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Sensitivity analysis of parametric weak vector equilibrium problems ☆

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ABSTRACT

In this paper, two implicit multifunction theorems are obtained. Then by using a set-valued gap function and the results obtained we study the sensitivity analysis of the solution mapping for a parametric weak vector equilibrium problem and obtain an explicit expression for the contingent derivative of the solution mapping.

1. Introduction

In this paper, we study the sensitivity analysis of the solution mapping for the following parametric weak vector equilibrium problem (for short, PWVEP): find $x \in K(p)$ such that

$$f(x, y, p) \notin -\operatorname{int} C, \quad \forall y \in K(p),$$

where $f : X \times X \times \Lambda \to Z$ is a vector-valued mapping, $K : \Lambda \rightrightarrows X$ is a set-valued mapping, $C \subseteq Z$ is a closed, convex and pointed cone with a nonempty interior $\operatorname{int} C$, X and Z are Banach spaces and Λ is a normed space. Throughout the paper, let the symbols 0_X, 0_Λ and 0_Z denote the origins of X, Λ and Z, respectively. For each $p \in \Lambda$, by $S(p)$ we denote the solution mapping of the PWVEP, i.e.,

$$S(p) := \{x \in K(p) \mid f(x, y, p) \notin -\operatorname{int} C,\ \forall y \in K(p)\}.$$

The PWVEP is a unified model of several problems, for example, the vector optimization problem (for short, VOP), the vector variational inequality problem (for short, VVI), the vector complementarity problem and the vector saddle point problem, etc.

In recent years, existence results for various types of vector equilibrium problems and the stability analysis of the solution mapping for the PWVEP have been investigated extensively (see [5,6,8,11,12,22,23,3,13,14,16,30]). In virtue of a nonlinear scalarization function Chen et al. [5] and Li et al. [22,23] discussed the existence of solutions for kinds of generalized vector quasi-equilibrium problems, respectively. By using a generalized KKM theorem Ding and Park [8] and Fakhar and Zafarani [11] established some existence results for generalized vector equilibrium problems, respectively. Making use of a scalarization technique Chen et al. [3] and Gong [13,14] studied the lower semicontinuity of the solution mapping S for the

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PWVEP. By a fixed point theorem Kimura and Yao [16] investigated the upper semicontinuity and the lower semicontinuity of the solution mapping S for the PWVEP.

Sensitivity analysis is not only theoretically interesting, but also practically important in optimization theory. Under suitable assumptions, it consists in the study of derivatives of perturbation maps. Until now, a number of useful results have been obtained (see [1,2,17–21,24,25,28,31–33]). Kuk et al. [18] and Tanino [33] first discussed the sensitivity analysis of the optimal value mapping for a VOP. Subsequently, Shi [31,32] made use of different conditions to study the sensitivity analysis of the optimal value mapping for the VOP. Later, by a class of set-valued gap functions, Li et al. [21] and Meng and Li [25] investigated the sensitivity analysis of a VVI and a Minty VVI, respectively. Recently, Li and Li [20] studied the second-order sensitivity analysis of a weak VVI. Based on these work, Li and Li [24] investigated the sensitivity analysis of the solution mapping for the VOP.

To the best of our knowledge, the sensitivity analysis of the solution mapping S for the PWVEP has not been studied until now. This paper aims to investigate the sensitivity analysis (mainly the contingent derivative property) of the solution mapping S for the PWVEP by using a so-called set-valued gap function which is similar to the set-valued gap functions in [21,25]. Since it is difficult to directly investigate the contingent derivative of S, our method is based on the following resolution: for $x \in X$ and $p \in \Lambda$,

$$S(p) = \{x \in K(p) \mid 0_Z \in V(p,x)\}, \tag{1}$$

where $V(p,x) := \min_{\operatorname{int} C} G(p,x)$ and $G(p,x) := \bigcup_{y \in K(p)} f(x,y,p) \cup \{0_Z\}$. At first, we investigate the relationships between the contingent derivative of S and the contingent derivative of V. Then, we study the contingent derivative of G under the Fréchet differentiability of f, and discuss the relationships between the contingent derivative of V and the contingent derivative of G.

In Section 2, when V is a general set-valued mapping in (1) we mainly discuss the relationships between the contingent derivative of S and the contingent derivative of V in infinite dimensional vector spaces. The contingent derivative of S has been studied in [2,19]. Levy [19] has obtained the following result:

$$DS(\hat{p},\hat{x})(p) \subset \{x \mid 0_Z \in DV(\hat{p},\hat{x},0_Z)(p,x)\}, \quad \forall p \in \operatorname{dom} DS(\hat{p},\hat{x}).$$

Aubin and Frankowska [2] obtained an explicit expression for the contingent derivative of S in the case when $X = Z$ and both X and Λ are finite dimensional vector spaces, however they assumed a regularity condition. To avoid this condition, in Theorem 2.1 we assume that S is Robinson metrically regular around $(\hat{p},\hat{x})$ and then obtain an explicit expression for the contingent derivative of S. The Robinson metric regularity of S has been investigated in [7,9,10,26,27]. Specially, Dontchev et al. [9] gave a sufficient condition of metric regularity of S, which is stronger than the Robinson metric regularity of S. By using a similar method we give a slight extension of a criteria of metric regularity of S (i.e., we get a sufficient condition of the Robinson metric regularity of S) without the hypothesis of upper semicontinuity of $p \mapsto d(0, V(p,\hat{x}))$ which is supposed in [9].

In Section 3, motivated by the work reported in [18,20,21,24,25,31–33], we firstly study the contingent derivative of G under the Fréchet differentiability of f and discuss the relationships between the contingent derivative of V and the contingent derivative of G. Then, by Theorem 2.1 in Section 2, we get the contingent derivative of (1). Finally, composing these results we obtain an explicit expression of the contingent derivative of the solution mapping S for the PWVEP.

2. Two implicit multifunction theorems

In this section, we study the following implicit multifunction which is introduced by Robinson (see [26,27]):

$$S(p) := \{x \in X \mid 0_Z \in V(p,x)\},$$

which is defined by the generalized equation $0_Z \in V(p,x)$, where $V : \Lambda \times X \rightrightarrows Z$ is a general set-valued mapping. In this section, we assume that X and Z are Banach spaces whose norms are both denoted by $\|\cdot\|$ and Λ is a metric space whose metric is denoted by ρ. Given a subset $A \subset X$, define the distance from $x \in X$ to A by $d(x,A) := \inf_{a \in A} \|x - a\|$ with the convention that $d(x,\emptyset) = \infty$.

At first, we recall the following concept which is important for this paper.

Definition 2.1. (See [26].) S is called Robinson metrically regular around $(\hat{p},\hat{x}) \in \operatorname{gph} S$ if there exist $\mu > 0$, $\gamma > 0$ and neighborhoods $U_{\hat{p}}$ of $\hat{p}$, $U_{\hat{x}}$ of $\hat{x}$ such that

$$d(x, S(p)) \leqslant \mu d(0_Z, V(p,x)), \quad \text{whenever } p \in U_{\hat{p}},\ x \in U_{\hat{x}},\ d(0_Z, V(p,x)) < \gamma. \tag{2}$$

Remark 2.1. (2) implies $S(p) \neq \emptyset$ for all $p \in U_{\hat{p}}$, $x \in U_{\hat{x}}$ with $d(0_Z, V(p,x)) < \gamma$. If V is lower semicontinuous at $(\hat{p},\hat{x},0_Z)$ in the sense that for every $\varepsilon > 0$ there exist neighborhoods $U_{\hat{p}}$ of $\hat{p}$ and $U_{\hat{x}}$ of $\hat{x}$ such that $V(p,x) \cap \varepsilon\mathbb{B} \neq \emptyset$ for all $p \in U_{\hat{p}}$ and $x \in U_{\hat{x}}$, then the constant $\gamma > 0$ and the related inequality in (2) can be omitted. For more details we refer the reader to [7].

Definition 2.2. (See [2].) Let $\hat{x} \in \operatorname{cl} K$, where $\operatorname{cl} K$ denotes the closure of K and K is a nonempty set of X. The contingent cone of K at $\hat{x}$ is the set $T(K, \hat{x}) := \limsup_{h\to 0} \frac{K-\hat{x}}{h}$. The adjacent cone of K at $\hat{x}$ is the set $T^\flat(K, \hat{x}) := \liminf_{h\to 0} \frac{K-\hat{x}}{h}$. K is said to be derivable at $\hat{x}$ if $T(K, \hat{x}) = T^\flat(K, \hat{x})$.

Equivalently, $x \in T(K, \hat{x})$ if and only if there exist sequences $t_n \downarrow 0$ and $\{x_n\} \subset X$ with $x_n \to x$ and $\hat{x} + t_n x_n \in K$, $\forall n$. $x \in T^\flat(K, \hat{x})$ if and only if for any sequence $t_n \downarrow 0$, there exists a sequence $\{x_n\} \subset X$ with $x_n \to x$ and $\hat{x} + t_n x_n \in K$, $\forall n$.

Let $F: X \rightrightarrows Z$ be a set-valued mapping. The effective domain, graph and inverse of F are defined by $\operatorname{dom} F := \{x \in X \mid F(x) \neq \emptyset\}$, $\operatorname{gph} F := \{(x, z) \in X \times Z \mid z \in F(x)\}$ and $F^{-1}(z) := \{x \in X \mid z \in F(x)\}$, respectively. The contingent derivative of F at $(\hat{x}, \hat{z}) \in \operatorname{gph} F$ is the set-valued map $DF(\hat{x}, \hat{z}): X \rightrightarrows Z$ whose graph is $T(\operatorname{gph} F, (\hat{x}, \hat{z}))$. The adjacent derivative of F at $(\hat{x}, \hat{z})$ is the set-valued map $D^\flat F(\hat{x}, \hat{z}): X \rightrightarrows Z$ whose graph is $T^\flat(\operatorname{gph} F, (\hat{x}, \hat{z}))$. F is said to be proto-differentiable at $(\hat{x}, \hat{z})$ if and only if $\operatorname{gph} F$ is derivable at $(\hat{x}, \hat{z})$, i.e., $T(\operatorname{gph} F, (\hat{x}, \hat{z})) = T^\flat(\operatorname{gph} F, (\hat{x}, \hat{z}))$ (see [28]). F is said to be semi-differentiable at $(\hat{x}, \hat{z})$ if and only if for any $z \in DF(\hat{x}, \hat{z})(x)$, any $t_n \downarrow 0$ and any $x_n \to x$, there exists a sequence $z_n \to z$ such that $\hat{z} + t_n z_n \in F(\hat{x} + t_n x_n)$. Semi-differentiability is a more exacting property than proto-differentiability. The relationship between them has been obtained by Rockafellar (see [28]). For more details we refer the reader to [28,29].

Now, we give an implicit multifunction theorem. Since the solution mapping of the PWVEP can be written as an implicit multifunction which is similar to S, the following Theorem 2.1 and Corollary 2.1 are very useful to investigate the contingent derivative of (1) for the PWVEP.

Theorem 2.1. *Suppose that S is Robinson metrically regular around $(\hat{p}, \hat{x}) \in \operatorname{gph} S$. Then one has that*

$$DS(\hat{p}, \hat{x})(p) = \{x \mid 0_Z \in DV(\hat{p}, \hat{x}, 0_Z)(p, x)\}, \quad \forall p \in \operatorname{dom} DS(\hat{p}, \hat{x}). \tag{3}$$

Moreover, if V is proto-differentiable at $(\hat{p}, \hat{x}, 0_Z)$, then S is proto-differentiable at $(\hat{p}, \hat{x})$.

Proof. It follows from Theorem 3.1 in [19] that the right-hand side of (3) includes the left-hand side. Let x be an element of the right-hand side of (3). Then there exist sequences $t_n \downarrow 0$ and $(p_n, x_n, z_n) \to (p, x, 0_Z)$ such that

$$t_n z_n \in V(\hat{p} + t_n p_n, \hat{x} + t_n x_n).$$

Since S is Robinson metrically regular around $(\hat{p}, \hat{x})$, there exist $\mu > 0$, $\gamma > 0$ and neighborhoods $U_{\hat{p}}$ of $\hat{p}$, $U_{\hat{x}}$ of $\hat{x}$ such that (2) holds. Because for sufficiently large n we get

$$\hat{p} + t_n p_n \in U_{\hat{p}}, \qquad \hat{x} + t_n x_n \in U_{\hat{x}} \quad \text{and} \quad d(0_Z, V(\hat{p} + t_n p_n, \hat{x} + t_n x_n)) \leqslant t_n \|z_n\| < \gamma,$$

one has

$$d(\hat{x} + t_n x_n, S(\hat{p} + t_n p_n)) \leqslant \mu d(0_Z, V(\hat{p} + t_n p_n, \hat{x} + t_n x_n)) \leqslant \mu t_n \|z_n\|.$$

Thus,

$$\hat{x} + t_n x_n \in S(\hat{p} + t_n p_n) + (\mu t_n \|z_n\| + t_n^2)\mathbb{B},$$

i.e., there exists $b_n \in \mathbb{B}$ such that

$$\hat{x} + t_n[x_n - (\mu\|z_n\| + t_n)b_n] \in S(\hat{p} + t_n p_n).$$

Since $(\mu\|z_n\| + t_n)\|b_n\| \to 0$, we get $x \in DS(\hat{p}, \hat{x})(p)$. Similar to the above proof we can easily get that S is proto-differentiable at $(\hat{p}, \hat{x})$ when V is proto-differentiable at $(\hat{p}, \hat{x}, 0_Z)$. This completes the proof. □

Let $S_1(p) := \{x \in K(p) \mid 0_Z \in V(p, x)\}$, where $K: \Lambda \rightrightarrows X$ is a set-valued mapping. It follows from Theorem 2.1 that we can easily get the following result.

Corollary 2.1. *Suppose that S_1 is Robinson metrically regular along with K around $(\hat{p}, \hat{x}) \in \operatorname{gph} S_1$, i.e., there exist $\mu > 0$, $\gamma > 0$ and neighborhoods $U_{\hat{p}}$ of $\hat{p}$, $U_{\hat{x}}$ of $\hat{x}$ such that*

$$d(x, S_1(p)) \leqslant \mu d(0_Z, V(p, x)), \quad \text{whenever } p \in U_{\hat{p}},\ x \in U_{\hat{x}} \cap K(p),\ d(0_Z, V(p, x)) < \gamma.$$

If K is semi-differentiable at $(\hat{p}, \hat{x}) \in \operatorname{gph} K$, then one has

$$DS_1(\hat{p}, \hat{x})(p) = \{x \in DK(\hat{p}, \hat{x})(p) \mid 0_Z \in DV(\hat{p}, \hat{x}, 0_Z)(p, x)\}, \quad \forall p \in \operatorname{dom} DS_1(\hat{p}, \hat{x}).$$

Moreover, if V is proto-differentiable at $(\hat{p}, \hat{x}, 0_Z)$, then S_1 is proto-differentiable at $(\hat{p}, \hat{x})$.

Next, we give another implicit multifunction theorem which is a refinement of Theorem 2.1 in [9]. The closed ball centered at x with radius r and the open ball are denoted by $\mathbb{B}_r(x)$ and $\mathbb{B}_r^\circ(x)$, respectively; $\mathbb{B}$ and $\mathbb{B}^\circ$ indicate the closed unit ball and the open unit ball, respectively. Recall that a set-valued mapping $F: X \rightrightarrows Z$ is said to have a locally closed graph at $(\hat{x}, \hat{z}) \in \operatorname{gph} F$ when $\operatorname{gph} F \cap [\mathbb{B}_r(\hat{x}) \times \mathbb{B}_r(\hat{z})]$ is a closed set for some $r > 0$. The partial contingent derivative $D_x V(p, x, z)$ of V is defined as the contingent derivative of the mapping $x \mapsto V(p, x)$ with p fixed. The inner norm (see [29]) of F is defined by $\|F\|^- := \sup_{x\in\mathbb{B}} \inf_{z\in F(x)} \|z\|$.

Theorem 2.2. *Assume that V has a locally closed graph at $(\hat{p}, \hat{x}, 0_Z) \in \operatorname{gph} V$. Then for every constant $\mu > 0$ satisfying*

$$\limsup_{(p,x,z) \xrightarrow{\operatorname{gph} V} (\hat{p},\hat{x},0_Z)} \left\| D_x V(p,x,z)^{-1} \right\|^{-} < \mu, \tag{4}$$

S is Robinson metrically regular around $(\hat{p}, \hat{x})$.

Proof. On the product space $Y := X \times Z$ we define the norm

$$\left|\!\left|\!\left| (x,z) \right|\!\right|\!\right| := \max\bigl\{ \|x\|, \mu\|z\| \bigr\}, \quad \forall (x,z) \in Y,$$

which makes Y a Banach space, and on the space $\Lambda \times Y$ we introduce the metric

$$\sigma\bigl((p_1, y_1), (p_2, y_2)\bigr) := \max\bigl\{ \rho(p_1, p_2), |||y_1 - y_2||| \bigr\}, \quad \forall p_1, p_2 \in \Lambda,\ y_1, y_2 \in Y,$$

which yields $\Lambda \times Y$ a metric space.

A constant μ satisfies (4) if and only if there exists $\eta > 0$ such that

$$\forall (p,x,z) \in \operatorname{gph} V \text{ with } \sigma\bigl((p,x,z), (\hat{p}, \hat{x}, 0_Z)\bigr) \leqslant 3\eta \text{ and } \forall v \in Z,$$
$$\exists u \in D_x V(p,x,z)^{-1}(v) \text{ with } \|u\| \leqslant \mu \|v\|.$$

By the assumption, we can always choose η smaller so that the set $\operatorname{gph} V \cap \mathbb{B}_{3\eta}(\hat{p}, \hat{x}, 0_Z)$ is closed. We also choose $\varepsilon > 0$ such that $0 < \varepsilon\mu < 1$ and γ with $0 < \gamma < \varepsilon\eta$. Set $U_{\hat{p}} := \mathbb{B}^{\circ}_{\frac{\gamma}{\varepsilon}}(\hat{p})$ and $U_{\hat{x}} := \mathbb{B}^{\circ}_{\frac{\gamma}{\varepsilon}}(\hat{x})$. Pick $p \in U_{\hat{p}}$, $x \in U_{\hat{x}}$ with $d(0_Z, V(p,x)) < \gamma$. Then, for a sufficiently small $c > 0$ we can find $z_c \in V(p,x)$ such that

$$\|z_c\| \leqslant d\bigl(0_Z, V(p,x)\bigr) + c < \gamma.$$

Thus,

$$\mu\|z_c\| < \mu\gamma < \mu\varepsilon\eta < \eta$$

and, consequently, $(p, x, z_c) \in \operatorname{gph} V \cap \mathbb{B}_\eta(\hat{p}, \hat{x}, 0_Z)$. Applying Lemma 2.2 of [9] for $(p,x,z) = (p,x,z_c)$, $s = \gamma$ and $y' = 0_Z$, we find $\bar{x}_c \in S(p)$ such that

$$\|x - \bar{x}_c\| \leqslant \frac{1}{\varepsilon}\|z_c\|.$$

Then

$$d\bigl(x, S(p)\bigr) \leqslant \|x - \bar{x}_c\| \leqslant \frac{1}{\varepsilon}\|z_c\| \leqslant \frac{1}{\varepsilon}\bigl(d\bigl(0_Z, V(p,x)\bigr) + c\bigr).$$

By the select of c, we make $c \downarrow 0$ and then we obtain

$$d\bigl(x, S(p)\bigr) \leqslant \frac{1}{\varepsilon} d\bigl(0_Z, V(p,x)\bigr).$$

Since $\frac{1}{\varepsilon}$ can be arbitrarily close to μ, we get that (2) holds. $\square$

Remark 2.2. When the conditions of Theorem 2.2 are valid and the function $p \mapsto d(0_Z, V(p, \hat{x}))$ is upper semicontinuous at $\hat{p}$, Dontchev et al. [9] have proved that the following result holds: there exist $\mu > 0$ and neighborhoods $U_{\hat{p}}$ of $\hat{p}$, $U_{\hat{x}}$ of $\hat{x}$ such that

$$d\bigl(x, S(p)\bigr) \leqslant \mu d\bigl(0_Z, V(p,x)\bigr), \quad \text{whenever } p \in U_{\hat{p}},\ x \in U_{\hat{x}}.$$

It is clear that in Theorem 2.2 getting rid of the upper semicontinuity of the function $p \mapsto d(0_Z, V(p, \hat{x}))$ we obtain (2). Though (2) is weaker than Dontchev's result, it follows from Theorem 2.1 that (2) is enough to discuss the contingent derivative of S.

3. Sensitivity analysis of the solution mapping for the PWVEP

In this section, we firstly study the contingent derivative of G under the Fréchet differentiability of f. Then, we discuss the relationships between the contingent derivative of V and the contingent derivative of G. Finally, making use of Theorem 2.1 and Corollary 2.1 we obtain a formula of the contingent derivative for S.

Let us start with some important definitions.

Definition 3.1. (See [15].) Let D be a nonempty subset of Z. $\min_C D$ is said to be the minimal point set of D, if $\min_C D := \{\hat{z} \in D\colon\ \nexists z \in D,\ \text{s.t.},\ z - \hat{z} \in -C \setminus \{0_Z\}\}$. $\min_{\operatorname{int} C} D$ is said to be the weakly minimal point set of D, if $\min_{\operatorname{int} C} D := \{\hat{z} \in D\colon\ \nexists z \in D,\ \text{s.t.},\ z - \hat{z} \in -\operatorname{int} C\}$.

Definition 3.2. (See [4].) A set-valued mapping $W : \Lambda \times X \rightrightarrows Z$ is said to be a gap function of the PWVEP for some parametric p, iff

$$0_Z \in W(p,x) \quad \text{iff} \quad x \text{ is a solution of the PWVEP} \quad \text{and} \quad W(p,x) \cap \operatorname{int} C = \emptyset, \quad \forall x \in K(p).$$

Proposition 3.1. *V is a gap function of the PWVEP.*

Proof. If $0_Z \in V(p,x)$, then by the definition of V we have $G(p,x) \cap (-\operatorname{int} C) = \emptyset$, namely,

$$f(x,y,p) \notin -\operatorname{int} C, \quad \forall y \in K(p).$$

Thus, $x \in S(p)$. If $x \in S(p)$, then $G(p,x) \cap (-\operatorname{int} C) = \emptyset$. Hence, $0_Z \in V(p,x)$.

For each $x \in K(p)$ and each $z \in V(p,x)$, by the definition of V and since $0_Z \in G(p,x)$, we get $z \notin \operatorname{int} C$. Thus, $V(p,x) \cap \operatorname{int} C = \emptyset$, $\forall x \in K(p)$. □

In what follows, let $\hat{p} \in \Lambda$, $\hat{x} \in S(\hat{p})$ and $\Omega(0_Z) := \{y \in K(\hat{p}) \mid f(\hat{x}, y, \hat{p}) = 0_Z\}$. Assume that X and Z are finite dimensional spaces.

Proposition 3.2. *Assume that K is compact (i.e., $\operatorname{gph} K$ is a compact set), for each $\bar{y} \in \Omega(0_Z)$ f is continuously Fréchet differentiable at $(\hat{x}, \bar{y}, \hat{p})$ and for each $\bar{y} \in \Omega(0_Z)$ the following equation holds:*

$$\ker \nabla_y f(\hat{x}, \bar{y}, \hat{p}) \cap DK(\hat{p}, \bar{y})(0_\Lambda) = \{0_Z\}. \tag{5}$$

Then for $(p,x) \in \operatorname{dom} DG(\hat{p}, \hat{x}, 0_Z)$ one has that

$$DG(\hat{p}, \hat{x}, 0_Z)(p,x) = \bigcup_{\bar{y} \in \Omega(0_Z)} \bigcup_{y \in DK(\hat{p}, \bar{y})(p)} \nabla f(\hat{x}, \bar{y}, \hat{p})(x, y, p) \cup \{0_Z\}.$$

Proof. Let $z \in DG(\hat{p}, \hat{x}, 0_Z)(p,x)$ and $(p,x) \in \operatorname{dom} DG(\hat{p}, \hat{x}, 0_Z)$. If $z = 0_Z$, it obviously holds. If $z \neq 0_Z$, then there exist sequences $t_n \downarrow 0$ and $(p_n, x_n, z_n) \to (p, x, z)$ such that

$$t_n z_n \in G(\hat{p} + t_n p_n, \hat{x} + t_n x_n).$$

By the definition of G there exists $\bar{y}_n \in K(\hat{p} + t_n p_n)$ such that

$$f(\hat{x} + t_n x_n, \bar{y}_n, \hat{p} + t_n p_n) = t_n z_n. \tag{6}$$

Since K is compact and $\hat{p} + t_n p_n \to \hat{p}$, we can assume, without loss of generality, that $\bar{y}_n \to \bar{y} \in K(\hat{p})$. By (6) we get $f(\hat{x}, \bar{y}, \hat{p}) = 0_Z$, i.e., $\bar{y} \in \Omega(0_Z)$. Because f is Fréchet differentiable at $(\hat{x}, \bar{y}, \hat{p})$, one has

$$f(\hat{x} + t_n x_n, \bar{y}_n, \hat{p} + t_n p_n) = \nabla f(\hat{x}, \bar{y}, \hat{p})(t_n x_n, \bar{y}_n - \bar{y}, t_n p_n) + o\big(\|t_n x_n\| + \|\bar{y}_n - \bar{y}\| + \|t_n p_n\|\big). \tag{7}$$

Then, by (6) we obtain

$$t_n z_n = \nabla f(\hat{x}, \bar{y}, \hat{p})(t_n x_n, \bar{y}_n - \bar{y}, t_n p_n) + o\big(\|t_n x_n\| + \|\bar{y}_n - \bar{y}\| + \|t_n p_n\|\big).$$

Thus,

$$\begin{aligned} &z_n - \nabla_x f(\hat{x}, \bar{y}, \hat{p})(x_n) - \nabla_p f(\hat{x}, \bar{y}, \hat{p})(p_n) \\ &\quad = \frac{\|\bar{y}_n - \bar{y}\|}{t_n}\left[\nabla_y f(\hat{x}, \bar{y}, \hat{p})\left(\frac{\bar{y}_n - \bar{y}}{\|\bar{y}_n - \bar{y}\|}\right) + \frac{o(\|t_n x_n\| + \|\bar{y}_n - \bar{y}\| + \|t_n p_n\|)}{\|t_n x_n\| + \|\bar{y}_n - \bar{y}\| + \|t_n p_n\|}\left(1 + \frac{t_n(\|x_n\| + \|p_n\|)}{\|\bar{y}_n - \bar{y}\|}\right)\right]. \end{aligned} \tag{8}$$

Now, we prove that $\{\frac{\bar{y}_n - \bar{y}}{t_n}\}$ is bounded. In fact, if not, then there exists a subsequence $\{\frac{\bar{y}_{n'} - \bar{y}}{t_{n'}}\}$ such that $\frac{\|\bar{y}_{n'} - \bar{y}\|}{t_{n'}} \to \infty$. Since X is a finite dimensional space, we can assume, without loss of generality, that $\frac{\bar{y}_{n'} - \bar{y}}{\|\bar{y}_{n'} - \bar{y}\|} \to y^*$ with $\|y^*\| = 1$. Thus,

$$\nabla_y f(\hat{x}, \bar{y}, \hat{p})\left(\frac{\bar{y}_{n'} - \bar{y}}{\|\bar{y}_{n'} - \bar{y}\|}\right) \to \nabla_y f(\hat{x}, \bar{y}, \hat{p})\big(y^*\big).$$

Because $\bar{y}_{n'} \in K(\hat{p} + t_{n'} p_{n'})$, one has that

$$\bar{y} + \|\bar{y}_{n'} - \bar{y}\| \frac{\bar{y}_{n'} - \bar{y}}{\|\bar{y}_{n'} - \bar{y}\|} \in K\left(\hat{p} + \|\bar{y}_{n'} - \bar{y}\| \frac{t_{n'}}{\|\bar{y}_{n'} - \bar{y}\|} p_{n'}\right).$$

Since $\|\bar{y}_{n'} - \bar{y}\| \to 0$ and $\frac{t_{n'}}{\|\bar{y}_{n'} - \bar{y}\|} p_{n'} \to 0_\Lambda$, $y^* \in DK(\hat{p}, \bar{y})(0_\Lambda) \setminus \{0_Z\}$. By using (5), we get

$$\nabla_y f(\hat{x}, \bar{y}, \hat{p})\big(y^*\big) \neq 0_Z.$$

Then, the right-hand side of (8) is unbounded, which contradicts the fact that the left-hand side of (8) is convergent. Thus, $\{\frac{\bar{y}_n-\bar{y}}{t_n}\}$ is bounded.

Set $y_n := \frac{\bar{y}_n-\bar{y}}{t_n}$. Since X is finite dimensional, we may assume, without loss of generality, that $y_n \to y$. It follows from $\bar{y}_n = \bar{y} + t_n y_n$ and $\bar{y}_n \in K(\hat{p} + t_n p_n)$ that $y \in DK(\hat{p}, \bar{y})(p)$. By using (6) and (7), we get

$$z_n = \nabla f(\hat{x}, \bar{y}, \hat{p})(x_n, y_n, p_n) + \frac{o(\|t_n x_n\| + \|t_n y_n\| + \|t_n p_n\|)}{t_n}.$$

Thus, $z = \nabla f(\hat{x}, \bar{y}, \hat{p})(x, y, p)$. Consequently,

$$DG(\hat{p}, \hat{x}, 0_Z)(p, x) \subset \bigcup_{\bar{y} \in \Omega(0_Z)} \bigcup_{y \in DK(\hat{p}, \bar{y})(p)} \nabla f(\hat{x}, \bar{y}, \hat{p})(x, y, p) \cup \{0_Z\}.$$

Let $z \in \bigcup_{\bar{y} \in \Omega(0_Z)} \bigcup_{y \in DK(\hat{p}, \bar{y})(p)} \nabla f(\hat{x}, \bar{y}, \hat{p})(x, y, p) \cup \{0_Z\}$. If $z = 0_Z$, then for any $t_n \downarrow 0$, any $(p_n, x_n) \to (p, x)$ it follows from the definition of G that

$$0_Z \in G(\hat{p} + t_n p_n, \hat{x} + t_n x_n).$$

Thus, $0_Z \in DG(\hat{p}, \hat{x}, 0_Z)(p, x)$. If $z \neq 0_Z$, then there exist $\bar{y} \in \Omega(0_Z)$ and $y \in DK(\hat{p}, \bar{y})(p)$ such that $z = \nabla f(\hat{x}, \bar{y}, \hat{p})(x, y, p)$. Then, there exist sequences $t_n \downarrow 0$ and $(p_n, y_n) \to (p, y)$ such that

$$\bar{y} + t_n y_n \in K(\hat{p} + t_n p_n).$$

Thus, for any $x_n \to x$ it follows from the definition of G that

$$f(\hat{x} + t_n x_n, \bar{y} + t_n y_n, \hat{p} + t_n p_n) \in G(\hat{p} + t_n p_n, \hat{x} + t_n x_n).$$

By using the Taylor expansion which is similar to (7), one has that

$$\nabla f(\hat{x}, \bar{y}, \hat{p})(x, y, p) \in DG(\hat{p}, \hat{x}, 0_Z)(p, x).$$

This completes the proof. □

Remark 3.1. It follows from the above proof and the proto-differentiability of K at $(\hat{p}, \hat{x})$ that G is proto-differentiable at $(\hat{p}, \hat{x}, 0_Z)$.

Before investigating the relationship between the contingent derivative of V and the contingent derivative of G, we introduce some important definitions. The set-valued mapping $F: X \rightrightarrows Z$ is said to be Lipschitz around $\hat{x} \in \operatorname{dom} F$ if and only if there exist a neighborhood U of $\hat{x}$ and a constant $c > 0$ such that

$$F(x_1) \subseteq F(x_2) + c\|x_1 - x_2\|\mathbb{B}, \quad \forall x_1, x_2 \in U.$$

F is said to be C-minicomplete around $\hat{x} \in \operatorname{dom} F$ if there exists a neighborhood U of $\hat{x}$ such that

$$F(x) \subset \min_{\operatorname{int} C} F(x) + C, \quad \forall x \in U.$$

Proposition 3.3. *Assume that the conditions of Proposition* 3.2 *hold. Suppose that K is proto-differentiable at $(\hat{p}, \hat{x})$ and K is Lipschitz around $\hat{p}$. Then $\forall (p, x) \in \operatorname{dom} DV(\hat{p}, \hat{x}, 0_Z)$ one has that*

$$DV(\hat{p}, \hat{x}, 0_Z)(p, x) \subset \min_{\operatorname{int} C} DG(\hat{p}, \hat{x}, 0_Z)(p, x).$$

Proof. Let $z \in DV(\hat{p}, \hat{x}, 0_Z)(p, x)$. If $z \notin \min_{\operatorname{int} C} DG(\hat{p}, \hat{x}, 0_Z)(p, x)$, then there exists $\bar{z} \in DG(\hat{p}, \hat{x}, 0_Z)(p, x)$ such that

$$\bar{z} - z \in -\operatorname{int} C. \tag{9}$$

Since $z \in DV(\hat{p}, \hat{x}, 0_Z)(p, x)$, there exist sequences $t_n \downarrow 0$ and $(p_n, x_n, z_n) \to (p, x, z)$ such that

$$0_Z + t_n z_n \in V(\hat{p} + t_n p_n, \hat{x} + t_n x_n). \tag{10}$$

Let us consider two possible cases for $\bar{z}$.

Case 1. $\bar{z} = 0_Z$. Then from (9) one has $z \in \operatorname{int} C$. It follows from (10) and $0_Z \in G(\hat{p} + t_n p_n, \hat{x} + t_n x_n)$ that $t_n z_n \notin \operatorname{int} C$. Thus $z \notin \operatorname{int} C$ which leads to a contradiction.

Case 2. $\bar{z} \neq 0_Z$. Because K is proto-differentiable at $(\hat{p}, \hat{x})$, by Remark 3.1 we obtain that G is proto-differentiable at $(\hat{p}, \hat{x}, 0_Z)$. Thus, for above $t_n \downarrow 0$ and $\bar{z} \in DG(\hat{p}, \hat{x}, 0_Z)(p, x)$ there exist sequences $(\bar{p}_n, \bar{x}_n, \bar{z}_n) \to (p, x, \bar{z})$ such that

$$0_Z + t_n \bar{z}_n \in G(\hat{p} + t_n \bar{p}_n, \hat{x} + t_n \bar{x}_n).$$

Then, there exists $\bar{y}_n \in K(\hat{p} + t_n \bar{p}_n)$ satisfying

$$0_Z + t_n \bar{z}_n = f(\hat{x} + t_n \bar{x}_n, \bar{y}_n, \hat{p} + t_n \bar{p}_n). \tag{11}$$

Since K is compact and $\hat{p}+t_n\bar{p}_n \to \hat{p}$, we might as well assume $\bar{y}_n \to \bar{y} \in K(\hat{p})$. Then from (11) one has $0_Z = f(\hat{x},\bar{y},\hat{p})$. Similar to the proof of Proposition 3.2, we get that $\|\frac{\bar{y}_n-\bar{y}}{t_n}\|$ is bounded. Set $y_n := \frac{\bar{y}_n-\bar{y}}{t_n}$ and there is no harm in assuming that $y_n \to y$. Since K is Lipschitz around $\hat{p}$ and $\bar{y}+t_n y_n \in K(\hat{p}+t_n\bar{p}_n)$, there exist a constant $c>0$ and $b_n \in \mathbb{B}$ such that

$$\bar{y}+t_n\big[y_n - c\|\bar{p}_n - p_n\|b_n\big] \in K(\hat{p}+t_n p_n).$$

Set $\tilde{y}_n := y_n - c\|\bar{p}_n - p_n\|b_n$. Then $\tilde{y}_n \to y$. It follows from Fréchet differentiability of f at $(\hat{x},\bar{y},\hat{p})$ and (11) that

$$\begin{aligned} t_n\bar{z}_n &= f(\hat{x},\bar{y},\hat{p}) + \nabla f(\hat{x},\bar{y},\hat{p})(t_n\bar{x}_n, t_n y_n, t_n\bar{p}_n) + o_1(t_n) \\ &= f(\hat{x},\bar{y},\hat{p}) + \nabla f(\hat{x},\bar{y},\hat{p})(t_n x_n, t_n\tilde{y}_n, t_n p_n) + o_2(t_n) \\ &\quad + t_n\nabla f(\hat{x},\bar{y},\hat{p})\big(\bar{x}_n - x_n, c\|\bar{p}_n - p_n\|b_n, \bar{p}_n - p_n\big) + o_1(t_n) - o_2(t_n), \end{aligned}$$

where $o_1(t_n) = o(\|t_n\bar{x}_n\| + \|t_n y_n\| + \|t_n\bar{p}_n\|)$ and $o_2(t_n) = o(\|t_n x_n\| + \|t_n\tilde{y}_n\| + \|t_n p_n\|)$. Thus,

$$\begin{aligned} & t_n\left[\bar{z}_n - \nabla f(\hat{x},\bar{y},\hat{p})\big(\bar{x}_n - x_n, c\|\bar{p}_n - p_n\|b_n, \bar{p}_n - p_n\big) - \frac{o_1(t_n)}{t_n} + \frac{o_2(t_n)}{t_n}\right] \\ &\quad = f(\hat{x}+t_n x_n, \bar{y}+t_n\tilde{y}_n, \hat{p}+t_n p_n) \in G(\hat{p}+t_n p_n, \hat{x}+t_n x_n). \end{aligned}$$

Then from (10) we obtain

$$f(\hat{x}+t_n x_n, \bar{y}+t_n\tilde{y}_n, \hat{p}+t_n p_n) - t_n z_n \notin -\operatorname{int} C,$$

namely,

$$\bar{z}_n - \nabla f(\hat{x},\bar{y},\hat{p})\big(\bar{x}_n - x_n, c\|\bar{p}_n - p_n\|b_n, \bar{p}_n - p_n\big) - \frac{o_1(t_n)}{t_n} + \frac{o_2(t_n)}{t_n} - z_n \notin -\operatorname{int} C.$$

Hence, $\bar{z} - z \notin -\operatorname{int} C$, which contradicts (9). This completes the proof. □

Proposition 3.4. *Let $0_Z \in \min_C G(\hat{p},\hat{x})$. Assume that the conditions of Proposition* 3.2 *hold and*

$$\bigcup_{\bar{y}\in\Omega(0_Z)} \nabla_y f(\hat{x},\bar{y},\hat{p})\big(DK(\hat{p},\bar{y})(0_\Lambda)\big) \cap (-C) = \{0_Z\}. \tag{12}$$

Then for $(p,x) \in \operatorname{dom} DG(\hat{p},\hat{x},0_Z)$ one has that

$$D(G+C)(\hat{p},\hat{x},0_Z)(p,x) = DG(\hat{p},\hat{x},0_Z)(p,x) + C$$

and

$$\min_{\operatorname{int} C} DG(\hat{p},\hat{x},0_Z)(p,x) \subset \min_{\operatorname{int} C} D(G+C)(\hat{p},\hat{x},0_Z)(p,x).$$

Proof. It follows from Proposition 3.2 and (12) that $DG(\hat{p},\hat{x},0_Z)(0_\Lambda,0_X) \cap (-C) = \{0_Z\}$. Then, similar to the proof of Lemma 4.1 in [21] we easily get the first result. The second result can be obtained from the first result and its proof is similar to the proof of Theorem 2.1 in [18]. □

Proposition 3.5. *Let the conditions of Proposition* 3.4 *hold. Assume that there exists a nonempty closed convex cone $\tilde{C}$ such that $\tilde{C} \subset \operatorname{int} C \cup \{0_Z\}$ and G is $\tilde{C}$-minicomplete around $(\hat{p},\hat{x})$. Then $\forall (p,x) \in \operatorname{dom} DV(\hat{p},\hat{x},0_Z)$ one has that*

$$\min_{\operatorname{int} C} DG(\hat{p},\hat{x},0_Z)(x,u) \subset DV(\hat{p},\hat{x},0_Z)(p,x).$$

Proof. It follows from Proposition 3.4 and the existence of $\tilde{C}$ that

$$D(G+\tilde{C})(\hat{p},\hat{x},0_Z)(p,x) = DG(\hat{p},\hat{x},0_Z)(p,x) + \tilde{C}$$

and

$$\min_{\operatorname{int} C} DG(\hat{p},\hat{x},0_Z)(p,x) \subset \min_{\operatorname{int} C} D(G+\tilde{C})(\hat{p},\hat{x},0_Z)(p,x).$$

Then, by using the similar proof of Theorem 2.1 in [18] we get

$$\min_{\operatorname{int} C} DG(\hat{p},\hat{x},0_Z)(p,x) = \min_{\operatorname{int} C} D(G+\tilde{C})(\hat{p},\hat{x},0_Z)(p,x). \tag{13}$$

It follows from $\tilde{C}$-minicompleteness of G around $(\hat{p},\hat{x})$ and Theorem 3.1 of [18] that the conclusion holds. □

It follows from the definitions of Robinson metric regularity and V that the following conclusion is obvious.

Proposition 3.6. *The following statements are equivalent*:

(i) *There exist neighborhoods $U_{\hat{x}}$ of $\hat{x}$, $U_{\hat{p}}$ of $\hat{p}$ and constants $M > 0$, $\gamma > 0$ such that*

$$\forall p \in U_{\hat{p}},\ x \in U_{\hat{x}} \text{ with } d\big(0_Z, V(p,x)\big) < \gamma \text{ and } \forall y \in K(p) \text{ with } f(x,y,p) \in V(p,x),$$
$$\text{there exists } x' \in S(p) \text{ satisfying } \big\|f(x,y,p)\big\| \geqslant M\big\|x - x'\big\|.$$

(ii) *S is Robinson metrically regular along with K around $(\hat{p}, \hat{x})$.*

Following from Corollary 2.1 and Propositions 3.2–3.6 we get the main result in this paper.

Theorem 3.1. *Assume that all the conditions of Propositions* 3.2–3.5 *hold and* (i) *or* (ii) *of Proposition* 3.6 *is satisfied. Then, when K is semi-differentiable at $(\hat{p}, \hat{x})$, one has that S is proto-differentiable at $(\hat{p}, \hat{x})$ and for $p \in \operatorname{dom} DS(\hat{p}, \hat{x})$*

$$DS(\hat{p},\hat{x})(p) = \big\{x \in DK(\hat{p},\hat{x})(p) \;\big|\; \nabla f(\hat{x},\bar{y},\hat{p})(x,y,p) \notin -\operatorname{int} C,\ \forall \bar{y} \in \Omega(0_Z),\ y \in DK(\hat{p},\bar{y})(p)\big\}.$$

Proof. It follows from Propositions 3.2–3.5 that

$$\begin{aligned} DV(\hat{p},\hat{x},0_Z)(p,x) &= \min_{\operatorname{int} C} DG(\hat{p},\hat{x},0_Z)(x,u) \\ &= \min_{\operatorname{int} C} \bigcup_{\bar{y}\in\Omega(0_Z)} \bigcup_{y \in DK(\hat{p},\bar{y})(p)} \nabla f(\hat{x},\bar{y},\hat{p})(x,y,p) \cup \{0_Z\}. \end{aligned}$$

Moreover, by using the condition (i) or (ii) of Proposition 3.6 and in virtue of Corollary 2.1, we get that

$$DS(\hat{p},\hat{x})(p) = \big\{x \in DK(\hat{p},\hat{x})(p) \;\big|\; 0_Z \in DV(\hat{p},\hat{x},0_Z)(p,x)\big\}, \quad \forall p \in \operatorname{dom} DS(\hat{p},\hat{x}).$$

Thus, the conclusion is obtained. □

The Robinson metric regularity of S is very important for the above theorem and the following examples illustrate that it is essential.

Example 3.1. Let $X = \Lambda = Z = \mathbb{R}$ and $C = \mathbb{R}_+$. Let $f(x,y,p) := x(y+p)$ and

$$K(p) := \begin{cases} [-p, p] & \text{if } p \geqslant 0, \\ [p, -p] & \text{if } p < 0. \end{cases}$$

Then, we can easily get

$$G(p,x) = \bigcup_{y\in K(p)} x(y+p) \cup \{0\}, \qquad V(p,x) = \begin{cases} \{0\} & \text{if } px \geqslant 0, \\ \{2px\} & \text{if } px < 0 \end{cases} \quad \text{and} \quad S(p) = \begin{cases} [0,p] & \text{if } p \geqslant 0, \\ [p,0] & \text{if } p < 0. \end{cases}$$

Let $\tilde{C} = \mathbb{R}_+$, $\hat{p} = 0.05$ and $\hat{x} = 0.05$. Then, $\Omega(0) = \{-0.05\}$ and $\bar{y} = -0.05$. For $\mu = 50$, $\gamma = 1$, $p \in U_{\hat{p}} := (0.01, 0.1)$ and

$$x \in U_{\hat{x}} \cap K(p) := (-0.05, 0.10) \cap [-p, p] = \begin{cases} [-p, p] & \text{if } 0.01 < p < 0.05, \\ (-0.05, p] & \text{if } 0.05 \leqslant p < 0.1 \end{cases}$$

we can verify that S is Robinson metrically regular along with K around $(\hat{p}, \hat{x})$. It is easy to check that other conditions of Theorem 3.1 hold. Thus, the conclusions of Theorem 3.1 are valid. By directly computing, we obtain $DK(\hat{p},\hat{x})(p) = (-\infty, p]$, $\nabla f(\hat{x},\bar{y},\hat{p})(x,y,p) = 0.05(y+p)$, $DK(\hat{p},\bar{y})(p) = [-p, \infty)$, $DS(\hat{p},\hat{x})(p) = (-\infty, p]$ and

$$\big\{x \in DK(\hat{p},\hat{x})(p) \;\big|\; \nabla f(\hat{x},\bar{y},\hat{p})(x,y,p) \notin -\operatorname{int} C,\ \forall \bar{y} \in \Omega(0_Z),\ y \in DK(\hat{p},\bar{y})(p)\big\} = (-\infty, p].$$

Example 3.2. Let $X = \Lambda = Z = \mathbb{R}$ and $C = \mathbb{R}_+$. Let $f(x,y,p) := x(p+y)$ and $K(p) := [-|p|, p]$. Then, for $x \in K(p)$ we can easily get

$$G(p,x) = \bigcup_{y\in K(p)} x(p+y) \cup \{0\}, \qquad V(p,x) = \begin{cases} \{2px\} & \text{if } p > 0,\ x < 0, \\ \{0\} & \text{otherwise} \end{cases} \quad \text{and} \quad S(p) = \begin{cases} [0,p] & \text{if } p \geqslant 0, \\ \{p\} & \text{if } p < 0. \end{cases}$$

Let $\tilde{C} = \mathbb{R}_+$, $\hat{p} = 0$ and $\hat{x} = 0$. Then, $\Omega(0) = \{0\}$ and $\bar{y} = 0$. If Robinson metric regularity of S is valid, then there exist $\mu > 0$, $\gamma > 0$ and neighborhoods $U_{\hat{p}}$ of $\hat{p}$, $U_{\hat{x}}$ of $\hat{x}$ such that

$$d\big(x, S(p)\big) \leqslant \mu d\big(0, V(p,x)\big), \quad \text{whenever } p \in U_{\hat{p}},\ x \in U_{\hat{x}} \cap K(p),\ d\big(0, V(p,x)\big) < \gamma,$$

which is clearly impossible when $p > 0$ and $x < 0$. While other conditions of Theorem 3.1 hold. By directly computing, we get $DK(\hat{p}, \hat{x})(p) = [-|p|, p]$, $\nabla f(\hat{x}, \bar{y}, \hat{p})(x, y, p) = 0$, $DK(\hat{p}, \bar{y})(p) = [-|p|, p]$, and thus one has

$$\left\{x \in DK(\hat{p}, \hat{x})(p) \;\middle|\; \nabla f(\hat{x}, \bar{y}, \hat{p})(x, y, p) \notin -\operatorname{int} C,\ \forall \bar{y} \in \Omega(0_Z),\ \ y \in DK(\hat{p}, \bar{y})(p)\right\} = \left[-|p|, p\right].$$

However,

$$DS(\hat{p}, \hat{x})(p) = \begin{cases} [0, p] & \text{if } p \geqslant 0, \\ \{p\} & \text{if } p < 0. \end{cases}$$

Thus, for $p > 0$ the conclusions of Theorem 3.1 do not hold.

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