



Monotone convergence theorems for Henstock–Kurzweil integrable functions and applications

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ABSTRACT

In this paper we prove monotone convergence theorems for Henstock–Kurzweil integrable functions from a compact real interval to an ordered Banach space. These theorems are then applied to prove existence results for solutions of a discontinuous functional integral equation containing Henstock–Kurzweil integrable functions.

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1. Introduction

The Henstock–Kurzweil integral provides a tool to construct a rigorous mathematical formulation for Feynman's path integral which plays an important role in quantum physics (see, e.g., [13,16]). The Henstock–Kurzweil integrals of Banach space-valued functions are used for that purpose in recent papers [7,8].

The main advantage of the Henstock–Kurzweil integral is its applicability for integration of highly oscillatory functions which occur in quantum theory and nonlinear analysis. It is also easy to understand because its definition requires no measure theory. Moreover, all Bochner integrable (in real-valued case Lebesgue integrable) functions are Henstock–Kurzweil (shortly HK) integrable, but not conversely. For instance, HK integrability encloses improper integrals. The real-valued function f defined on $[0, 1]$ by $f(0) = 0$ and $f(t) = t^2 \cos(1/t^2)$ is differentiable on $[0, 1]$, and f' is HK integrable. But f' is not Lebesgue integrable on $[0, 1]$. More generally, let t be called a singular point of the domain interval of a real-valued function being not Lebesgue integrable on any interval that contains t . Then (cf. [15]) there exist HK “integrable functions on an interval that admit a set of singular points with its measure as close as possible but not equal to that of the whole interval”.

On the other hand, the lack of a Banach space structure for the class of HK integrable functions even in real-valued case (see [2]) is noticed in [8, p. 13] as “the most important factor preventing for wide spread use of the Henstock–Kurzweil integral in engineering, mathematics and physics”.

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In this paper we prove monotone convergence theorems for HK integrable functions from a compact real interval to an ordered Banach space. These theorems make possible, as we shall demonstrate, to solve equations which contain HK integrable Banach space-valued functions and discontinuous nonlinearities. In this sense these theorems, together with the order structure, compensate the above stated lack of a Banach space structure.

The paper is organized as follows. In Section 2 we present basic concepts of ordered normed spaces, and a preliminary theory of HK integrable functions from a compact real interval $[a, b]$ into a Banach space X . In Section 3 we prove two monotone convergence theorems for these functions. These theorems are then applied in Section 4 to derive existence results for least and/or greatest solutions of a Urysohn integral equation in ordered Banach spaces. The functions in that equation are assumed to be HK integrable with respect to independent variables. They are also allowed to be discontinuous and depend functionally on the unknown function. Results are illustrated by examples.

2. Preliminaries

We shall first present some basic properties of ordered normed spaces.

A closed subset X_+ of a normed space X is called an *order cone* if $X_+ + X_+ \subseteq X_+$, $X_+ \cap (-X_+) = \{0\}$ and $cX_+ \subseteq X_+$ for each $c \geq 0$. It is easy to see that the order relation $\leq$, defined by

$$x \leq y \quad \text{if and only if} \quad y - x \in X_+,$$

is a partial ordering in X , and that $X_+ = \{y \in X \mid 0 \leq y\}$. The space X , equipped with this partial ordering, is called an *ordered normed space*. The order interval $[y, z] = \{x \in X \mid y \leq x \leq z\}$ is a closed subset of X . We say that an order cone X_+ of a normed space X is *normal* if there is such a constant $\gamma \geq 1$ that

$$0 \leq x \leq y \quad \text{in } X \quad \text{implies} \quad \|x\| \leq \gamma \|y\|. \quad (2.1)$$

X_+ is called *regular* if all increasing and order bounded sequences of X_+ converge, and *fully regular* if all increasing and (norm) bounded sequences of X_+ converge. As for the proof of the following result, see, e.g., [9, Theorems 2.2.1 and 2.4.5].

Lemma 2.1. *Let X_+ be an order cone of a Banach space X . If X_+ is fully regular, it is also regular, and if X_+ is regular, it is also normal. Converse holds if X is weakly sequentially complete.*

Next we study Henstock–Kurzweil integrability of functions from a compact real interval $[a, b]$ to a Banach space X .

We say that $D = \{(\xi_i, I_i)\}$ is a *K-partition* of $[a, b]$ if $\{I_i\}$ is a finite collection of closed intervals I_i whose union is $[a, b]$ and which are non-overlapping, i.e. their interiors are pairwise disjoint, and if $\xi_i \in I_i$ for every i . D is called a *partial K-partition* of $[a, b]$ if $\bigcup_i I_i$ is a proper subset of $[a, b]$. Given a function δ from $[a, b]$ to $(0, \infty)$, called *gauge* of $[a, b]$, we say that a K-partition $D = \{(\xi_i, I_i)\}$ is δ -fine if $I_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for every i . The length of I_i is denoted by $|I_i|$.

A function $u : [a, b] \rightarrow X$ is *Henstock–Kurzweil* (shortly *HK*) *integrable* if there exists an element of X , denoted by ${}^K \int_a^b u(s) ds$ and called a *Henstock–Kurzweil integral* of u over $[a, b]$, having the following property: For every $\epsilon > 0$ there is such a gauge δ of $[a, b]$ that

$$\left\| \sum_i u(\xi_i) |I_i| - {}^K \int_a^b u(s) ds \right\| < \epsilon$$

whenever $D = \{(\xi_i, I_i)\}$ is a δ -fine K-partition of $[a, b]$.

If A is a subset of $[a, b]$, χ_A is the characteristic function of A , and if $\chi_A f$ is HK integrable on $[a, b]$, we say that f is HK integrable on A , and define

$${}^K \int_A f(s) ds = {}^K \int_a^b \chi_A(s) f(s) ds.$$

If f is HK integrable on $[a, b]$, it is HK integrable on every subinterval of $[a, b]$, and the Henstock–Kurzweil integral of f is linear and additive over non-overlapping subintervals of $[a, b]$ (cf. [18]).

The proofs for the results of the next lemma can be found, e.g., from [18].

Lemma 2.2.

- (a) *The a.e. equal functions are HK integrable and their integrals are equal if one of these functions is HK integrable.*
- (b) *A Bochner integrable function $f : [a, b] \rightarrow X$ is HK integrable, and $\int_I f(s) ds = {}^K \int_I f(s) ds$ whenever I is a closed subinterval of $[a, b]$.*

The following variant of Saks–Henstock lemma is a consequence of [18, Lemma 3.4.1].

Lemma 2.3. If $f : [a, b] \rightarrow X$ is HK integrable, then for every $\epsilon > 0$ there exists such a gauge $\delta : [a, b] \rightarrow (0, \infty)$ that if $D = \{(\xi_i, I_i)\}$ is any δ -fine K-partition or partial K-partition of $[a, b]$, then

$$\left\| \sum_i \left(f(\xi_i) |I_i| - {}^K \int_{I_i} f(s) ds \right) \right\| \leq \epsilon.$$

If $f : [a, b] \rightarrow X$ is HK integrable and $c \in X$, then the relation

$$\tilde{f}(t) = c + {}^K \int_a^t f(s) ds, \quad t \in [a, b],$$

defines a function $\tilde{f} \in C([a, b], X)$, which is called a *primitive* of f .

The next result plays an important role in the theory of HK integrable functions in ordered Banach spaces.

Lemma 2.4. Let X be an ordered Banach space, and let $f_{\pm} : [a, b] \rightarrow X$ be HK integrable. If $f_{-}(s) \leq f_{+}(s)$ for a.e. $s \in [a, b]$, and if I is a closed subinterval of $[a, b]$, then

$${}^K \int_I f_{-}(s) ds \leq {}^K \int_I f_{+}(s) ds. \quad (2.2)$$

Proof. By Lemma 2.2(a) we may assume that $f_{-}(s) \leq f_{+}(s)$ for all $s \in [a, b]$. Denoting $f = f_{+} - f_{-}$, then $f(s)$ belongs to the order cone X_{+} of X for all $s \in [a, b]$. Let $I = [c, d]$ be a closed subinterval of $[a, b]$. f is HK integrable on I . To prove that ${}^K \int_I f(s) ds \in X_{+}$, notice first that ${}^K \int_I f(s) ds = 0 \in X_{+}$ if $c = d$. Assume next that $c < d$. According to the definition of HK integrability we can choose for each $n \in \mathbb{N}$ a function $\delta_n : [c, d] \rightarrow (0, \infty)$, partitions $\{t_i^n\}_{i=1}^{m_n}$ of $[c, d]$ and points ξ_i^n so that $\xi_i^n \in [t_{i-1}^n, t_i^n] \subset (\xi_i^n - \delta(\xi_i^n), \xi_i^n + \delta(\xi_i^n))$, and that

$$\left\| \sum_{i=1}^{m_n} {}^K \int_{t_{i-1}^n}^{t_i^n} f(s) ds - f(\xi_i^n)(t_i^n - t_{i-1}^n) \right\| < \frac{1}{n}.$$

Denoting $y_n = \sum_{i=1}^{m_n} f(\xi_i^n)(t_i^n - t_{i-1}^n)$, $n \in \mathbb{N}$, we obtain

$$\left\| {}^K \int_I f(s) ds - y_n \right\| = \left\| {}^K \int_I f(s) ds - \sum_{i=1}^{m_n} f(\xi_i^n)(t_i^n - t_{i-1}^n) \right\| < \frac{1}{n}, \quad n \in \mathbb{N}.$$

Thus ${}^K \int_I f(s) ds = \lim_{n \rightarrow \infty} y_n \in X_{+}$, since X_{+} is closed, and since $y_n \in X_{+}$ for every $n \in \mathbb{N}$. Consequently,

$$0 \leq {}^K \int_I f(s) ds = {}^K \int_I f_{+}(s) ds - {}^K \int_I f_{-}(s) ds.$$

This proves the assertion. $\square$

3. Monotone convergence theorems

In this section we prove monotone convergence theorems for HK integrable functions in the case when X is an ordered Banach space. Denote by $HK([a, b], X)$ the space of all HK integrable functions from $[a, b]$ to X , ordered by a.e. pointwise ordering. We say that a sequence of functions $f_n \in HK([a, b], X)$, $n \in \mathbb{N}$, is *increasing* if $f_n \leq f_{n+1}$ for every $n \in \mathbb{N}$, *decreasing* if $f_{n+1} \leq f_n$ for every $n \in \mathbb{N}$, *monotone* if it is increasing or decreasing, and *a.e. pointwise bounded* if $\sup_n \|f_n(s)\| < \infty$ for a.e. $s \in [a, b]$.

Theorem 3.1. Let X be a Banach space ordered by a fully regular order cone. Assume that $(f_n)_{n=1}^{\infty}$ is a monotone and a.e. pointwise bounded sequence in $HK([a, b], X)$, and that the sequence of integrals ${}^K \int_a^b f_n(s) ds$, $n \in \mathbb{N}$, is bounded. Then there exists such a function $f \in HK([a, b], X)$ that $f(s) = \lim_{n \rightarrow \infty} f_n(s)$ for a.e. $s \in [a, b]$, and that

$$\lim_{n \rightarrow \infty} {}^K \int_a^b f_n(s) ds = {}^K \int_a^b f(s) ds. \quad (3.1)$$

Proof. Assume first that $(f_n)_{n=1}^\infty$ is increasing and a.e. pointwise bounded. Since X_+ is fully regular, then $f(s) = \lim_{n \rightarrow \infty} f_n(s)$ exists for a.e. $s \in [a, b]$. If $I = [c, d]$ is a closed subinterval of $[a, b]$, we have for each $n \in \mathbb{N}$,

$$0 \leq \int_I^K (f_n(s) - f_0(s)) ds \leq \left(\int_a^c^K + \int_c^d^K + \int_d^b^K \right) (f_n(s) - f_0(s)) ds = \int_a^b^K (f_n(s) - f_0(s)) ds.$$

Since X_+ is fully regular, it is also normal by Lemma 2.1. Thus the above inequalities imply by (2.1) that

$$\left\| \int_I^K (f_n(s) - f_0(s)) ds \right\| \leq \gamma \left\| \int_a^b^K (f_n(s) - f_0(s)) ds \right\|, \quad n \in \mathbb{N},$$

where $\gamma \geq 1$. Thus

$$\left\| \int_I^K f_n(s) ds \right\| \leq \left\| \int_I^K f_0(s) ds \right\| + \gamma \left\| \int_a^b^K f_n(s) ds \right\| + \gamma \left\| \int_a^b^K f_0(s) ds \right\|, \quad n \in \mathbb{N}.$$

This result, and the hypothesis that the sequence of integrals $\int_a^b^K f_n(s) ds$, $n \in \mathbb{N}$, is bounded, implies that sequence of integrals $\int_I^K f_n(s) ds$, $n \in \mathbb{N}$, is bounded. It is also increasing by Lemma 2.4. Because the order cone of X is fully regular, then the sequence of integrals $\int_I^K f_n(s) ds$, $n \in \mathbb{N}$, converges, i.e., the limit

$$F(I) := \lim_{n \rightarrow \infty} F_n(I), \quad F_n(I) := \int_I^K f_n(s) ds, \quad n \in \mathbb{N}, \quad (3.2)$$

exists for every closed subinterval I of $[a, b]$. It is easy to see that (3.2) defines an additive X -valued interval function F .

Let $\epsilon > 0$ be given, and let $\gamma \geq 1$ be such a constant that (2.1) holds. Choose an $n_\epsilon \in \mathbb{N}$ so that

$$\|F[a, b] - F_n[a, b]\| \leq \frac{\epsilon}{\gamma} \quad \text{for } n \geq n_\epsilon. \quad (3.3)$$

There exist by Lemma 2.3 gauges $\delta_n : [a, b] \rightarrow (0, \infty)$, $n \in \mathbb{N}$, for which

$$\left\| \sum_i (f_n(\xi_i)|I_i| - F_n(I_i)) \right\| \leq \frac{\epsilon}{2^n} \quad (3.4)$$

whenever $D = \{(\xi_i, I_i)\}$ is a δ_n -fine K-partition or partial K-partition of $[a, b]$. By Lemma 2.2(a) we may assume that for every $\xi \in [a, b]$,

$$f(\xi) = \lim_{n \rightarrow \infty} f_n(\xi).$$

Thus, for every fixed $\xi \in [a, b]$ there exists an $n(\xi) \in \mathbb{N}$ such that

$$\|f(\xi) - f_n(\xi)\| < \epsilon \quad \text{when } n \geq n(\xi). \quad (3.5)$$

Define a gauge $\delta : [a, b] \rightarrow (0, \infty)$ by

$$\delta(\xi) = \delta_{\max\{n(\xi), n_\epsilon\}}(\xi), \quad \xi \in [a, b]. \quad (3.6)$$

Let $D = \{(\xi_i, I_i)\}$ be a δ -fine K-partition of $[a, b]$.

For every i and $n_i = \max\{n(\xi_i), n_\epsilon\}$ we have

$$f(\xi_i)|I_i| - F(I_i) = (f(\xi_i) - f_{n_i}(\xi_i))|I_i| + f_{n_i}(\xi_i)|I_i| - F_{n_i}(I_i) + F_{n_i}(I_i) - F(I_i). \quad (3.7)$$

Summing both sides of (3.7) over i , and denoting $k = \min_i \{n_i\}$ and $m = \max_i \{n_i\}$ we obtain

$$\sum_i (f(\xi_i)|I_i| - F(I_i)) = \sum_i (f(\xi_i) - f_{n_i}(\xi_i))|I_i| + \sum_{n=k}^m \sum_{n_i=n} (f_{n_i}(\xi_i)|I_i| - F_{n_i}(I_i)) + \sum_i (F_{n_i}(I_i) - F(I_i)). \quad (3.8)$$

Because

$$0 \leq \sum_i (F(I_i) - F_{n_i}(I_i)) \leq \sum_i (F(I_i) - F_{n_\epsilon}(I_i)) = F[a, b] - F_{n_\epsilon}[a, b],$$

it follows from (2.1) and (3.3) that

$$\left\| \sum_i (F(I_i) - F_{n_i}(I_i)) \right\| \leq \gamma \|F[a, b] - F_{n_\epsilon}[a, b]\| \leq \epsilon. \quad (3.9)$$

Since (3.4) holds also for δ_n -fine partial K-partitions of $[a, b]$, we have

$$\left\| \sum_{n_i=n} (f_{n_i}(\xi_i)|I_i| - F_{n_i}(I_i)) \right\| \leq \frac{\epsilon}{2^n}, \quad k \leq n \leq m. \quad (3.10)$$

It then follows from (3.5), (3.8), (3.9) and (3.10) that

$$\begin{aligned} \left\| \sum_i f(\xi_i)|I_i| - F([a, b]) \right\| &= \left\| \sum_i (f(\xi_i)|I_i| - F(I_i)) \right\| \\ &\leq \sum_i \|f(\xi_i) - f_{n_i}(\xi_i)\| |I_i| + \sum_{n=k}^m \left\| \sum_{n_i=n} (f_{n_i}(\xi_i)|I_i| - F_{n_i}(I_i)) \right\| \\ &\quad + \left\| \sum_i (F_{n_i}(I_i) - F(I_i)) \right\| \leq \epsilon(b-a) + \sum_{n=k}^m \frac{\epsilon}{2^n} + \epsilon < (b-a+3)\epsilon. \end{aligned}$$

This proves that f is HK integrable, and that $F([a, b]) = {}^K \int_a^b f(s) ds$. Moreover,

$$F([a, b]) = \lim_{n \rightarrow \infty} F_n([a, b]) = \lim_{n \rightarrow \infty} {}^K \int_a^b f_n(s) ds,$$

so that (3.1) is valid.

If $(f_n)_{n=1}^\infty$ is decreasing and a.e. pointwise bounded, and if the integrals ${}^K \int_a^b f_n(s) ds$, $n \in \mathbb{N}$ form a bounded sequence, then the sequence $(-f_n)_{n \in \mathbb{N}}$ is increasing and a.e. pointwise bounded, and the sequence of integrals ${}^K \int_a^b (-f_n(s)) ds$, $n \in \mathbb{N}$ is bounded. Applying the above proof to $(-f_n)_{n=1}^\infty$ we see that the assertions hold also in this case. $\square$

Example 3.1. Denote

$$l_2(\mathbb{N} \times \mathbb{N}) = \left\{ x = (x_{i,j})_{i,j=0}^\infty : x_{i,j} \in \mathbb{R}, \|x\|_2 = \left(\sum_{i,j=0}^\infty |x_{i,j}|^2 \right)^{\frac{1}{2}} < \infty \right\}. \quad (3.11)$$

The Banach space $X := (l_2(\mathbb{N} \times \mathbb{N}), \|\cdot\|_2)$ is weakly sequentially complete. The set

$$X_+ := \{x = (x_{i,j})_{i,j=0}^\infty : x_{i,j} \geq 0, (i, j) \in \mathbb{N} \times \mathbb{N}\}$$

is a normal order cone of X , and it induces on X the pointwise ordering:

$$x = (x_{i,j})_{i,j=0}^\infty \leq y = (y_{i,j})_{i,j=0}^\infty \quad \text{iff} \quad x_{i,j} \leq y_{i,j} \quad \text{for all } (i, j) \in \mathbb{N} \times \mathbb{N}.$$

The characteristic functions $e_{i,j}$ of the singletons $\{(i, j)\}$ in $\mathbb{N} \times \mathbb{N}$ form an orthonormal basis for $l_2(\mathbb{N} \times \mathbb{N})$. Define functions $g_i : [0, 1] \rightarrow l_2(\mathbb{N} \times \mathbb{N})$, $i \in \mathbb{N}$, by

$$g_0(t) = \begin{cases} (0)_{i,j=0}^\infty, & t = 0, \\ \left(\frac{1}{(i+1)(j+1)} \left(2t \cos\left(\frac{1}{t^2}\right) + \frac{2}{t} \sin\left(\frac{1}{t^2}\right) \right) \right)_{i,j=0}^\infty, & t \in (0, 1], \end{cases} \quad (3.12)$$

and

$$g_i(t) = \begin{cases} 2^i e_{i,j}, & \frac{j}{2^i} \leq t < \frac{j}{2^i} + \frac{1}{2^{2i}}, \quad i = 1, 2, \dots, \quad j = 0, \dots, 2^i - 1, \\ 0, & \text{otherwise.} \end{cases} \quad (3.13)$$

The functions $f_n : [0, 1] \rightarrow l_2(\mathbb{N} \times \mathbb{N})$, $n \in \mathbb{N}$, defined by

$$f_n(t) = \sum_{i=0}^n g_i(t), \quad t \in [0, 1], \quad n \in \mathbb{N}, \quad (3.14)$$

form an increasing sequence of HK integrable functions. Moreover, $\|{}^K \int_0^1 g_0(s) ds\|_2 = \frac{\pi^2 \cos(1)}{6}$, and (cf. [5, Example 3.2]) $\|{}^K \int_0^1 \sum_{i=1}^n g_i(s) ds\|_2 \leq 1$ for every $n = 1, 2, \dots$. Thus the sequence of integrals ${}^K \int_0^1 f_n(s) ds$ is bounded. The sequence (f_n) is also a.e. pointwise bounded. Consequently, there exists by Theorem 3.1 such an HK integrable function $f_0 : [0, 1] \rightarrow X$ that $f_0(s) = \lim_{n \rightarrow \infty} f_n(s)$ for a.e. $s \in [0, 1]$, and that $\lim_{n \rightarrow \infty} {}^K \int_0^1 f_n(s) ds = {}^K \int_0^1 f_0(s) ds$.

The next monotone convergence theorem is a special case of [12, Theorem 3.1].

Theorem 3.2. Let X be a Banach space ordered by regular cone. Assume that $(f_n)_{n=1}^\infty$ is a monotone sequence in $HK([a, b], X)$, that $f_\pm \in HK([a, b], X)$, and that $f_- \leq f_n \leq f_+$ for every $n \in \mathbb{N}$. Then there exists such a function $f \in HK([a, b], X)$ that $f(s) = \lim_{n \rightarrow \infty} f_n(s)$ for a.e. $s \in [a, b]$, and that (3.1) holds.

Proof. The given hypotheses and Lemma 2.4 imply that for every closed subinterval I of $[a, b]$ the sequence of integrals ${}^K \int_I f_n(s) ds$, $n \in \mathbb{N}$, is monotone, and that ${}^K \int_I f_-(s) ds \leq {}^K \int_I f_n(s) ds \leq {}^K \int_I f_+(s) ds$ for every $n \in \mathbb{N}$. Because the order cone of X is regular, then the sequence of integrals ${}^K \int_I f_n(s) ds$, $n \in \mathbb{N}$, converges, i.e., the limit in (3.2) exists for every closed subinterval I of $[a, b]$. The rest of the proof is the same as that of Theorem 3.1. $\square$

Remark 3.1. Theorems 3.1 and 3.2 can be considered as generalizations of [14, Theorem 4.1] to Banach space-valued functions.

In monotone convergence theorems [4, Theorems 14 and 15] it is assumed that $0 \leq f_n$, which implies by [12, Corollary 4.1] that f_n 's are McShane integrable if X is weakly sequentially complete. Moreover, the inequality $I - \epsilon < I_\nu$, where ϵ is a positive number and $I, I_\nu \in X$, is used in the proofs. Thus these proofs are correct only when $X = \mathbb{R}$.

In [3,10] a monotone convergence theorem has been proved for order bounded sequences of HL (Henstock–Lebesgue) integrable functions which are also Bochner integrable, but not necessarily HK integrable (see [18]). Also a dominated convergence theorem has been proved in [3,10] for order bounded sequences of HL integrable functions. On the other hand, in Example 3.1 the functions f_n , $n \in \mathbb{N}$, are HL-integrable but f_0 is not (see [5, Example 3.2]). Thus Theorem 3.1 does not hold for HL integrable functions.

4. Application to an integral equation

Let X be an ordered Banach space, and let $[a, b]$ be a compact real interval.

Definition 4.1. We say that an a.e. pointwise bounded sequence $(f_n)_{n=0}^\infty$ of $HK([a, b], X)$ is HK-bounded if the sequence of integrals ${}^K \int_a^b f_n(s) ds$, $n \in \mathbb{N}_0$, is bounded.

In this section we apply the monotone convergence theorems presented in Theorems 3.1 and 3.2 to derive existence results for least and/or greatest solutions the functional integral equation

$$u(t) = h(t) + {}^K \int_a^b f(t, s, u(s), u) ds, \quad t \in [a, b], \quad (4.1)$$

where $h \in HK([a, b], X)$, and $f : [a, b] \times [a, b] \times X \times HK([a, b], X) \rightarrow X$.

Definition 4.2. We say that $u \in HK([a, b], X)$ is a lower solution of (4.1) if

$$u(t) \leq h(t) + {}^K \int_a^b f(t, s, u(s), u) ds \quad \text{for a.e. } t \in [a, b]. \quad (4.2)$$

If the reversed inequality holds in (4.2) for a.e. $t \in [a, b]$, we say that u is an upper solution of (4.1). If equality holds in (4.2) for a.e. $t \in [a, b]$, we say that u is a solution of (4.1).

We impose the following hypotheses on the function f .

- (f0) If $u \in HK([a, b], X)$, then $f(t, \cdot, u(\cdot), u) \in HK([a, b], X)$ for every $t \in [a, b]$, and ${}^K \int_a^b f(\cdot, s, u(s), u) ds \in HK([a, b], X)$.
- (f1) If $u \leq v$ in $HK([a, b], X)$, then $f(t, \cdot, u(\cdot), u) \leq f(t, \cdot, v(\cdot), v)$ for every $t \in [a, b]$.
- (f2) If $(u_n)_{n=0}^\infty$ is a monotone sequence in $HK([a, b], X)$ which converges a.e. pointwise to $u \in HK([a, b], X)$, then $f(t, s, u_n(s), u_n) \rightarrow f(t, s, u(s), u)$ for every $t \in [a, b]$ and for a.e. $s \in [a, b]$.
- (f3) If $(u_n)_{n=0}^\infty$ is a monotone sequence in $HK([a, b], X)$, then the sequence of functions ${}^K \int_a^b f(\cdot, s, u_n(s), u_n) ds$, $n \in \mathbb{N}_0$, is HK-bounded.

Our first existence result for the integral equation (4.1) reads as follows.

Proposition 4.1. Assume that the hypotheses (f0)–(f3) are valid, and that the order cone of X is fully regular.

(a) If (4.1) has a lower solution u_- , then the successive approximations:

$$u_{n+1}(t) = h(t) + K \int_a^b f(t, s, u_n(s), u_n) ds, \quad t \in [a, b], \quad n \in \mathbb{N}_0, \quad u_0 = u_-, \quad (4.3)$$

form in $[u_-] = \{u \in HK([a, b], X) : u_- \leq u\}$ an increasing sequence $(u_n)_{n=0}^\infty$ that converges a.e. pointwise to the least solution of (4.1) in $[u_-]$.

(b) If (4.1) has an upper solution u_+ , then the successive approximations:

$$v_{n+1}(t) = h(t) + K \int_a^b f(t, s, v_n(s), v_n) ds, \quad t \in [a, b], \quad n \in \mathbb{N}_0, \quad v_0 = u_+, \quad (4.4)$$

form in $(u_+) = \{u \in HK([a, b], X) : u \leq u_+\}$ a decreasing sequence $(v_n)_{n=0}^\infty$ that converges a.e. pointwise to the greatest solution of (4.1) in (u_+) .

Proof. (a) Assume that u_- is a lower solution of (4.1). The hypothesis (f0) and (4.3) imply that $u_n \in HK([a, b], X)$ for every n . The hypothesis (f1) and Lemma 2.4 imply by induction that

$$u_n(t) \leq u_{n+1}(t) \quad \text{for all } n \in \mathbb{N}_0 \text{ and } t \in [a, b].$$

In view of the hypothesis (f3) and (4.3) the sequence $(u_n)_{n=0}^\infty$ is HK-bounded. Because the order cone of X is fully regular, it follows from Theorem 3.1 that the a.e. pointwise limit u_* of $(u_n)_{n=0}^\infty$ exists and belongs to $HK([a, b], X)$.

By the hypotheses (f0) and (f1) the sequence $(f(t, \cdot, u_n(\cdot), u_n))_{n=0}^\infty$ belongs to $HK([a, b], X)$, and is increasing for every fixed $t \in [a, b]$. Moreover, $u_* = \sup_n u_n$ by [11, Proposition 1.1.3], whence

$$f(t, \cdot, u_-(\cdot), u_-) \leq f(t, \cdot, u_n(\cdot), u_n) \leq f(t, \cdot, u_*(\cdot), u_*), \quad n \in \mathbb{N}_0.$$

Because the order cone of X is regular by Lemma 2.1, there exists by Theorem 3.2 for every $t \in [a, b]$ such a function $f_t \in HK([a, b], X)$ that $f(t, s, u_n(s), u_n) \rightarrow f_t(s)$ for a.e. $s \in [a, b]$, and that

$$K \int_a^b f(t, s, u_n(s), u_n) ds \rightarrow K \int_a^b f_t(s) ds.$$

The hypothesis (f2) ensures that $f_t = f(t, \cdot, u_*(\cdot), u_*)$ for every $t \in [a, b]$, so that

$$K \int_a^b f(t, s, u_n(s), u_n) ds \rightarrow K \int_a^b f(t, s, u_*(s), u_*) ds, \quad t \in [a, b].$$

It then follows from (4.3) as $n \rightarrow \infty$ that u_* is a solution of (4.1). By induction one can show that if u is any solution of (4.1) in $[u_-]$, then $u_n \leq u$ for every n . Thus $u_* = \sup_n u_n \leq u$, so that u_* is the least solution of (4.1) in $[u_-]$.

(b) By similar reasonings one can show that if u_+ is an upper solution of (4.1), then the sequence (v_n) defined in (4.4) is decreasing, and converges a.e. pointwise to a function $u^* \in HK([a, b], X)$, and that u^* is the greatest solution of (4.1) in (u_+) . $\square$

Example 4.1. Let $f_0 : [0, 1] \rightarrow l_2(\mathbb{N} \times \mathbb{N})$ be the function constructed in Example 3.1, let $f_1 : l_2(\mathbb{N} \times \mathbb{N}) \rightarrow l_2(\mathbb{N} \times \mathbb{N})$ be increasing, continuous and bounded from below, and let $f_2 : HK([0, 1], l_2(\mathbb{N} \times \mathbb{N})) \rightarrow l_2(\mathbb{N} \times \mathbb{N})$ be increasing, bounded from below, and $f_2(u_n) \rightarrow f_2(u)$ whenever (u_n) is a monotone sequence in $HK([a, b], X)$ and $u_n(s) \rightarrow u(s)$ for a.e. $s \in [a, b]$. Then for every function $k \in HK(J, (0, \infty))$ the equation

$$f(t, s, x, u) = k(t)(f_0(s) + f_1(x) + f_2(u)) \quad (4.5)$$

defines a function $f : J \times J \times l_2(\mathbb{N} \times \mathbb{N}) \times HK(J, l_2(\mathbb{N} \times \mathbb{N})) \rightarrow l_2(\mathbb{N} \times \mathbb{N})$ which satisfies the hypotheses (f0)–(f3) when $X = l_2(\mathbb{N} \times \mathbb{N})$. Thus the integral equation

$$u(t) = h(t) + \int_0^1 k(t)(f_0(s) + f_1(u(s)) + f_2(u)) ds, \quad t \in J := [0, 1], \quad (4.6)$$

has by Proposition 4.1(a) the least solution for all choices of $h \in HK(J, l_2(\mathbb{N} \times \mathbb{N}))$. This solution is obtained by Proposition 4.1 as a limit of successive approximations.

Next we replace the hypothesis (f3) by the following hypothesis:

(lu) Eq. (4.1) has a lower solution u_- and an upper solution u_+ , and $u_- \leq u_+$.

Assuming the order cone of X is regular we shall prove the following result.

Proposition 4.2. Assume that the hypotheses (f0)–(f2) and (lu) are valid, and that the order cone of X is regular. Denote $[u_-, u_+] = \{u \in HK([a, b], X) : u_- \leq u \leq u_+\}$, where $u_{\pm}$ are as in the hypothesis (lu).

- (a) The successive approximations u_n , $n \in \mathbb{N}_0$, defined by (4.3), form in $[u_-, u_+]$ an increasing sequence $(u_n)_{n=0}^{\infty}$ which converges a.e. pointwise on $[a, b]$ to a function $u_* \in HK([a, b], X)$ that is the least solution of (4.1) in $[u_-, u_+]$.
- (b) The successive approximations v_n , $n \in \mathbb{N}_0$, defined by (4.4), form in $[u_-, u_+]$ a decreasing sequence $(v_n)_{n=0}^{\infty}$ which converges a.e. pointwise on $[a, b]$ to a function $u^* \in HK([a, b], X)$ that is the greatest solution of (4.1) in $[u_-, u_+]$.

Proof. (a) Let u_n , $n \in \mathbb{N}_0$, be defined by (4.3). The hypothesis (f0) ensures that $u_n \in HK([a, b], X)$ for every n . The hypotheses (f1), (lu) and Lemma 2.4 imply by induction that

$$u_-(t) \leq u_n(t) \leq u_{n+1}(t) \leq u_+(t), \quad n \in \mathbb{N}_0, \text{ for a.e. } t \in [a, b].$$

Because the order cone of X is regular, it follows from Theorem 3.2 that the a.e. pointwise limit u_* of (u_n) exists and belongs to $HK([a, b], X)$.

The hypotheses (f0), (f1) and (lu) imply that for a.e. $t \in [a, b]$ the sequence $(f(t, \cdot, u_n(\cdot), u_n))_{n=0}^{\infty}$ belongs to $HK([a, b], X)$, and that

$$f(t, \cdot, u_-(\cdot), u_-) \leq f(t, \cdot, u_n(\cdot), u_n) \leq f(t, \cdot, u_{n+1}(\cdot), u_{n+1}) \leq f(t, \cdot, u_+(\cdot), u_+), \quad n \in \mathbb{N}_0.$$

In view of these results and Theorem 3.2 there exists for a.e. $t \in [a, b]$ such a function $f_t \in HK([a, b], X)$ that $f(t, s, u_n(s), u_n) \rightarrow f_t(s)$ for a.e. $s \in [a, b]$, and that

$$K \int_a^b f(t, s, u_n(s), u_n) ds \rightarrow K \int_a^b f_t(s) ds.$$

Since the hypothesis (f2) is valid, then $f_t = f(t, \cdot, u_*(\cdot), u_*)$, so that

$$K \int_a^b f(t, s, u_n(s), u_n) ds \rightarrow K \int_a^b f(t, s, u_*(s), u_*) ds \quad \text{for a.e. } t \in [a, b].$$

It then follows from (4.3) as $n \rightarrow \infty$ that u_* is a solution of (4.1). By induction one can show that if u is any solution of (4.1) in $[u_-, u_+]$, then $u_n \leq u$ for every n . Thus $u_* = \sup_n u_n \leq u$, so that u_* is the least solution of (4.1) in $[u_-, u_+]$.

(b) By similar reasonings one can show that the sequence (v_n) defined in (4.4) is decreasing, and converges a.e. pointwise to the greatest solution u^* of (4.1) in $[u_-, u_+]$. $\square$

The next result is a consequence of Proposition 4.2.

Corollary 4.1. Assume that the order cone of X is regular. Let the hypotheses (f0), (f1), (f2) and the following hypothesis hold.

(f4) There exist $f_{\pm} \in HK([a, b], X)$, $f_- \leq f_+$ such that $f_- \leq K \int_a^b f(\cdot, s, u(s), u) ds \leq f_+$ for all $u \in HK([a, b], X)$.

Then the integral equation (4.1) has least and greatest solutions for every $h \in HK([a, b], X)$.

Proof. Let $h \in HK([a, b], X)$ be given. The hypothesis (f4) ensures that the hypothesis (lu) holds when $u_{\pm} = h + f_{\pm}$. Thus the hypotheses of Proposition 4.2 are valid, so that Eq. (4.1) has least and greatest solutions u_* and u^* in $[u_-, u_+]$. The hypotheses (f0) and (f1), (f4) and Lemma 2.4 imply that if $u \in HK([a, b], X)$, then

$$u_-(t) \leq h(t) + K \int_a^b f(t, s, u(s), u) ds \leq u_+(t) \quad \text{for a.e. } t \in [a, b].$$

Thus all the solutions of (4.1) belong to the order interval $[u_-, u_+]$. In particular, u_* and u^* are least and greatest of all the solutions of (4.1), which concludes the proof. $\square$

Example 4.2. Let $f_0 : [0, 1] \rightarrow l_2(\mathbb{N} \times \mathbb{N})$ be the function constructed in Example 3.1. Let $g_1 : l_2(\mathbb{N} \times \mathbb{N}) \rightarrow l_2(\mathbb{N} \times \mathbb{N})$ be increasing, continuous and order bounded, and let $g_2 : HK([0, 1], l_2(\mathbb{N} \times \mathbb{N})) \rightarrow l_2(\mathbb{N} \times \mathbb{N})$ be increasing and order bounded. Assume also that $g_2(u_n) \rightarrow g_2(u)$ whenever (u_n) is a monotone sequence in $HK([a, b], X)$ which converges a.e. pointwise to $u \in HK([a, b], X)$. Then for every function $k \in HK(J, (0, \infty))$ the equation

$$f(t, s, x, u) = k(t)(f_0(s) + g_1(x) + g_2(u)) \quad (4.7)$$

defines a function $f : J \times J \times l_2(\mathbb{N} \times \mathbb{N}) \times HK(J, l_2(\mathbb{N} \times \mathbb{N})) \rightarrow l_2(\mathbb{N} \times \mathbb{N})$ which satisfies the hypotheses (f0)–(f2) and (f4) when $X = l_2(\mathbb{N} \times \mathbb{N})$. Thus the integral equation

$$u(t) = h(t) + \int_0^1 k(t)(f_0(s) + g_1(u(s)) + g_2(u)) ds, \quad t \in J := [0, 1], \quad (4.8)$$

has by Corollary 4.1 the least and greatest solutions for all choices of $h \in HK(J, l_2(\mathbb{N} \times \mathbb{N}))$. These solutions are obtained by Proposition 4.2 as limits of successive approximations.

Remark 4.1. The mapping $f(t, \cdot, u(\cdot), u)$ is neither Bochner integrable Henstock–Lebesgue (HL) integrable for any choices of $t \in J$ and $u : J \rightarrow l_2(\mathbb{N} \times \mathbb{N})$ when f is defined by (4.7) (see [5, Example 3.2]).

It follows from [12, Corollary 4.1] that if $u \leq v$ in $HK([a, b], X)$, and if u or v is McShane integrable, then both are McShane integrable. Thus the solution u_* (respectively u^*) of Eq. (4.1) is McShane integrable if u_- (respectively u_+) is. In particular, all the results of this section remain valid if HK integrability is replaced by McShane integrability.

The space $HK([a, b], X)$ contains also those functions $u : [a, b] \rightarrow X$ which are Bochner integrable on every closed subinterval $[c, d]$ of (a, b) , and for which the limits of the Bochner integral $\int_c^d u(s) ds$ when $c \rightarrow a+$ and $d \rightarrow b-$ exist (cf. [18, Theorem 3.4.5] and remark after it). In particular, the integral in (4.1) can be replaced by the improper integral $\int_{a+}^{b-}$.

The following spaces are examples of Banach spaces having fully regular order cones (cf. [11]):

- A finite-dimensional normed space ordered by any closed cone.
- A reflexive (e.g., a uniformly convex) Banach space ordered by a normal order cone.
- A reflexive Banach lattice.
- A Banach lattice which is uniformly monotone in the sense defined in [1, XV, 14].
- A separable Hilbert space whose order cone is generated by an orthonormal basis.
- A Hilbert space H with such an order cone H_+ that $(x|y) \geq 0$ for all $x, y \in H_+$.
- A Hilbert space H whose order cone is $H_+ = \{x \in H \mid (x|\bar{e}) \geq c\|x\|_2\}$, where $\bar{e}$ is a unit vector of H and $c \in (0, 1)$.
- Sequence spaces l^p , $1 \leq p < \infty$, normed by p -norm and ordered componentwise.
- Function spaces $L^p(\Omega)$, $1 \leq p < \infty$, normed by p -norm and ordered a.e. pointwise, where Ω is a measure space.
- Function spaces $L^p([a, b], X)$, $1 \leq p < \infty$, ordered a.e. pointwise, where X is any of the spaces listed above.

In the sequence space (c_0) , normed by a sup-norm the componentwise ordering is induced by the cone of all nonnegative sequences. This cone is regular, but not fully regular.

As for other results on integral equations including non-absolutely integrable Banach space-valued functions, see, e.g., [3,6,17,19–21].

References

- [1] G. Birkhoff, Lattice Theory, Amer. Math. Soc. Colloq. Publ., vol. XXV, Amer. Math. Soc., Providence, RI, 1940.
- [2] B. Bongiorno, Relatively weakly compact sets in the Denjoy space, J. Math. Study 27 (1994) 37–43.
- [3] S. Carl, S. Heikkilä, Fixed Point Theory in Ordered Spaces and Applications, Springer, ISBN 978-1-4419-7584-3, 2011.
- [4] M. Federson, The monotone convergence theorem for multidimensional abstract Kurzweil vector integrals, Czechoslovak Math. J. 52 (2002) 429–437.
- [5] M. Federson, Some peculiarities of the Henstock and Kurzweil integrals of Banach space-valued functions, Real Anal. Exchange 29 (1) (2003/2004) 1–22.
- [6] M. Federson, M. Bianconi, Linear Fredholm integral equations and the integral of Kurzweil, J. Appl. Anal. 8 (1) (2002) 81–108.
- [7] T.L. Gill, W.W. Zachary, Foundations of relativistic quantum theory I: Feynman's operator calculus and the Dyson conjectures, J. Math. Phys. 43 (2002).
- [8] T.L. Gill, W.W. Zachary, Constructive representation theory for the Feynman operator calculus, arXiv:math-ph/0701039, 2007.
- [9] D. Guo, Y.J. Cho, J. Zhu, Partial Ordering Methods in Nonlinear Problems, Nova Science Publishers, Inc., New York, 2004.
- [10] S. Heikkilä, M. Kumpulainen, S. Seikkala, Convergence theorems for HL integrable vector-valued functions with applications, Nonlinear Anal. 70 (5) (2009) 1939–1955.
- [11] S. Heikkilä, V. Lakshmikantham, Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations, Marcel Dekker, Inc., New York, 1994.
- [12] S. Heikkilä, Guojun Ye, Convergence and comparison results for Henstock–Kurzweil and McShane integrable vector-valued functions, Southeast Asian Bull. Math. 34 (2010).
- [13] R. Henstock, The General Theory of Integration, Clarendon Press, Oxford, 1991.
- [14] P.-Y. Lee, Landzhou Lectures on Henstock Integration, World Scientific, Singapore, 1989.
- [15] P.-Y. Lee, J. Lu, On singularity of Henstock integrable functions, Real Anal. Exchange 25 (2) (1999) 795–798.
- [16] P. Muldowney, A General Theory of Integration in Function Spaces, Pitman Res. Notes Math., John Wiley & Sons, New York, 1987.

- [17] B.-R. Satco, Nonlinear Volterra integral equations in Henstock integrability setting, *Electron. J. Differential Equations* 2008 (39) (2008) 1–9.
- [18] S. Schwabik, Ye Guoju, *Topics in Banach Space Integration*, World Scientific, Singapore, 2005.
- [19] A. Sikorska-Nowak, On the existence of nonlinear integral equations in Banach spaces and Henstock–Kurzweil integrals, *Ann. Polon. Math.* 83 (3) (2004) 257–267.
- [20] A. Sikorska-Nowak, Existence of solutions of nonlinear integral equations in Banach spaces and Henstock–Kurzweil integrals, *Comment. Math.* 47 (2) (2007) 227–238.
- [21] A. Sikorska-Nowak, Nonlinear integral equations in Banach spaces and Henstock–Kurzweil–Pettis integrals, *Dynam. Systems Appl.* 17 (2008) 97–108.