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Multiple solutions for a class of biharmonic equations with a nonlinearity concave at the origin ☆

Jian Zhang [a,*], Zhongli Wei [a,b]

[a] School of Mathematics, Shandong University, Jinan, Shandong 250100, People's Republic of China
[b] Department of Mathematics, Shandong Jianzhu University, Jinan, Shandong 250101, People's Republic of China

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ABSTRACT

In this paper, we investigate the existence of multiple solutions for a class of biharmonic equations where the nonlinearity involves a concave term at the origin. The solutions are obtained from the versions of mountain pass lemma and linking theorem.

1. Introduction

In this paper, we are concerned with the multiplicity of solutions for the following biharmonic problem:

$$\begin{cases} \Delta^2 u + a\Delta u = -\lambda |u|^{q-2}u + f(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ^2 is the biharmonic operator, $\Omega \subset R^s$ is a bounded smooth domain with smooth boundary $\partial\Omega$ and $s \in N$. $a < \lambda_1$ (λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$), λ is a real parameter and $1 < q < 2$. We assume that $f(x,u)$ satisfies some of the following assumptions:

(f_1) $f \in C(\overline{\Omega} \times R, R)$.
(f_2) There exists $C > 0$ such that $|f(x,u)| \leqslant C(1 + |u|^{p-1})$ for $x \in \Omega$ and $u \in R$, where $2 < p < 2^{**}$, $2^{**} = \frac{2s}{s-4}$ for $s > 4$ and $2^{**} = \infty$ for $s \leqslant 4$.
(f_3) There exist $c_1 > 0$ and $r_0 > 0$ such that $|f(x,u)| \leqslant c_1|u|$ for $x \in \Omega$ and $|u| \leqslant r_0$.
(f_4) $\lim_{u\to\pm\infty} \frac{f(x,u)}{u} = b^{\pm}$ uniformly for $x \in \Omega$.
(f_5) $H(x,u) \geqslant L(x) \in L^1(\Omega)$ and $\lim_{|u|\to\infty} H(x,u) = +\infty$ a.e. $x \in \Omega$, where $H(x,u) = \frac{1}{2}f(x,u)u - F(x,u)$ and $F(x,u) = \int_0^u f(x,s)\,ds$.
(f_6) There exist $0 < \mu < 2^{**}$, $c_2 > 0$ and $D > 0$ such that $H(x,u) \geqslant c_2|u|^{\mu}$ for $x \in \Omega$ and $|u| \geqslant D$.

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* Corresponding author.
E-mail address: chjmath@yahoo.com.cn (J. Zhang).

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Biharmonic equations have been studied by many authors. In [5], Lazer and Mckenna considered the biharmonic problem:

$$\begin{cases} \Delta^2 u + a\Delta u = d\big[(u+1)^+ - 1\big], & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $u^+ = \max\{u, 0\}$ and $d \in R$. They pointed out that this type of nonlinearity furnishes a model to study traveling waves in suspension bridges. In [6], the authors got $2k - 1$ solutions when $N = 1$ and $d > \lambda_k(\lambda_k - c)$ (λ_k is the sequence of the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$) via the global bifurcation method. In [14], a negative solution of (1.2) was obtained when $d \geqslant \lambda_1(\lambda_1 - c)$ by a degree argument. If the nonlinearity $d[(u+1)^+ - 1]$ is replaced by a general function $f(x, u)$, one has the following problem:

$$\begin{cases} \Delta^2 u + c\Delta u = f(x, u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In [7,8], the authors proved the existence of two or three solutions of problem (1.3) for a more general nonlinearity f by using a variational method. In [16], positive solutions of problem (1.3) were got when f satisfies the local superlinearity and sublinearity.

On the other hand, there has been considerable amount of papers on elliptic problems involving concave terms. We refer the reader to [1–4,9,10,13,15] and the references therein. In particular, de Paiva and Massa [4] considered the following problem:

$$\begin{cases} -\Delta u = -\lambda |u|^{q-2}u + au + g(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset R^N$ is an open bounded domain with smooth boundary $\partial\Omega$, $a \in R$, $\lambda > 0$ is a real parameter, $1 < q < 2$ and $g : R \to R$ is a function of class C^1. Moreover, g satisfies some of the following assumptions:

(g_0) $g(0) = 0$.
(g_1) $g'(0) = 0$ and $a \in [\lambda_k, \lambda_{k+1})$.
(g_2) (i) $G(u) \geqslant 0$, where $G(u) = \int_0^u g(s)\,ds$.
(ii) $G(u) \leqslant C + C|u|^p$ with $2 < p < 2^* = \frac{2N}{N-2}$.
(g_3) $\lim_{s\to\pm\infty} \frac{as+g(s)}{s} = b^{\pm} \in (\lambda_{k+1}, +\infty]$.
(g_4) (i) There exist $\bar{t} > 0$ and $\mu < \frac{1}{2}$ such that $[\frac{a}{2}t^2 + G(t)] \leqslant \mu t[at + g(t)]$ for $|t| > \bar{t}$.
(ii) $\mu(p-1) < \frac{N+2}{2N}$.
(g_4') $b^{\pm} \in R$ but $(b^+, b^-) \notin \sum$, where we denote by $\sum$ the Fučík spectrum of the operator.
(g_4'') (i) There exist $\bar{t} > 0$ and $\mu < \frac{1}{2}$ such that $[\frac{a}{2}t^2 + G(t)] \leqslant \mu t[at + g(t)]$ for $t > \bar{t}$.
(ii) $b^- \in R$ but $b^- \neq \lambda_1$.
(iii) There exists $\alpha \in [0, 1)$ such that $\lim_{s\to-\infty} \frac{as+g(s)-\lambda|s|^{q-2}s-b^-s}{|s|^\alpha} = 0$ and $\mu(p-1) < \min\{\frac{1}{\alpha+1}, \frac{N+2}{2N}\}$.

They proved the following two theorems.

Theorem A. *Assume that g satisfies (g_0), (g_2)(ii), (g_3) with $k \geqslant 0$ and one of the (g_4)′s, then for all $\lambda > 0$, problem (1.4) has at least two nontrivial solutions.*

Theorem B. *Assume that g satisfies (g_0)–(g_3) with $k \geqslant 1$ and one of the (g_4)′s, then there exists $\lambda^* > 0$, such that problem (1.4) has at least three nontrivial solutions for $\lambda \in (0, \lambda^*)$.*

Our aim in the present paper is to improve and generalize the result obtained in [4] to problem (1.1). We note that $u \in H^2(\Omega)$ does not imply that $u^{\pm} \in H^2(\Omega)$, where $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$. Thus, the method in [4] cannot be applied directly. On the other hand, in order to have the (PS) condition for the corresponding functional, the authors in [4] assumed one of the (g_4)′s. However, the assumptions in our paper are different from (g_4). For the case $b^{\pm} = +\infty$, (g_4)(i) is replaced by (f_6) with $\mu > \frac{N}{2}(p-2)$. For the case $b^{\pm} \in R$, (f_5) replaces (g_4)(ii). Then it is difficult to derive the boundedness of the (PS) sequence for the corresponding functional. The mountain pass lemma and linking theorem without the (PS) condition must be applied to overcome the difficulty. Besides, for the case $b^{\pm} \in R$, by weakening (g_2)(i), we get a theorem different from results in [4].

Before stating our main results we give some notations. Throughout this paper, we denote by C a universal positive constant unless otherwise specified and we set $L^s(\Omega)$ the usual Lebesgue space equipped with the norm $\|u\|_s := (\int_\Omega |u|^s\,dx)^{\frac{1}{s}}$, $1 \leqslant s < \infty$. Let λ_k ($k = 1, 2, \ldots$) denote the eigenvalues and φ_k ($k = 1, 2, \ldots$) the corresponding normalized eigenfunctions of the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Here, we repeat each eigenvalue according to its (finite) multiplicity. Then, $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ and $\lambda_k \to \infty$ as $k \to \infty$.

Our main results are stated as follows:

Theorem 1.1. *Assume that f satisfies (f_1) and (f_3)–(f_4) with $\lambda_1(\lambda_1 - a) < b^+ < +\infty$ or $\lambda_1(\lambda_1 - a) < b^- < +\infty$. Then, given $\lambda > 0$, problem (1.1) has at least one nontrivial solution.*

Theorem 1.2. *Assume that f satisfies (f_1) and (f_3)–(f_5) with $\lambda_{k+1}(\lambda_{k+1} - a) < b^\pm < +\infty$ for some $k \in N$. Moreover, $F(x,u) \geqslant \frac{1}{2}\lambda_m(\lambda_m - a)u^2$ and $\limsup_{u\to 0} \frac{F(x,u)}{u^2} < \frac{1}{2}\lambda_{m+1}(\lambda_{m+1} - a)$ for some $m \in N$, $m \leqslant k$. Then, there exists $\lambda^* > 0$, such that for $0 < \lambda < \lambda^*$, problem (1.1) has at least three nontrivial solutions.*

In our next result we establish the multiplicity of solutions for problem (1.1) by weakening $F(x,u) \geqslant \frac{1}{2}\lambda_m(\lambda_m - a)u^2$. For doing that we assume a stronger version of (f_3).

Theorem 1.3. *Assume that f satisfies (f_1) and (f_4)–(f_5) with $\lambda_{k+1}(\lambda_{k+1} - a) < b^\pm < +\infty$ for some $k \in N$. Moreover, for some $m \in N$, $m \leqslant k$, $F(x,u) \geqslant \frac{1}{2}\lambda_m(\lambda_m - a)u^2 - W_0(x)$ and there exist $L, \delta_0 > 0$, such that for $|u| \leqslant L$, $F(x,u) \leqslant [\frac{1}{2}\lambda_{m+1}(\lambda_{m+1} - a) - \delta_0]u^2$. Here, $\lambda_m < \lambda_{m+1}$ and $W_0(x) \in L^1(\Omega)$. Then, given $\lambda > 0$, there exists $L^* > 0$, such that for $L > L^*$, problem (1.1) has at least three nontrivial solutions.*

In the case $b^\pm = +\infty$, we establish the following version of Theorem 1.2.

Theorem 1.4. *Assume that $s \geqslant 5$, f satisfies (f_1)–(f_4) and (f_6) with $b^\pm = +\infty$. Moreover, $F(x,u) \geqslant \frac{1}{2}\lambda_m(\lambda_m - a)u^2$ and $\limsup_{u\to 0} \frac{F(x,u)}{u^2} < \frac{1}{2}\lambda_{m+1}(\lambda_{m+1} - a)$ for some $m \in N$. Then, there exists $\lambda^{**} > 0$, such that for $0 < \lambda < \lambda^{**}$ and $\mu > \frac{s}{4}(p-2)$, problem (1.1) has at least three nontrivial solutions.*

2. Preliminary lemmas

Let $\Omega \subset R^s$ be a bounded smooth domain, $H = H^2(\Omega) \cap H_0^1(\Omega)$ be the Hilbert space equipped with the inner product

$$(u,v)_H = \int_\Omega \Delta u \Delta v \, dx,$$

which induces the norm

$$\|u\|_H = \left(\int_\Omega |\Delta u|^2 \, dx \right)^{\frac{1}{2}}.$$

Note that $\mu_k = \lambda_k^2$, $k = 1, 2, \ldots$, are eigenvalues of the eigenvalue problem

$$\begin{cases} \Delta^2 u = \mu u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

φ_k, $k = 1, 2, \ldots$ are the corresponding eigenfunctions. Furthermore, the set of $\{\varphi_k\}$ is an orthogonal base on the Hilbert space H.

We observe that $\{\frac{\varphi_k}{\|\varphi_k\|_H}\}_{k=1}^\infty$ is an orthonormal basis of H. Then, for $u \in H$, we can write that

$$u = \sum_{k=1}^\infty c_k \frac{\varphi_k}{\|\varphi_k\|_H}$$

for $c_k = (u, \frac{\varphi_k}{\|\varphi_k\|_H})_H$, the series converging in H. In addition,

$$\|u\|_H^2 = \sum_{k=1}^\infty c_k^2. \tag{2.1}$$

Denote $u_m = \sum_{k=1}^m c_k \frac{\varphi_k}{\|\varphi_k\|_H}$, where $m \in N$. Thus,

$$\lim_{m\to\infty} \|u_m - u\|_H = 0.$$

Recall that $H_0^1(\Omega)$ is the Hilbert space equipped with the inner product

$$(u,v)_{H_0^1} = \int_\Omega \nabla u \nabla v \, dx,$$

which induces the norm

$$\|u\|_{H_0^1} = \left(\int_\Omega |\nabla u|^2 \, dx \right)^{\frac{1}{2}}.$$

We note that for $u \in H$,

$$\int_\Omega |\nabla u|^2 \, dx \leqslant C \int_\Omega |\Delta u|^2 \, dx.$$

Thus,

$$\lim_{m\to\infty} \|u_m - u\|_{H_0^1} = 0.$$

Now, we rewrite that

$$u = \sum_{k=1}^\infty c_k \frac{\|\varphi_k\|_{H_0^1}}{\|\varphi_k\|_H} \frac{\varphi_k}{\|\varphi_k\|_{H_0^1}},$$

the series converging in $H_0^1(\Omega)$. Observe that $\{\frac{\varphi_k}{\|\varphi_k\|_{H_0^1}}\}_{k=1}^\infty$ is an orthonormal basis of $H_0^1(\Omega)$, we have

$$\|u\|_{H_0^1}^2 = \sum_{k=1}^\infty c_k^2 \frac{\|\varphi_k\|_{H_0^1}^2}{\|\varphi_k\|_H^2}. \tag{2.2}$$

Combining (2.1)–(2.2), we obtain that for $u \in H$,

$$\|u\|_H^2 \geqslant \lambda_1 \|u\|_{H_0^1}^2. \tag{2.3}$$

For $a < \lambda_1$, define a norm $u \in H$ as follows:

$$\|u\| = \left(\int_\Omega |\Delta u|^2 \, dx - a \int_\Omega |\nabla u|^2 \, dx \right)^{\frac{1}{2}}.$$

From (2.3), the norm $\|.\|$ is an equivalent norm on H. Throughout this paper, we use the norm $\|.\|$ unless stated otherwise. It is well known that $\wedge_k = \lambda_k(\lambda_k - a)$, $k = 1, 2, \ldots$, are eigenvalues of the eigenvalue problem

$$\begin{cases} \Delta^2 u + a\Delta u = \wedge u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

φ_k, $k = 1, 2, \ldots$ are the corresponding eigenfunctions. Furthermore, the set of $\{\varphi_k\}$ is an orthogonal basis on the Hilbert space H.

For $u \in H$, denote

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{\lambda}{q} \int_\Omega |u|^q \, dx - \int_\Omega F(x, u) \, dx$$

and

$$I^\pm(u) = \frac{1}{2}\|u\|^2 + \frac{\lambda}{q} \int_\Omega \left|u^\pm\right|^q dx - \int_\Omega F\left(x, u^\pm\right) dx,$$

where $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$. Under the assumptions (f_1)–(f_2), we have $I, I^\pm \in C^1(H)$.

Recall that a sequence $\{u_n\}$ is a $(C)_c$ sequence for the functional I if $I(u_n) \to c$ and $(1 + \|u_n\|)I'(u_n) \to 0$. If any $(C)_c$ sequence $\{u_n\}$ has a convergent subsequence, we say that I satisfies the $(C)_c$ condition.

In order to prove our main results, we need the following theorems.

Theorem 2.1. *(See [11].) Let E be a real Banach space with its dual space E^* and suppose that $I \in C^1(E, R)$ satisfies the condition*

$$\max\left\{I(0), I(u_1)\right\} \leqslant \alpha < \beta \leqslant \inf_{\|u\|=\rho} I(u)$$

for some $\rho > 0$ and $u_1 \in E$ with $\|u_1\| > \rho$. Let c be characterized by

$$c := \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} I\left(\gamma(t)\right),$$

where $\Gamma := \{\gamma \in C([0, 1], E);\ \gamma(0) = 0,\ \gamma(1) = u_1\}$. Then there exists a $(C)_c$ sequence $\{u_n\}$ for the functional I satisfying $c \geqslant \beta$.

Definition C. (See [12].) Let E be a Banach space and let Φ be the set of all continuous maps $\Gamma = \Gamma(t)$ from $E \times [0,1]$ to E such that

1. $\Gamma(0) = I$.
2. For each $t \in [0,1)$, $\Gamma(t)$ is a homeomorphism of E onto E and $\Gamma^{-1}(t) \in C(E \times [0,1), E)$.
3. $\Gamma(1)E$ is a single point in E and $\Gamma(t)A$ converges uniformly to $\Gamma(1)E$ as $t \to 1$ for each bounded set $A \subset E$.
4. For each $t_0 \in [0,1)$ and each bounded set $A \subset E$

$$\sup_{0 \leqslant t \leqslant t_0, u \in A} \left\{ \left\| \Gamma(t)u \right\| + \left\| \Gamma^{-1}(t)u \right\| \right\} < \infty.$$

Definition D. (See [12].) We say that A links B $[hm]$ if A, B are subsets of E such that $A \cap B = \emptyset$ and, for each $\Gamma(t) \in \Phi$, there is $t \in (0,1]$ such that $\Gamma(t)A \cap B \neq \emptyset$.

The following proposition provides an example of A links B $[hm]$.

Proposition E. *(See [12].) Let E be a real Hilbert space, E_1, E_2 be two closed subspace of E such that*

$$E = E_1 \oplus E_2, \qquad \dim E_2 < +\infty.$$

Consider $e \in E_1$, $\|e\| = 1$. Let R, ρ be positive numbers and set

$$S = E_1 \cap S_\rho, \qquad Q = \left\{ u + v;\ u \in E_2,\ v = te,\ t \geqslant 0,\ \|u + v\| \leqslant R \right\}.$$

Then, if $R > \rho$, ∂Q links S $[hm]$.

Theorem 2.2. *(See [12].) Let E be a real Hilbert space and assume that $I \in C^1(E, R)$ satisfies the condition*

$$\sup_{u \in \partial Q} I(u) < \inf_{u \in S} I(u),$$

where ∂Q, S are defined in Proposition E. Set

$$c := \inf_{\Gamma \in \Phi} \sup_{0 \leqslant s \leqslant 1,\, u \in \partial Q} I\big(\Gamma(s)u\big),$$

where Φ is defined in Definition C. Then, if c is finite, there exists a $(C)_c$ sequence $\{u_n\}$ for the functional I satisfying $c \geqslant \inf_{u \in S} I(u)$.

Lemma 2.3. *Assume that f satisfies (f_1)–(f_3). Then given $\lambda > 0$, there exist $\rho_1, \beta_1 > 0$, such that*

$$\inf_{u \in H,\, \|u\| = \rho_1} I^+(u) \geqslant \beta_1 > 0.$$

Proof. By (f_1)–(f_3), there holds

$$\left| F(x,u) \right| \leqslant C\big(|u|^2 + |u|^p\big). \tag{2.4}$$

Thus, from (2.4),

$$\begin{aligned} I^+(u) &= \frac{1}{2}\|u\|^2 + \frac{\lambda}{q} \int_\Omega \left|u^+\right|^q \mathrm{d}x - \int_\Omega F\big(x, u^+\big)\, \mathrm{d}x \\ &\geqslant \frac{1}{2}\|u\|^2 - C \int_\Omega \left|u^+\right|^2 \mathrm{d}x - C \int_\Omega |u|^p \,\mathrm{d}x + \frac{\lambda}{q} \int_\Omega \left|u^+\right|^q \mathrm{d}x \\ &\geqslant \frac{1}{2}\|u\|^2 - C\|u\|^p - C \int_\Omega \left|u^+\right|^2 \mathrm{d}x + \frac{\lambda}{q} \int_\Omega \left|u^+\right|^q \mathrm{d}x. \end{aligned}$$

Hence, for $\|u\|$ small enough,

$$I^+(u) \geqslant \frac{1}{3}\|u\|^2 - C \int_\Omega \left|u^+\right|^2 \mathrm{d}x + \frac{\lambda}{q} \int_\Omega \left|u^+\right|^q \mathrm{d}x.$$

(2.3) implies that

$$\|u\|^2 \geqslant (\lambda_1 - a)\|u\|^2_{H_0^1}. \tag{2.5}$$

Then we can choose $i \in N$ and $\mu \in (\frac{(\lambda_1-a)\lambda_i}{4}, \frac{(\lambda_1-a)\lambda_{i+1}}{4})$, such that

$$\begin{aligned} I^+(u) &\geqslant \frac{1}{12}\|u\|^2 + \frac{\lambda_1 - a}{4}\|u\|^2_{H_0^1} - \mu \int_\Omega |u^+|^2 \, dx + \frac{\lambda}{q}\int_\Omega |u^+|^q \, dx \\ &\geqslant \frac{1}{12}\|u\|^2 + \frac{\lambda_1 - a}{4}\|u^+\|^2_{H_0^1} - \mu \int_\Omega |u^+|^2 \, dx + \frac{\lambda}{q}\int_\Omega |u^+|^q \, dx. \end{aligned} \tag{2.6}$$

Let $X_j := span\{\varphi_j\}$, $j \in N$ and set $G_i := X_1 \oplus X_2 \oplus \cdots \oplus X_i$, $i \in N$, where $\oplus$ means the orthogonal sum of the subspace. We note that $H_0^1(\Omega) = G_i \oplus G_i^\perp$. Thus, u^+ can be decomposed as $u^+ = v + w$, where $v \in G_i$ and $w \in G_i^\perp$. Observe that for $v \in G_i$, there holds

$$\|v\|^2_{H_0^1} \geqslant \lambda_1 \int_\Omega v^2 \, dx,$$

and for $w \in G_i^\perp$, there holds

$$\|w\|^2_{H_0^1} \geqslant \lambda_{i+1} \int_\Omega w^2 \, dx.$$

Therefore,

$$\begin{aligned} I^+(u) &\geqslant \frac{1}{12}\|u\|^2 + \frac{1}{4}\left[(\lambda_1 - a) - \frac{4\mu}{\lambda_{i+1}}\right]\|w\|^2_{H_0^1} - \frac{1}{4}\left[\frac{4\mu}{\lambda_1} - (\lambda_1 - a)\right]\|v\|^2_{H_0^1} + \frac{\lambda}{q}\int_\Omega |u^+|^q \, dx \\ &:= \frac{1}{12}\|u\|^2 + \xi\|w\|^2_{H_0^1} - \eta\|v\|^2_{H_0^1} + \frac{\lambda}{q}\int_\Omega |u^+|^q \, dx, \end{aligned} \tag{2.7}$$

where $\xi, \eta > 0$.

It suffices to show that there exists $\rho_1 > 0$ small enough, such that for $\|u\| = \rho_1$,

$$I_1^+(u) := \xi\|w\|^2_{H_0^1} - \eta\|v\|^2_{H_0^1} + \frac{\lambda}{q}\int_\Omega |u^+|^q \, dx \geqslant 0. \tag{2.8}$$

Seeking a contradiction we suppose that there exist $u_n \neq 0$ satisfying $\|u_n\| \to 0$ as $n \to \infty$ and $I_1^+(u_n) < 0$. By (2.5), $\|u_n^+\|_{H_0^1} \to 0$ as $n \to \infty$. Decompose u_n^+ as $u_n^+ = v_n + w_n$, where $v_n \in G_i$ and $w_n \in G_i^\perp$, we have

$$I_1^+(u_n) = \xi\|w_n\|^2_{H_0^1} - \eta\|v_n\|^2_{H_0^1} + \frac{\lambda}{q}\int_\Omega |u_n^+|^q \, dx < 0. \tag{2.9}$$

Then $u_n^+ \neq 0$ in $H_0^1(\Omega)$. Let $z_n = \frac{u_n^+}{\|u_n^+\|_{H_0^1}}$. Up to a subsequence, we get that

$$\begin{aligned} &z_n \rightharpoonup z \quad \text{weakly in } H_0^1(\Omega), \\ &z_n \to z \quad \text{strongly in } L^t(\Omega),\ 1 \leqslant t < 2^*, \\ &z_n(x) \to z(x) \quad \text{a.e. } x \in \Omega. \end{aligned}$$

Dividing $\|u_n^+\|^q_{H_0^1}$ in both sides of (2.9),

$$\frac{\xi\|w_n\|^2_{H_0^1} - \eta\|v_n\|^2_{H_0^1}}{\|u_n^+\|^2_{H_0^1}}\|u_n^+\|^{2-q}_{H_0^1} + \frac{\lambda}{q}\int_\Omega |z_n|^q \, dx < 0.$$

Let $n \to \infty$, there holds

$$\int_\Omega |z|^q \, dx \leqslant 0,$$

in view of $\|u_n^+\|_{H_0^1} \to 0$ as $n \to \infty$. Thus, $z = 0$ a.e. $x \in \Omega$. Then we have

$$\frac{\|v_n\|_{H_0^1}^2}{\|u_n^+\|_{H_0^1}^2} \leqslant \frac{C\|v_n\|_2^2}{\|u_n^+\|_{H_0^1}^2} \leqslant C\|z_n\|_2^2 \to 0, \quad \text{as } n \to \infty, \tag{2.10}$$

using the equivalence of all norms on the finite dimensional space. Choosing n sufficient large, we obtain that

$$\begin{aligned} I_1^+(u_n) &= \xi\|w_n\|_{H_0^1}^2 - \eta\|v_n\|_{H_0^1}^2 + \frac{\lambda}{q}\int_\Omega |u_n^+|^q\,dx \\ &\geqslant \xi\|u_n^+\|_{H_0^1}^2 - (\xi+\eta)\|v_n\|_{H_0^1}^2 \\ &= \left[\xi - (\xi+\eta)\frac{\|v_n\|_{H_0^1}^2}{\|u_n^+\|_{H_0^1}^2}\right]\|u_n^+\|_{H_0^1}^2 \geqslant 0, \end{aligned}$$

in view of (2.10). Thus we get a contradiction.

Therefore, we can choose $\rho_1 > 0$ small enough, such that for $\|u\| = \rho_1$,

$$I^+(u) \geqslant \frac{1}{12}\|u\|^2 + I_1^+(u) \geqslant \frac{1}{12}\rho_1^2 := \beta_1 > 0. \qquad \square$$

Using a similar argument as Lemma 2.3, we have the following Lemma 2.4 and Lemma 2.5.

Lemma 2.4. *Assume that f satisfies (f_1)–(f_3). Then given $\lambda > 0$, there exist $\rho_2, \beta_2 > 0$, such that*

$$\inf_{u\in H, \|u\|=\rho_2} I^-(u) \geqslant \beta_2 > 0.$$

Lemma 2.5. *Assume that f satisfies (f_1)–(f_3). Then given $\lambda > 0$, there exist $\rho, \beta > 0$, such that*

$$\inf_{u\in H, \|u\|=\rho} I(u) \geqslant \beta > 0.$$

Now we are ready to prove our main results.

3. Proof of main results

Proof of Theorem 1.1. It is easy to see that $I^+(0) = 0$. We note that (f_1) and (f_4) with $\wedge_1 < b^+ < +\infty$ imply (f_2). Then from Lemma 2.3, given $\lambda > 0$, there exist $\rho_1, \beta_1 > 0$, such that

$$\inf_{\|u\|=\rho_1} I^+(u) \geqslant \beta_1 > 0.$$

On the other hand, (f_1) and (f_4) imply that

$$\lim_{u\to+\infty} \frac{F(x,u)}{u^2} = \frac{1}{2}b^+ > \frac{1}{2}\wedge_1.$$

Then, there exists $\epsilon_0 > 0$, such that

$$F(x,u^+) \geqslant \frac{1}{2}(\wedge_1 + \epsilon_0)|u^+|^2 - C.$$

Thus,

$$I^+(u) \leqslant \frac{1}{2}\|u\|^2 + \frac{\lambda}{q}\int_\Omega |u^+|^q\,dx - \frac{1}{2}(\wedge_1+\epsilon_0)\int_\Omega |u^+|^2\,dx + C\,meas(\Omega).$$

Choosing $u = t\varphi_1$, where $t > 0$ and φ_1 is the eigenfunction associated to $\wedge_1$, we have

$$I^+(t\varphi_1) \leqslant -\frac{1}{2}\epsilon_0 t^2\int_\Omega |\varphi_1|^2\,dx + \frac{\lambda}{q}t^q\int_\Omega |\varphi_1|^q\,dx + C\,meas(\Omega) \to -\infty, \quad \text{as } t \to +\infty.$$

Let t_1 be such that $\|t_1\varphi_1\| > \rho_1$ and $I^+(t_1\varphi_1) < 0$. Define

$$c^+ := \inf_{\gamma\in\Gamma^+}\max_{0\leqslant t\leqslant 1} I^+(\gamma(t)),$$

where $\Gamma^+ := \{\gamma \in C([0,1], H);\ \gamma(0) = 0,\ \gamma(1) = t_1\varphi_1\}$. It follows from Theorem 2.1 that there exists a sequence $\{u_n\} \subset H$, such that

$$I^+(u_n) \to c^+ \geqslant \beta_1, \quad \text{as } n \to +\infty, \tag{3.1}$$

and

$$\big(1 + \|u_n\|\big) I^{+'}(u_n) \to 0, \quad \text{as } n \to +\infty. \tag{3.2}$$

We claim that the sequence $\{u_n\}$ is bounded in H.

(f_3) implies that $f(x,0) = 0$. Thus, by (3.2),

$$o(1) = \big(I^{+'}(u_n), u_n\big) = \|u_n\|^2 + \lambda \int_\Omega \big|u_n^+\big|^q \,\mathrm{d}x - \int_\Omega f\big(x, u_n^+\big) u_n^+ \,\mathrm{d}x. \tag{3.3}$$

Seeking a contradiction we suppose that $\|u_n\| \to \infty$. Let $z_n = \frac{u_n}{\|u_n\|}$. Up to a subsequence, we get that

$$\begin{aligned} &z_n \rightharpoonup z \quad \text{weakly in } H,\\ &z_n \to z \quad \text{strongly in } L^t(\Omega),\ 1 \leqslant t < 2^*,\\ &z_n(x) \to z(x) \quad \text{a.e. } x \text{ in } \Omega. \end{aligned}$$

We claim that

$$z \neq 0 \quad \text{in } H. \tag{3.4}$$

Otherwise, $z = 0$ in H. Dividing $\|u_n\|^2$ in both sides of (3.3), we get that

$$o(1) = 1 - \int_\Omega \frac{f(x, u_n^+) u_n^+}{\|u_n\|^2} \,\mathrm{d}x. \tag{3.5}$$

(f_1) and (f_3)–(f_4) with $\wedge_1 < b^+ < +\infty$ imply that there exists $C' > 0$, such that

$$\big|f\big(x, u_n^+\big)\big| \leqslant C' u_n^+. \tag{3.6}$$

Combining (3.5)–(3.6), we have

$$\begin{aligned} 1 &= \int_\Omega \frac{f(x, u_n^+) u_n^+}{\|u_n\|^2} \,\mathrm{d}x + o(1)\\ &\leqslant C' \int_\Omega \big|z_n^+\big|^2 \,\mathrm{d}x + o(1)\\ &\leqslant C' \int_\Omega |z_n|^2 \,\mathrm{d}x + o(1), \end{aligned}$$

where $z_n^+ = \frac{u_n^+}{\|u_n\|}$. Let $n \to \infty$, we get a contradiction. Thus, (3.4) is proved.

Set

$$p_n(x) = \begin{cases} \frac{f(x, u_n^+(x))}{u_n^+(x)} & \text{for } x \in \Omega \text{ with } u_n(x) > 0,\\ 0 & \text{for } x \in \Omega \text{ with } u_n(x) \leqslant 0. \end{cases}$$

From $I^{+'}(u_n) = o(1)$,

$$\int_\Omega [\Delta u_n \Delta\varphi - a\nabla u_n \nabla\varphi] \,\mathrm{d}x + \lambda \int_\Omega \big|u_n^+\big|^{q-2} u_n^+ \varphi \,\mathrm{d}x - \int_\Omega f\big(x, u_n^+\big)\varphi \,\mathrm{d}x = o(1),$$

for all $\varphi \in H$. Dividing $\|u_n\|$ in both sides of the above equality, there holds

$$\int_\Omega [\Delta z_n \Delta\varphi - a\nabla z_n \nabla\varphi] \,\mathrm{d}x - \int_\Omega p_n z_n^+ \varphi \,\mathrm{d}x = o(1). \tag{3.7}$$

We note that $\{z_n\}$ is bounded in $H_0^1(\Omega)$. Thus, $z_n \rightharpoonup z$ weakly in $H_0^1(\Omega)$, which implies that $z_n^+ \to z^+$ strongly in $L^2(\Omega)$ and $z_n^+(x) \to z^+(x)$ a.e. $x \in \Omega$. By (3.6), $|p_n(x)| \leqslant C'$ for $x \in \Omega$. Then we have

$$\left|\int\limits_{\{x\in\Omega, z^+(x)=0\}} p_n z_n^+ \varphi \,\mathrm{d}x\right| \leqslant C' \int\limits_{\{x\in\Omega, z^+(x)=0\}} z_n^+ |\varphi| \,\mathrm{d}x = o(1) + C' \int\limits_{\{x\in\Omega, z^+(x)=0\}} z^+ |\varphi| \,\mathrm{d}x = o(1). \tag{3.8}$$

On the other hand, since $z_n^+(x) \to z^+(x)$ a.e. $x \in \Omega$, we have $\lim_{n\to\infty} u_n^+(x) = +\infty$ for a.e. $x \in \{x \in \Omega,\ z^+(x) > 0\}$, which implies that $\lim_{n\to\infty} p_n(x) = b^+$ for a.e. $x \in \{x \in \Omega,\ z^+(x) > 0\}$. Besides, $|p_n(x)| \leqslant C'$ for $x \in \Omega$. Using the Lebesgue's Dominated Convergence Theorem, we obtain that

$$\begin{aligned}
\left|\int\limits_{\{x\in\Omega, z^+(x)>0\}} \big(p_n - b^+\big) z_n^+ \varphi \,\mathrm{d}x\right| &\leqslant \int\limits_{\{x\in\Omega, z^+(x)>0\}} \big|p_n - b^+\big| |\varphi| z_n^+ \,\mathrm{d}x \\
&\leqslant \Bigg(\int\limits_{\{x\in\Omega, z^+(x)>0\}} \big|p_n - b^+\big|^2 \varphi^2 \,\mathrm{d}x\Bigg)^{\frac{1}{2}} \Bigg(\int\limits_{\{x\in\Omega, z^+(x)>0\}} \big(z_n^+\big)^2 \,\mathrm{d}x\Bigg)^{\frac{1}{2}} \\
&\leqslant C\Bigg(\int\limits_{\{x\in\Omega, z^+(x)>0\}} \big|p_n - b^+\big|^2 \varphi^2 \,\mathrm{d}x\Bigg)^{\frac{1}{2}} = o(1),
\end{aligned}$$

which implies that

$$\begin{aligned}
\int\limits_{\{x\in\Omega, z^+(x)>0\}} p_n z_n^+ \varphi \,\mathrm{d}x &= \int\limits_{\{x\in\Omega, z^+(x)>0\}} \big(p_n - b^+\big) z_n^+ \varphi \,\mathrm{d}x + \int\limits_{\{x\in\Omega, z^+(x)>0\}} b^+ z_n^+ \varphi \,\mathrm{d}x \\
&= o(1) + \int\limits_{\{x\in\Omega, z^+(x)>0\}} b^+ z^+ \varphi \,\mathrm{d}x.
\end{aligned} \tag{3.9}$$

Therefore, from (3.8)–(3.9),

$$\begin{aligned}
\int\limits_\Omega p_n z_n^+ \varphi \,\mathrm{d}x &= \int\limits_{\{x\in\Omega, z^+(x)=0\}} p_n z_n^+ \varphi \,\mathrm{d}x + \int\limits_{\{x\in\Omega, z^+(x)>0\}} p_n z_n^+ \varphi \,\mathrm{d}x \\
&= o(1) + b^+ \int\limits_{\{x\in\Omega, z^+(x)>0\}} z^+ \varphi \,\mathrm{d}x \\
&= o(1) + b^+ \int\limits_\Omega z^+ \varphi \,\mathrm{d}x.
\end{aligned} \tag{3.10}$$

Combining (3.7), (3.10) and letting $n \to \infty$, there holds

$$\int\limits_\Omega [\Delta z \Delta\varphi - a\nabla z \nabla\varphi] \,\mathrm{d}x = b^+ \int\limits_\Omega z^+ \varphi \,\mathrm{d}x. \tag{3.11}$$

We claim that

$$meas\big\{x \in \Omega,\ z^+(x) \neq 0\big\} > 0. \tag{3.12}$$

Otherwise, $z^+(x) = 0$ for a.e. $x \in \Omega$. Taking $\varphi = z$ in (3.11), we have $z = 0$ in H, a contradiction with (3.4). Thus, (3.12) is proved. Note that $z^+ \geqslant 0$, combining with (3.11)–(3.12) and the maximum principle, we have $z > 0$ in Ω. Taking $\varphi = \varphi_1$ in (3.11), we obtain that

$$\int\limits_\Omega [\Delta z \Delta\varphi_1 - a\nabla z \nabla\varphi_1] \,\mathrm{d}x = b^+ \int\limits_\Omega z\varphi_1 \,\mathrm{d}x.$$

On the other hand, since $\varphi_1 > 0$ is the eigenfunction associated to $\wedge_1$ and $z > 0$, we have

$$\int\limits_\Omega [\Delta z \Delta\varphi_1 - a\nabla z \nabla\varphi_1] \,\mathrm{d}x = \wedge_1 \int\limits_\Omega z\varphi_1 \,\mathrm{d}x,$$

this is impossible since $b^+ > \wedge_1$. Then $\{u_n\}$ is bounded in H. Combining with (3.1)–(3.2), we have $u_n \to u_+$ strongly in H, $I^+(u_+) = c^+ \geqslant \beta_1$ and $I^{+\prime}(u_+) = 0$. Then u_+ is a nontrivial solution of problem (1.1). Similarly, for $\wedge_1 < b^- < +\infty$, we can find $u_- \neq 0$, such that $I^-(u_-) = c^- > 0$ and $I^{-\prime}(u_-) = 0$. $\square$

Proof of Theorem 1.2. We will prove that there exists $\lambda^* > 0$, such that for $0 < \lambda < \lambda^*$, problem (1.1) has a nontrivial solution.

Let $X_j := span\{\varphi_j\}$, $j \in N$ and set $F_m := X_1 \oplus X_2 \oplus \cdots \oplus X_m$, $m \in N$, where $\oplus$ means the orthogonal sum of the subspace. Then $H = F_m \oplus F_m^\perp$. Note that (f_1), (f_4) with $\wedge_{k+1} < b^\pm < +\infty$ and $\limsup_{u\to 0} \frac{F(x,u)}{u^2} < \frac{1}{2}\wedge_{m+1}$ imply that there exists ϵ_0', $C > 0$, such that

$$F(x,u) \leqslant \frac{1}{2}\big(\wedge_{m+1} - \epsilon_0'\big)u^2 + C|u|^p.$$

Thus, for $u \in F_m^\perp$,

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \int_\Omega F(x,u)\,dx \\ &\geqslant \frac{1}{2}\|u\|^2 - \frac{1}{2}\big(\wedge_{m+1} - \epsilon_0'\big)\int_\Omega u^2\,dx - C\int_\Omega |u|^p\,dx \\ &\geqslant \frac{1}{2}\left(1 - \frac{\wedge_{m+1} - \epsilon_0'}{\wedge_{m+1}}\right)\|u\|^2 - C\|u\|^p. \end{aligned}$$

Choosing $r > 0$ small enough, there holds

$$\inf_{u\in F_m^\perp,\ \|u\|=r} I(u) \geqslant \alpha > 0, \tag{3.13}$$

independent of $\lambda > 0$.

For $u \in F_m$, we have

$$\begin{aligned} I(u) &\leqslant \frac{\wedge_m}{2}\int_\Omega u^2\,dx + \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \int_\Omega F(x,u)\,dx \\ &= \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \int_\Omega \left(F(x,u) - \frac{1}{2}\wedge_m u^2\right)dx. \end{aligned} \tag{3.14}$$

From

$$\lim_{u\to\pm\infty} \frac{f(x,u)}{u} = b^\pm > \wedge_{k+1},$$

we obtain that

$$\lim_{u\to\pm\infty} \frac{F(x,u)}{u^2} = \frac{1}{2}b^\pm > \frac{1}{2}\wedge_{k+1}.$$

Thus, there exists ϵ_0, $R_0 > 0$, such that for $|u| \geqslant R_0$, there holds

$$\frac{F(x,u)}{u^2} \geqslant \frac{1}{2}(\wedge_m + \epsilon_0). \tag{3.15}$$

Since $F(x,u) \geqslant \frac{1}{2}\wedge_m u^2$, together with (3.14)–(3.15), we have

$$I(u) \leqslant \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \frac{1}{2}\epsilon_0 \int_{\{x\in\Omega,\ |u(x)|\geqslant R_0\}} u^2\,dx.$$

Then, for $u \in F_m$ with $\|u\| = 1$, there holds

$$I(tu) \leqslant \frac{\lambda}{q}t^q\int_\Omega |u|^q\,dx - \frac{1}{2}\epsilon_0 t^2 \int_{\{x\in\Omega,\ |tu(x)|\geqslant R_0\}} u^2\,dx. \tag{3.16}$$

From [4], we know that there exists $\epsilon > 0$, such that

$$meas\big\{x \in \Omega,\ \big|u(x)\big| \geqslant \epsilon\big\} \geqslant \epsilon \tag{3.17}$$

for every $u \in F_m$ with $\|u\| = 1$. In addition, for $t \geqslant \frac{R_0}{\epsilon}$,

$$\big\{x \in \Omega,\ \big|u(x)\big| \geqslant \epsilon\big\} \subset \big\{x \in \Omega,\ \big|tu(x)\big| \geqslant R_0\big\}. \tag{3.18}$$

Thus, for $u \in F_m$ with $\|u\| = 1$ and $t \geqslant \frac{R_0}{\epsilon}$, (3.16)–(3.18) imply that

$$I(tu) \leqslant \frac{\lambda}{q} C t^q - \frac{1}{2}\epsilon_0 \epsilon^3 t^2.$$

Direct calculation shows that

$$\sup_{t \geqslant \frac{R_0}{\epsilon}} I(tu) \leqslant \sup_{t \geqslant 0} \left[\frac{\lambda}{q} C t^q - \frac{1}{2}\epsilon_0 \epsilon^3 t^2\right] \leqslant C\lambda^{\frac{2}{2-q}}. \tag{3.19}$$

On the other hand, for $u \in F_m$ with $\|u\| = 1$,

$$\sup_{0 \leqslant t \leqslant \frac{R_0}{\epsilon}} I(tu) \leqslant \sup_{0 \leqslant t \leqslant \frac{R_0}{\epsilon}} \left[\frac{\lambda}{q} t^q \int_\Omega |u|^q \, dx\right] \leqslant C\lambda. \tag{3.20}$$

Combining (3.19)–(3.20), we obtain that for $u \in F_m$ with $\|u\| = 1$,

$$\sup_{t \geqslant 0} I(tu) \leqslant \max\left\{C\lambda^{\frac{2}{2-q}}, C\lambda\right\}.$$

That is,

$$\sup_{u \in F_m} I(u) \leqslant \max\left\{C\lambda^{\frac{2}{2-q}}, C\lambda\right\}.$$

Therefore, there exists $\lambda^* > 0$, such that for $0 < \lambda < \lambda^*$,

$$\sup_{u \in F_m} I(u) < \alpha. \tag{3.21}$$

For $u \in F_{m+1}$, we have

$$I(u) \leqslant \frac{1}{2} \wedge_{m+1} \int_\Omega u^2 \, dx + \frac{\lambda}{q} \int_\Omega |u|^q \, dx - \int_\Omega F(x, u) \, dx. \tag{3.22}$$

(f_1) and (f_4) with $b^{\pm} > \wedge_{k+1}$ imply that there exists $\epsilon_0'' > 0$, such that

$$F(x, u) \geqslant \frac{1}{2}\left(\wedge_{m+1} + \epsilon_0''\right) u^2 - C. \tag{3.23}$$

Combining (3.22)–(3.23), for $u \in F_{m+1}$,

$$I(u) \leqslant -\frac{1}{2}\epsilon_0'' \int_\Omega u^2 \, dx + \frac{\lambda}{q} \int_\Omega |u|^q \, dx + C \leqslant -C\|u\|^2 + C\lambda\|u\|^q + C,$$

using the equivalence of all norms on the finite dimensional space. Therefore, for $0 < \lambda < \lambda^*$, choosing $R > r$ large enough, there holds

$$\sup_{u \in F_{m+1},\, \|u\| = R} I(u) < 0. \tag{3.24}$$

Consequently, (3.13), (3.21) and (3.24) imply that there exists $\lambda^* > 0$, such that for $0 < \lambda < \lambda^*$,

$$\inf_{u \in F_m^{\perp}, \|u\| = r} I(u) > \sup_{u \in \partial Q} I(u),$$

where

$$Q := \left\{u + v;\ u \in F_m,\ v = t\varphi_{m+1},\ t \geqslant 0,\ \|u + v\| \leqslant R\right\}.$$

Define

$$c := \inf_{\Gamma \in \Phi} \sup_{0 \leqslant s \leqslant 1,\, u \in \partial Q} I\big(\Gamma(s)(u)\big),$$

where Φ is defined in Definition C. It follows from Theorem 2.2 that there exists a sequence $\{u_n\} \subset H$, such that

$$I(u_n) \to c \geqslant \alpha, \quad \text{as } n \to +\infty, \tag{3.25}$$

and

$$\big(1 + \|u_n\|\big) I'(u_n) \to 0, \quad \text{as } n \to +\infty. \tag{3.26}$$

We claim that

$$\{u_n\} \quad \text{is bounded in } H. \tag{3.27}$$

Seeking a contradiction we suppose that $\|u_n\| \to \infty$. Let $w_n = \frac{u_n}{\|u_n\|}$. Up to a subsequence, we get that

$$\begin{aligned}
&w_n \rightharpoonup w \quad \text{weakly in } H,\\
&w_n \to w \quad \text{strongly in } L^t(\Omega),\ 1 \leqslant t < 2^*,\\
&w_n(x) \to w(x) \quad \text{a.e. } x \text{ in } \Omega.
\end{aligned}$$

Now, we consider the two possible cases.

Case 1. $w = 0$ in H.

From $o(1) = (I'(u_n), u_n)$, we have

$$o(1) = \|u_n\|^2 + \lambda \int_\Omega |u_n|^q \,dx - \int_\Omega f(x, u_n)u_n \,dx.$$

Dividing $\|u_n\|^2$ in both sides of the above equality, we get that

$$o(1) = 1 - \int_\Omega \frac{f(x, u_n)u_n}{\|u_n\|^2} \,dx. \tag{3.28}$$

(f_1) and (f_3)–(f_4) with $\wedge_{k+1} < b^{\pm} < +\infty$ imply that

$$\left| f(x, u_n)u_n \right| \leqslant C|u_n|^2. \tag{3.29}$$

Combining (3.28)–(3.29), we have

$$1 = \int_\Omega \frac{f(x, u_n)u_n}{\|u_n\|^2} \,dx + o(1) \leqslant C \int_\Omega |w_n|^2 \,dx + o(1).$$

Let $n \to \infty$, we get a contradiction.

Case 2. $w \neq 0$ in H.

(3.25)–(3.26) imply that

$$\begin{aligned}
c + o(1) &= I(u_n) - \frac{1}{2}\big(I'(u_n), u_n\big)\\
&= \int_\Omega \left[\frac{1}{2} f(x, u_n)u_n - F(x, u_n)\right] dx + \lambda\left(\frac{1}{q} - \frac{1}{2}\right) \int_\Omega |u_n|^q \,dx\\
&\geqslant \int_\Omega \left[\frac{1}{2} f(x, u_n)u_n - F(x, u_n)\right] dx.
\end{aligned}$$

Set $\Omega_1 := \{x \in \Omega,\ w(x) \neq 0\}$. Thus, for $x \in \Omega_1$, $|u_n(x)| \to +\infty$ as $n \to \infty$. By (f_5), we obtain that

$$c + o(1) \geqslant \int_{\Omega/\Omega_1} L(x) \,dx + \int_{\Omega_1} H(x, u_n) \,dx. \tag{3.30}$$

Since $meas(\Omega_1) > 0$ and for a.e. $x \in \Omega_1$, $\lim_{n\to\infty} H(x, u_n) = +\infty$, using Fatou's lemma, we obtain that

$$\lim_{n\to\infty} \int_{\Omega_1} H(x, u_n) \,dx = +\infty,$$

which contradicts (3.30). Thus, (3.27) is proved. Combining with (3.25)–(3.26), we have $u_n \to u_0$ strongly in H, $I(u_0) = c \geqslant \alpha > 0$ and $I'(u_0) = 0$. Thus, there exists $\lambda^* > 0$, such that for $0 < \lambda < \lambda^*$, u_0 is a nontrivial solution of problem (1.1).

Furthermore, from the proof of Theorem 1.1, we know that given $\lambda > 0$, problem (1.1) has nontrivial solutions $u_{\pm}$. We remark that u_+ and u_- may be the same.

Note that $I(0)=0$. Lemma 2.5 implies that given $\lambda>0$, there exist $\rho,\beta>0$, such that

$$\inf_{\|u\|=\rho} I(u)\geqslant\beta>0.$$

On the other hand, from (f_1) and (3.15),

$$F(x,u)\geqslant\frac{1}{2}(\wedge_1+\epsilon_0)u^2-C.$$

Thus,

$$I(u)\leqslant\frac{1}{2}\|u\|^2+\frac{\lambda}{q}\int_\Omega |u|^q\,\mathrm{d}x-\frac{1}{2}(\wedge_1+\epsilon_0)\int_\Omega u^2\,\mathrm{d}x+C\,meas(\Omega).$$

Choosing $u=t\varphi_1$, where $t>0$ and φ_1 is the eigenfunction associated to $\wedge_1$, we have

$$\lim_{t\to\infty} I(t\varphi_1)=-\infty.$$

Let t' be such that $\|t'\varphi_1\|>\rho$ and $I(t'\varphi_1)<0$. Define

$$c^*:=\inf_{\gamma\in\Gamma}\max_{0\leqslant t\leqslant 1} I\big(\gamma(t)\big),$$

where $\Gamma:=\{\gamma\in C([0,1],H);\ \gamma(0)=0,\ \gamma(1)=t'\varphi_1\}$. It follows from Theorem 2.1 that there exists a sequence $\{u_n\}\subset H$, such that

$$I(u_n)\to c^*\geqslant\beta,\quad \text{as } n\to+\infty, \tag{3.31}$$

and

$$\big(1+\|u_n\|\big)I^{+'}(u_n)\to 0,\quad \text{as } n\to+\infty. \tag{3.32}$$

Then (3.27) holds. Combining with (3.31)–(3.32), we have $u_n\to u^*$ strongly in H, $I(u^*)=c^*\geqslant\beta>0$ and $I'(u^*)=0$. Thus, given $\lambda>0$, u^* is a nontrivial solution of problem (1.1).

We claim that u_+, u^* are distinct or u_-, u^* are distinct.

If not, then $u_\pm=u^*=v$. From $I'(v)=I^{\pm'}(v)=0$,

$$\int_\Omega\big(\lambda|v|^{q-2}v\varphi-f(x,v)\varphi\big)\,\mathrm{d}x=\int_\Omega\big(\lambda\big|v^\pm\big|^{q-2}v^\pm\varphi-f\big(x,v^\pm\big)\varphi\big)\,\mathrm{d}x, \tag{3.33}$$

for all $\varphi\in H$. We note that

$$\begin{aligned}&\int_\Omega\big(\lambda|v|^{q-2}v\varphi-f(x,v)\varphi\big)\,\mathrm{d}x\\&\quad=\int_{\{x\in\Omega,\ v(x)>0\}}\big(\lambda|v|^{q-2}v\varphi-f(x,v)\varphi\big)\,\mathrm{d}x+\int_{\{x\in\Omega,\ v(x)<0\}}\big(\lambda|v|^{q-2}v\varphi-f(x,v)\varphi\big)\,\mathrm{d}x\\&\quad=\int_\Omega\big(\lambda\big|v^+\big|^{q-2}v^+\varphi-f\big(x,v^+\big)\varphi\big)\,\mathrm{d}x+\int_\Omega\big(\lambda\big|v^-\big|^{q-2}v^-\varphi-f\big(x,v^-\big)\varphi\big)\,\mathrm{d}x.\end{aligned}$$

Together with (3.33), there holds

$$\int_\Omega\big(\lambda|v|^{q-2}v\varphi-f(x,v)\varphi\big)\,\mathrm{d}x=0,\quad \text{for all } \varphi\in H.$$

In view of $(I'(v),\varphi)=0$ for all $\varphi\in H$, we can conclude that $v=0$ in H, a contradiction. Thus, the claim is proved.

Without loss of generality, we may assume that u_+ and u^* are distinct. For $0<\lambda<\lambda^*$, we will show that u_0 and u_+, u^* are distinct.

Note that

$$I^+(u_+)=c^+=\inf_{\gamma\in\Gamma^+}\max_{0\leqslant t\leqslant 1} I^+\big(\gamma(t)\big),$$

where $\Gamma^+=\{\gamma\in C([0,1],H);\ \gamma(0)=0,\ \gamma(1)=t_1\varphi_1\}$. Since $\gamma^+(t):=tt_1\varphi_1$, $t\in[0,1]$ belongs to Γ^+ and $\gamma^+[0,1]\subset F_m$, we have

$$c^+=I^+(u_+)\leqslant\max_{0\leqslant t\leqslant 1} I^+\big(\gamma^+(t)\big)=\max_{0\leqslant t\leqslant 1} I\big(\gamma^+(t)\big)\leqslant\sup_{u\in F_m} I(u)<\alpha\leqslant I(u_0)=c.$$

Thus, u_0 and u_+ are distinct. Similarly, u_0 and u^* are distinct. $\square$

Proof of Theorem 1.3. Given $\lambda > 0$, we will show that there exists $L^* > 0$, such that for $L > L^*$, problem (1.1) has a nontrivial solution. The assumptions of Theorem 1.3 imply that there exists $F_0 > 0$, such that $|F(x,u)| \leqslant F_0 u^2$. Then, for $|u| > L$,

$$\left|F(x,u)\right| < \frac{F_0}{L^{p-2}}|u|^p.$$

We note that for $|u| \leqslant L$,

$$F(x,u) \leqslant \left(\frac{1}{2}\wedge_{m+1} - \delta_0\right)u^2.$$

Thus, we obtain that

$$F(x,u) \leqslant \left(\frac{1}{2}\wedge_{m+1} - \delta_0\right)u^2 + \frac{F_0}{L^{p-2}}|u|^p. \tag{3.34}$$

(3.34) implies that

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{\lambda}{q}\int_\Omega |u|^q \,\mathrm{d}x - \int_\Omega F(x,u)\,\mathrm{d}x \\ &\geqslant \frac{1}{2}\|u\|^2 - \left(\frac{1}{2}\wedge_{m+1} - \delta_0\right)\int_\Omega u^2\,\mathrm{d}x - \frac{F_0}{L^{p-2}}\int_\Omega |u|^p\,\mathrm{d}x. \end{aligned} \tag{3.35}$$

For simplicity, we may assume that $0 < \delta_0 < \frac{1}{2}\wedge_{m+1}$. Thus, for $u \in F_m^\perp$,

$$\begin{aligned} I(u) &\geqslant \frac{1}{2}\left(1 - \frac{\wedge_{m+1} - 2\delta_0}{\wedge_{m+1}}\right)\|u\|^2 - \frac{F_0}{L^{p-2}}\int_\Omega |u|^p\,\mathrm{d}x \\ &\geqslant \frac{1}{2}\left(1 - \frac{\wedge_{m+1} - 2\delta_0}{\wedge_{m+1}}\right)\|u\|^2 - \frac{SF_0}{L^{p-2}}\|u\|^p. \end{aligned} \tag{3.36}$$

Choosing $\|u\| = (\frac{2\delta_0}{pSF_0\wedge_{m+1}})^{\frac{1}{p-2}}L$, there holds

$$\inf_{u\in F_m^\perp,\,\|u\|=(\frac{2\delta_0}{pSF_0\wedge_{m+1}})^{\frac{1}{p-2}}L} I(u) \geqslant \left(\frac{p-2}{2}\right)\left(\frac{1}{SF_0}\right)^{\frac{2}{p-2}}\left(\frac{2\delta_0}{p\wedge_{m+1}}\right)^{\frac{p}{p-2}}L^2. \tag{3.37}$$

On the other hand, using a similar argument as Theorem 1.2, we obtain that

$$\sup_{u\in F_m} I(u) \leqslant \max\left\{C\lambda^{\frac{2}{2-q}}, C\lambda\right\} + \int_\Omega W_0(x)\,\mathrm{d}x. \tag{3.38}$$

(3.37)–(3.38) imply that there exists $L^* > 0$, such that for $L > L^*$,

$$\inf_{u\in F_m^\perp,\,\|u\|=(\frac{2\delta_0}{pSF_0\wedge_{m+1}})^{\frac{1}{p-2}}L} I(u) > \sup_{u\in F_m} I(u). \tag{3.39}$$

Fix $L > L^*$, as in the proof of Theorem 1.2, we may choose M large enough such that $M > (\frac{2\delta_0}{pSF_0\wedge_{m+1}})^{\frac{1}{p-2}}L$ and

$$\sup_{u\in F_{m+1},\,\|u\|=M} I(u) < 0. \tag{3.40}$$

Therefore, combining (3.39)–(3.40), we obtain that there exists $L^* > 0$, such that for $L > L^*$,

$$\inf_{u\in F_m^\perp,\,\|u\|=(\frac{2\delta_0}{pSF_0\wedge_{m+1}})^{\frac{1}{p-2}}L} I(u) > \sup_{u\in\partial Q_1} I(u),$$

where

$$Q_1 := \left\{u + v;\ u \in F_m,\ v = t\varphi_{m+1},\ t \geqslant 0,\ \|u+v\| \leqslant M\right\}.$$

Arguing as in the proof of Theorem 1.2, we obtain that given $\lambda > 0$, there exists $L^* > 0$, such that for $L > L^*$, problem (1.1) has a nontrivial solution. Moreover, we claim that for $L > L^*$, problem (1.1) has at least three nontrivial solutions. The proof is similar to the proof of Theorem 1.2. We omit the details. □

Proof of Theorem 1.4. In view of the proof of Theorem 1.1 and 1.2, we only need to prove that the $(C)_c$ sequence of I, $I^{\pm}$ is bounded under assumptions of Theorem 1.4.

For $\{u_n\}$ satisfying

$$c+o(1)=I^+(u_n) \tag{3.41}$$

and

$$o(1)=\big(1+\|u_n\|\big)I^{+\prime}(u_n), \tag{3.42}$$

we will prove that $\|u_n\|$ is bounded in H.

(3.41)–(3.42) imply that

$$\begin{aligned} c+o(1)&=I^+(u_n)-\big(I^{+\prime}(u_n),u_n\big)\\ &=\int_\Omega\left[\frac{1}{2}f\big(x,u_n^+\big)u_n^+-F\big(x,u_n^+\big)\right]\mathrm{d}x+\lambda\left(\frac{1}{q}-\frac{1}{2}\right)\int_\Omega\big|u_n^+\big|^q\,\mathrm{d}x\\ &\geqslant\int_\Omega\left[\frac{1}{2}f\big(x,u_n^+\big)u_n^+-F\big(x,u_n^+\big)\right]\mathrm{d}x. \end{aligned} \tag{3.43}$$

By (f_1) and (f_6), we have

$$\frac{1}{2}f\big(x,u^+\big)u^+-F\big(x,u^+\big)\geqslant C\big|u^+\big|^\mu-C. \tag{3.44}$$

Combining (3.43)–(3.44), there holds

$$c+o(1)\geqslant C\int_\Omega\big|u_n^+\big|^\mu\,\mathrm{d}x-C,$$

from which we have the estimate

$$\big\|u_n^+\big\|_\mu\leqslant C. \tag{3.45}$$

On the other hand, by $o(1)=(I^{+'}(u_n),u_n)$ and (f_1)–(f_2), we have

$$\|u_n\|^2\leqslant o(1)+\int_\Omega f\big(x,u_n^+\big)u_n^+\,\mathrm{d}x\leqslant C\int_\Omega\big(\big|u_n^+\big|+\big|u_n^+\big|^p\big)\,\mathrm{d}x+C\leqslant C\int_\Omega\big(|u_n|+\big|u_n^+\big|^p\big)\,\mathrm{d}x+C. \tag{3.46}$$

Observe that $\frac{s}{4}(p-2)<(p-1)\frac{2s}{s+4}<p$, we will consider two cases.

Case 1. $\mu\geqslant(p-1)\frac{2s}{s+4}$.

From (3.45)–(3.46),

$$\begin{aligned} \|u_n\|^2&\leqslant C\|u_n\|+C\int_\Omega\big|u_n^+\big|^{p-1}|u_n|\,\mathrm{d}x+C\\ &\leqslant C\|u_n\|+C\|u_n\|_{2^{**}}\big\|u_n^+\big\|^{p-1}_{(p-1)\frac{2s}{s+4}}+C\\ &\leqslant C\|u_n\|+C\|u_n\|\big\|u_n^+\big\|^{p-1}_\mu+C\\ &\leqslant C\|u_n\|+C, \end{aligned}$$

which implies that $\|u_n\|\leqslant C$.

Case 2. $\frac{s}{4}(p-2)<\mu<p$.

If $0 < \mu < p < 2^{**}$ and $t \in (0,1)$ are such that $\frac{1}{p} = \frac{1-t}{\mu} + \frac{t}{2^{**}}$, then $\forall\ u \in L^{\mu}(\Omega) \cap L^{**}(\Omega)$, we have

$$\int_{\Omega} |u|^p \,\mathrm{d}x = \int_{\Omega} |u|^{(1-t)p} |u|^{tp} \,\mathrm{d}x \leqslant \|u\|_{\mu}^{(1-t)p} \|u\|_{2^{**}}^{tp}. \tag{3.47}$$

Combining (3.45)–(3.47), there holds

$$\begin{aligned} \|u_n\|^2 &\leqslant C\|u_n\| + C\|u_n\|_p^p + C \\ &\leqslant C\|u_n\| + C\|u_n\|_{\mu}^{(1-t)p} \|u_n\|_{2^{**}}^{tp} + C \\ &\leqslant C\|u_n\| + C\|u_n\|^{tp} + C. \end{aligned} \tag{3.48}$$

Note that the condition $\mu > \frac{s}{4}(p-2)$ is equivalent to $tp < 2$, we conclude from (3.48) that $\|u_n\| \leqslant C$. Thus, the $(C)_c$ sequence of I^+ is bounded under assumptions of Theorem 1.4. Similarly, we can prove that the $(C)_c$ sequence of I, I^- is bounded. The details are omitted. □

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References

[1] A. Ambrosetti, J.G. Azorero, I. Peral, Multiplicity results for some nonlinear elliptic equations, J. Funct. Anal. 137 (1996) 219–242.
[2] A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, J. Funct. Anal. 122 (1994) 519–543.
[3] T. Bartsch, M. Willem, On an elliptic equation with concave and convex nonlinearities, Proc. Amer. Math. Soc. 123 (1995) 3555–3561.
[4] F.O. de Paiva, E. Massa, Multiple solutions for some elliptic equations with a nonlinearity concave at the origin, Nonlinear Anal. 66 (2007) 2940–2946.
[5] A.C. Lazer, P.J. Mckenna, Large amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis, SIAM Rev. 32 (1990) 537–578.
[6] A.C. Lazer, P.J. Mckenna, Global bifurcation and a theorem of Tarantello, J. Math. Anal. Appl. 181 (1994) 648–655.
[7] A.M. Micheletti, A. Pistoia, Multiplicity results for a fourth-order semilinear elliptic problem, Nonlinear Anal. 31 (1998) 895–908.
[8] A.M. Micheletti, A. Pistoia, Nontrivial solutions for some fourth order semilinear elliptic problems, Nonlinear Anal. 34 (1998) 509–523.
[9] K. Perera, Multiplicity results for some elliptic problems with concave nonlinearities, J. Differential Equations 140 (1997) 133–141.
[10] K. Perera, Critical groups of pair of critical points produced by linking subsets, J. Differential Equations 140 (1997) 142–160.
[11] M. Schechter, A variation of the mountain pass lemma and applications, J. Lond. Math. Soc. 44 (1991) 491–502.
[12] M. Schechter, Minimax systems, J. Math. Anal. Appl. 345 (2008) 431–454.
[13] L.S. Shi, X.J. Chang, Multiple solutions to p-Laplacian problems with concave nonlinearities, J. Math. Anal. Appl. 363 (2010) 155–160.
[14] G. Tarantello, A note on a semilinear elliptic problem, Differential Integral Equations 5 (1992) 561–566.
[15] S.P. Wu, H.T. Yang, A class of resonant elliptic problems with sublinear nonlinearities at origin and at infinity, Nonlinear Anal. 45 (2001) 925–935.
[16] G. Xu, J. Zhang, Existence results for some fourth-order nonlinear elliptic problems of local superlinearity and sublinearity, J. Math. Anal. Appl. 281 (2003) 633–640.