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Metric adjusted skew information and uncertainty relation

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ABSTRACT

We show that an uncertainty relation for Wigner–Yanase–Dyson skew information proved by Yanagi (2010) [10] can hold for an arbitrary quantum Fisher information under some conditions. This is a refinement of the result of Gibilisco and Isola (2011) [4].

1. Introduction

Wigner–Yanase skew information

$$\begin{aligned} I_\rho(H) &= \frac{1}{2} Tr\big[\big(i\big[\rho^{1/2}, H\big]\big)^2\big] \\ &= Tr\big[\rho H^2\big] - Tr\big[\rho^{1/2} H \rho^{1/2} H\big] \end{aligned}$$

was defined in [9]. This quantity can be considered as a kind of the degree for non-commutativity between a quantum state ρ and an observable H. Here we denote the commutator by $[X, Y] = XY - YX$. This quantity was generalized by Dyson

$$\begin{aligned} I_{\rho,\alpha}(H) &= \frac{1}{2} Tr\big[\big(i\big[\rho^\alpha, H\big]\big)\big(i\big[\rho^{1-\alpha}, H\big]\big)\big] \\ &= Tr\big[\rho H^2\big] - Tr\big[\rho^\alpha H \rho^{1-\alpha} H\big], \quad \alpha \in [0,1] \end{aligned}$$

which is known as the Wigner–Yanase–Dyson skew information. Recently it is shown that these skew informations are connected to special choices of quantum Fisher information in [3]. The family of all quantum Fisher informations is parametrized by a certain class of operator monotone functions $\mathcal{F}_{op}$ which were justified in [7]. The Wigner–Yanase skew information and Wigner–Yanase–Dyson skew information are given by the following operator monotone functions

$$\begin{aligned} f_{WY}(x) &= \left(\frac{\sqrt{x}+1}{2}\right)^2, \\ f_{WYD}(x) &= \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)}, \quad \alpha \in (0,1), \end{aligned}$$

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respectively. In particular the operator monotonicity of the function f_{WYD} was proved in [8]. On the other hand the uncertainty relation related to Wigner–Yanase skew information was given by Luo [6] and the uncertainty relation related to Wigner–Yanase–Dyson skew information was given by Yanagi [10], respectively. In this paper we generalize these uncertainty relations to the uncertainty relations related to quantum Fisher informations.

2. Operator monotone functions

Let $M_n = M_n(\mathbb{C})$ (resp. $M_{n,sa} = M_{n,sa}(\mathbb{C})$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices), endowed with the Hilbert–Schmidt scalar product $\langle A, B\rangle = Tr(A^*B)$. Let $\mathcal{D}_n$ be the set of strictly positive elements of M_n and $\mathcal{D}_n^1 \subset \mathcal{D}_n$ be the set of strictly positive density matrices, that is $\mathcal{D}_n^1 = \{\rho \in M_n \mid Tr\,\rho = 1,\ \rho > 0\}$. If it is not otherwise specified, from now on we shall treat the case of faithful states, that is $\rho > 0$.

A function $f : (0, +\infty) \to \mathbb{R}$ is said operator monotone if, for any $n \in \mathbb{N}$, and $A, B \in M_n$ such that $0 \leqslant A \leqslant B$, the inequalities $0 \leqslant f(A) \leqslant f(B)$ hold. An operator monotone function is said symmetric if $f(x) = xf(x^{-1})$ and normalized if $f(1) = 1$.

Definition 2.1. $\mathcal{F}_{op}$ is the class of functions $f : (0, +\infty) \to (0, +\infty)$ such that

(1) $f(1) = 1$,
(2) $tf(t^{-1}) = f(t)$,
(3) f is operator monotone.

Example 2.1. Examples of elements of $\mathcal{F}_{op}$ are given by the following list

$$f_{RLD}(x) = \frac{2x}{x+1}, \qquad f_{WY}(x) = \left(\frac{\sqrt{x}+1}{2}\right)^2, \qquad f_{BKM}(x) = \frac{x-1}{\log x},$$
$$f_{SLD}(x) = \frac{x+1}{2}, \qquad f_{WYD}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^{\alpha}-1)(x^{1-\alpha}-1)}, \quad \alpha \in (0,1).$$

Remark 2.1. Any $f \in \mathcal{F}_{op}$ satisfies

$$\frac{2x}{x+1} \leqslant f(x) \leqslant \frac{x+1}{2}, \quad x > 0.$$

For $f \in \mathcal{F}_{op}$ define $f(0) = \lim_{x \to 0} f(x)$. We introduce the sets of regular and non-regular functions

$$\mathcal{F}_{op}^r = \{f \in \mathcal{F}_{op} \mid f(0) \neq 0\}, \qquad \mathcal{F}_{op}^n = \{f \in \mathcal{F}_{op} \mid f(0) = 0\}$$

and notice that trivially $\mathcal{F}_{op} = \mathcal{F}_{op}^r \cup \mathcal{F}_{op}^n$.

Definition 2.2. For $f \in \mathcal{F}_{op}^r$ we set

$$\tilde{f}(x) = \frac{1}{2}\left[(x+1) - (x-1)^2\frac{f(0)}{f(x)}\right], \quad x > 0.$$

Theorem 2.1. *(See [1,3,5].) The correspondence $f \to \tilde{f}$ is a bijection between $\mathcal{F}_{op}^r$ and $\mathcal{F}_{op}^n$.*

3. Means, Fisher information and metric adjusted skew information

In Kubo–Ando theory of matrix means one associates a mean to each operator monotone function $f \in \mathcal{F}_{op}$ by the formula

$$m_f(A, B) = A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2},$$

where $A, B \in \mathcal{D}_n$. Using the notion of matrix means one may define the class of monotone metrics (also said quantum Fisher informations) by the following formula

$$\langle A, B\rangle_{\rho,f} = Tr\left(A \cdot m_f(L_\rho, R_\rho)^{-1}(B)\right),$$

where $L_\rho(A) = \rho A$, $R_\rho(A) = A\rho$. In this case one has to think of A, B as tangent vectors to the manifold $\mathcal{D}_n^1$ at the point ρ (see [7,3]).

Definition 3.1. For $A \in M_{n,sa}$, we define as follows

$$I^f_\rho(A) = \frac{f(0)}{2}\langle i[\rho, A], i[\rho, A]\rangle_{\rho,f},$$
$$C^f_\rho(A) = Tr\big(m_f(L_\rho, R_\rho)(A) \cdot A\big),$$
$$U^f_\rho(A) = \sqrt{V_\rho(A)^2 - \big(V_\rho(A) - I^f_\rho(A)\big)^2}.$$

The quantity $I^f_\rho(A)$ is known as metric adjusted skew information.

Proposition 3.1. *Let $A_0 = A - Tr(\rho A)I$. The following hold*:

(1) $I^f_\rho(A) = I^f_\rho(A_0) = Tr(\rho A_0^2) - Tr(m_{\tilde{f}}(L_\rho, R_\rho)(A_0) \cdot A_0) = V_\rho(A) - C^{\tilde{f}}_\rho(A_0)$,
(2) $J^f_\rho(A) = Tr(\rho A_0^2) + Tr(m_{\tilde{f}}(L_\rho, R_\rho)(A_0) \cdot A_0) = V_\rho(A) + C^{\tilde{f}}_\rho(A_0)$,
(3) $0 \leqslant I^f_\rho(A) \leqslant U^f_\rho(A) \leqslant V_\rho(A)$,
(4) $U^f_\rho(A) = \sqrt{I^f_\rho(A) \cdot J^f_\rho(A)}$.

Remark 3.1. $I^f_\rho(A)$ is identified in [2] with $Cov_\rho(A, A) - q\, Cov^F_\rho(A, A)$.

4. The main result

Theorem 4.1. *For $f \in \mathcal{F}^r_{op}$, if*

$$\frac{x+1}{2} + \tilde{f}(x) \geqslant 2f(x), \tag{4.1}$$

then it holds

$$U^f_\rho(A) \cdot U^f_\rho(B) \geqslant f(0)\big|Tr\big(\rho[A, B]\big)\big|^2, \tag{4.2}$$

where $A, B \in M_{n,sa}$.

In order to prove Theorem 4.1, we use several lemmas.

Lemma 4.1. *If* (4.1) *holds, then the following inequality is satisfied*

$$\left(\frac{x+y}{2}\right)^2 - m_{\tilde{f}}(x, y)^2 \geqslant f(0)(x-y)^2.$$

Proof. By (4.1) we have

$$\frac{x+y}{2} + m_{\tilde{f}}(x, y) \geqslant 2m_f(x, y). \tag{4.3}$$

Since

$$\begin{aligned} m_{\tilde{f}}(x, y) &= y\tilde{f}\left(\frac{x}{y}\right) \\ &= \frac{y}{2}\left\{\frac{x}{y} + 1 - \left(\frac{x}{y} - 1\right)^2 \frac{f(0)}{f(x/y)}\right\} \\ &= \frac{x+y}{2} - \frac{f(0)(x-y)^2}{2m_f(x, y)}, \end{aligned}$$

we have

$$\begin{aligned} \left(\frac{x+y}{2}\right)^2 - m_{\tilde{f}}(x, y)^2 &= \left\{\frac{x+y}{2} - m_{\tilde{f}}(x, y)\right\}\left\{\frac{x+y}{2} + m_{\tilde{f}}(x, y)\right\} \\ &= \frac{f(0)(x-y)^2}{2m_f(x, y)}\left\{\frac{x+y}{2} + m_{\tilde{f}}(x, y)\right\} \\ &\geqslant f(0)(x-y)^2 \quad \big(\text{by (4.3)}\big). \qquad \square \end{aligned}$$

Lemma 4.2. *Let $\{|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle\}$ be a basis of eigenvectors of ρ, corresponding to the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. We put $a_{jk} = \langle\phi_j|A_0|\phi_k\rangle$, $b_{jk} = \langle\phi_j|B_0|\phi_k\rangle$. By Corollary* 6.1 *in* [1],

$$I_\rho^f(A) = \frac{1}{2}\sum_{j,k}(\lambda_j + \lambda_k)a_{jk}a_{kj} - \sum_{j,k} m_{\tilde f}(\lambda_j, \lambda_k)a_{jk}a_{kj},$$

$$J_\rho^f(A) = \frac{1}{2}\sum_{j,k}(\lambda_j + \lambda_k)a_{jk}a_{kj} + \sum_{j,k} m_{\tilde f}(\lambda_j, \lambda_k)a_{jk}a_{kj},$$

$$\left(U_\rho^f(A)\right)^2 = \frac{1}{4}\left(\sum_{j,k}(\lambda_j + \lambda_k)|a_{jk}|^2\right)^2 - \left(\sum_{j,k} m_{\tilde f}(\lambda_j, \lambda_k)|a_{jk}|^2\right)^2.$$

Proof of Theorem 4.1. Since

$$Tr\big(\rho[A, B]\big) = Tr\big(\rho[A_0, B_0]\big) = \sum_{j,k}(\lambda_j - \lambda_k)a_{jk}b_{kj},$$

we have

$$\begin{aligned}
f(0)\big|Tr\big(\rho[A, B]\big)\big|^2 &\leqslant \left(\sum_{j,k} f(0)^{1/2}|\lambda_j - \lambda_k||a_{jk}||b_{kj}|\right)^2 \\
&\leqslant \left(\sum_{j,k}\left\{\left(\frac{\lambda_j + \lambda_k}{2}\right)^2 - m_{\tilde f}(\lambda_j, \lambda_k)^2\right\}^{1/2}|a_{jk}||b_{kj}|\right)^2 \\
&\leqslant \left(\sum_{j,k}\left\{\frac{\lambda_j + \lambda_k}{2} - m_{\tilde f}(\lambda_j, \lambda_k)\right\}|a_{jk}|^2\right) \times \left(\sum_{j,k}\left\{\frac{\lambda_j + \lambda_k}{2} + m_{\tilde f}(\lambda_j, \lambda_k)\right\}|b_{kj}|^2\right) \\
&= I_\rho^f(A)J_\rho^f(B).
\end{aligned}$$

We also have

$$I_\rho^f(B)J_\rho^f(A) \geqslant f(0)\big|Tr\big(\rho[A, B]\big)\big|^2.$$

Hence we have the final result (4.2). □

By putting

$$f_{WYD}(x) = \alpha(1 - \alpha)\frac{(x - 1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)}, \quad \alpha \in (0, 1),$$

we obtain the following uncertainty relation:

Corollary 4.1. *(See [10].) For $A, B \in M_{n,sa}$,*

$$U_\rho^{f_{WYD}}(A)U_\rho^{f_{WYD}}(B) \geqslant \alpha(1 - \alpha)\big|Tr\big(\rho[A, B]\big)\big|^2.$$

Proof. Since

$$f_{WYD}(x) = \alpha(1 - \alpha)\frac{(x - 1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)},$$

it is clear that

$$\tilde f_{WYD}(x) = \frac{1}{2}\left\{x + 1 - \left(x^\alpha - 1\right)\left(x^{1-\alpha} - 1\right)\right\}.$$

By Lemma 3.3 in [10] we have for $0 \leqslant \alpha \leqslant 1$ and $x > 0$,

$$(1 - 2\alpha)^2(x - 1)^2 - \left(x^\alpha - x^{1-\alpha}\right)^2 \geqslant 0.$$

Then we can rewrite as follows

$$\left(x^{2\alpha} - 1\right)\left(x^{2(1-\alpha)} - 1\right) \geqslant 4\alpha(1 - \alpha)(x - 1)^2.$$

Thus

$$\begin{aligned}\frac{x+1}{2}+\tilde{f}_{WYD}(x) &= x+1-\frac{1}{2}\left(x^{\alpha}-1\right)\left(x^{1-\alpha}-1\right)\\ &= \frac{1}{2}\left(x^{\alpha}+1\right)\left(x^{1-\alpha}+1\right)\\ &\geqslant 2\alpha(1-\alpha)\frac{(x-1)^2}{(x^{\alpha}-1)(x^{1-\alpha}-1)}\\ &= 2f_{WYD}(x).\end{aligned}$$

It follows from Theorem 4.1 that we can give the aimed result. □

Remark 4.1. In [4], the following result was given. Even if (4.1) does not necessarily hold, then

$$U_{\rho}^{f}(A)U_{\rho}^{f}(B)\geqslant f(0)^2\left|Tr\left(\rho[A,B]\right)\right|^2, \tag{4.4}$$

where $A, B \in M_{n,sa}$. Since $f(0) < 1$, it is easy to show (4.4) is weaker than (4.2).

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