



The cardinality of certain $\mu_{M,D}$ -orthogonal exponentials

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ABSTRACT

The self-affine measures $\mu_{M,D}$ corresponding to the case (i) $M = pI_3$, $D = \{0, e_1, e_2, e_3\}$ in the space \mathbb{R}^3 and the case (ii) $M = pI_2$, $D = \{0, e_1, e_2, e_1 + e_2\}$ in the plane \mathbb{R}^2 are non-spectral, where $p > 1$ is odd, I_n is the $n \times n$ identity matrix, and e_1, \dots, e_n are the standard basis of unit column vectors in \mathbb{R}^n . One of the non-spectral problem on $\mu_{M,D}$ is to estimate the number of orthogonal exponentials in $L^2(\mu_{M,D})$ and to find them. In the present paper we show that, in both cases (i) and (ii), there are at most 4 mutually orthogonal exponentials in $L^2(\mu_{M,D})$ each, and the number 4 is the best.

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1. Introduction

Let $M \in M_n(\mathbb{Z})$ be an expanding integer matrix, that is, all the eigenvalues of the integer matrix M have modulus > 1 . Associated with a finite subset $D \subset \mathbb{Z}^n$, there exists a unique non-empty compact set $T := T(M, D)$ such that $MT = \bigcup_{d \in D} (T + d)$. More precisely, $T(M, D)$ is the attractor (or invariant set) of the iterated function system (IFS) $\{\phi_d(x) = M^{-1}(x + d)\}_{d \in D}$. Let $|D|$ be the cardinality of D . Relating to the IFS $\{\phi_d\}_{d \in D}$, there exists a unique probability measure $\mu := \mu_{M,D}$ satisfying the self-affine identity

$$\mu = \frac{1}{|D|} \sum_{d \in D} \mu \circ \phi_d^{-1}. \quad (1.1)$$

Such a measure $\mu_{M,D}$ is supported on $T(M, D)$ (cf. [4,7]), and is called a *self-affine measure*.

For a probability measure μ of compact support on \mathbb{R}^n , we call μ a *spectral measure* if there exists a discrete set $\Lambda \subset \mathbb{R}^n$ such that $E_\Lambda := \{e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda\}$ forms an orthogonal basis for $L^2(\mu)$. The set Λ is then called a *spectrum* for μ . Spectral measure is a natural generalization of spectral set introduced by Fuglede [5]. The spectrality or non-spectrality of a self-affine measure $\mu_{M,D}$ has been received much attention in recent years (see e.g., [1–3,6,9,11] and references cited therein). The non-spectral problem on the self-affine measure consists of the following two classes:

- (I) There are at most a finite number of orthogonal exponentials in $L^2(\mu_{M,D})$, that is, $\mu_{M,D}$ -orthogonal exponentials contain at most finite elements. The main questions here are to estimate the number of orthogonal exponentials in $L^2(\mu_{M,D})$ and to find them.
- (II) There are natural infinite families of orthogonal exponentials, but none of them forms an orthogonal basis in $L^2(\mu_{M,D})$. The main question is whether some of these families can be combined to form larger collections of orthogonal exponentials.

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Except the case that there might be no more than two orthogonal exponentials, the problem on a non-spectral measure $\mu_{M,D}$ in fact falls into one of the above two classes (see [2, Section 3]). Let $|\det(M)| = m = p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r}$ be the standard prime factorization, where $p_1 < p_2 < \cdots < p_r$ are prime numbers and $b_j > 0$. We denote by $W(m)$ the non-negative integer combination of p_1, p_2, \dots, p_r . For non-spectral measure $\mu_{M,D}$, it has been an interesting topic to examine the maximal cardinality of orthogonal exponentials. Under the condition $|D| \notin W(m)$, it could happen that there exists at most a finite number of orthogonal complex exponentials for a non-spectral measure. The first result of this kind is probably due to Jorgensen and Pedersen [8] in which they proved that for the Middle Third Cantor set with its Hausdorff measure of dimension $\ln 2 / \ln 3$, no three exponentials are mutually orthogonal. A more detailed analysis on this was given and many new examples were constructed in a recent paper by Dutkay and Jorgensen [2]. The known results in this direction provide some supportive evidence that the following conjecture should be true, although we cannot prove it.

Conjecture 1. For an expanding integer matrix $M \in M_n(\mathbb{Z})$ and a finite digit set $D \subset \mathbb{Z}^n$, if $|D| \notin W(m)$, then $\mu_{M,D}$ is a non-spectral measure and the non-spectral problem on this $\mu_{M,D}$ falls in the class (I).

More recently, the author [10–12] proved Conjecture 1 for a class of planar self-affine measures with three-elements digit set. Conjecture 1 is still open for the four-elements digit set, even in the following cases:

- (i) $M = pI_3$, $D = \{0, e_1, e_2, e_3\}$ in the space \mathbb{R}^3 ;
- (ii) $M = pI_2$, $D = \{0, e_1, e_2, e_1 + e_2\}$ in the plane \mathbb{R}^2 ,

where $p > 1$ is odd, I_n is the $n \times n$ identity matrix, and e_1, \dots, e_n are the standard basis of unit column vectors in \mathbb{R}^n . In the case (i), Dutkay and Jorgensen [2, Theorem 5.1(iii)] obtained that there are at most 256 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, Yuan [13] reduced the number 256 to 7. In the case (ii), Yuan [13] obtained that there are at most 5 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. In the present paper we show that for the above expanding integer matrix M and the above four-elements digit set D in (i) or (ii), there are at most 4 mutually orthogonal exponentials in $L^2(\mu_{M,D})$, and the number 4 is the best. In fact, we prove the following Theorems 1 and 2.

Theorem 1. Let $p \in 2\mathbb{Z} + 1$ with $|p| > 1$. For the self-affine measure $\mu_{M,D}$ corresponding to

$$M = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad (1.2)$$

there are at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number 4 is the best.

Theorem 2. Let $p_1, p_2 \in 2\mathbb{Z} + 1$ with $|p_1| > 1$ and $|p_2| > 1$. For the self-affine measure $\mu_{M,D}$ corresponding to

$$M = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \quad (1.3)$$

there are at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number 4 is the best.

These generalize the above mentioned results and solve the corresponding non-spectral problem (Conjecture 1) on self-affine measure $\mu_{M,D}$.

2. Proof of Theorem 1

If λ_j ($j = 1, 2, 3, 4, 5$) $\in \mathbb{R}^3$ are such that the exponential functions

$$e^{2\pi i \langle \lambda_1, x \rangle}, \quad e^{2\pi i \langle \lambda_2, x \rangle}, \quad e^{2\pi i \langle \lambda_3, x \rangle}, \quad e^{2\pi i \langle \lambda_4, x \rangle}, \quad e^{2\pi i \langle \lambda_5, x \rangle}$$

are mutually orthogonal in $L^2(\mu_{M,D})$, then the differences $\lambda_j - \lambda_k$ ($1 \leq j \neq k \leq 5$) are in the zero set $Z(\hat{\mu}_{M,D})$ of the Fourier transform $\hat{\mu}_{M,D}$ (see [10, p. 163], [11, p. 3140]). That is, we have

$$\lambda_j - \lambda_k \in Z(\hat{\mu}_{M,D}) \quad (1 \leq j \neq k \leq 5). \quad (2.1)$$

By characterizing the zero set $Z(\hat{\mu}_{M,D})$, we will deduce a contradiction below.

First, for the given digit set D in (1.2), we have

$$\Theta_0 := \{\xi \in \mathbb{R}^3 : m_D(\xi) = 0\} = A_1 \cup A_2 \cup A_3, \quad (2.2)$$

where

$$m_D(\xi) = \frac{1}{4} \{1 + e^{2\pi i \xi_1} + e^{2\pi i \xi_2} + e^{2\pi i \xi_3}\}, \quad \xi = (\xi_1, \xi_2, \xi_3)^t \in \mathbb{R}^3, \quad (2.3)$$

$$A_1 = \left\{ \begin{pmatrix} 1/2 + k_1 \\ a + k_2 \\ 1/2 + a + k_3 \end{pmatrix} : a \in \mathbb{R}, k_1, k_2, k_3 \in \mathbb{Z} \right\} \subset \mathbb{R}^3, \quad (2.4)$$

$$A_2 = \left\{ \begin{pmatrix} 1/2 + a + k_1 \\ 1/2 + k_2 \\ a + k_3 \end{pmatrix} : a \in \mathbb{R}, k_1, k_2, k_3 \in \mathbb{Z} \right\} \subset \mathbb{R}^3, \quad (2.5)$$

and

$$A_3 = \left\{ \begin{pmatrix} a + k_1 \\ 1/2 + a + k_2 \\ 1/2 + k_3 \end{pmatrix} : a \in \mathbb{R}, k_1, k_2, k_3 \in \mathbb{Z} \right\} \subset \mathbb{R}^3. \quad (2.6)$$

The set $A_1 \cup A_2 \cup A_3$ can be divided into several disjoint sets. Let

$$Z_1(a) := \left\{ \begin{pmatrix} 1/2 + k_1 \\ a + k_2 \\ 1/2 + a + k_3 \end{pmatrix} : k_1, k_2, k_3 \in \mathbb{Z} \right\} \subset \mathbb{R}^3, \quad (2.7)$$

$$Z_2(a) := \left\{ \begin{pmatrix} 1/2 + a + k_1 \\ 1/2 + k_2 \\ a + k_3 \end{pmatrix} : k_1, k_2, k_3 \in \mathbb{Z} \right\} \subset \mathbb{R}^3, \quad (2.8)$$

$$Z_3(a) := \left\{ \begin{pmatrix} a + k_1 \\ 1/2 + a + k_2 \\ 1/2 + k_3 \end{pmatrix} : k_1, k_2, k_3 \in \mathbb{Z} \right\} \subset \mathbb{R}^3, \quad (2.9)$$

and

$$B = \mathbb{R} \setminus (\mathbb{Z} \cup (\mathbb{Z} + 1/2)). \quad (2.10)$$

Then, for $j = 1, 2, 3$, we have

$$A_j = \bigcup_{a \in \mathbb{R}} Z_j(a) = \left(\bigcup_{a \in \mathbb{Z}} Z_j(a) \right) \cup \left(\bigcup_{a \in \mathbb{Z} + 1/2} Z_j(a) \right) \cup \left(\bigcup_{a \in B} Z_j(a) \right). \quad (2.11)$$

It follows from (2.7), (2.8) and (2.9) that

$$\bigcup_{a \in \mathbb{Z}} Z_1(a) = \bigcup_{a \in \mathbb{Z} + 1/2} Z_3(a); \quad \bigcup_{a \in \mathbb{Z}} Z_2(a) = \bigcup_{a \in \mathbb{Z} + 1/2} Z_1(a); \quad \bigcup_{a \in \mathbb{Z}} Z_3(a) = \bigcup_{a \in \mathbb{Z} + 1/2} Z_2(a). \quad (2.12)$$

We use the symbols B_j and \tilde{B}_j ($j = 1, 2, 3$) to denote the following sets:

$$B_j = \bigcup_{a \in \mathbb{Z}} Z_j(a) \quad \text{and} \quad \tilde{B}_j = \bigcup_{a \in B} Z_j(a) \quad (j = 1, 2, 3). \quad (2.13)$$

From (2.2), (2.11), (2.12) and (2.13), we get the desired representation that the zero set Θ_0 is given by

$$\Theta_0 = B_1 \cup B_2 \cup B_3 \cup \tilde{B}_1 \cup \tilde{B}_2 \cup \tilde{B}_3, \quad (2.14)$$

where

$$B_1, B_2, B_3, \tilde{B}_1, \tilde{B}_2, \tilde{B}_3 \text{ are mutually disjoint and } \bigcup_{j=1}^3 (B_j \cup \tilde{B}_j) \cap \mathbb{Z}^3 = \emptyset. \quad (2.15)$$

Secondly, for the matrix M given by (1.2), one can verify that

$$M(B_j) \subseteq B_j \quad (j = 1, 2, 3) \quad (2.16)$$

hold. On the other hand, we have the following inclusion relations:

$$M(\tilde{B}_1) \subseteq B_1 \cup \tilde{B}_1 \cup B_2; \quad M(\tilde{B}_2) \subseteq B_2 \cup \tilde{B}_2 \cup B_3; \quad M(\tilde{B}_3) \subseteq B_3 \cup \tilde{B}_3 \cup B_1. \quad (2.17)$$

In fact, for any given $(x, y, z)^t \in M(\tilde{B}_1)$, we see that there exist $\tilde{a} \in B$, $\tilde{k}_1, \tilde{k}_2, \tilde{k}_3 \in \mathbb{Z}$ such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p/2 + p\tilde{k}_1 \\ p\tilde{a} + p\tilde{k}_2 \\ p/2 + p\tilde{a} + p\tilde{k}_3 \end{pmatrix}. \quad (2.18)$$

If $p\tilde{a} \in \mathbb{Z}$, then $(x, y, z)^t \in B_1$, which shows $M(\tilde{B}_1) \subseteq B_1$. If $p\tilde{a} \notin \mathbb{Z}$ and $p\tilde{a} \notin \mathbb{Z} + 1/2$, then $(x, y, z)^t \in \tilde{B}_1$, which shows $M(\tilde{B}_1) \subseteq \tilde{B}_1$. If $p\tilde{a} \notin \mathbb{Z}$ and $p\tilde{a} \in \mathbb{Z} + 1/2$, then $(x, y, z)^t \in B_2$, which shows $M(\tilde{B}_1) \subseteq B_2$. Hence $M(\tilde{B}_1) \subseteq B_1 \cup \tilde{B}_1 \cup B_2$ holds. Similarly, the other two inclusion relations in (2.17) hold.

It follows from (2.14), (2.16) and (2.17) that

$$M(\Theta_0) \subseteq \Theta_0. \quad (2.19)$$

Therefore, from [11, p. 3129], the zero set $Z(\hat{\mu}_{M,D})$ can be represented as

$$\begin{aligned} Z(\hat{\mu}_{M,D}) &= \bigcup_{j=1}^{\infty} M^j(\Theta_0) = M(\Theta_0) = M(B_1) \cup M(B_2) \cup M(B_3) \cup M(\tilde{B}_1) \cup M(\tilde{B}_2) \cup M(\tilde{B}_3) \\ &:= Z_1 \cup Z_2 \cup Z_3 \cup \tilde{Z}_1 \cup \tilde{Z}_2 \cup \tilde{Z}_3, \end{aligned} \quad (2.20)$$

where

$$\begin{aligned} Z_1 &= M(B_1), & Z_2 &= M(B_2), & Z_3 &= M(B_3), \\ \tilde{Z}_1 &= M(\tilde{B}_1), & \tilde{Z}_2 &= M(\tilde{B}_2), & \tilde{Z}_3 &= M(\tilde{B}_3). \end{aligned} \quad (2.21)$$

We need the following Lemma 1 which can be verified directly.

Lemma 1. The sets Z_j and \tilde{Z}_j ($j = 1, 2, 3$) given by (2.21) satisfy the following properties:

- (a) $Z_1, Z_2, Z_3, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$ are mutually disjoint;
- (b) $Z(\hat{\mu}_{M,D}) \cap \mathbb{Z}^3 = \bigcup_{j=1}^3 (Z_j \cup \tilde{Z}_j) \cap \mathbb{Z}^3 = \emptyset$;
- (c) $Z_j \pm Z_j \subseteq \mathbb{Z}^3$, $Z_j = -Z_j$, $\tilde{Z}_j = -\tilde{Z}_j$ ($j = 1, 2, 3$);
- (d) $(Z_j - \tilde{Z}_j) \cap Z(\hat{\mu}_{M,D}) = (\tilde{Z}_j - Z_j) \cap Z(\hat{\mu}_{M,D}) = \emptyset$ ($j = 1, 2, 3$);
- (e) $(\tilde{Z}_j - \tilde{Z}_k) \cap Z(\hat{\mu}_{M,D}) = \emptyset$ ($1 \leq j \neq k \leq 3$);
- (f) $Z_j - Z_k \subseteq Z_l$ ($1 \leq j \neq k \neq l \leq 3$);
- (g) $\tilde{Z}_1 - \tilde{Z}_1 \subseteq (\mathbb{R}^3 \setminus Z(\hat{\mu}_{M,D})) \cup Z_3$, $\tilde{Z}_2 - \tilde{Z}_2 \subseteq (\mathbb{R}^3 \setminus Z(\hat{\mu}_{M,D})) \cup Z_1$, $\tilde{Z}_3 - \tilde{Z}_3 \subseteq (\mathbb{R}^3 \setminus Z(\hat{\mu}_{M,D})) \cup Z_2$.

Now, from (2.1) and (2.20), we see that the following ten differences:

$$\begin{aligned} \lambda_2 - \lambda_1, & \quad \lambda_3 - \lambda_1, & \lambda_4 - \lambda_1, & \quad \lambda_5 - \lambda_1, \\ & \lambda_3 - \lambda_2, & \lambda_4 - \lambda_2, & \quad \lambda_5 - \lambda_2, \\ & & \lambda_4 - \lambda_3, & \quad \lambda_5 - \lambda_3, \\ & & & \quad \lambda_5 - \lambda_4 \end{aligned} \quad (2.22)$$

belong to the union of the six sets $Z_1, Z_2, Z_3, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$. Let

$$P = Z_1 \cup Z_2 \cup Z_3 \quad \text{and} \quad \tilde{P} = \tilde{Z}_1 \cup \tilde{Z}_2 \cup \tilde{Z}_3. \quad (2.23)$$

Then

$$\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1 \in P \cup \tilde{P} \quad \text{and} \quad P \cap \tilde{P} = \emptyset. \quad (2.24)$$

We divide the proof of Theorem 1 into the following three cases according to (2.24).

Case 1. 4 – 0 (or 0 – 4) distribution. That is, the four differences in (2.24) belong to one set P or \tilde{P} .

Case 2. 3 – 1 (or 1 – 3) distribution. That is, one of the two sets P and \tilde{P} contains three differences in (2.24), the other set contains one difference (the remainder one) in (2.24).

Z_1	Z_2	Z_3	\tilde{Z}_1	\tilde{Z}_2	\tilde{Z}_3
	$\lambda_2 - \lambda_1$	$\lambda_3 - \lambda_1$	$\lambda_4 - \lambda_1$ $\lambda_5 - \lambda_1$		

Box 1.

Case 3. 2 – 2 distribution. That is, each of the two sets P and \tilde{P} contains two differences in (2.24).

We will complete the proof of Theorem 1 by showing that each case is impossible.

2.1. Case 1: 4 – 0 (or 0 – 4) distribution

In this case, if $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1 \in P = Z_1 \cup Z_2 \cup Z_3$, then, by Lemma 1(a), there exists at least one set, say Z_1 , which contains two differences in (2.24). This is impossible. For example, if $\lambda_2 - \lambda_1 \in Z_1$ and $\lambda_5 - \lambda_1 \in Z_1$, then, by Lemma 1(c),

$$\lambda_5 - \lambda_2 = (\lambda_5 - \lambda_1) - (\lambda_2 - \lambda_1) \in Z_1 - Z_1 \subseteq \mathbb{Z}^3,$$

which contradicts (2.1) and Lemma 1(b).

If $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1 \in \tilde{P} = \tilde{Z}_1 \cup \tilde{Z}_2 \cup \tilde{Z}_3$, then, by Lemma 1(e), the four differences must belong to one of the three sets \tilde{Z}_1, \tilde{Z}_2 and \tilde{Z}_3 . This is impossible. To see this, we prove the following fact.

Claim 1. For each $j \in \{1, 2, 3\}$, \tilde{Z}_j cannot contain any three differences of the four differences $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1$.

For example, if $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_4 - \lambda_1 \in \tilde{Z}_1$, then $\lambda_3 - \lambda_2, \lambda_4 - \lambda_2 \in \tilde{Z}_1 - \tilde{Z}_1$. By Lemma 1(g) and (2.1), we have $\lambda_3 - \lambda_2, \lambda_4 - \lambda_2 \in Z_3$ and

$$\lambda_4 - \lambda_3 = (\lambda_4 - \lambda_2) - (\lambda_3 - \lambda_2) \in Z_3 - Z_3 \subseteq \mathbb{Z}^3,$$

which contradicts (2.1) and Lemma 1(b). The other cases can be proved in the same manner. Therefore Case 1 is impossible.

2.2. Case 2: 3 – 1 (or 1 – 3) distribution

From Lemma 1(e) and Claim 1, the set \tilde{P} cannot contain any three differences in (2.24). Hence, in this case, the set P contains three differences in (2.24), the set \tilde{P} contains the remainder one difference in (2.24). If $P = Z_1 \cup Z_2 \cup Z_3$ contains three differences in (2.24), then, by Lemma 1(b) and (c), each Z_j ($j = 1, 2, 3$) will contain one difference. It follows from Lemma 1(d) that the set \tilde{P} cannot contain any differences in (2.24). This is impossible because the set \tilde{P} contains the remainder one difference in (2.24).

2.3. Case 3: 2 – 2 distribution

In this case, we may assume without essential loss of generality that

$$\lambda_2 - \lambda_1, \lambda_3 - \lambda_1 \in P = Z_1 \cup Z_2 \cup Z_3 \tag{2.25}$$

and

$$\lambda_4 - \lambda_1, \lambda_5 - \lambda_1 \in \tilde{P} = \tilde{Z}_1 \cup \tilde{Z}_2 \cup \tilde{Z}_3. \tag{2.26}$$

By Lemma 1(e) and (2.26), $\lambda_4 - \lambda_1$ and $\lambda_5 - \lambda_1$ must be in one of the three sets \tilde{Z}_1, \tilde{Z}_2 and \tilde{Z}_3 . We may assume that the two differences in (2.26) belong to the set \tilde{Z}_1 . If $\lambda_4 - \lambda_1$ and $\lambda_5 - \lambda_1$ belong to the set \tilde{Z}_1 , then, by Lemma 1(d) and (2.25), $\lambda_2 - \lambda_1$ and $\lambda_3 - \lambda_1$ will be in $Z_2 \cup Z_3$. By Lemma 1(a) and (c), we may assume $\lambda_2 - \lambda_1 \in Z_2$ and $\lambda_3 - \lambda_1 \in Z_3$. Hence we only consider the following typical case.

Typical case. $\lambda_2 - \lambda_1 \in Z_2, \lambda_3 - \lambda_1 \in Z_3, \lambda_4 - \lambda_1 \in \tilde{Z}_1, \lambda_5 - \lambda_1 \in \tilde{Z}_1$.

The other cases can be proved in the same way.

In this typical case, each set contains elements (or differences) in Box 1.

The other elements in (2.22) are also in certain boxes. By Lemma 1(f), (g), we have

$$\lambda_3 - \lambda_2 = (\lambda_3 - \lambda_1) - (\lambda_2 - \lambda_1) \in Z_3 - Z_2 \Rightarrow \lambda_3 - \lambda_2 \in Z_1 \tag{2.27}$$

and

$$\lambda_5 - \lambda_4 = (\lambda_5 - \lambda_1) - (\lambda_4 - \lambda_1) \in \tilde{Z}_1 - \tilde{Z}_1 \Rightarrow \lambda_5 - \lambda_4 \in Z_3. \tag{2.28}$$

Z_1	Z_2	Z_3	\tilde{Z}_1	\tilde{Z}_2	\tilde{Z}_3
	$\lambda_2 - \lambda_1$	$\lambda_3 - \lambda_1$	$\lambda_4 - \lambda_1$ $\lambda_5 - \lambda_1$		
$\lambda_3 - \lambda_2$		$\lambda_5 - \lambda_4$			

Box 2.

From (2.27) and (2.28), Box 1 becomes Box 2.

The other four elements $\lambda_4 - \lambda_2, \lambda_5 - \lambda_2, \lambda_4 - \lambda_3, \lambda_5 - \lambda_3$ in (2.22) are also in certain small boxes of Box 2. It follows from Lemma 1 that this is impossible. For example, by Lemma 1(a), (b), (c), (d), (e), we see that $\lambda_4 - \lambda_2$ cannot belong to the sets (or small boxes) $Z_1, Z_2, Z_3, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$, a contradiction. Also, $\lambda_5 - \lambda_2$ cannot belong to the sets (or small boxes) $Z_1, Z_2, Z_3, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$, another contradiction. Therefore Case 3 is also impossible.

Hence any set of $\mu_{M,D}$ -orthogonal exponentials contains at most 4 elements. One can obtain many such orthogonal systems which contain 4 elements. For instance, the exponential function system E_S with S given by

$$S = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ p/2 \\ p/2 \end{pmatrix}, \begin{pmatrix} p/2 \\ 0 \\ p/2 \end{pmatrix}, \begin{pmatrix} p/2 \\ p/2 \\ 0 \end{pmatrix} \right\} \quad (2.29)$$

is the four elements orthogonal system in $L^2(\mu_{M,D})$. This shows that the number 4 is the best. The proof of Theorem 1 is complete.

3. Proof of Theorem 2

If λ_j ($j = 1, 2, 3, 4, 5$) $\in \mathbb{R}^2$ are such that the exponential functions

$$e^{2\pi i \langle \lambda_1, x \rangle}, \quad e^{2\pi i \langle \lambda_2, x \rangle}, \quad e^{2\pi i \langle \lambda_3, x \rangle}, \quad e^{2\pi i \langle \lambda_4, x \rangle}, \quad e^{2\pi i \langle \lambda_5, x \rangle}$$

are mutually orthogonal in $L^2(\mu_{M,D})$, then the differences $\lambda_j - \lambda_k$ ($1 \leq j \neq k \leq 5$) are in the zero set $Z(\hat{\mu}_{M,D})$ of the Fourier transform $\hat{\mu}_{M,D}$. That is, we have

$$\lambda_j - \lambda_k \in Z(\hat{\mu}_{M,D}) \quad (1 \leq j \neq k \leq 5). \quad (3.1)$$

It is different from Theorem 1 that, in this case, M and D are given by (1.3). First, for the given digit set D in (1.3), we have

$$\Theta_0 := \{\xi \in \mathbb{R}^2: m_D(\xi) = 0\} = A_1 \cup A_2, \quad (3.2)$$

where

$$m_D(\xi) = \frac{1}{4} \{1 + e^{2\pi i \xi_1} + e^{2\pi i \xi_2} + e^{2\pi i (\xi_1 + \xi_2)}\}, \quad \xi = (\xi_1, \xi_2)^t \in \mathbb{R}^2, \quad (3.3)$$

$$A_1 = \left\{ \begin{pmatrix} 1/2 + k \\ a \end{pmatrix} : a \in \mathbb{R}, k \in \mathbb{Z} \right\} \subset \mathbb{R}^2, \quad (3.4)$$

and

$$A_2 = \left\{ \begin{pmatrix} a \\ 1/2 + k \end{pmatrix} : a \in \mathbb{R}, k \in \mathbb{Z} \right\} \subset \mathbb{R}^2. \quad (3.5)$$

It can be proved that

$$M^j(A_1) \subseteq M(A_1) \quad \text{and} \quad M^j(A_2) \subseteq M(A_2) \quad (3.6)$$

hold for $j = 1, 2, \dots$. So we have

$$Z(\hat{\mu}_{M,D}) = \bigcup_{j=1}^{\infty} M^j(\Theta_0) = \bigcup_{j=1}^{\infty} M^j(A_1 \cup A_2) = M(A_1) \cup M(A_2). \quad (3.7)$$

The set $M(A_1) \cup M(A_2)$ can be divided into three disjoint sets. In fact, we have

$$Z(\hat{\mu}_{M,D}) = M(A_1) \cup M(A_2) = Z_1 \cup Z_2 \cup \tilde{Z}_1, \quad (3.8)$$

where

$$Z_1 = \left\{ \begin{pmatrix} p_1/2 + p_1 k \\ ap_2 \end{pmatrix} : a \in \mathbb{R} \setminus (\mathbb{Z} + 1/2), k \in \mathbb{Z} \right\} \subset \mathbb{R}^2, \quad (3.9)$$

$$Z_2 = \left\{ \begin{pmatrix} p_1/2 + p_1 k_1 \\ p_2/2 + p_2 k_2 \end{pmatrix} : k_1 \in \mathbb{Z}, k_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2, \quad (3.10)$$

$$\tilde{Z}_1 = \left\{ \begin{pmatrix} ap_1 \\ p_2/2 + p_2 k \end{pmatrix} : a \in \mathbb{R} \setminus (\mathbb{Z} + 1/2), k \in \mathbb{Z} \right\} \subset \mathbb{R}^2. \quad (3.11)$$

We need the following Lemma 2 which can be verified directly.

Lemma 2. The sets Z_1, Z_2 and \tilde{Z}_1 given by (3.9)–(3.11) satisfy the following properties:

- (a) Z_1, Z_2, \tilde{Z}_1 are mutually disjoint;
- (b) $Z(\hat{\mu}_{M,D}) \cap \mathbb{Z}^2 = (Z_1 \cup Z_2 \cup \tilde{Z}_1) \cap \mathbb{Z}^2 = \emptyset$;
- (c) $Z_2 \pm Z_2 \subseteq \mathbb{Z}^2, Z_j = -Z_j$ ($j = 1, 2$), $\tilde{Z}_1 = -\tilde{Z}_1$;
- (d) $\tilde{Z}_1 - \tilde{Z}_1 \subseteq (\mathbb{R}^2 \setminus Z(\hat{\mu}_{M,D})) \cup Z_1, Z_1 - Z_1 \subseteq (\mathbb{R}^2 \setminus Z(\hat{\mu}_{M,D})) \cup \tilde{Z}_1, Z_1 - Z_2 \subseteq (\mathbb{R}^2 \setminus Z(\hat{\mu}_{M,D})) \cup \tilde{Z}_1, \tilde{Z}_1 - Z_2 \subseteq (\mathbb{R}^2 \setminus Z(\hat{\mu}_{M,D})) \cup Z_1$.

Now, from (3.1) and (3.8), we see that the following ten differences:

$$\begin{aligned} \lambda_2 - \lambda_1, \quad \lambda_3 - \lambda_1, \quad \lambda_4 - \lambda_1, \quad \lambda_5 - \lambda_1, \\ \lambda_3 - \lambda_2, \quad \lambda_4 - \lambda_2, \quad \lambda_5 - \lambda_2, \\ \lambda_4 - \lambda_3, \quad \lambda_5 - \lambda_3, \\ \lambda_5 - \lambda_4 \end{aligned} \tag{3.12}$$

belong to the union of the three sets Z_1, Z_2, \tilde{Z}_1 . In particular, we have

$$\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1 \in Z_1 \cup Z_2 \cup \tilde{Z}_1. \tag{3.13}$$

By Lemma 2(b), (c), the set Z_2 contains at most one difference in (3.13). So we divide the proof of Theorem 2 into the following two cases.

Case 1. The set Z_2 contains one difference in (3.13).

Case 2. The set Z_2 contains no differences in (3.13).

3.1. Case 1

In this case, we may assume that the set Z_2 contains the difference $\lambda_2 - \lambda_1$. Then

$$\lambda_3 - \lambda_1, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1 \in Z_1 \cup \tilde{Z}_1. \tag{3.14}$$

There are two cases that we need to prove the impossibility.

(1) 3 – 0 (or 0 – 3) distribution in (3.14). If the three differences $\lambda_3 - \lambda_1, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1$ are in one set Z_1 or \tilde{Z}_1 , say Z_1 , then we have

$$\begin{aligned} \lambda_4 - \lambda_3 &= (\lambda_4 - \lambda_1) - (\lambda_3 - \lambda_1) \in Z_1 - Z_1, \\ \lambda_5 - \lambda_3 &= (\lambda_5 - \lambda_1) - (\lambda_3 - \lambda_1) \in Z_1 - Z_1, \\ \lambda_5 - \lambda_4 &= (\lambda_5 - \lambda_1) - (\lambda_4 - \lambda_1) \in Z_1 - Z_1, \end{aligned} \tag{3.15}$$

which shows, by Lemma 2(d) and (3.1), that

$$\lambda_4 - \lambda_3, \lambda_5 - \lambda_3, \lambda_5 - \lambda_4 \in \tilde{Z}_1. \tag{3.16}$$

From (3.16), we also have

$$\lambda_5 - \lambda_4 = (\lambda_5 - \lambda_3) - (\lambda_4 - \lambda_3) \in \tilde{Z}_1 - \tilde{Z}_1, \tag{3.17}$$

which shows, by Lemma 2(d) and (3.1), that $\lambda_5 - \lambda_4 \in Z_1$, a contradiction. Hence this case is impossible.

(2) 2 – 1 (or 1 – 2) distribution in (3.14). If two differences among the three differences $\lambda_3 - \lambda_1, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1$ are in one set Z_1 or \tilde{Z}_1 , say Z_1 , the remainder one will be in the set \tilde{Z}_1 . We may assume that

$$\lambda_3 - \lambda_1, \lambda_4 - \lambda_1 \in Z_1 \quad \text{and} \quad \lambda_5 - \lambda_1 \in \tilde{Z}_1. \tag{3.18}$$

Note that $\lambda_2 - \lambda_1 \in Z_2$, we have

$$\begin{aligned}\lambda_4 - \lambda_3 &= (\lambda_4 - \lambda_1) - (\lambda_3 - \lambda_1) \in Z_1 - Z_1, \\ \lambda_3 - \lambda_2 &= (\lambda_3 - \lambda_1) - (\lambda_2 - \lambda_1) \in Z_1 - Z_2, \\ \lambda_4 - \lambda_2 &= (\lambda_4 - \lambda_1) - (\lambda_2 - \lambda_1) \in Z_1 - Z_2,\end{aligned}\tag{3.19}$$

which shows, by Lemma 2(d) and (3.1), that

$$\lambda_4 - \lambda_3, \lambda_3 - \lambda_2, \lambda_4 - \lambda_2 \in \tilde{Z}_1.\tag{3.20}$$

From (3.20), we also have

$$\lambda_3 - \lambda_2 = (\lambda_4 - \lambda_2) - (\lambda_4 - \lambda_3) \in \tilde{Z}_1 - \tilde{Z}_1,\tag{3.21}$$

which shows, by Lemma 2(d) and (3.1), that $\lambda_3 - \lambda_2 \in Z_1$, a contradiction. Hence this case is impossible. The other cases can be proved in the same way.

3.2. Case 2

In this case, the set Z_2 contains no differences in (3.13) and

$$\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1 \in Z_1 \cup \tilde{Z}_1.\tag{3.22}$$

There are three cases that we need to prove the impossibility.

(3) 4 – 0 (or 0 – 4) distribution in (3.22). The reason is the same as Case (i).

(4) 3 – 1 (or 1 – 3) distribution in (3.22). The reason is the same as Case (i).

(5) 2 – 2 distribution in (3.22). We may assume that

$$\lambda_2 - \lambda_1, \lambda_3 - \lambda_1 \in Z_1 \quad \text{and} \quad \lambda_4 - \lambda_1, \lambda_5 - \lambda_1 \in \tilde{Z}_1.\tag{3.23}$$

That is, from (3.9) and (3.11), we have

$$\lambda_2 - \lambda_1 = \begin{pmatrix} p_1/2 + p_1 k_{21} \\ a_{21} p_2 \end{pmatrix} \quad \text{with } k_{21} \in \mathbb{Z} \text{ and } a_{21} \in \mathbb{R} \setminus (\mathbb{Z} + 1/2),\tag{3.24}$$

$$\lambda_3 - \lambda_1 = \begin{pmatrix} p_1/2 + p_1 k_{31} \\ a_{31} p_2 \end{pmatrix} \quad \text{with } k_{31} \in \mathbb{Z} \text{ and } a_{31} \in \mathbb{R} \setminus (\mathbb{Z} + 1/2),\tag{3.25}$$

$$\lambda_4 - \lambda_1 = \begin{pmatrix} \tilde{a}_{41} p_1 \\ p_2/2 + p_2 \tilde{k}_{41} \end{pmatrix} \quad \text{with } \tilde{k}_{41} \in \mathbb{Z} \text{ and } \tilde{a}_{41} \in \mathbb{R} \setminus (\mathbb{Z} + 1/2),\tag{3.26}$$

$$\lambda_5 - \lambda_1 = \begin{pmatrix} \tilde{a}_{51} p_1 \\ p_2/2 + p_2 \tilde{k}_{51} \end{pmatrix} \quad \text{with } \tilde{k}_{51} \in \mathbb{Z} \text{ and } \tilde{a}_{51} \in \mathbb{R} \setminus (\mathbb{Z} + 1/2).\tag{3.27}$$

It follows from Lemma 2(d) that

$$\lambda_3 - \lambda_2 = (\lambda_3 - \lambda_1) - (\lambda_2 - \lambda_1) = \begin{pmatrix} p_1(k_{31} - k_{21}) \\ (a_{31} - a_{21})p_2 \end{pmatrix} \in \tilde{Z}_1 \quad \text{with } a_{31} - a_{21} = \frac{1}{2} + k_{32} \text{ for some } k_{32} \in \mathbb{Z},\tag{3.28}$$

and

$$\lambda_5 - \lambda_4 = (\lambda_5 - \lambda_1) - (\lambda_4 - \lambda_1) = \begin{pmatrix} (\tilde{a}_{51} - \tilde{a}_{41})p_1 \\ p_2(\tilde{k}_{51} - \tilde{k}_{41}) \end{pmatrix} \in Z_1 \quad \text{with } \tilde{a}_{51} - \tilde{a}_{41} = \frac{1}{2} + \tilde{k}_{54} \text{ for some } \tilde{k}_{54} \in \mathbb{Z}.\tag{3.29}$$

Now, the other four elements $\lambda_4 - \lambda_2, \lambda_5 - \lambda_2, \lambda_4 - \lambda_3, \lambda_5 - \lambda_3$ in (3.12) are also in the union of the three sets Z_1, Z_2, \tilde{Z}_1 . This will deduce an impossible result easily. For example, from (3.24) and (3.26), we have

$$\lambda_4 - \lambda_2 = (\lambda_4 - \lambda_1) - (\lambda_2 - \lambda_1) = \begin{pmatrix} \tilde{a}_{41} p_1 - p_1/2 - p_1 k_{21} \\ p_2/2 + \tilde{k}_{41} p_2 - a_{21} p_2 \end{pmatrix},\tag{3.30}$$

which does not belong to the union of the three sets Z_1, Z_2, \tilde{Z}_1 . In fact, if the difference (3.30) belongs to the set Z_1 , then $\tilde{a}_{41} \in \mathbb{Z}$, which shows $\tilde{a}_{51} \in \mathbb{Z} + 1/2$ by (3.29), a contradiction. The same reason shows that the difference (3.30) does not

belong to the set Z_2 . If the difference (3.30) belongs to the set \tilde{Z}_1 , then $a_{21} \in \mathbb{Z}$, which shows $a_{31} \in \mathbb{Z} + 1/2$ by (3.28), a contradiction. Hence this case is also impossible. The other cases can be proved in the same way.

In a word, the above discussion shows that any set of $\mu_{M,D}$ -orthogonal exponentials contains at most 4 elements. One can obtain many such orthogonal systems which contain 4 elements. For instance, the exponential function system E_S with S given by

$$S = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ p_2/2 \end{pmatrix}, \begin{pmatrix} p_1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} p_1/2 \\ p_2/2 \end{pmatrix} \right\} \quad (3.31)$$

is the four elements orthogonal system in $L^2(\mu_{M,D})$. This shows that the number 4 is the best. The proof of Theorem 2 is complete.

4. A concluding remark

In Theorem 1, the four elements digit set D in (1.2) can be replaced by a more general digit set $D_1 = \{0, d_1, d_2, d_3\} \subset \mathbb{R}^3$ if d_1, d_2 and d_3 are three linearly independent vectors in \mathbb{R}^3 . In fact, for the given M and D in (1.2), we can write D_1 and $Z(\hat{\mu}_{M,D_1})$ as

$$D_1 = P(D) \quad \text{and} \quad Z(\hat{\mu}_{M,D_1}) = P^{*-1}(Z(\hat{\mu}_{M,D})),$$

where $P = [d_1, d_2, d_3]$ is an invertible 3×3 matrix whose column vectors are d_1, d_2 and d_3 . So it follows from Theorem 1 that μ_{M,D_1} -orthogonal exponentials contain at most 4 elements and the number 4 is the best.

Similarly, for any 2×2 expanding matrix $M_1 \in M_2(\mathbb{R})$ and any digit set $D_1 = \{0, d_1, d_2, d_1 + d_2\} \subset \mathbb{R}^2$, if $P = [d_1, d_2]$ is an invertible 2×2 matrix such that $P^{-1}M_1P = M$, where M is given in (1.3), then μ_{M_1,D_1} -orthogonal exponentials contain at most 4 elements and the number 4 is the best.

Finally, it should be pointed out that the method used to prove Theorems 1 and 2 in [13] is not completeness. The proof there contains some big gaps. For example, on the page 397, the solution (2.8) of the equation $1 + w_1 + w_2 + w_3 = 0$ with $|w_1| = |w_2| = |w_3| = 1$ in [13] does not contain the root $\{w_1, w_2, w_3\} = \{-1, -1, 1\}$. If $a = 0$ is added, the representation (2.9) of the zero set $Z(\hat{\mu}_{M,D})$ in [13] should contain $a \in (-1, 1)$. Such a (or a_1, a_2, a_3) may take $a = 0$ in the proof, so t_1 and t_2 may be equal if $t \in Z_i$ or \tilde{Z}_i , this case is not covered there. Even so, our present results improve the corresponding results and solve Conjecture 1 in the above mentioned cases.

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References

- [1] D.E. Dutkay, D. Han, Q. Sun, On the spectra of a Cantor measure, *Adv. Math.* 221 (2009) 251–276.
- [2] D.E. Dutkay, P.E.T. Jorgensen, Analysis of orthogonality and of orbits in affine iterated function systems, *Math. Z.* 256 (2007) 801–823.
- [3] D.E. Dutkay, P.E.T. Jorgensen, Duality questions for operators, spectrum and measures, available on <http://arxiv.org/abs/0809.3274v1>, 2008.
- [4] K.J. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, second ed., John Wiley & Sons, Inc., 2003.
- [5] B. Fuglede, Commuting self-adjoint partial differential operators and a group theoretic problem, *J. Funct. Anal.* 16 (1974) 101–121.
- [6] T.-Y. Hu, K.-S. Lau, Spectral property of the Bernoulli convolutions, *Adv. Math.* 219 (2008) 554–567.
- [7] J.E. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.* 30 (1981) 713–747.
- [8] P.E.T. Jorgensen, S. Pedersen, Dense analytic subspaces in fractal L^2 -spaces, *J. Anal. Math.* 75 (1998) 185–228.
- [9] J.-L. Li, Spectral sets and spectral self-affine measures, PhD thesis, The Chinese University of Hong Kong, November, 2004.
- [10] J.-L. Li, Orthogonal exponentials on the generalized plane Sierpinski gasket, *J. Approx. Theory* 153 (2008) 161–169.
- [11] J.-L. Li, Non-spectral problem for a class of planar self-affine measures, *J. Funct. Anal.* 255 (2008) 3125–3148.
- [12] J.-L. Li, Non-spectrality of planar self-affine measures with three-elements digit set, *J. Funct. Anal.* 257 (2009) 537–552.
- [13] Y.-B. Yuan, Orthogonal exponentials on the generalized three dimension Sierpinski gasket, *J. Math. Anal. Appl.* 349 (2009) 395–402.