{f(a) - f(aq)}{a} \tag{1.4}$$

and the Leibniz rule for D_q is referred to the following identity

$$D_q^n\{f(a)g(a)\} = \sum_{k=0}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} D_q^k\{f(a)\} D_q^{n-k}\{g(aq^k)\}. \tag{1.5}$$

The following relations are easily verified.

Proposition 1.1. *Let k be a nonnegative integer. Then we have*

$$D_q^k\left\{\frac{1}{(at;q)_\infty}\right\} = \frac{t^k}{(at;q)_\infty},$$
$$D_q^k\{(at;q)_\infty\} = (-t)^k q^{\binom{n}{2}} (atq^k;q)_\infty,$$
$$D_q^k\left\{\frac{(av;q)_\infty}{(at;q)_\infty}\right\} = t^k (v/t;q)_k \frac{(avq^k;q)_\infty}{(at;q)_\infty}.$$

We recall that Chen and Gu [6] introduced the Cauchy operator

$$\mathbb{T}(a,b;D_q) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} (bD_q)^n, \tag{1.6}$$

as the basis of parameter augmentation which serves as a method for proving extensions of the Askey–Wilson integral, the Askey–Roy integral and so on.

Liu [12] established two general q-exponential operator identities by solving two simple q-difference equations. Zhu [15] established the following q-exponential operator identity by solving a simple q-difference equation.

Proposition 1.2. *Let $f(a,b,c)$ be a three variables analytic function in a neighborhood of $(a,b,c) = (0,0,0) \in \mathcal{C}^3$, satisfying the q-difference equation*

$$(c-b)f(a,b,c) = abf(a,bq,cq) - bf(a,b,cq) + (c-ab)f(a,bq,c). \tag{1.7}$$

Then we have

$$f(a,b,c) = \mathbb{T}(a,b;D_q)\{f(a,0,c)\}. \tag{1.8}$$

Proof. We write (1.7) in the form

$$c\{f(a,b,c) - f(a,bq,c)\} = b\{f(a,b,c) - f(a,b,cq) - af(a,bq,c) + af(a,bq,cq)\}. \tag{1.9}$$

Now we begin to solve this q-difference equation. From the theory of several complex variables (see, for example, [14]), we may assume that

$$f(a,b,c) = \sum_{n=0}^{\infty} A_n(a,c) b^n \tag{1.10}$$

and then substitute the above equation into (1.9) to obtain

$$c\sum_{n=0}^{\infty}(1-q^n)A_n(a,c)b^n = \sum_{n=0}^{\infty}\{A_n(a,c) - A_n(a,cq) - aq^nA_n(a,c) + aq^nA_n(a,cq)\}b^{n+1}.$$

Equating coefficients of b^n, we readily find that, for each integer $n \geqslant 1$,

$$A_n(a,c)=\frac{1-aq^{n-1}}{1-q^n}D_{q,c}\{A_{n-1}(a,c)\}.$$

By iteration, we easily deduce that

$$A_n(a,c)=\frac{(a;q)_n}{(q;q)_n}D_{q,c}^n\{A_0(a,c)\}. \tag{1.11}$$

It remains to calculate $A_0(a,c)$. Putting $b=0$ in (1.10), we immediately deduce that $A_0(a,c)=f(a,0,c)$. Substituting (1.11) back into (1.10), we find that

$$f(a,b,c)=\sum_{n=0}^{\infty}\frac{(a;q)_n(bD_q)^n}{(q;q)_n}\{f(a,0,c)\}=\mathbb{T}(a,b;D_q)\{f(a,0,c)\},$$

which completes the proof of proposition. □

If we take $a=0$ and then substitute c with a in Proposition 1.2, it reduces to Theorem 1 of [12]. Proposition 1.2 tell us that if a analytic function $f(a,b,c)$ in three variables a, b and c satisfies q-difference equation (1.7), then we can recover $f(a,b,c)$ from its special case $f(a,0,c)$. To get $f(a,b,c)$ we should use the Cauchy operator $\mathbb{T}(a,b;D_q)$ to act on $f(a,0,c)$.

In Section 2, we verify four operator identities.

In Section 3, we use the operator identities to obtain a generating function for Rogers–Szegö polynomials for $h_n(x,y|q)$. And it can be stated in the equivalent forms in terms of the continuous big q-Hermite polynomial.

In Section 4, applying the technique of parameter augmentation to two multiple generalizations of q-Chu–Vandermonde summation theorem given by Milne, we obtain two multiple generalizations of the Kalnins–Miller transformation which extend the results of Zhang [16].

2. Cauchy operator identities

In fact, Proposition 1.2 contain the following two operator identities as special cases.

Theorem 2.1. *We have*

$$\mathbb{T}(a,b;D_q)\left\{\frac{1}{(ct;q)_\infty}\right\}=\frac{(abt;q)_\infty}{(bt,ct;q)_\infty}, \tag{2.1}$$

provided $|bt|<1$.

$$\mathbb{T}(a,b;D_q)\left\{\frac{1}{(cs,ct;q)_\infty}\right\}=\frac{(abt;q)_\infty}{(bt,cs,ct;q)_\infty}{}_2\phi_1\left(\begin{matrix}a,ct\\abt\end{matrix};q,bs\right), \tag{2.2}$$

provided $\max\{|bs|,|bt|\}<1$.

Proof. We first prove (2.1). Using the identity, $(x;q)_\infty=(1-x)(xq;q)_\infty$, by direct calculation, we find that

$$f(a,b,c):=\frac{(abt;q)_\infty}{(bt,ct;q)_\infty}$$

satisfies the functional equation

$$(c-b)f(a,b,c)=abf(a,bq,cq)-bf(a,b,cq)+(c-ab)f(a,bq,c).$$

And the identity (1.8) becomes

$$\frac{(abt;q)_\infty}{(bt,ct;q)_\infty}=\mathbb{T}(a,b;D_q)\left\{\frac{1}{(ct;q)_\infty}\right\}$$

which is (2.1). Similarly we can verify that

$$f(a,b,c):=\frac{(abt;q)_\infty}{(bt,cs,ct;q)_\infty}{}_2\phi_1\left(\begin{matrix}a,ct\\abt\end{matrix};q,bs\right)$$

satisfies the functional equation

$$(c-b)f(a,b,c)=abf(a,bq,cq)-bf(a,b,cq)+(c-ab)f(a,bq,c).$$

And the identity (1.8) becomes

$$\mathbb{T}(a,b;D_q)\left\{\frac{1}{(cs,ct;q)_\infty}\right\}=\frac{(abt;q)_\infty}{(bt,cs,ct;q)_\infty}{}_2\phi_1\left(\begin{matrix}a,ct\\abt\end{matrix};q,bs\right)$$

which is (2.2). □

We can verify the following operator identity by using (2.1) directly.

Theorem 2.2. *We have*

$$\mathbb{T}(a,b;D_q)\left\{\frac{(cv;q)_\infty}{(ct;q)_\infty}\right\}=\frac{(cv;q)_\infty}{(ct;q)_\infty}{}_2\phi_1\left(\begin{matrix}a,v/t\\cv\end{matrix};q,bt\right), \tag{2.3}$$

provided $|bt|<1$.

Proof. Recall the operator identity in (2.1), namely

$$\mathbb{T}(a,b;D_q)\left\{\frac{1}{(ct;q)_\infty}\right\}=\frac{(abt;q)_\infty}{(bt,ct;q)_\infty}. \tag{2.4}$$

We now introduce the following linear transform

$$L\{t^n\}=(v/t;q)_nt^n,\quad n=0,1,2,\dots,\infty.$$

By the q-binomial theorem, we find that

$$\begin{aligned}L\left\{\frac{1}{(ct;q)_\infty}\right\}&=\sum_{n=0}^\infty\frac{c^n}{(q;q)_n}L\{t^n\}\\&=\sum_{n=0}^\infty\frac{c^n}{(q;q)_n}(v/t;q)_nt^n\\&=\frac{(cv;q)_\infty}{(ct;q)_\infty}.\end{aligned}$$

Employing the same type argument as the above, we have

$$L\left\{\frac{(abt;q)_\infty}{(bt,ct;q)_\infty}\right\}=\frac{(cv;q)_\infty}{(ct;q)_\infty}\sum_{n=0}^\infty\frac{(a,v/t;q)_n}{(q,cv;q)_n}(bt)^n. \tag{2.5}$$

Applying the operator L to both sides of (2.4) and then use the above two equations, we conclude that

$$\mathbb{T}(a,b;D_q)\left\{\frac{(cv;q)_\infty}{(ct;q)_\infty}\right\}=\frac{(cv;q)_\infty}{(ct;q)_\infty}{}_2\phi_1\left(\begin{matrix}a,v/t\\cv\end{matrix};q,bt\right),$$

which is (2.3). Thus we complete the proof of theorem. □

By using (2.2), we can verify the following operator identity.

Theorem 2.3.

$$\mathbb{T}(a,b;D_q)\left\{\frac{(cv;q)_\infty}{(cs,ct;q)_\infty}\right\}=\frac{(abt,cv;q)_\infty}{(bt,ct,cs;q)_\infty}{}_3\phi_2\left(\begin{matrix}a,ct,v/s\\abt,cv\end{matrix};q,bs\right), \tag{2.6}$$

provided $\max\{|bs|,|bt|\}<1$.

Proof. Recall the operator identity in (2.2), namely

$$\mathbb{T}(a,b;D_q)\left\{\frac{1}{(cs,ct;q)_\infty}\right\}=\frac{(abt;q)_\infty}{(bt,cs,ct;q)_\infty}{}_2\phi_1\left(\begin{matrix}a,ct\\abt\end{matrix};q,bs\right).$$

It can be rewritten as

$$\mathbb{T}(a,b;D_q)\left\{\frac{1}{(cs,ct;q)_\infty}\right\}=\frac{(abt;q)_\infty}{(bt,ct;q)_\infty}\sum_{n=0}^\infty\sum_{k=0}^n\frac{c^{n-k}b^k(a,ct;q)_k}{(q;q)_{n-k}(q,abt;q)_k}s^n. \tag{2.7}$$

We now introduce the following linear transform

$$L\{s^n\} = (v/s;q)_n s^n, \quad n = 0, 1, 2, \ldots, \infty.$$

By the q-binomial theorem, we find that

$$\begin{aligned} L\left\{\frac{1}{(cs;q)_\infty}\right\} &= \sum_{n=0}^{\infty} \frac{c^n}{(q;q)_n} L\{s^n\} \\ &= \sum_{n=0}^{\infty} \frac{c^n}{(q;q)_n} (v/s;q)_n s^n \\ &= \frac{(cv;q)_\infty}{(cs;q)_\infty}. \end{aligned}$$

Applying the operator L to both sides of (2.7) and then use the above equation, we have

$$\begin{aligned} \mathbb{T}(a,b;D_q)\left\{\frac{(cv;q)_\infty}{(cs,ct;q)_\infty}\right\} &= \frac{(abt;q)_\infty}{(bt,ct;q)_\infty} \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{c^{n-k}b^k(a,ct;q)_k}{(q;q)_{n-k}(q,abt;q)_k}(v/s;q)_n s^n \\ &= \sum_{k=0}^{n} \frac{(a,ct,v/s;q)_k(bs)^k}{(q,abt;q)_k} \sum_{n=0}^{\infty} \frac{(vq^k/s;q)_{n-k}(cs)^{n-k}}{(q;q)_{n-k}} \\ &= \sum_{k=0}^{\infty} \frac{(a,ct,v/s;q)_k(bs)^k}{(q,abt;q)_k} \sum_{n=0}^{\infty} \frac{(vq^k/s;q)_n(cs)^n}{(q;q)_n} \\ &= \sum_{k=0}^{\infty} \frac{(a,ct,v/s;q)_k(bs)^k}{(q,abt;q)_k} \frac{(cvq^k;q)_\infty}{(cs;q)_\infty} \\ &= \frac{(abt,cv;q)_\infty}{(bt,ct,cs;q)_\infty} \sum_{k=0}^{\infty} \frac{(a,ct,v/s;q)_k(bs)^k}{(q,abt,cv;q)_k}, \end{aligned}$$

which is (2.6). Thus we complete the proof of theorem. □

3. The bivariate Rogers–Szegö

The bivariate Rogers–Szegö polynomials are introduced by Chen, Fu and Zhang [5], as defined by

$$h_n(x,y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(x,y). \tag{3.1}$$

Setting $y = 0$, the polynomials $h_n(x,y|q)$ reduce to the classical Rogers–Szegö polynomials $h_n(x|y)$ defined by

$$h_n(x|y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k. \tag{3.2}$$

The continuous big q-Hermite polynomials [11] are defined by

$$H_n(x,a|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \left(ae^{i\theta};q\right)_k e^{i(n-2k)\theta}, \quad x = \cos\theta.$$

We observe that the bivariate Rogers–Szegö polynomials $h_n(x,y|q)$ are equivalent to the continuous big q-Hermite polynomials owing to the following relation

$$H_n(x,a|q) = e^{in\theta} h_n\left(e^{-2i\theta}, ae^{-i\theta}\middle|q\right), \quad x = \cos\theta. \tag{3.3}$$

The polynomials $h_n(x,y|q)$ have the generating function

$$\sum_{n=0}^{\infty} h_n(x,y|q)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(t,xt;q)_\infty}, \quad |t|<1,\ |xt|<1. \tag{3.4}$$

A direct calculation shows that

$$D_q^k\{a^n\}=\begin{cases} a^{n-k}(q;q)_n/(q;q)_{n-k}, & 0\leqslant k\leqslant n;\\ 0, & k>n.\end{cases} \tag{3.5}$$

From the identity (3.5), we can easily establish the following lemma.

Lemma 3.1. *We have*

$$\mathbb{T}(a,b;D_q)\{c^n\}=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}(a;q)_k b^k c^{n-k}. \tag{3.6}$$

From (3.1) and (3.6), we can easily obtain

$$h_n(x,y|q)=\lim_{c\to 1}\mathbb{T}(y/x,x;D_q)\{c^n\}. \tag{3.7}$$

Carlitz [4] studied generating functions for Rogers–Szegö polynomials systematically and gave a formula

$$\sum_{n=0}^{\infty}h_{m+n}(a|q)h_n(b|q)\frac{z^n}{(q;q)_n}=\frac{(az;q)_m(abz^2;q)_\infty}{(abz^2;q)_m(z,az,bz,abz;q)_\infty}{}_2\phi_1\left(\begin{matrix} q^{-m},bz\\ q^{1-m}/(az)\end{matrix};q,\frac{q}{z}\right), \tag{3.8}$$

where $m\in \mathrm{N}$ and $\max\{|z|,|az|,|bz|,|abz|\}<1$.

Cao [3] used the q-exponential operator to prove (3.8). In this section, we will use the Cauchy operator to derive (3.8) for $h_n(x,y|q)$.

Theorem 3.1. *We have*

$$\sum_{n=0}^{\infty}h_{m+n}(x,y|q)h_n(u,v|q)\frac{z^n}{(q;q)_n}=\sum_{i=0}^{m}\begin{bmatrix} m\\ i\end{bmatrix}a^i(b/a;q)_i\frac{(buzq^i,vzq^i;q)_\infty}{(auz,uzq^i,zq^i;q)_\infty}{}_3\phi_2\left(\begin{matrix} bq^i/a,uzq^i,v\\ buzq^i,vzq^i\end{matrix};q,az\right), \tag{3.9}$$

where $\max\{|az|,|auz|\}<1$.

Proof. By Lemma 3.1, the left side of (3.9) can be written as

$$\begin{aligned}\sum_{n=0}^{\infty}\lim_{c\to 1}\mathbb{T}(b/a,a;D_q)\{c^{m+n}\}h_n(u,v|q)\frac{z^n}{(q;q)_n}&=\lim_{c\to 1}\mathbb{T}(b/a,a;D_q)\left\{c^m\sum_{n=0}^{\infty}h_n(u,v|q)\frac{(cz)^n}{(q;q)_n}\right\}\\ &=\lim_{c\to 1}\mathbb{T}(b/a,a;D_q)\left\{c^m\frac{(cvz;q)_\infty}{(cz,cuz;q)_\infty}\right\}.\end{aligned}$$

In view of (1.6) and (1.5), the above sum equals

$$\begin{aligned}&\lim_{c\to 1}\sum_{n=0}^{\infty}\frac{(b/a;q)_n}{(q;q)_n}a^nD_q^n\left\{c^m\frac{(cvz;q)_\infty}{(cz,cuz;q)_\infty}\right\}\\ &=\lim_{c\to 1}\sum_{n=0}^{\infty}\frac{(b/a;q)_n}{(q;q)_n}a^n\sum_{i=0}^{n}q^{i(i-n)}\begin{bmatrix} n\\ i\end{bmatrix}D_q^i\{c^m\}D_q^{n-i}\left\{\frac{(cq^ivz;q)_\infty}{(cq^iz,cq^iuz;q)_\infty}\right\}.\end{aligned}$$

In view of (3.5), the above sum equals

$$\begin{aligned}&\lim_{c\to 1}\sum_{n=0}^{\infty}\frac{(b/a;q)_n}{(q;q)_n}a^n\sum_{i=0}^{n}q^{i(i-n)}\begin{bmatrix} n\\ i\end{bmatrix}\frac{(q;q)_m}{(q;q)_{m-i}}c^{m-i}D_q^{n-i}\left\{\frac{(cq^ivz;q)_\infty}{(cq^iz,cq^iuz;q)_\infty}\right\}\\ &=\lim_{c\to 1}\sum_{i=0}^{n}\frac{(q;q)_ma^ic^{m-i}(b/a;q)_i}{(q;q)_i(q;q)_{m-i}}\sum_{n=0}^{\infty}\frac{(bq^i/a)_{n-i}}{(q;q)_{n-i}}q^{i(i-n)}a^{n-i}D_q^{n-i}\left\{\frac{(cq^ivz;q)_\infty}{(cq^iz,cq^iuz;q)_\infty}\right\}\\ &=\lim_{c\to 1}\sum_{i=0}^{m}\begin{bmatrix} m\\ i\end{bmatrix}c^{m-i}a^i(b/a;q)_i\sum_{n=0}^{\infty}\frac{(bq^i/a;q)_n}{(q;q)_n}q^{-in}a^nD_q^n\left\{\frac{(cq^ivz;q)_\infty}{(cq^iz,cq^iuz;q)_\infty}\right\}\\ &=\lim_{c\to 1}\sum_{i=0}^{m}\begin{bmatrix} m\\ i\end{bmatrix}c^{m-i}a^i(b/a;q)_i\sum_{n=0}^{\infty}\frac{(bq^i/a;q)_n}{(q;q)_n}\left(aq^{-i}D_q\right)^n\left\{\frac{(cq^ivz;q)_\infty}{(cq^iz,cq^iuz;q)_\infty}\right\}.\end{aligned}$$

In view of (1.6), the above sum equals

$$\lim_{c\to 1}\sum_{i=0}^{\infty}\begin{bmatrix} m \\ i \end{bmatrix} c^{m-i}a^i(b/a;q)_i\mathbb{T}\left(bq^i/a, aq^{-i}; D_q\right)\left\{\frac{(cq^ivz;q)_\infty}{(cq^iz, cq^iuz;q)_\infty}\right\}$$

$$= \lim_{c\to 1}\sum_{i=0}^{\infty}\begin{bmatrix} m \\ i \end{bmatrix} c^{m-i}a^i(b/a;q)_i\frac{(buzq^i, cvzq^i;q)_\infty}{(auz, cuzq^i, czq^i;q)_\infty}{}_3\phi_2\left(\begin{matrix} bq^i/a, cuzq^i, v \\ buzq^i, cvzq^i \end{matrix}; q, az\right)$$

$$= \sum_{i=0}^{\infty}\begin{bmatrix} m \\ i \end{bmatrix} a^i(b/a;q)_i\frac{(buzq^i, vzq^i;q)_\infty}{(auz, uzq^i, zq^i;q)_\infty}{}_3\phi_2\left(\begin{matrix} bq^i/a, uzq^i, v \\ buzq^i, vzq^i \end{matrix}; q, az\right),$$

where $\max\{|az|, |auz|\} < 1$. This complete the proof of theorem. □

Remark 3.1. Setting $b = 0$, $v = 0$ and $u = b$, (3.9) reduce to (3.8).

From the above theorem and (1.3), we get the following equivalent formula for $H_n(x, a|q)$.

Corollary 3.1. *We have*

$$\sum_{n=0}^{\infty}H_{m+n}(x, a|q)H_n(u, b|q)\frac{z^n}{(q;q)_n} = e^{im\theta}\sum_{j=0}^{m}\begin{bmatrix} m \\ j \end{bmatrix} a^j(b/a;q)_j\frac{(bze^{i(\theta-\beta)}q^j, bze^{i(\theta+2\beta)}q^j;q)_\infty}{(aze^{i(\theta-\beta)}, ze^{i(\theta-\beta)}q^j, ze^{i(\theta+\beta)}q^j;q)_\infty}$$
$$\times {}_3\phi_2\left(\begin{matrix} bq^j/a, ze^{i(\theta-\beta)}q^j, be^{i\beta} \\ bze^{i(\theta-\beta)}q^j, bze^{i(\theta+2\beta)}q^j \end{matrix}; q, aze^{i(\theta+\beta)}\right),$$

where $x = \cos\theta$, $u = \cos\beta$ *and* $\max\{|aze^{i(\theta-\beta)}|, |aze^{i(\theta+\beta)}q^j|\} < 1$.

4. The $U(n+1)$ generations of the Kalnins–Miller transformation

Proposition 4.1 *(The $U(n+1)$ generations of the q-Chu–Vandermonde summation theorem). (See [13, Theorem 5.10].) Let b, c and $x_1, \ldots, x_n$ be indeterminate, let N_i be nonnegative integers for $i = 1, 2, \ldots, n$ with $n \geqslant 1$. Suppose that none of the denominators in the following identity vanishes. Then*

$$\left\{b^{N_1+\cdots+N_n}\prod_{i=1}^{n}\frac{(\frac{x_i}{x_n}c/b;q)_{N_i}}{(\frac{x_i}{x_n}c;q)_{N_i}}\right\} = \sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}}\left\{\prod_{1\leqslant r<s\leqslant n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]\prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{(q\frac{x_r}{x_s};q)_{y_r}}\right]\right.$$
$$\left.\times\prod_{i=1}^{n}\left[\left(\frac{x_i}{x_n}c;q\right)_{y_i}^{-1}\right](b;q)_{y_1+\cdots+y_n}q^{y_1+2y_2+\cdots+ny_n}\right\}. \tag{4.1}$$

Proof. See [13]. □

Theorem 4.1 *(The $U(n+1)$ generalization of the fourth Kalnins–Miller transformation). Let b, c, x, y and $x_1, \ldots, x_n$ be indeterminate, let N_i be nonnegative integers for $i = 1, 2, \ldots, n$ with $n \geqslant 1$. Suppose that none of the denominators in the following identity vanishes, and that* $\max\{|dx|, |dy|, |dyq^{y_1+\cdots+y_n}|, |dxq^{y_1+\cdots+y_n}|\} < 1$. *Then*

$$\sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}}\left\{\prod_{1\leqslant r<s\leqslant n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]\prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{(q\frac{x_r}{x_s};q)_{y_r}}\right]\prod_{i=1}^{n}\left[\left(\frac{x_i}{x_n}cx;q\right)_{y_i}^{-1}\right]\right.$$
$$\left.\times\frac{(bx, dx;q)_{y_1+\cdots+y_n}}{(adx;q)_{y_1+\cdots+y_n}}{}_2\phi_1\left(\begin{matrix} a, bxq^{y_1+\cdots+y_n} \\ adxq^{y_1+\cdots+y_n} \end{matrix}; q, dy\right)q^{y_1+2y_2+\cdots+ny_n}\right\}$$
$$= \frac{(dx, ady;q)_\infty}{(dy, adx;q)_\infty}\left(\frac{x}{y}\right)^{N_1+\cdots+N_n}\prod_{i=1}^{n}\frac{(\frac{x_i}{x_n}cy;q)_{N_i}}{(\frac{x_i}{x_n}cx;q)_{N_i}}\sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}}\left\{\prod_{1\leqslant r<s\leqslant n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]\right.$$

$$\times \prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{(q\frac{x_r}{x_s};q)_{y_r}}\right]\prod_{i=1}^{n}\left[\left(\frac{x_i}{x_n}cy;q\right)_{y_i}^{-1}\right]$$

$$\times \frac{(by,dy;q)_{y_1+\cdots+y_n}}{(ady;q)_{y_1+\cdots+y_n}}{}_2\phi_1\left(\begin{matrix}a, byq^{y_1+\cdots+y_n}\\ adyq^{y_1+\cdots+y_n}\end{matrix};q,dx\right)q^{y_1+2y_2+\cdots+ny_n}\Bigg\}. \tag{4.2}$$

Proof. Replacing (b,c) by (bx,cx) and (by,cy), respectively, in Proposition 4.1, we have

$$\left\{(bx)^{N_1+\cdots+N_n}\prod_{i=1}^{n}\frac{(\frac{x_i}{x_n}c/b;q)_{N_i}}{(\frac{x_i}{x_n}cx;q)_{N_i}}\right\} = \sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}}\Bigg\{\prod_{1\leqslant r<s\leqslant n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]\prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{(q\frac{x_r}{x_s};q)_{y_r}}\right]$$

$$\times \prod_{i=1}^{n}\left[\left(\frac{x_i}{x_n}cx;q\right)_{y_i}^{-1}\right](bx;q)_{y_1+\cdots+y_n}q^{y_1+2y_2+\cdots+ny_n}\Bigg\} \tag{4.3}$$

and

$$\left\{(by)^{N_1+\cdots+N_n}\prod_{i=1}^{n}\frac{(\frac{x_i}{x_n}c/b;q)_{N_i}}{(\frac{x_i}{x_n}cy;q)_{N_i}}\right\} = \sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}}\Bigg\{\prod_{1\leqslant r<s\leqslant n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]\prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{(q\frac{x_r}{x_s};q)_{y_r}}\right]$$

$$\times \prod_{i=1}^{n}\left[\left(\frac{x_i}{x_n}cy;q\right)_{y_i}^{-1}\right](by;q)_{y_1+\cdots+y_n}q^{y_1+2y_2+\cdots+ny_n}\Bigg\}. \tag{4.4}$$

Comparing (4.3) and (4.4), we immediately obtain

$$\sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}}\Bigg\{\prod_{1\leqslant r<s\leqslant n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]\prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{(q\frac{x_r}{x_s};q)_{y_r}}\right]\prod_{i=1}^{n}\left[\left(\frac{x_i}{x_n}cx;q\right)_{y_i}^{-1}\right](bx;q)_{y_1+\cdots+y_n}q^{y_1+2y_2+\cdots+ny_n}\Bigg\}$$

$$=\left(\frac{x}{y}\right)^{N_1+\cdots+N_n}\prod_{i=1}^{n}\frac{(\frac{x_i}{x_n}cy;q)_{N_i}}{(\frac{x_i}{x_n}cx;q)_{N_i}}\sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}}\Bigg\{\prod_{1\leqslant r<s\leqslant n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]$$

$$\times \prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{(q\frac{x_r}{x_s};q)_{y_r}}\right]\prod_{i=1}^{n}\left[\left(\frac{x_i}{x_n}cy;q\right)_{y_i}^{-1}\right](by;q)_{y_1+\cdots+y_n}q^{y_1+2y_2+\cdots+ny_n}\Bigg\}. \tag{4.5}$$

We rewrite (4.5) as

$$\sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}}\Bigg\{\prod_{1\leqslant r<s\leqslant n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]\prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{(q\frac{x_r}{x_s};q)_{y_r}}\right]$$

$$\times \prod_{i=1}^{n}\left[\left(\frac{x_i}{x_n}cx;q\right)_{y_i}^{-1}\right]\frac{1}{(by,bxq^{y_1+\cdots+y_n};q)_\infty}q^{y_1+2y_2+\cdots+ny_n}\Bigg\}$$

$$=\left(\frac{x}{y}\right)^{N_1+\cdots+N_n}\prod_{i=1}^{n}\frac{(\frac{x_i}{x_n}cy;q)_{N_i}}{(\frac{x_i}{x_n}cx;q)_{N_i}}\sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}}\Bigg\{\prod_{1\leqslant r<s\leqslant n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]\prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{(q\frac{x_r}{x_s};q)_{y_r}}\right]\prod_{i=1}^{n}\left[\left(\frac{x_i}{x_n}cy;q\right)_{y_i}^{-1}\right]$$

$$\times \frac{1}{(bx,byq^{y_1+\cdots+y_n};q)_\infty}q^{y_1+2y_2+\cdots+ny_n}\Bigg\}. \tag{4.6}$$

Applying the operator $\mathbb{T}(a, d; D_q)$ with respect to the variable b to both sides of the equation and using (2.2), we get

$$\sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}} \Bigg\{ \prod_{1\leqslant r<s\leqslant n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]\prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{(q\frac{x_r}{x_s};q)_{y_r}}\right]$$
$$\times\prod_{i=1}^{n}\left[\left(\frac{x_i}{x_n}cx;q\right)_{y_i}^{-1}\right]\frac{(adxq^{y_1+\cdots+y_n};q)_\infty}{(dxq^{y_1+\cdots+y_n},by,bxq^{y_1+\cdots+y_n};q)_\infty}$$
$$\times{}_2\phi_1\left(\begin{matrix}a, bxq^{y_1+\cdots+y_n}\\ adxq^{y_1+\cdots+y_n}\end{matrix};q,dy\right)q^{y_1+2y_2+\cdots+ny_n}\Bigg\}$$
$$=\left(\frac{x}{y}\right)^{N_1+\cdots+N_n}\prod_{i=1}^{n}\frac{(\frac{x_i}{x_n}cy;q)_{N_i}}{(\frac{x_i}{x_n}cx;q)_{N_i}}\sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}}\Bigg\{\prod_{1\leqslant r<s\leqslant n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]$$
$$\times\prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{(q\frac{x_r}{x_s};q)_{y_r}}\right]\prod_{i=1}^{n}\left[\left(\frac{x_i}{x_n}cy;q\right)_{y_i}^{-1}\right]\frac{(adyq^{y_1+\cdots+y_n};q)_\infty}{(dyq^{y_1+\cdots+y_n},bx,byq^{y_1+\cdots+y_n};q)_\infty}$$
$$\times{}_2\phi_1\left(\begin{matrix}a, byq^{y_1+\cdots+y_n}\\ adyq^{y_1+\cdots+y_n}\end{matrix};q,dx\right)q^{y_1+2y_2+\cdots+ny_n}\Bigg\}.$$

We obtain the theorem after using (1.1). □

Remark 4.1. If we take $a = 0$ in Theorem 4.1, we get Theorem 3.2 of [16].

Proposition 4.2 *(The $U(n+1)$ generations of the q-Chu–Vandermonde summation theorem). (See [13, Theorem 5.26].) Let b, c and $x_1, \ldots, x_n$ be indeterminate, let N_i be nonnegative integers for $i = 1, 2, \ldots, n$ with $n \geqslant 1$. Suppose that none of the denominators in the following identity vanishes. Then*

$$\left\{\left[(c;q)_{N_1+\cdots+N_n}^{-1}\prod_{i=1}^{n}\left(\frac{x_i}{x_n}c/b;q\right)_{N_i}\right]\left[b^{N_1+\cdots+N_n}qe_2(N_1,\ldots,N_n)\prod_{i=1}^{n}\left(\frac{x_n}{x_i}\right)^{N_i}\right]\right\}$$
$$=\sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}}\Bigg\{\prod_{1\leqslant r<s\leqslant n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]\prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{(q\frac{x_r}{x_s};q)_{y_r}}\right]$$
$$\times\prod_{i=1}^{n}\left[\left(\frac{x_n}{x_i}bq^{y_1+\cdots+y_n-y_i};q\right)_{y_i}\right](c;q)_{y_1+\cdots+y_n}^{-1}q^{y_1+2y_2+\cdots+ny_n}\Bigg\}, \tag{4.7}$$

where $e_2(N_1, \ldots, N_n)$ is the second elementary symmetric function of $\{N_1, \ldots, N_n\}$.

Proof. See [13]. □

Theorem 4.2 *(The $U(n+1)$ generalization of the first Kalnins–Miller transformation). Let b, c, x, y and $x_1, \ldots, x_n$ be indeterminate, let N_i be nonnegative integers for $i = 1, 2, \ldots, n$ with $n \geqslant 1$. Suppose that none of the denominators in the following identity vanishes and that* $\max\{|dxq^{N_1+\cdots+N_n}|, |dyq^{N_1+\cdots+N_n}|, |dyq^{y_1+\cdots+y_n}|, |dxq^{y_1+\cdots+y_n}|\} < 1$. *Then*

$$\sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}}\Bigg\{\prod_{1\leqslant r<s\leqslant n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]\prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{(q\frac{x_r}{x_s};q)_{y_r}}\right]q^{y_1+2y_2+\cdots+ny_n}$$
$$\times\prod_{i=1}^{n}\left[\left(\frac{x_n}{x_i}bxq^{y_1+\cdots+y_n-y_i};q\right)_{y_i}\right]\frac{1}{(cx,ady;q)_{y_1+\cdots+y_n}}$$
$$\times{}_2\phi_1\left(\begin{matrix}a, cyq^{y_1+\cdots+y_n}\\ adyq^{y_1+\cdots+y_n}\end{matrix};q,dxq^{N_1+\cdots+N_n}\right)\Bigg\}$$

$$= \left(\frac{x}{y}\right)^{N_1+\cdots+N_n} \frac{(cy, dy; q)_{N_1+\cdots+N_n}}{(ady, cx; q)_{N_1+\cdots+N_n}} \sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}} \left\{ \prod_{1\leqslant r<s\leqslant n} \left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] \right.$$

$$\times \prod_{r,s=1}^{n} \left[\frac{(\frac{x_r}{x_s}q^{-N_s}; q)_{y_r}}{(q\frac{x_r}{x_s}; q)_{y_r}}\right] q^{y_1+2y_2+\cdots+ny_n} \prod_{i=1}^{n} \left[\left(\frac{x_n}{x_i} bq^{y_1+\cdots+y_n-y_i}; q\right)_{y_i}\right]$$

$$\left. \times \frac{1}{(cy, dy; q)_{y_1+\cdots+y_n}} {}_2\phi_1\left(\begin{matrix} a, cyq^{N_1+\cdots+N_n} \\ adyq^{N_1+\cdots+N_n}\end{matrix}; q, dxq^{y_1+\cdots+y_n}\right)\right\}.$$

Proof. Similar to the proof of Theorem 3.1. □

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