



# Complementation and decompositions in some weakly Lindelöf Banach spaces

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## ABSTRACT

Let  $\Gamma$  denote an uncountable set. We consider the questions if a Banach space  $X$  of the form  $C(K)$  of a given class (1) has a complemented copy of  $c_0(\Gamma)$  or (2) for every  $c_0(\Gamma) \subseteq X$  has a complemented  $c_0(E)$  for an uncountable  $E \subseteq \Gamma$  or (3) has a decomposition  $X = A \oplus B$  where both  $A$  and  $B$  are nonseparable. The results concern a superclass of the class of nonmetrizable Eberlein compacts, namely  $K$ s such that  $C(K)$  is Lindelöf in the weak topology and we restrict our attention to  $K$ s scattered of countable height. We show that the answers to all these questions for these  $C(K)$ s depend on additional combinatorial axioms which are independent of  $ZFC \pm CH$ . If we assume the  $P$ -ideal dichotomy, for every  $c_0(\Gamma) \subseteq C(K)$  there is a complemented  $c_0(E)$  for an uncountable  $E \subseteq \Gamma$ , which yields the positive answer to the remaining questions. If we assume  $\clubsuit$ , then we construct a nonseparable weakly Lindelöf  $C(K)$  for  $K$  of height  $\omega + 1$  where every operator is of the form  $cI + S$  for  $c \in \mathbb{R}$  and  $S$  with separable range and conclude from this that there are no decompositions as above which yields the negative answer to all the above questions. Since, in the case of a scattered compact  $K$ , the weak topology on  $C(K)$  and the pointwise convergence topology coincide on bounded sets, and so the Lindelöf properties of these two topologies are equivalent, many results concern also the space  $C_p(K)$ .

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## 1. Introduction

This paper is concerned with two directions of research presented in the literature. The first direction provides conditions under which nonseparable Banach spaces have rich structure of complemented canonical subspaces which in turn implies that the entire space has many operators and decompositions. The second direction is represented by constructions of nonseparable Banach spaces with few operators in the sense that there are only operators of the form  $T = cI + S$  where  $c \in \mathbb{R}$  and  $S$  has a separable range. We restrict ourselves to the class of Banach spaces  $C(K)$  of real-valued continuous functions on a Hausdorff compact, scattered space of countable height. Recall that a compact  $K$  is scattered if and only if every nonempty  $L \subseteq K$  has an isolated point. By the Sierpiński and Mazurkiewicz theorem all scattered compact metrizable spaces are homeomorphic to ordinal intervals  $[0, \alpha]$  with the order topology where  $\alpha$  is a countable ordinal. On the other hand the class of nonmetrizable scattered compact spaces is very far from its classification.

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Scattered compact spaces play special role in functional analysis, namely a  $C(K)$  is an Asplund space if and only if  $K$  is scattered [8, Theorem 12.29]. There are many other analytic characterizations of this class of Banach spaces (see [16]). If  $K$  is scattered we can define its Cantor–Bendixson derivative  $X^{(\alpha)}$  for each ordinal  $\alpha$  by the inductive conditions:  $K^{(0)} = K$ ,  $K^{(\alpha+1)} = (K^{(\alpha)})'$  and  $K^{(\lambda)} = \bigcap_{\alpha < \lambda} K^{(\alpha)}$  where  $X'$  is the set of all nonisolated points of  $X$ . The minimal ordinal  $\alpha$  such that  $K^{(\alpha)} = \emptyset$  is called the height of  $K$  and denoted  $ht(K)$ . In particular we will use the fact that if  $L \subseteq K$  is a closed subset of a scattered  $K$ , then  $L$  is scattered and  $ht(L) \leq ht(K)$ . Also it is well known that scattered compact spaces are totally disconnected.

Amir and Lindenstrauss introduced in [1] a quite successful generalization of reflexive and separable spaces, namely the weakly compactly generated spaces, WCG. As proved in [1] there are many operators with separable ranges on such spaces, many complemented separable subspaces and nice extension results hold. In [2] the question of nonseparable complemented subspaces, in particular copies of  $c_0(\Gamma)$  for uncountable  $\Gamma$  in WCG spaces was addressed. A special case of [2, Theorems 1.1, 1.2] is that every copy of  $c_0(\omega_1)$  in a WCG Banach space is complemented and for every copy of  $c_0(\Gamma)$  in a WCG Banach space and  $\Gamma$  of uncountable cofinality there is  $E \subseteq \Gamma$  of the same cardinality as  $\Gamma$  such that  $c_0(E)$  is complemented. Of course complemented copies of  $c_0(\Gamma)$  yield not only projections but also a rich algebra of operators, for example obtained by composing the projection with operators on  $c_0(\Gamma)$  induced by permutations of  $\Gamma$ .

These results of [2] were obtained for a superclass  $\mathcal{V}$  of WCG spaces namely spaces with Valdivia compact dual ball. We address similar questions for another superclass of WCG spaces of the form  $C(K)$ , namely Banach spaces of the form  $C(K)$  which are Lindelöf in the weak topology, also called weakly Lindelöf. We restrict our attention to  $K$ s scattered of countable height for which  $C(K)$  necessarily contain a copy of  $c_0(\omega_1)$  for  $K$  nonmetrizable. Note that it is well known that every WCG space is weakly Lindelöf [8, Theorem 12.34]. The first example of a weakly Lindelöf Banach space which is not WCG was obtained in [18]. It is of the form  $C(K)$  where  $K$  is the ladder system space. We recall this example in details as it is a fundamental example of the class of Banach spaces we consider in this paper. Here we use the notation valid for the entire paper:  $S(\omega_1)$  is the set of all countable ordinals which are successor ordinals and  $L(\omega_1)$  is the set of all countable ordinals which are limit ordinals.

**Example** (*The ladder system space*). (See [18], IV.7.1 of [5].) For each  $\alpha \in L(\omega_1)$  choose  $S_\alpha \subseteq S(\omega_1)$  of order type  $\omega$  such that the only accumulation point of  $S_\alpha$  in the order topology is  $\alpha$ . The ladder system space is the Stone space of the Boolean subalgebra of  $\wp(\omega_1)$  generated by finite subsets of  $S(\omega_1)$  and the ladders  $S_\alpha \cup \{\alpha\}$  for  $\alpha < \omega_1$ . Its points can be identified with elements of  $\omega_1$  or with one extra ultrafilter which contains all cofinite subsets of  $S(\omega_1)$  and all complements of  $S_\alpha \cup \{\alpha\}$ s. This point will be denoted by  $\omega_1$  and the underlying set of the space will be identified with  $[0, \omega_1]$ . Of course such a topology depends on the choice of the ladders ( $S_\alpha$ :  $\alpha < \omega_1$ ). It is well known that such a space is scattered and of height three.

This example does not fall in the class  $\mathcal{V}$ , on the other hand  $C([0, \omega_1])$  is in  $\mathcal{V}$  but is not weakly Lindelöf (and  $[0, \omega_1]$  has uncountable height). It is easy and probably well known to see that already the above space has an uncomplemented copy of  $c_0(\omega_1)$  (see 2.7). So our positive results must be weaker from a version of Sobczyk's theorem.

If  $C(K)$  is nonseparable WCG, then  $K$  is a nonmetrizable Eberlein compact, and so, it contains  $c_0(\omega_1)$  by [8, Theorem 12.9]. The same holds for nonseparable weakly Lindelöf  $C(K)$  for  $K$  scattered of countable height. Namely,  $K$  has to be uncountable and nonseparable by [17] and so  $K \setminus K^{(1)}$  is an uncountable set of isolated points in  $K$  which easily gives rise to an isometric copy of  $c_0(\omega_1)$ . Thus, we always have copies of  $c_0(\omega_1)$  in our spaces.

Our results are that if  $K$  is nonmetrizable scattered compact of countable height such that  $C(K)$  is weakly Lindelöf then the questions:

- (1) Must  $C(K)$  have a complemented copy of  $c_0(\Gamma)$  for  $\Gamma$  uncountable?
- (2) If  $\Gamma$  is uncountable and  $c_0(\Gamma) \subseteq C(K)$ , is there a complemented  $c_0(E)$  for an uncountable  $E \subseteq \Gamma$ ?
- (3) Is there a decomposition  $C(K) = A \oplus B$  where both  $A$  and  $B$  are nonseparable?

all depend on additional combinatorial axioms which are independent from  $\text{ZFC} \pm \text{CH}$ . We could note here that if the height of  $K$  is finite, all these questions have positive answers.

In the following, second section, we prove some useful facts about weakly Lindelöf  $C(K)$ s which do not require any additional set-theoretic assumptions. It is well known that for bounded sets in  $C(K)$  for  $K$  scattered, the weak topology coincides with the topology of pointwise convergence. This implies that for these  $K$ s each of these topologies has the Lindelöf property if and only if the other has it as well. So the results of this section also refer to  $C_p(K)$  spaces which are weakly Lindelöf. [5] contains many interesting results on the Lindelöf property of  $C_p(K)$  spaces, in particular Chapter IV.7 of [5] refers to compact scattered  $K$ s. In this section we note that  $C(K)$  has uncomplemented copies of  $c_0(\omega_1)$  where  $K$  is the ladder system space but also there are always complemented copies of  $c_0(\omega_1)$  in any weakly Lindelöf  $C(K)$  for  $K$  compact scattered nonmetrizable and of finite height. Recall that if  $C(K)$  is WCG and  $K$  is of finite height and weight  $< \omega_\omega$ , then, by a result of Godefroy, Kalton and Lancien [9],  $C(K)$  is already isomorphic to a  $c_0(\Gamma)$  for some  $\Gamma$ . Here the assumption on the weight is essential by a result of Marciszewski [14]. Our results suggest also the possibility of a characterization of compact  $K$ s of countable height such that  $C(K)$  is weakly Lindelöf (possibly of weights  $< \omega_\omega$ ) as those  $K$ s where closures of countable sets are countable (see Question 2.6).

The first axiom we consider, in Section 3, is the  $P$ -ideal dichotomy. A  $P$ -ideal  $I$  of subsets of a set  $X$  is a family of subsets which is closed under taking subsets and finite unions and for every sequence of elements  $A_n \in I$  for  $n \in \mathbb{N}$  there is an  $A \in I$  such that  $A_n \setminus A$  is at most finite for each  $n \in \mathbb{N}$ .

**Definition 1.1.** (See [25].) The  $P$ -ideal dichotomy is the following statement: For every  $P$ -ideal  $I$  of countable subsets of some set  $S$ , either

- (1) there is an uncountable  $A \subseteq S$  such that  $[A]^\omega \subseteq I$ , or
- (2)  $S$  can be decomposed into countably many sets orthogonal to  $I$ , that is such  $A_n$ s that  $[A_n]^\omega \cap I = [A_n]^{<\omega}$  for each  $n \in \mathbb{N}$ .

We prove that it implies positive answers to all the above questions (1)–(3) because it gives the positive answer to question (2) (see 3.2). It is known (see [25]) that the  $P$ -ideal dichotomy is consistent with  $\text{ZFC} \pm \text{CH}$ . We only use the  $P$ -ideal dichotomy for  $S$  of cardinality  $\omega_1$  which is considerably weaker than the full strength of the axiom. On the other hand in Section 4 we consider  $\clubsuit$ , also consistent with  $\text{ZFC} \pm \text{CH}$  (see [15,22]) which implies the negative answers to the above questions because assuming  $\clubsuit$  we construct a space  $K_0$  which does not have decompositions as in question (3) (see 4.11). Moreover  $\clubsuit$  already implies the existence of  $K$ s of finite height which constitute counterexamples to question (2). Recall the formulation of this axiom:

**Definition 1.2.** (See [15].)  $\clubsuit$  is the following sentence: There is a sequence  $(S_\alpha)_{\alpha \in L(\omega_1)}$  such that for each  $\alpha \in L(\omega_1)$ :

- (1)  $S_\alpha \subseteq \alpha$ ,
- (2)  $S_\alpha$  converges to  $\alpha$  in the order topology,
- (3) for every uncountable  $X \subseteq \omega_1$  there is  $\alpha \in L(\omega_1)$  such that  $S_\alpha \subseteq X$ .

Actually, our construction of a  $C(K_0)$  from  $\clubsuit$  goes further and is related to the subject of nonseparable Banach spaces with few operators, i.e., Banach spaces  $X$  on which the only operators are of the form  $T = cI + S$  where  $I$  is the identity on  $X$ ,  $c$  is a scalar and  $S$  is an operator with separable range. Our construction has this property and this is how we prove the nonexistence of the decompositions of our  $C(K_0)$  into two nonseparable factors. In fact, we can even control the factors. As shown in 4.11, one factor is isomorphic to  $c_0$  or  $C_0(\omega^\omega)$  and the other is isomorphic to  $C(K_0)$ . This is related to few decompositions as in [12] and [4].

The first construction of a Banach space (not of the form  $C(K)$ )  $X$  on which the only operators are of the form  $T = cI + S$  where  $I$  is the identity on  $X$ ,  $c$  is a scalar and  $S$  is an operator with a separable range is due to Shelah [21] and was obtained under  $\diamond$ . In [23] a weaker, modified version of it was given which did not require any additional set-theoretic assumptions and use Todorćević's anti-Ramsey results from [24]. H. Wark in [26] modified this example obtaining a reflexive space (and so WCG) with this property. All the above constructions (as ours) have a transfinite basis of length  $\omega_1$ , which in the case of a WCG space is a necessary condition. Thus, there are lots of projections on separable subspaces. Finally Argyros, Lopez-Abad and Todorćević obtained in [3] reflexive spaces with transfinite basis of length  $\omega_1$  where all operators are of the above form where  $S$  is strictly singular.

In [11] we investigated the versions of the above property of having few operators in WCG spaces with transfinite bases of length bigger than  $\omega_1$ . The spaces of [21,23,26,3,11] are not of the form  $C(K)$ . We do not know if such  $C(K)$ s with few operators in the above sense exist in ZFC (if they are of the form  $C(K)$  they cannot be WCG but  $K$  must be scattered). In [12] we obtain such a space of the form  $C(K)$  assuming CH. As shown in Section 3, CH does not imply the existence of such a  $C(K)$  which is additionally weakly Lindelöf.

## 2. The Lindelöf property in the weak topology in ZFC

**Lemma 2.1.** If  $L \subseteq K$  are compact spaces such that  $C(K)$  is weakly Lindelöf, then  $C(L)$  is weakly Lindelöf as well.

**Proof.** Under the assumption of the lemma the Banach space  $C(L)$  is a quotient of the Banach space  $C(K)$ . So the canonical map  $T : C(K) \rightarrow C(L)$  given by  $T(f) = f|_L$  is continuous with respect to the weak topologies. Now it is enough to recall (Theorem 3.8.7 of [7]) that a continuous image of a Lindelöf space is a Lindelöf space.  $\square$

**Lemma 2.2.** If  $K$  is a compact scattered space of countable height such that  $C(K)$  is weakly Lindelöf, then the closures of countable sets in  $K$  are countable.

**Proof.** If for any countable  $X \subseteq K$  its closure  $L$  is uncountable, we obtain a separable subspace  $L$  of  $K$  which is scattered and of countable height. By a result of Pol (Theorem 2 of [17]),  $C(L)$  is not weakly Lindelöf, and hence  $C(K)$  is not weakly Lindelöf by 2.1.  $\square$

By a theorem of Reznichenko (IV.8.16 of [5]), Martin's axiom and the negation of CH imply that for any compact scattered  $K$  such that  $C(K)$  is weakly Lindelöf we have closures in of countable sets countable (this is equivalent to  $\aleph_0$ -monolithicity of [5] for compact spaces).

**Lemma 2.3.** *If  $K$  is a compact scattered space of countable height such that  $C(K)$  is weakly Lindelöf, then  $K$  is a Frechet topological space.*

**Proof.** First, let us prove that  $K$  has countable tightness. Let  $A \subseteq K$  and  $x \in \bar{A}$ . Working in the scattered compact space  $\bar{A}$  of countable height, we prove by induction on the height of the point  $y$  that there is a countable  $B_y \subseteq A$  such that  $y \in \bar{B}_y$ . The inductive step follows from the fact that each nonisolated point in a scattered space of countable height is the limit of a convergent sequence of points of smaller heights.

Now, knowing that  $K$  has countable tightness suppose that  $x \in K$  is in the closure  $\bar{A}$ . So we may w.l.o.g. assume that  $A$  is countable. By 2.2  $\bar{A}$  is countable, compact and hence is a metrizable compact space and so there is a sequence of elements of  $A$  convergent to  $x$ , which completes the proof.  $\square$

Following [18], we will say that a space  $X$  has the strong condensation property if and only if every uncountable subset  $A \subseteq X$  has an uncountable subset  $B \subseteq A$  which is concentrated around a point  $x$  of  $X$ , which means that for every neighborhood  $V$  of  $x$ , the set  $B \setminus V$  is at most countable. We will use the following version of a lemma from [18]. Here  $C(K, \{0, 1\})$  is the family of all characteristic functions of clopen sets of  $K$  with the pointwise convergence topology (which coincides for scattered  $K$ s on bounded sets with the weak topology).

**Lemma 2.4.** (See Lemma 4 of [18].) *Suppose that  $K$  is a scattered compact space of weight  $\omega_1$  and  $x \in K$ . Let  $G = \{f \in C(K, \{0, 1\}) : f(x) = 0\}$ . If there is a set  $E \subseteq G$  such that*

- (1) *for every  $f \in G$  there exist  $f_1, \dots, f_m \in E$  such that  $f = f_1 + \dots + f_m$ ,*
- (2) *the space  $E$  has the strong condensation property,*

*then the space  $C(K)$  is weakly Lindelöf.*

**Theorem 2.5.** *Suppose that  $K$  is a compact scattered space of countable height and weight  $\omega_1$  where closures of countable sets are countable and there is only one point  $x_0 \in K$  which does not have a countable neighborhood. Then  $C(K)$  is weakly Lindelöf.*

**Proof.** First note that, by the compactness, each clopen set  $a \subseteq K$  missing  $x_0$  must be at most countable. We will closely follow [18]. To use 2.4 consider  $G = \{f \in C(K, \{0, 1\}) : f(x_0) = 0\}$ . Let  $E = G$ , i.e.,  $G$  is the set of characteristic functions of clopen sets missing  $x_0$ . So, we need to prove that  $E$  has the strong condensation property.

**Claim 1.** *If  $a_\xi$ s for  $\xi < \omega_1$  are clopen sets all containing a point  $x \in K \setminus \{x_0\}$ , then there is a clopen  $a \subseteq K$  such that  $x \in a \subseteq a_\xi$  for uncountably many  $\xi$ s.*

**Proof.** By the hypothesis about  $x_0$  each point in  $K \setminus \{x_0\}$  has a countable, compact, and so, metrizable neighborhood. Thus, it has a countable clopen basis, so one element of such a basis works for uncountably many  $\xi$ s.  $\square$

For a subset  $a \subseteq K$  define

$$ht^*(a) = \min\{\alpha \leq ht(K) : a \cap K^{(\alpha)} = \emptyset\}.$$

Note that if  $a$  is closed, then  $ht^*(a)$  is a successor ordinal, that  $ht^*(a) \leq ht(K)$  and  $a \cap K^{(ht^*(a)-1)}$  is finite.

**Claim 2.** *Suppose that  $\{a_\xi : \xi < \omega_1\}$  is a sequence of clopen sets missing  $x_0$  such that  $ht^*(a_\xi) = \alpha$  for each  $\xi \in \omega_1$ . Then there is an uncountable  $X \subseteq \omega_1$  and clopen sets  $a, b_\xi, c_\xi$  for  $\xi \in X$  such that*

- (1)  $a_\xi = a \cup b_\xi \cup c_\xi$ ,
- (2)  $b_\xi, c_\xi$  and  $a$  are pairwise disjoint for each  $\xi \in X$ ,
- (3)  $b_\xi$ s are pairwise disjoint for all  $\xi \in X$ ,
- (4)  $ht^*(c_\xi) < \alpha$  for all  $\xi \in X$ .

**Proof.** Let  $F_\xi = a_\xi \cap K^{(\alpha-1)}$ . As we noted above,  $F_\xi$ s are finite. Applying the  $\Delta$ -system lemma (Chapter 2, 1.5 of [13]) we may assume that there is a finite  $F \subseteq K^{(\alpha-1)}$  such that  $F = F_\xi \cap F_\eta$  for distinct  $\xi, \eta \in \omega_1$ . By Claim 1 there is a clopen  $a' \subseteq K$  such that  $F \subseteq a' \subseteq a_\xi$ . Note that proving the claim for  $a_\xi \setminus a'$  would be sufficient as we would add  $a'$  to  $a$  to cover the case of the original family. Thus w.l.o.g. we may assume that  $F_\xi$ 's are pairwise disjoint.

Now, by induction on  $\alpha < \omega_1$  we construct a sequence of clopen sets  $b_\alpha$  and  $\xi_\alpha \in \omega_1$  such that

- (1)  $F_{\xi_\alpha} \subseteq b_\alpha \subseteq a_{\xi_\alpha}$ ,
- (2)  $b_\alpha$ s are countable and pairwise disjoint.

The inductive step  $\alpha < \omega_1$  follows from the fact that  $\bigcup_{\beta < \alpha} b_\beta$  is countable, and hence has countable closure by the hypothesis on  $K$ , so there is an  $F_{\xi_\alpha}$  outside of this closure. We pick  $b_\alpha$  as its clopen neighborhood included in  $a_{\xi_\alpha}$  disjoint from  $\bigcup_{\beta < \alpha} b_\beta$ . The set  $b_\alpha$  is countable because it is missing  $x_0$ . Now put  $c_\alpha = a_{\xi_\alpha} \setminus b_\alpha$ . This completes the proof of Claim 2.  $\square$

Let  $\{\chi_{a_\xi} : \xi < \omega_1\}$  be a subset of  $E$  such that  $a_\xi$ s are all distinct. As the height of  $K$  is countable, we may assume w.l.o.g. that for all  $\xi \in \omega_1$  we have  $ht^*(a_\xi) = \alpha$  for some  $\alpha \leq ht(K)$ . By induction on  $\alpha$  we will prove that  $\{\chi_{a_\xi} : \xi < \omega_1\}$  has an uncountable set which is concentrated around a point in  $E$ . Suppose it is true for  $\beta < \alpha$  and assume  $a_\xi = a \cup b_\xi \cup c_\xi$  as in Claim 2. By the inductive hypothesis w.l.o.g. we may assume that all  $\chi_{c_\xi}$ s are concentrated around some  $\chi_c$ . By the disjointness of all  $c_\xi$ s with  $a$  we have  $a \cap c = \emptyset$ . Now, note that  $\{\chi_{a_\xi} : \xi < \omega_1\}$  is concentrated in the topology of pointwise convergence around  $\chi_{a \cup c}$ . This follows from Claim 2.  $\square$

**Question 2.6.** Suppose that  $K$  is compact scattered of countable height such that closures of countable sets are countable. Is it true that  $C(K)$  is weakly Lindelöf? Consistently? For  $K$  of weight  $< \omega_\omega$ ?

**Theorem 2.7.** There is a compact scattered space of height three, such that  $C(K)$  is weakly Lindelöf and has an uncomplemented copy of  $c_0(\omega_1)$ .

**Proof.** Consider as  $K$  the ladder system space which is scattered and of height three and the  $C(K)$  has the weak Lindelöf property (see [18,5]). Of course  $\{1_{\{\xi\}} : \xi \in S(\omega_1)\}$  generates a copy of  $c_0(\omega_1)$ . Call this copy  $X \subseteq C(K)$ . Let  $C_0(K) = \{f \in C(K) : f(\omega_1) = 0\}$ . It is a hyperplane of  $C(K)$  containing  $X$  and so, complemented in  $C(K)$ , hence it is enough to prove that there is no projection from  $C_0(K)$  onto  $X$ . We will use the fact that the dual to  $C_0(K)$  is isometric to the space of Radon measures on  $[0, \omega_1]$  which vanish on  $\{\omega_1\}$ .

Suppose  $P : C_0(K) \rightarrow X$  is a projection onto  $X$ . For  $\xi \in S(\omega_1)$  consider the measures  $\mu_\xi = P^*(\delta_\xi)$ . Note that  $P^*(\delta_\xi)(\{\xi\}) = P(1_{\{\xi\}})(\xi) = 1_{\{\xi\}}(\xi) = 1$  and  $P^*(\delta_\xi)(\{\eta\}) = P(1_{\{\eta\}})(\xi) = 1_{\{\eta\}}(\xi) = 0$  for any distinct  $\xi, \eta \in S(\omega_1)$ . That is, for each  $\xi \in S(\omega_1)$  we have  $\mu_\xi(\{\xi\}) = 1$  and  $|\mu_\xi|(S(\omega_1) \setminus \{\xi\}) = 0$ , since all measures are atomic.

Using the standard closure argument and the fact that the supports of Radon measures in the dual to  $C_0(K)$  are countable and do not contain  $\omega_1$ , it is easy to find an  $\alpha \in L(\omega_1)$  such that for each  $\xi < \alpha$  we have  $|\mu_\xi|([(\alpha, \omega_1)]) = 0$ . Now note that for  $\xi < \alpha$  we have

$$P(1_{S_\alpha \cup \{\alpha\}})(\xi) = \mu_\xi(S_\alpha \cup \{\alpha\}) = \mu_\xi(S_\alpha) = 1_{S_\alpha}(\xi).$$

However there is no such function in  $X$  which completes the proof of the theorem.  $\square$

Note that three is the smallest possible height of a scattered space where we can have the above result. This is because in the case of height two,  $K^{(1)} = K'$  must be finite, and so  $K$  is finite union of one-point compactifications of discrete spaces and so, for example, a version of Sobczyk's theorem obtained in [2] applies. Also a similar argument and the result of Godefroy, Kalton and Lancien (Theorem 4.8 of [9]) implies that there cannot be such  $C(K)$  which is WCG.

However any weakly Lindelöf  $C(K)$  for  $K$  nonmetrizable of finite height also has a complemented copy of  $c_0(\omega_1)$ :

**Theorem 2.8.** For every weakly Lindelöf  $C(K)$  with  $K$  compact nonmetrizable scattered of finite height there is a complemented copy of  $c_0(\omega_1)$  in  $C(K)$ .

**Proof.** Take a point  $x \in K$  of the smallest height which does not have a countable neighborhood. The compactness and the fact that  $K$  is nonmetrizable and so, uncountable imply the existence of such an  $x$ . Say  $x \in K^{(n+1)}$ . Let  $V$  be a neighborhood of  $x$  witnessing that it is isolated in  $K^{(n+1)}$ . Let  $m < n+1$  be the biggest integer such that  $V \cap K^{(m)}$  is uncountable and let  $U$  be the union of some countable neighborhoods of the countably many points of  $[K^{(m+1)} \setminus K^{(n+1)}] \cap V$ . Take  $\{x_\xi : \xi \in \omega_1\}$  from  $V \cap (K^{(m)} \setminus U)$ . This means that  $x$  is the unique accumulation point of  $\{x_\xi : \xi \in \omega_1\}$ . Now by induction on  $\alpha < \omega_1$  we construct a sequence of clopen sets  $U_\alpha$  and  $\xi_\alpha \in \omega_1$  such that

- (1)  $x_{\xi_\alpha} \in U_\alpha \subseteq V$ ,
- (2)  $U_\alpha$ s are countable and pairwise disjoint.

The inductive step  $\alpha < \omega_1$  follows from the fact that  $\bigcup_{\beta < \alpha} U_\beta$  is countable, and hence has countable closure by 2.2, so there is  $\xi_\alpha$  outside of this closure. We pick  $U_\alpha$  as its clopen neighborhood included in  $V$  which moreover may be assumed to be countable by the choice of  $x$ .

Now, consider the space generated by  $\{\chi_{U_\alpha} : \alpha < \omega_1\}$ . It is isomorphic to  $c_0(\omega_1)$  because for every  $(t_\alpha)_{\alpha < \omega_1} \in c_0(\omega_1)$  the function  $\sum_{\alpha \in \omega_1} t_\alpha \chi_{U_\alpha}$  is in  $C(K)$ . Now if  $f \in C(K)$ , then  $(f(x_{\xi_\alpha}) - f(x))_{\alpha \in \omega_1}$  is in  $c_0(\omega_1)$  because  $x$  is the only accumulation point of  $\{x_\xi : \xi \in \omega_1\}$ .

Define an operator

$$P(f) = \sum_{\alpha \in \omega_1} (f(x_{\xi_\alpha}) - f(x)) \chi_{U_\alpha}.$$

It is the required projection.  $\square$

### 3. A consequence of the $P$ -ideal dichotomy

**Lemma 3.1.** Suppose that  $K$  is an uncountable compact scattered space where closures of countable sets are countable,  $Y \subseteq K$  and  $x_0 \in K$ . Then

$$I_{x_0, Y} = \{A \in [Y]^{\leq \omega}; A \text{ does not have accumulation points different than } x_0\}$$

is a  $P$ -ideal of countable subsets of  $Y$ .

**Proof.** Note that closed subsets of  $K$  which miss  $x_0$  may intersect an element of  $I$  only on a finite set.

Let  $A_n \in I$  for  $n \in \mathbb{N}$ . Letting  $X \subseteq K$  be the closure of  $\bigcup_{n \in \mathbb{N}} A_n$ , by the hypothesis we can enumerate  $X \setminus \{x_0\}$  as  $\{y_n : n \in \mathbb{N}\}$ . Let  $U_n$  be an open neighborhood of  $y_n$  such that  $x_0 \notin \overline{U_n}$ . Put

$$A = \bigcup_{k \in \mathbb{N}} (A_k \setminus (U_1 \cup \dots \cup U_k)).$$

First we will prove that  $A_n \setminus A$  is finite for each  $n \in \mathbb{N}$ :

$$\begin{aligned} A_n \setminus A &= A_n \setminus \bigcup_{k \in \mathbb{N}} (A_k \setminus (U_1 \cup \dots \cup U_k)) \\ &= \bigcap_{k \in \mathbb{N}} [A_n \setminus (A_k \setminus (U_1 \cup \dots \cup U_n))] \subseteq A_n \setminus (A_n \setminus (U_1 \cup \dots \cup U_n)) \\ &= A_n \cap (U_1 \cup \dots \cup U_n) = (A_n \cap U_1) \cup \dots \cup (A_n \cap U_n). \end{aligned}$$

But the last set is finite because  $A_n \cap \overline{U_i}$  is finite for each  $i, n \in \mathbb{N}$  by the first sentence of this proof.

Now let us prove that  $A \in I$ . Note that  $A$  is a subset of the closed set  $\{y_n : n \in \mathbb{N}\} \cup \{x_0\}$  so the only accumulation points of  $A$  could be  $y_n$ s or  $x_0$ , but for each  $n \in \mathbb{N}$  the intersection of  $A$  with the neighborhood  $U_n$  of  $y_n$  is finite, hence  $A \in I$ , as required.  $\square$

**Theorem 3.2.** Assume the  $P$ -ideal dichotomy. Let  $K$  be an uncountable scattered compact space of countable height such that  $C(K)$  is weakly Lindelöf. Suppose  $T : c_0(\omega_1) \rightarrow C(K)$  is an isomorphism onto its image. Then there is an uncountable  $E \subseteq \omega_1$  such that  $T[c_0(E)]$  is complemented in  $C(K)$ .

**Proof.** Let  $f_\xi = T(1_{\{\xi\}})$  and let  $x_\xi \in K$  and  $\varepsilon > 0$  be such that  $|f_\xi(x_\xi)| > \varepsilon$  holds for each  $\xi \in \omega_1$ .

Note that  $\|1_{\{\xi_1\}} + \dots + 1_{\{\xi_n\}}\| = 1$  and for each  $x \in K$  we have  $\|T(1_{\{\xi_1\}} + \dots + 1_{\{\xi_n\}})\| \geq |\sum_{1 \leq i \leq n} f_{\xi_i}(x)|$ , so given  $x \in K$  can be equal to at most finitely many  $x_\xi$ s, and so we may assume that all  $x_\xi$ 's are distinct.

**Claim.** There is an uncountable  $E_0 \subseteq \omega_1$  such that  $\{x_\xi : \xi \in E_0\}$  is discrete and its closure is the one point compactification of  $\{x_\xi : \xi \in E_0\}$ .

**Proof.** Consider  $K_0 \subseteq K$  equal to the closure of  $\{x_\xi : \xi \in \omega_1\}$  in  $K$ . It is, by 2.1, like  $K$ , a compact scattered space of countable height such that  $C(K_0)$  is weakly Lindelöf. By compactness and the fact that  $K_0$  is uncountable, there must be a point  $x_\infty \in K_0$  without a countable neighborhood. Assume that  $x_\infty$  is such a point of the smallest possible height. Consider a clopen neighborhood  $U$  of  $x_\infty$  consisting of points of smaller heights. Note that  $U$  contains uncountably many points  $x_\xi$ s because otherwise the density of  $\{x_\xi : \xi \in \omega_1\}$  in  $K_0$  would contradict the countability of closures of countable sets 2.2.

The considerations of the previous paragraph and 2.1 imply that by taking  $U$  instead of  $K$  we may assume that  $K$  has only one point  $x_\infty$  which does not have countable neighborhoods.

Now we will use the  $P$ -ideal dichotomy to obtain  $E_0$  such that the only accumulation point of  $\{x_\xi : \xi \in E_0\}$  is  $x_\infty$ . Let  $Y = \{x_\xi : \xi \in \omega_1\}$  and consider the ideal  $I = I_{x_\infty, Y}$ . It is impossible to have an uncountable subset  $A$  of  $Y$  such that  $A \perp I$ , because the only complete accumulation point of an uncountable set can be  $x_\infty$  (other points have countable neighborhoods) but by 2.3 this would give a countable infinite  $B \subseteq A$  such that  $B \in I$ . So, the second alternative of the  $P$ -ideal dichotomy

fails, thus the first must hold, that is, there is an uncountable  $X \subseteq Y$  such that all countable subsets of  $X$  are in the ideal. By 2.3, this means that  $x_\infty$  is the only accumulation point of  $X$  and so  $X \cup \{x_\infty\}$  is the one-point compactification of  $X$ . Let  $E_0$  be such that  $X = \{x_\xi : \xi \in E_0\}$  which completes the proof of the claim.  $\square$

It is clear that  $\{x_\xi : \xi \in E_0\} \cup \{x_\infty\}$  is homeomorphic to a one-point compactification of  $\omega_1 = |X|$ .

Note that for all but countably many  $\xi \in E_0$  we must have  $f_\xi(x_\infty) = 0$ , because otherwise there would be  $f_{\xi_1}, \dots, f_{\xi_n}$  such that  $|f_{\xi_1}(x_\infty) + \dots + f_{\xi_n}(x_\infty)| > \|T\|$  which would contradict the fact that  $1_{\{\xi_1, \dots, \xi_n\}} = 1_{\xi_1} + \dots + 1_{\xi_n}$  has norm one in  $c_0(\omega_1)$ . Let  $E_1 \subseteq E_0$  be cocountable such that  $f_\xi(x_\infty) = 0$  for all  $\xi \in E_1$ .

A similar argument shows that for any countable  $F \subseteq E_1$  for all but countably many  $\xi \in E_1$  we have  $f_\xi|_F = 0$ .

Since  $\{x_\xi : \xi \in E_1\} \cup \{x_\infty\}$  is homeomorphic to one-point compactification of  $\{x_\xi : \xi \in E_1\}$  it follows from the choice of  $E_1$  that for each  $\xi \in E_1$  we have that  $f_\xi(x_\eta) = 0$  for all but countably many  $\eta \in E_1$ .

The last two observations imply that by induction we may construct an uncountable set  $E \subseteq E_1$  such that  $f_\xi(x_\eta) = 0$  for any distinct  $\xi, \eta \in E$ .

Now we claim that  $T[c_0(E)]$  is complemented in  $C(K)$ . Note that for each  $f \in C(K)$  the sequence  $(\frac{f(x_\xi) - f(x_\infty)}{f_\xi(x_\xi)})_{\xi \in E}$  is in  $c_0(E) \subseteq c_0(\omega_1)$ . Consider the operator:

$$P(f) = T \left[ \left( \frac{f(x_\xi) - f(x_\infty)}{f_\xi(x_\xi)} \right)_{\xi \in E} \right]$$

for  $f \in C(K)$ . Then  $P(f_\eta) = T(1_{\{\eta\}}) = f_\eta$  for  $\eta \in E$  and hence  $P$  is the identity on its image and so a projection.  $\square$

**Corollary 3.3.** Assume the  $P$ -ideal dichotomy. Let  $K$  be an uncountable scattered compact space of countable height such that  $C(K)$  is weakly Lindelöf. Then  $C(K)$  has complemented copies of  $c_0(\omega_1)$ .

**Proof.**  $K \setminus K^{(1)}$  of such a space must be uncountable by 2.2. So it is enough to note that the closure of finite linear combinations of characteristic functions of elements of  $K \setminus K^{(1)}$  is a copy of  $c_0(\omega_1)$  in  $C(K)$ . Now apply 3.2.  $\square$

#### 4. A construction of a weakly Lindelöf $C(K)$ with few operators

Although the main result of this section concerns a space of height  $\omega + 1$ , let us first note that  $\clubsuit$  already has an impact on scattered spaces of finite height. The following result shows that the conclusion of 3.2 cannot be obtained in  $\text{ZFC} \pm \text{CH}$  alone. Also compare it with 2.7.

**Theorem 4.1.** Assume  $\clubsuit$ . There is a compact scattered  $K$  of height three such that  $C(K)$  is weakly Lindelöf and there is a copy of  $c_0(\omega_1)$  in the  $C(K)$  such that for no uncountable  $E \subseteq \omega_1$  the space  $c_0(E)$  is complemented in  $C(K)$ .

**Proof.** Let  $S_\alpha$ s be from  $\clubsuit$ , see Definition 1.2. For all  $\alpha \in L(\omega_1)$  define

$$T_\alpha = \{\xi + 1 : \xi \in S_\alpha\}.$$

It is clear that:

- (1)  $T_\alpha \subseteq \alpha$  and  $T_\alpha \subseteq S(\omega_1)$ ,
- (2)  $T_\alpha$  converges to  $\alpha$ ,
- (3) for every uncountable  $X \subseteq S(\omega_1)$  there is  $\alpha \in L(\omega_1)$  such that  $T_\alpha \subseteq X$ .

Let  $K$  be the ladder system space obtained using the above  $T_\alpha$ s. The proof is similar to that of 2.7. We will use the same notation with the exception that instead of  $X$  we will consider  $c_0(E)$  equal to the closure of  $\{1_{\{\xi\}} : \xi \in E\}$  for any uncountable  $E \subseteq S(\omega_1)$ . Also we will shorten some arguments which are already in the proof of 2.7. Of course  $c_0(E)$  is a copy of  $c_0(\omega_1)$ . Again it is enough to prove that there is no projection from  $C_0(K)$  onto  $c_0(E)$ .

Suppose  $P_E : C_0(K) \rightarrow c_0(E)$  is a projection onto  $c_0(E)$ . For  $\xi \in E$  consider the Radon measures on  $K$  given by  $\mu_\xi = P_E^*(\delta_\xi)$ . For  $\xi \in E$  we have  $\mu_\xi(\{\xi\}) = 1$  and  $\mu_\xi(\{\eta\}) = 0$  for  $\eta \in E \cup \{\omega_1\}$  and  $\xi \in E$  distinct than  $\eta$ .

Using the standard closure argument and the fact that Radon measures on scattered spaces have countable carriers, it is easy to find a closed and unbounded  $C_E \subseteq L(\omega_1)$  such that for each  $\alpha \in C_E$  and for each  $\xi \in E \cap \alpha$  we have  $|\mu_\xi|([\alpha, \omega_1]) = 0$ . Thin out  $E$  to an uncountable  $E_1 \subseteq E$  such that the only accumulation points of  $E_1$  in the order topology are in  $C_E$ , for example by choosing at most one element of  $E$  between any two consecutive points of  $C_E$ . Now apply (3) to find  $\alpha \in L(\omega_1)$  such that  $T_\alpha \subseteq E_1$ . By the choice of  $E_1$  we have that  $\alpha \in C_E$  since  $T_\alpha$  converges to  $\alpha$  in the order topology. Now for  $\xi \in T_\alpha$

$$P_E(\chi_{T_\alpha \cup \{\alpha\}})(\xi) = P_E^*(\delta_\xi)(T_\alpha \cup \{\alpha\}) = P_E^*(\delta_\xi)(\{\xi\}) = 1.$$

As  $T_\alpha$  is infinite,  $P_E(\chi_{T_\alpha \cup \{\alpha\}})$  does not belong to  $c_0(E)$ , a contradiction.  $\square$

To describe the main construction of this section, we will need more terminology and some lemmas. Let  $\alpha \leq \omega_1$ . Put  $F_0(\alpha) = \alpha = \{\beta : \beta < \alpha\}$  and for  $n > 0$  let  $F_{n+1}(\alpha)$  consist of all finite sequences of elements of  $F_n(\alpha)$ . Finally, define  $F(\alpha) = \bigcup_{n \in \mathbb{N}} F_n(\alpha)$ . For  $x \in F(\alpha)$  such that  $x \in F_n(\alpha)$ , define by induction on  $n \in \mathbb{N}$  the *support* of  $x$  denoted  $\text{supp}(x)$  as the union of all sets  $\text{supp}(y)$  where  $y$  is a term of the sequence  $x$  with  $\text{supp}(x) = \{x\}$  for  $x \in F_0(\alpha)$ . If  $x, y \in F(\alpha)$ ,  $\alpha \leq \omega_1$ , then we say that  $x < y$  if and only if  $\xi < \eta$  for every  $\xi \in \text{supp}(x)$  and  $\eta \in \text{supp}(y)$ . If  $S \subseteq F(\alpha)$ , we say that it is *consecutive* if and only if  $x < y$  or  $y < x$  whenever  $x$  and  $y$  are two distinct elements of  $S$ . If  $S \subseteq F(\alpha)$  is infinite, we say that it *converges* to  $\gamma \in L(\omega_1)$  if and only if  $S$  is consecutive and for every  $\beta < \gamma$  the set

$$\{x \in S : \text{supp}(x) \not\subseteq (\beta, \gamma)\} \text{ is finite.}$$

If  $F \in F_1(\alpha)$  for some  $\alpha < \omega_1$ , abusing the notation, we may identify it with the set of its terms.

**Lemma 4.2.** *If  $S \subseteq F(\omega_1)$  is uncountable made of elements with pairwise disjoint supports, then there is an uncountable  $S' \subseteq S$  which is consecutive.*

**Definition 4.3.**  $\clubsuit'$  is the following sentence: There is a sequence  $(S'_\alpha)_{\alpha \in L(\omega_1)}$  such that for each  $\alpha \in L(\omega_1)$ :

- (1)  $S'_\alpha \subseteq F(\alpha)$ ,
- (2)  $S'_\alpha$  converges to  $\alpha$ ,
- (3) for every uncountable consecutive  $X \subseteq F(\omega_1)$  there is  $\alpha \in L(\omega_1)$  such that  $S'_\alpha \subseteq X$ .

Similar axiom as  $\clubsuit'$  was considered, for example, in [10] in a Boolean algebraic context not distant from ours.

**Lemma 4.4.**  $\clubsuit$  and  $\clubsuit'$  are equivalent.

**Proof.** There is a bijection  $\phi : \omega_1 \rightarrow F(\omega_1)$ . Using the standard closure argument one can show that the set  $C_\phi$  of all  $\alpha \in \omega_1$  such that  $\phi[\alpha] = F(\alpha)$  is closed and unbounded in  $\omega_1$  (Chapter 2, §6 of [13]).

Define  $S'_\alpha = \phi[S_\alpha]$  if such an  $S'_\alpha$  is included in  $F(\alpha)$  and converges to  $\alpha$  and otherwise  $S'_\alpha$  is any sequence in  $F(\alpha)$  convergent to  $\alpha$ . Let  $X \subseteq F(\omega_1)$  be uncountable and consecutive. Consider  $Y' = \phi^{-1}[X]$  and

$$Y = \{\min(Y' \cap [\alpha, \beta)) : \alpha, \beta \in C_\phi, [\alpha, \beta) \cap C_\phi = \{\alpha\}, Y' \cap [\alpha, \beta) \neq \emptyset\},$$

that is, we allow in  $Y \subseteq Y'$  just one element of  $Y'$  from the interval between two consecutive elements of  $C_\phi$ .

Now use  $\clubsuit$  to find  $S_\alpha \subseteq Y$ . Note that the construction of  $Y$  guarantees that the elements of  $\phi[S_\alpha] \subseteq X$  have their supports included and unbounded in  $\alpha$ . But this, together with the hypothesis that  $X$  is consecutive and  $S_\alpha$  converges to  $\alpha$ , implies that  $\phi[S_\alpha]$  converges to  $\alpha$  and so  $S'_\alpha = \phi[S_\alpha]$  and  $S'_\alpha \subseteq X$ , as required in  $\clubsuit'$ .

The other direction is clear.  $\square$

**Lemma 4.5.** *Suppose that  $K$  is a compact space,  $(\mu_\alpha)_{\alpha \in \omega_1}$  is a sequence of Radon measures on  $K$  such that for each finite  $\Delta \subseteq K$  we have  $|\mu_\alpha|(\Delta) = 0$  for all but countably many  $\alpha \in \omega_1$ . Assume that  $\varepsilon > 0$  and that  $F'_\alpha$ s for  $\alpha \in \omega_1$  are finite subsets of  $K$  such that  $|\mu_\alpha|(K \setminus F'_\alpha) < \varepsilon$  for each  $\alpha \in \omega_1$ . Then there is an uncountable  $X \subseteq \omega_1$  and  $F_\alpha \subseteq F'_\alpha$  for  $\alpha \in X$  such that  $|\mu_\alpha|(K \setminus F_\alpha) < \varepsilon$  and the sequence  $(F_\alpha)_{\alpha \in X}$  is pairwise disjoint.*

**Proof.** Apply the  $\Delta$ -system lemma (Chapter 2, 1.5 of [13]) to  $(F'_\alpha)_{\alpha \in \omega_1}$  obtaining an uncountable  $X' \subseteq \omega_1$  such that  $(F'_\alpha)_{\alpha \in X'}$  forms a  $\Delta$ -system with root  $\Delta$ . Now take such a cocountable  $X \subseteq X'$  that  $|\mu_\alpha|(\Delta) = 0$  for  $\alpha \in X$  and put  $F_\alpha = F'_\alpha \setminus \Delta$ .  $\square$

**Lemma 4.6.** *Suppose  $K$  is a compact space and  $x_0 \in K$  is the only point in  $K$  which does not have countable neighborhoods. Let  $C_0(K) = \{f \in C(K) : f(x_0) = 0\}$ . Let  $T : C_0(K) \rightarrow C_0(K)$  be a bounded linear operator and  $T^*$  its adjoint. For  $\alpha \in \omega_1$  let  $x_\alpha$ 's be distinct points of  $K \setminus \{x_0\}$ . Then for each finite  $\Delta \subseteq K \setminus \{x_0\}$ , for all but countably many  $\alpha \in \omega_1$  we have  $|T^*(\delta_{x_\alpha})|(\Delta) = 0$ .*

**Proof.** If the lemma is false, then w.l.o.g. we may assume that  $|T^*(\delta_{x_\alpha})|(\Delta) > \varepsilon$  for some  $\varepsilon > 0$  and for all  $\alpha \in \omega_1$  and moreover that  $\Delta$  is a singleton  $\{y\}$ .

Let  $V$  be a countable and so metrizable neighborhood of  $y$ . For each  $\alpha \in \omega_1$  find an open  $V_\alpha \subseteq V$  from a fixed countable basis at  $y$  of  $V$  such that  $|\mu_\alpha|(V_\alpha \setminus \{y\}) < \varepsilon/2$  where  $\mu_\alpha = T^*(\delta_{x_\alpha})$ . For uncountable set  $X \subseteq \omega_1$  the set  $V_\alpha$  is the same, say  $U$ .

Find a function  $f \in C(K)$  of norm one and such that  $\text{supp}(f) \subseteq U$  and  $f(y) = 1$ . Then  $|\int f d\mu_\alpha| > \varepsilon - \varepsilon/2 = \varepsilon/2$  for  $\alpha \in X$ . In other words  $|T(f)(x_\alpha)| > \varepsilon/2$  for all  $\alpha \in X$ . But  $x_0$  can be the only complete accumulation point of  $\{x_\alpha : \alpha \in X\}$  since other points have countable neighborhoods. So  $T(f)(x_0) \neq 0$ , contradicting the hypothesis that  $T$  is into  $C_0(K)$ .  $\square$

Now we proceed to the main result of this section. First we will construct a topology on  $[0, \omega_1]$  whose properties will be proved in what follows. The construction is based on Ostaszewski's construction [15] the way it is presented in [19]. The



difference is that we want to take care of operators and so we need to work with  $F(\omega_1)$  and  $\clubsuit'$  instead of  $\clubsuit$ , moreover we want to make sure that the space has height  $\omega + 1$  unlike the Ostaszewski space. However we do not need CH.

By induction on  $\gamma < \omega_1$  we will construct a topology  $\tau_\gamma$  on  $[0, \gamma]$  such that  $[0, \gamma]$  with the topology  $\tau_\gamma$  is:

- (1) 0-dimensional, locally compact,
- (2) scattered of height not bigger than  $\omega$ ,
- (3)  $\tau_\gamma$  refines the order topology on  $[0, \gamma]$ .

If  $\gamma < \gamma' < \omega_1$ , then  $[0, \gamma]$  with  $\tau_\gamma$  is an open subspace of  $[0, \gamma']$  with  $\tau_{\gamma'}$ , that  $\tau_{\gamma'}$  is a conservative extension of  $\tau_\gamma$  (see [19, 2.3]).

The topology is given by local bases  $\mathcal{B}(\gamma)$  at  $\gamma$ . If  $\gamma$  is a successor then we put  $\mathcal{B}(\gamma) = \{\{\gamma\}\}$ . If  $\gamma$  is a limit ordinal, then let  $\tau'_\gamma$  be the topology on  $[0, \gamma)$  defined as the simple limit  $\sum_{\beta < \gamma} \tau_\beta$  of the topologies  $\tau_\beta$  for  $\beta < \gamma$  (see [19, 2.3]). It easily follows from the inductive assumption that  $[0, \gamma)$  with  $\tau'_\gamma$  is 0-dimensional, locally compact, scattered of height not bigger than  $\omega$  and  $\tau'_\gamma$  refines the order topology on  $[0, \gamma)$ .

In the case of  $\gamma \in L(\omega_1)$  we consider  $S'_\gamma$  from  $\clubsuit'$ . In this case we also put  $\mathcal{B}(\gamma) = \{\{\gamma\}\}$  unless  $S'_\gamma$  consists only of quadruples from  $\omega_1 \times \omega_1 \times \omega_1^{<\omega} \times \omega_1^{<\omega}$  such that the first two terms of all elements of  $S'_\gamma$  are distinct and have their heights in  $\tau'_\gamma$  uniformly bounded by some  $k \in \mathbb{N}$ . If this is the case, then  $\mathcal{B}(\gamma)$  and its description requires some introductory comments. As  $S'_\gamma$  converges to  $\gamma$ , we can enumerate it as  $s_n(\gamma) = (\xi_n(\gamma), \eta_n(\gamma), F_n(\gamma), H_n(\gamma))$  for  $n \in \mathbb{N}$ , so that  $m < n$  implies  $\text{supp}(s_m(\gamma)) < \text{supp}(s_n(\gamma))$ . Now we will use the inductive hypotheses (1)–(3) to construct two sequences of  $\tau'_\gamma$ -open sets  $W_n^0(\gamma)$  and  $W_n^1(\gamma)$  for  $n \in \mathbb{N}$  such that

- $\xi_n(\gamma) \in W_n^0(\gamma)$ ,  $\eta_n(\gamma) \in W_n^1(\gamma)$ ,
- the heights in  $\tau'_\gamma$  of all points of  $W_n^0(\gamma)$  and  $W_n^1(\gamma)$  are not bigger than  $k$ ,
- $W_n^0(\gamma)$  and  $W_n^1(\gamma)$  are compact,
- $W_n^0(\gamma) \subseteq (\max(\text{supp}(s_{n-1}(\gamma))), \xi_n(\gamma)]$ ,  $W_n^1(\gamma) \subseteq (\max(\text{supp}(s_{n-1}(\gamma))), \eta_n(\gamma)]$ ,
- $W_n^0(\gamma) \cap ((\{\eta_n(\gamma)\} \cup F_n(\gamma) \cup H_n(\gamma)) \setminus \{\xi_n(\gamma)\}) = \emptyset$  and  $W_n^1(\gamma) \cap ((\{\xi_n(\gamma)\} \cup F_n(\gamma) \cup H_n(\gamma)) \setminus \{\eta_n(\gamma)\}) = \emptyset$ .

The construction is straightforward, as all intervals  $(\beta, \beta']$  with  $\beta < \beta' < \gamma$  must be  $\tau'_\gamma$  open by the fact that  $\tau'_\gamma$  refines the order topology. Above, abusing the notation, we identified  $H_n(\gamma)$ s and  $F_n(\gamma)$ s with the sets of their terms.

For  $n \in \mathbb{N}$  put  $W_n(\gamma) = W_n^0(\gamma) \cup W_n^1(\gamma)$  for  $n$  odd and  $W_n(\gamma) = W_n^0(\gamma)$  for  $n$  even. Note that, in particular  $W_n(\gamma)$ s are pairwise disjoint, compact and open. Now we are ready to define  $\mathcal{B}(\gamma)$  completing the definition of  $\tau_\gamma$ .  $\mathcal{B}(\gamma)$  is the collection of sets of the form

$$V_m(\gamma) = \bigcup \{W_n(\gamma) : n > m\} \cup \{\gamma\}$$

where  $m \in \mathbb{N}$ .

This completes the inductive step. We check that (1)–(3) hold for  $\tau_\gamma$ . (1) is standard, e.g. as in [19, 5.3]. (2) follows from the fact that the height of the only new point cannot be bigger than  $k + 1 \in \mathbb{N}$  because all points of  $V_m(\gamma)$ s have heights not bigger than  $k$ . (3) follows from the fact that for every  $\beta < \gamma$  there will be  $V_m(\gamma)$  such that  $V_m(\gamma) \subseteq (\beta, \gamma]$ . The final topology  $\tau_{\omega_1}$  on  $[0, \omega_1]$  is the one-point compactification of  $[0, \omega_1)$  with the topology given as the simple limit  $\sum_{\gamma < \omega_1} \tau_\gamma$ .  $[0, \omega_1]$  with this topology will be denoted  $K_0$ .

The construction implies that we have the following two lemmas:

**Lemma 4.7.** Suppose  $\gamma < \omega_1$ . Then we have:

- (1)  $\xi_n(\gamma) \in V_m(\gamma)$  iff  $n > m$ ,
- (2) for  $n \in \mathbb{N}$  odd we have  $\eta_n(\gamma) \in V_m(\gamma)$  iff  $n > m$ ,
- (3) for  $n \in \mathbb{N}$  even and any  $m \in \mathbb{N}$  we have  $\eta_n(\gamma) \notin V_m(\gamma)$ ,
- (4)  $V_m(\gamma) \cap ((F_n(\gamma) \cup H_n(\gamma)) \setminus \{\xi_n(\gamma), \eta_n(\gamma)\}) = \emptyset$  for each  $n, m \in \mathbb{N}$ .

**Lemma 4.8.**  $K_0$  is a compact, scattered space of height  $\leq \omega + 1$  whose topology restricted to  $[0, \omega_1)$  refines the order topology of  $\omega_1$ . The only point of  $K_0$  which does not have a countable neighborhood is  $\omega_1$ . Closures of countable sets in  $K_0$  are countable.

**Proof.** The properties follow from the construction. The statement on countable closures follows from the fact that for  $\alpha < \omega_1$  the subspaces  $[0, \alpha] \cup \{\omega_1\}$  are closed in  $K_0$  as  $(\alpha + 1, \omega_1)$  is open in the order topology.  $\square$

Now we proceed to the properties of the  $C_0(K_0) = \{f \in C(K_0) : f(\omega_1) = 0\}$ . Let  $T : C_0(K_0) \rightarrow C_0(K_0)$  be an operator. For each  $\alpha \in \omega_1$  define  $c_\alpha^T = T^*(\delta_\alpha)(\{\alpha\})$ . We will often skip the superscript  $T$  if it is clear from the context.

**Lemma 4.9.** Suppose  $T : C_0(K_0) \rightarrow C_0(K_0)$  is a linear operator. There is  $c \in \mathbb{R}$ , such that eventually for  $\alpha < \omega_1$  we have  $c_\alpha^T = c$ .

**Proof.** Fix  $T : C_0(K_0) \rightarrow C_0(K_0)$  and suppose that the lemma does not hold. We will obtain a contradiction from this assumption. First note that by the regularity of the cardinal  $\omega_1$  and by the fact that uncountable sets of the reals have uncountably many concentration points, there must be two disjoint uncountable  $A, B \subseteq \omega_1$  and two rationals  $p, q$  such that for all  $\xi \in A$  and for all  $\eta \in B$

$$c_\xi < p < q < c_\eta.$$

For each  $\eta \in B$  choose a finite  $G_\eta \subseteq \omega_1$ , such that

$$|T^*(\delta_\eta)|(K \setminus G_\eta) < \frac{q-p}{6}.$$

We may assume that  $G_\eta$ s form a  $\Delta$ -system with root  $G \subseteq \gamma$  for some  $\gamma < \omega_1$ . Now we construct recursively two sequences  $(\xi_\alpha)_{\alpha \in \omega_1}, (\eta_\alpha)_{\alpha \in \omega_1} \subseteq \omega_1 \setminus \gamma$ , which for each  $\alpha \in \omega_1$  satisfy the following

$$\begin{aligned} \xi_\alpha &\in A, & \eta_\alpha &\in B, \\ \xi_\alpha, \eta_\alpha &< \xi_{\alpha'}, \eta_{\alpha'} & \text{ if } \alpha < \alpha' < \omega_1, \\ T^*(\delta_{\xi_\alpha})(\{\eta_\alpha\}) &= 0, \\ |T^*(\delta_{\eta_\alpha})(\{\xi_\alpha\})| &< \frac{q-p}{6}. \end{aligned}$$

This can be done in the following way. First choose  $\xi_\alpha \in A$  bigger than the previously chosen terms of the sequences and bigger than  $\gamma$ . Note that there is at most one  $\eta \in B$  such that  $\xi_\alpha \in G_\eta$ . So now choose  $\eta_\alpha \in B$  outside the countable support of  $T^*(\delta_{\xi_\alpha})$  and  $\eta_\alpha \neq \eta$ . This completes the construction of the sequences.

Now again use the fact that the measures  $T^*(\delta_{\xi_\alpha}), T^*(\delta_{\eta_\alpha}) \in M_0(K_0)$  are atomic to find for all  $\alpha \in \omega_1$  two finite sets  $F_\alpha, H_\alpha \subseteq K_0$  such that

$$\begin{aligned} \{\xi_\alpha, \eta_\alpha\} \cap (F_\alpha \cup H_\alpha) &= \emptyset, \\ |T^*(\delta_{\xi_\alpha})(K_0 \setminus (F_\alpha \cup \{\xi_\alpha, \eta_\alpha\}))| &< \frac{q-p}{3}, \\ |T^*(\delta_{\eta_\alpha})(K_0 \setminus (H_\alpha \cup \{\xi_\alpha, \eta_\alpha\}))| &< \frac{q-p}{6}. \end{aligned}$$

By Lemmas 4.5 and 4.6 we may assume that  $F_\alpha$ s are pairwise disjoint and  $H_\alpha$ s are pairwise disjoint. By thinning-out and Lemma 4.2 we may assume that  $X = \{(\xi_\alpha, \eta_\alpha, F_\alpha, H_\alpha); \alpha \in \omega_1\}$  is consecutive.

Applying  $\clubsuit'$  we obtain  $\gamma \in \text{Lim}(\omega_1)$  for which  $S'_\gamma \subseteq X$ . As in the construction of  $K_0$  we have an enumeration  $S'_\gamma = \{(\xi_n(\gamma), \eta_n(\gamma), F_n(\gamma), H_n(\gamma)); n \in \mathbb{N}\}$  in the increasing order.

By 4.7(4)  $V_m(\gamma) \cap (F_n(\gamma) \cup H_n(\gamma)) = \emptyset$  for each  $m, n \in \mathbb{N}$ . So, for all  $n \in \mathbb{N}$  we obtain

$$\begin{aligned} |T(\chi_{V_0(\gamma)})(\xi_n(\gamma))| &= |T^*(\delta_{\xi_n(\gamma)})(\chi_{V_0(\gamma)})| \\ &= |T^*(\delta_{\xi_n(\gamma)})(\{\xi_n(\gamma)\}) + 0 + T^*(\delta_{\xi_n(\gamma)})(V_0(\gamma) \cap K_0 \setminus (F_n(\gamma) \cup \{\xi_n(\gamma), \eta_n(\gamma)\}))| \\ &\leq c_{\xi_n(\gamma)} + |T^*(\delta_{\xi_n(\gamma)})(K_0 \setminus (F_n(\gamma) \cup \{\xi_n(\gamma), \eta_n(\gamma)\}))| < p + \frac{q-p}{3} = \frac{2p+q}{3}, \end{aligned}$$

where the first equality makes sense since the set  $V_0(\gamma)$  is clopen and so its characteristic function is continuous and 0 stands for  $T^*(\delta_{\xi_n(\gamma)})(\{\eta_n(\gamma)\})$  since  $T^*(\delta_{\xi_\alpha})(\{\eta_\alpha\}) = 0$  for each  $\alpha < \omega_1$  and  $S'_\gamma \subseteq X$ . On the other hand we have

$$\begin{aligned} |T(\chi_{V_0(\gamma)})(\eta_n(\gamma))| &= |T^*(\delta_{\eta_n(\gamma)})(\chi_{V_0(\gamma)})| \\ &\geq |T^*(\delta_{\eta_n(\gamma)})(\{\eta_n(\gamma)\})| - |T^*(\delta_{\eta_n(\gamma)})(\{\xi_n(\gamma)\})| \\ &\quad - |T^*(\delta_{\eta_n(\gamma)})(K_0 \setminus (H_n(\gamma) \cup \{\xi_n(\gamma), \eta_n(\gamma)\}))| \\ &\geq c_{\eta_n} - \frac{q-p}{6} - \frac{q-p}{6} > q - \frac{q-p}{3} = \frac{2q+p}{3}. \end{aligned}$$

However, the sequences  $(\xi_{2n+1}(\gamma))_{n \in \mathbb{N}}$  and  $(\eta_{2n+1}(\gamma))_{n \in \mathbb{N}}$  converge in the topology  $\tau_{\omega_1}$  to  $\gamma$  by 4.7(1) and (2), so the continuity of  $T(\chi_{V_0(\gamma)}) \in C_0(K_0)$  and the above inequalities imply

$$\begin{aligned} |T(\chi_{V_0(\gamma)})(\gamma)| &= \lim_{n \rightarrow \infty} |T(\chi_{V_0(\gamma)})(\xi_{2n+1}(\gamma))| \\ &\leq \frac{2p+q}{3} < \frac{2q+p}{3} \leq \lim_{n \rightarrow \infty} |T(\chi_{V_0(\gamma)})(\eta_{2n+1}(\gamma))| = |T(\chi_{V_0(\gamma)})(\gamma)|, \end{aligned}$$

which brings the hypothesis that the sequence  $(c_\alpha)_{\alpha \in \omega_1}$  is not eventually constant to a contradiction.  $\square$

**Theorem 4.10.** Every operator  $T : C(K_0) \rightarrow C(K_0)$  is of the form  $T = cI + S$  where  $S : C(K_0) \rightarrow C(K_0)$  has separable range.

**Proof.** As  $K_0$  contains convergent sequences, it has a complemented copy of  $c_0$  and so the hyperplanes of  $C(K_0)$ , in particular  $C_0(K_0)$  are isomorphic with the entire  $C(K_0)$ . Thus, it is enough to prove the theorem for operators on  $C_0(K_0)$  instead of  $C(K_0)$ .

Suppose that  $T : C_0(K_0) \rightarrow C_0(K_0)$ . Let  $c \in \mathbb{R}$  be given by 4.9. Note that considering  $T - cI$  we may w.l.o.g. assume that  $c = 0$ , that is, that there is  $\alpha_0 < \omega_1$  such that  $T^*(\delta_{\alpha})(\{\alpha\}) = 0$  for all  $\alpha_0 < \alpha < \omega_1$ . It will be enough to show that  $T$  has separable range. Suppose not, and let us arrive at a contradiction. Define

$$C_\alpha = \{f \in C_0(K_0) : f|(\alpha, \omega_1] = 0\},$$

and note that it is a separable closed subspace of  $C_0(K_0)$ .

It follows from our hypothesis that the image of  $T$  is not included in any  $C_\alpha$ , and so there is a strictly increasing sequence  $(\xi_\alpha : \alpha < \omega_1)$  such that  $\alpha < \xi_\alpha < \omega_1$  and there are  $f_\alpha \in C_0(K_0)$  such that  $|T(f_\alpha)(\xi_\alpha)| > 0$ .

As  $T(f_\alpha)(\xi_\alpha) = T^*(\delta_{\xi_\alpha})(f_\alpha)$ , possibly by thinning-out the sequence, we can find  $\eta_\alpha < \omega_1$  and  $\varepsilon > 0$  such that  $|T^*(\delta_{\xi_\alpha})(\{\eta_\alpha\})| > \varepsilon$ . Note that  $\xi_\alpha \neq \eta_\alpha$  by the hypothesis that  $c_\alpha = 0$ .

Note that the set of all  $\eta_\alpha$ 's is uncountable because otherwise the same  $\eta_\alpha$  would be repeated for uncountably many  $\alpha \in \omega_1$  contradicting 4.6. So, we may assume that all the  $\eta_\alpha$ s are distinct. Let  $F_\alpha \subseteq \omega_1$  be such a finite set that  $\xi_\alpha, \eta_\alpha \notin F_\alpha$  and  $|T^*(\delta_{\xi_\alpha})(K_0 \setminus (F_\alpha \cup \{\xi_\alpha, \eta_\alpha\}))| < \varepsilon/3$ .

Consider the sequence  $X = \{(\xi_\alpha, \eta_\alpha, F_\alpha, F_\alpha) : \alpha \in \omega_1\}$ . Applying 4.5 and 4.6 we may assume that  $F_\alpha$ s are pairwise disjoint. Applying 4.2 we may assume that it is consecutive.

Now use  $\clubsuit$ , to find  $\gamma \in L(\omega_1)$  such that  $S'_\gamma \subseteq X$ . As in the construction of  $K_0$  we have an enumeration of  $S'_\gamma$  in the increasing order as  $(\xi_n(\gamma), \eta_n(\gamma), F_n(\gamma), F_n(\gamma)) : n \in \mathbb{N}$ . We have

$$\begin{aligned} |T(\chi_{V_0(\gamma)})(\xi_n(\gamma))| &= |T^*(\delta_{\xi_n(\gamma)})(V_0(\gamma))| \\ &= |T^*(\delta_{\xi_n(\gamma)})(V_0(\gamma) \cap \{\eta_n(\gamma)\}) + T^*(\delta_{\xi_n(\gamma)})(V_0(\gamma) \setminus (F_n \cup \{\xi_n(\gamma), \eta_n(\gamma)\}))| \end{aligned}$$

and so if  $n \in \mathbb{N}$  is even 4.7(3) gives that

$$|T(\chi_{V_0(\gamma)})(\xi_n(\gamma))| = |T^*(\delta_{\xi_n(\gamma)})(V_0(\gamma) \setminus (F_n(\gamma) \cup \{\xi_n(\gamma), \eta_n(\gamma)\}))| < \varepsilon/3.$$

On the other hand, if  $n \in \mathbb{N}$  is odd 4.7(2) gives that

$$\begin{aligned} |T(\chi_{V_0(\gamma)})(\xi_n(\gamma))| &> |T^*(\delta_{\xi_n(\gamma)})(\eta_n(\gamma))| \\ &\quad - |T^*(\delta_{\xi_n(\gamma)})(V_0(\gamma) \setminus (F_n(\gamma) \cup \{\xi_n(\gamma), \eta_n(\gamma)\}))| > \varepsilon - \varepsilon/3 = 2\varepsilon/3. \end{aligned}$$

However, the sequences  $(\xi_{2n}(\gamma))_{n \in \mathbb{N}}$  and  $(\xi_{2n+1}(\gamma))_{n \in \mathbb{N}}$  converge in the topology  $\tau_{\omega_1}$  to  $\gamma$  by 4.7(1), so the continuity of  $T(\chi_{V_0(\gamma)}) \in C_0(K_0)$  and the above inequalities yield a contradiction.  $\square$

**Theorem 4.11.** Assume  $\clubsuit$ . There is a compact nonmetrizable, scattered space  $K_0$  of height  $\omega + 1$  such that  $C(K_0)$  is Lindelöf in the weak topology and every operator  $T$  on  $K_0$  is of the form  $T = cI + S$  where  $S$  has its range included in a separable complemented subspace isomorphic to  $c_0$  or to  $C_0(\omega^\omega)$ . In particular all copies of  $c_0(\omega_1)$  included in  $C(K_0)$  are not complemented and all decompositions  $C(K_0) = A \oplus B$  where  $A$  and  $B$  are infinite dimensional are such that  $A \sim c_0$  or  $A \sim C_0(\omega^\omega)$  and  $B \sim C(K_0)$  or  $B \sim C_0(\omega^\omega)$  or  $B \sim c_0$  and  $A \sim C(K_0)$ .

**Proof.** The weak Lindelöf property follows from 4.8 and 2.5. We will often use in this proof the fact that the hyperplanes of  $C(K)$ s for  $K$  scattered are isomorphic to the entire  $C(K)$ , this is a consequence of a well-known result that  $c_0$  is complemented in such spaces for  $K$  infinite.

**Claim 1.** There is no complemented copy of  $c_0(\omega_1)$  in  $C(K_0)$ .

**Proof.** We will show that every projection in  $C(K_0)$  has separable or coseparable range. Since  $c_0(\omega_1)$  can be decomposed into two complemented nonseparable subspaces, this will prove the claim. Let  $P : C(K_0) \rightarrow C(K_0)$  be a projection, i.e.,  $P^2 = P$ . We know that  $P = cI + S$  where  $c \in \mathbb{R}$  and  $S$  has a separable range. Thus we get  $(c^2 - c)I = -2cS - S^2 + S$  which has separable range, and so  $c = 0$  or  $c = 1$ . In other words  $P$  has separable range or  $I - P$  has separable range which completes the proof of the claim.  $\square$

It follows from 2.8 and 4.8 that  $ht(K_0)$  is actually equal to  $\omega + 1$ .

Now, for  $0 \leq \alpha < \beta < \omega_1$ ,  $\alpha$  being isolated in  $K_0$  (for example a successor or 0) consider a linear closed subspace of  $C(K_0)$  given by

$$C_{\alpha, \beta} = \{f \in C(K_0) : f|([0, \alpha) \cup (\beta, \omega_1]) = \text{constant}\}.$$

We assume that  $[0, 0) = \emptyset$ .

**Claim 2.** For each  $0 \leq \alpha < \beta < \omega_1$  such that  $\alpha$  is isolated, the space  $C_{\alpha,\beta}$  is complemented in  $C(K_0)$  and isomorphic to  $C([\alpha, \beta] \cup \{\omega_1\})$  which is isomorphic to  $c_0$  or to  $C_0(\omega^\omega)$ . Moreover  $\{f \in C_{\alpha,\beta} : f(\omega_1) = 0\}$  is complemented in  $C_0(K_0)$ .

**Proof.** Let  $r : K_0 \rightarrow K_0$  given by  $r(\xi) = \xi$  for  $\xi \in [\alpha, \beta]$  and  $r(\xi) = \omega_1$  otherwise. Since  $[0, \alpha] \cup (\beta, \omega_1)$  is open and, by the hypothesis that  $\alpha$  is isolated, also  $[\alpha, \beta]$  is open,  $r$  is a continuous retraction which witnesses that  $C_{\alpha,\beta}$  is complemented in  $C(K_0)$ . Also the induced projection sends elements of  $C_0(K_0)$  into  $\{f \in C_{\alpha,\beta} : f(\omega_1) = 0\}$  as required in the last part of the claim.

$C_{\alpha,\beta}$  is isomorphic to  $C(K)$  where  $K$  is obtained from  $K_0$  by identifying all points of  $K_0$  belonging to  $[0, \beta] \cup (\alpha, \omega_1]$  with  $\omega_1$ . Again, since both  $[0, \alpha] \cup (\beta, \omega_1)$  and  $[\alpha, \beta]$  are open, this  $K$  is homeomorphic to  $[\alpha, \beta] \cup \{\omega_1\}$ . It is also a compact scattered space of height  $\leq \omega + 1$ , and so homeomorphic to a countable successor ordinal less than  $\omega(\omega^\omega)$ . By the Bessaga–Pełczyński classification of the  $C(K)$ s for  $K$  being a countable ordinal [6] such spaces are isomorphic to  $c_0$  or  $C_0(\omega^\omega)$  which completes the proof of the claim.  $\square$

Now let us prove that any separable subspace  $X$  of  $C(K_0)$  is included in a complemented copy of  $c_0$  or  $C_0(\omega^\omega)$ . As basic neighborhoods of any  $\alpha < \omega_1$  in  $K_0$  are included in  $\alpha + 1$ , any function in  $C(K_0)$  is eventually constant. Considering a dense countable subset of  $X$  we note that there is  $\alpha < \omega_1$  such that  $X \subseteq C_{0,\alpha}$  and so Claim 2 implies that  $X$  is included in a complemented copy of  $c_0$  or  $C_0(\omega^\omega)$ .

So, we are left with the statement about the decompositions.

**Claim 3.** For every  $0 < \alpha < \omega_1$  the space

$$B_\alpha = \{f \in C(K_0) : f|_{[0, \alpha] \cup \{\omega_1\}} \equiv 0\}$$

contains a copy of  $C_0(\omega^\omega)$  which is complemented in  $C(K_0)$ .

**Proof.** As  $[\alpha + 1, \omega_1]$  is closed in  $K_0$  and  $[\alpha + 1, \omega_1)$  is open in  $K_0$ , we have that  $B_\alpha$  is isometric to  $C_0([\alpha + 1, \omega_1])$  and so isomorphic to  $C([\alpha + 1, \omega_1])$  where  $[\alpha + 1, \omega_1]$  is considered with the topology inherited from  $K_0$ . But  $[\alpha + 1, \omega_1]$  is a scattered compact space of height  $\leq \omega + 1$  as a subspace of  $K_0$ . Moreover, by Claim 1,  $B_\alpha$  cannot have a complemented copy of  $c_0(\omega_1)$  and hence by Theorem 2.8, the height of  $[\alpha + 1, \omega_1]$  is actually  $\omega + 1$ . Since  $K_0$  is a Frechet space by Lemma 2.3, we can find  $\alpha < \beta < \omega_1$  such that  $[\alpha + 1, \beta] \cup \{\omega_1\}$  is of height  $\omega + 1$ . However, this space is countable, and so, by the Mazurkiewicz–Sierpiński theorem it is homeomorphic to a countable ordinal which must be in the interval  $[\omega^\omega, \omega(\omega^\omega)]$ . The Bessaga–Pełczyński classification gives that  $C_0([\alpha + 1, \beta] \cup \{\omega_1\}) \sim C([\alpha + 1, \beta] \cup \{\omega_1\}) \sim C_0(\omega^\omega)$ . By Claim 2, a copy of  $C_0([\alpha + 1, \beta] \cup \{\omega_1\})$  which is included in  $B_\alpha$  is complemented in  $C_0(K_0)$  which is complemented in  $C(K_0)$  as required.  $\square$

Now consider any decomposition  $C(K_0) = A \oplus B$ . By the previous considerations, one of the factors, say  $A$ , is included in a complemented copy of  $C_0(\omega^\omega)$  or  $c_0$  of the form  $C_{0,\alpha}$  for some  $\alpha < \omega_1$ . But complemented subspaces of  $C_0(\omega^\omega)$  are  $c_0$  or  $C_0(\omega^\omega)$  (Corollary 5.10 of [20]) and so,  $A$  is isomorphic to one of these spaces.

**Claim 4.**  $B$  contains  $B_\alpha$  for some  $\alpha < \omega_1$ .

**Proof.** If  $P$  is a projection from  $C(K_0)$  on the first separable factor  $A$  included in  $C_{0,\beta}$ , consider the measures  $P^*(\delta_\gamma)$  for  $\gamma \in [0, \beta] \cup \{\omega_1\}$  and find  $\alpha < \omega_1$  such that the carriers of the measures  $P^*(\delta_\gamma)$  for  $\gamma \in [0, \beta] \cup \{\omega_1\}$  miss  $(\alpha, \omega_1)$ . Then,  $P$  restricted to  $B_\alpha$  is zero, and so  $B_\alpha \subseteq B$  which completes the proof of the claim.  $\square$

Now, suppose  $A \sim C_0(\omega^\omega)$ . By Claims 3 and 4,  $B \sim A' \oplus C$  for some  $C$  where  $A'$  is isomorphic to  $A$ . So,

$$C(K_0) = A \oplus B \sim A \oplus A' \oplus C \sim A' \oplus C \sim B.$$

This is since  $C_0(\omega^\omega)$  is isomorphic to its square.

A similar argument with a version of Claim 3 with  $c_0$  instead of  $C_0(\omega^\omega)$  (using the fact that  $c_0$  is complemented in  $C_0(\omega^\omega)$ ) works in the case when  $A \sim c_0$  to get the same conclusion that  $B \sim C(K_0)$ .  $\square$

An example of a scattered  $K$  (of height three, but with  $C(K)$  which is not Lindelöf) where all nontrivial decompositions of  $C(K)$  have one factor isomorphic to the  $C(K)$  and the other to  $c_0$  was obtained under CH in [12]. Also Argyros and Raikoftsalis have constructed in [4] a separable Banach space  $X$  (necessarily not of the form  $C(K)$ ) where all nontrivial decompositions are of the form  $c_0 \oplus X$ .

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