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On the existence of solutions for a class of second-order differential inclusions and applications ☆

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ABSTRACT

In this paper, by extending the method of guiding functions, we give sufficient conditions for the existence of solutions to the problem

$$u'' \in Q(u), \quad u(0) = u(1) = 0, \tag{1.1}$$

where Q is a multivalued map whose values are not necessarily convex. It is shown how abstract results can be applied to study feedback control systems and to describe a motion of a particle in a 1-D potential.

1. Introduction

Various aspects of the theory of second-order differential inclusions attract the attention of many researchers (see., e.g., [1–5,9,11,12,15,23]). In this paper we consider the boundary value problem of the form (1.1) for second-order differential inclusions which arises naturally from some physical and control problems. An appropriate approach to study the existence of solutions to problem (1.1) is the using of the fixed point theory by the exploiting the fact that the set of solutions of (1.1) coincides with the fixed point set of the corresponding integral multioperator. However, this approach involves the necessity to evaluate the topological degree of this multioperator which, in general, should be not an easy problem.

For the purposes of investigation of periodic solutions of the first-order differential equations and inclusions, one of the effective tools is the method of guiding functions developed by M.A. Krasnosel'skii, A.I. Perov and others (see, e.g., [20–22]). The main idea of this method may be expressed by the fact that the topological degree of the integral multioperator can be evaluated through the topological index of the special, "guiding" function of the equation or the inclusion under consideration.

In this paper we extend this method to study problem (1.1). The paper is organized in the following way. In the next section we recall some notions and notation from multivalued analysis. In Section 3, using a guiding function of the form $V(x) = \delta x$, $x \in \mathbb{R}$ and $\delta > 0$, we obtain the sufficient conditions for the existence of solutions to (1.1) in $\mathbb{R}$. The result is extended to the case of a general integral guiding function. In Section 4, we apply the abstract result to the investigation of a boundary value problem for a system whose feedback law is expressed in the form of a differential inclusion. Another application deals with a model of motion of a particle in a 1-D potential.

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In the last section, we present the generalization of our method to the case of problem (1.1) considered in an infinite-dimensional Hilbert space. As an example, we consider a feedback control system governed by partial differential equation and inclusion.

2. Preliminaries

Let X, Y be metric spaces. Denote by $P(Y)[C(Y), K(Y)]$ the collection of all nonempty [resp., nonempty closed, nonempty compact] subsets of Y. For a Banach space E by symbols $Cv(E)\,[Kv(E)]$ we denote the collection of all nonempty convex closed [resp., nonempty convex compact] subsets of E.

Definition 1. (See, e.g., [6,14,19].) A multivalued map (multimap) $F : X \to P(Y)$ is said to be upper semicontinuous (u.s.c.) if for every open subset $V \subset Y$ the set

$$F_+^{-1}(V) = \{x \in X\colon\ F(x) \subset V\}$$

is open in X. A u.s.c. multimap F is said to be completely continuous, if it maps every bounded subset $X_1 \subset X$ into a relatively compact subset $F(X_1)$ of Y.

For a multimap $F : X \to P(Y)$, by Γ_F^X we denote its graph, i.e.,

$$\Gamma_F^X = \{(x, y) \in X \times Y\colon\ y \in F(x)\}.$$

The multimap F is said to be *closed* if its graph is a closed subset of $X \times Y$, while the metric in $X \times Y$ is defined in a natural way as

$$\rho\big((x, y), (x', y')\big) = \max\{\rho_X(x, x'), \rho_Y(y, y')\}.$$

Notice that a closed multimap F which is locally compact, i.e., each point $x \in X$ has a neighborhood $U(x)$ such that the set $F(U(x))$ is relatively compact, is u.s.c.

A set $M \in K(Y)$ is said to be *aspheric* (or UV^∞, or ∞*-proximally connected*) (see, e.g., [14,24]), if for every $\varepsilon > 0$ there exists $\delta > 0$ such that each continuous map $\sigma : S^n \to O_\delta(M)$, $n = 0, 1, 2, \ldots,$ can be extended to a continuous map $\tilde\sigma : B^{n+1} \to O_\varepsilon(M)$, where $S^n = \{x \in \mathbb{R}^{n+1}\colon\ \|x\| = 1\}$, $B^{n+1} = \{x \in \mathbb{R}^{n+1}\colon\ \|x\| \leqslant 1\}$, and $O_\delta(M)$ $[O_\varepsilon(M)]$ denotes the δ-neighborhood [resp. ε-neighborhood] of the set M.

Definition 2. (See [17].) A nonempty compact space A is said to be an R_δ*-set* if it can be represented as the intersection of a decreasing sequence of compact, contractible spaces.

Definition 3. (See [14].) A u.s.c. multimap $\Sigma : X \to K(Y)$ is said to be a J*-multimap* ($\Sigma \in J(X, Y)$) if every value $\Sigma(x)$, $x \in X$, is an aspheric set.

Now let us recall (see, e.g., [7,14]) that a metric space X is called *the absolute retract* (*the AR-space*) [resp., *the absolute neighborhood retract* (*the ANR-space*)] provided for each homeomorphism h taking it onto a closed subset of a metric space X', the set $h(X)$ is the retract of X' [resp., of its open neighborhood in X']. Notice that the class of *ANR*-spaces is broad enough: in particular, a finite-dimensional compact set is the *ANR*-space if and only if it is locally contractible. In turn, it means that compact polyhedrons and compact finite-dimensional manifolds are the *ANR*-spaces. The union of a finite number of convex closed subsets in a normed space is also the *ANR*-space.

Proposition 1. *(See [14].) Let Z be an ANR-space. In each of the following cases a u.s.c. multimap $\Sigma : X \to K(Z)$ is a J-multimap: for each $x \in X$ the value $\Sigma(x)$ is*

(a) *a convex set*;
(b) *a contractible set*;
(c) *an R_δ-set*;
(d) *an AR-space.*

In particular, every continuous map $\sigma : X \to Z$ is a J-multimap.

Definition 4. Let $\mathcal{O} \subseteq X$. By $CJ(\mathcal{O}, X)$ we will denote the collection of all multimaps $F : \mathcal{O} \to K(X)$ that may be represented in the form of composition $F = f \circ G$, where $G \in J(\mathcal{O}, Y)$ and $f : Y \to X$ is a continuous map. The composition $f \circ G$ will be called the decomposition of F. We will denote $F = (f \circ G)$.

It has to be noted that a multimap can admit different decompositions (see [14]).

Now, let X be a Banach space and $U \subset X$ be an open bounded subset and $F = (f \circ G) \in CJ(\overline{U}, X)$ be a completely continuous multimap such that $x \notin F(x)$ for $x \in \partial U$. Then the topological degree $\deg(i - F, \overline{U})$ of the corresponding multivalued vector field $(i - F)(x) = x - F(x)$ is well defined and has all usual properties of the Leray–Shauder topological degree (see, e.g., [14]).

3. The existence problem

Denote by $\mathcal{C}$ the space of all continuous functions $C[0,1]$ and by $\mathcal{L}^2$ the space of all square-integrable functions $L_2[0,1]$. For every $u \in \mathcal{C}$ and $f \in \mathcal{L}^2$ their corresponding norms are:

$$\|u\|_{\mathcal{C}} = \max_{t\in[0,1]} \left|u(t)\right| \quad \text{and} \quad \|f\|_2 = \left(\int_0^1 \left|f(s)\right|^2 ds \right)^{\frac{1}{2}}.$$

The symbol $\langle \cdot,\cdot \rangle$ denotes the inner product in $\mathcal{L}^2$ and $B_{\mathcal{C}}(0,R)$ denotes the ball in $\mathcal{C}$ of radius R centered at the origin. For an integer $k > 0$ the *Sobolev space* $W_2^k[0,1]$ is defined as

$$W_2^k[0,1] = \left\{u \in \mathcal{L}^2\colon\ u^{(m)} \in \mathcal{L}^2 \text{ for all } 1 \leqslant m \leqslant k\right\},$$

where $u^{(m)}$ denotes the distributional derivative of u of order m. The norm of an element $u \in W_2^k[0,1]$ is defined as

$$\|u\|_W = \sum_{m=0}^{k} \left\|u^{(m)}\right\|_2.$$

By $\mathring{W}_2^k[0,1]$ we will denote the subset of $W_2^k[0,1]$ consisting of all functions vanishing at the end-points of $[0,1]$, i.e.,

$$\mathring{W}_2^k[0,1] = \left\{u \in W_2^k[0,1]\colon\ u(0) = u(1) = 0\right\}.$$

Let us recall (see, e.g., [8]) that according to the Sobolev embedding theorem the space $W_2^k[0,1]$ is compactly embedded into $\mathcal{C}$.

Define a continuous integral operator $j : \mathcal{L}^2 \to \mathcal{C}$ by

$$(jf)(t) = \int_0^1 G(t,s) f(s)\, ds,$$

where

$$G(t,s) = \begin{cases} t(s-1) & \text{if } 0 \leqslant t \leqslant s, \\ s(t-1) & \text{if } s \leqslant t \leqslant 1. \end{cases}$$

Notice that the operator j in fact acts into $\mathring{W}_2^2[0,1]$ and, for any $f \in \mathcal{L}^2$, the boundary value problem

$$u''(t) = f(t) \quad \text{for a.e. } t \in [0,1],$$
$$u(0) = u(1) = 0$$

may be written in the form: $u = jf$ (see, e.g., [16, Chapter XII, Sec. 4]). By applying the Arzela–Ascoli theorem, it is easy to see also that the operator j transforms bounded sets into relatively compact ones.

In this section we will consider the existence of solutions to the following boundary value problem for the operator-differential inclusion

$$\begin{aligned} &u'' \in Q(u), \\ &u(0) = u(1) = 0, \end{aligned} \tag{3.1}$$

where $Q : \mathcal{C} \to C(\mathcal{L}^2)$ is a multimap satisfying the following conditions:

$(Q1)$ the composition $j \circ Q$ belongs to the class $CJ(\mathcal{C}, \mathcal{C})$;

$(Q2)$ there are constants $p, q > 0$ such that

$$\left\|Q(u)\right\|_2 \leqslant q\left(1 + \|u\|_2^p\right)$$

for all $u \in \mathcal{C}$, where

$$\left\|Q(u)\right\|_2 = \sup\left\{\|f\|_2\colon\ f \in Q(u)\right\}.$$

By a solution to problem (3.1) we mean a function $u \in \overset{\circ}{W}{}_2^2[0,1]$ such that there is a function $f \in Q(u)$ satisfying

$$u''(t) = f(t) \quad \text{for a.e. } t \in [0,1].$$

Remark 1. Let us mention that the class of multimaps Q satisfying condition $(Q1)$ is large enough. For example, for every CJ-multimap Q the multimap $j \circ Q$ is a CJ-multimap. Moreover, there are multimaps Q which are not CJ-multimaps while $j \circ Q$ are CJ-multimaps. For example, let $F : [0,1] \times \mathbb{R} \to Kv(\mathbb{R})$ be a multimap satisfying the following conditions:

$(F1)$ for every $y \in \mathbb{R}$ the multifunction $F(\cdot, y) : [0,1] \to Kv(\mathbb{R})$ has a measurable selection;
$(F2)$ for a.e. $t \in [0,1]$ the multimap $F(t,\cdot) : \mathbb{R} \to Kv(\mathbb{R})$ is upper semicontinuous;
$(F3)$ for every bounded subset $\Omega \subset \mathbb{R}$ there is a function $\gamma_\Omega \in \mathcal{L}^2$ such that

$$\left|F(t,y)\right| \leqslant \gamma_\Omega(t)$$

for all $y \in \Omega$ and a.e. $t \in [0,1]$, where $|F(t,y)| = \max\{|z|:\ z \in F(t,y)\}$.

It is well known that under conditions $(F1)$–$(F3)$ the superposition multioperator

$$\mathcal{P}_F : \mathcal{C} \multimap \mathcal{L}^2,$$
$$\mathcal{P}_F(x) = \left\{ f \in \mathcal{L}^2:\ f(s) \in F\big(s, x(s)\big) \text{ for a.e. } s \in [0,1] \right\},$$

is well defined, it is closed and has convex closed values (see, e.g., [6,19]).

Set $Q : \mathcal{C} \to Cv(\mathcal{L}^2)$, $Q(x) = \mathcal{P}_F(x)$. From Theorem 1.5.30 [6] it follows that the multimap $j \circ Q$ is closed. By virtue of $(F3)$, for every bounded subset $U \subset \mathcal{C}$ the set $Q(U)$ is bounded in $\mathcal{L}^2$, therefore the set $j(Q(U))$ is a relatively compact set in $\mathcal{C}$. Hence, $j \circ Q$ is a u.s.c. multimap with compact convex values and so, it belongs to the class $J(\mathcal{C},\mathcal{C}) \subset CJ(\mathcal{C},\mathcal{C})$.

The main result of this section is the following assertion.

Theorem 1. *Let conditions $(Q1)$–$(Q2)$ hold. Assume that there exists $N > 0$ such that for every $u \in \mathcal{C}$, $\|u\|_2 > N$, the following relation holds*

$$\langle u, f\rangle > 0 \quad \textit{for all } f \in Q(u).$$

Then problem (3.1) *has a solution.*

Proof. Problem (3.1) can be substituted by the following inclusion

$$u \in j \circ Q(u).$$

From $(Q1)$ it follows that the multioperator $j \circ Q : \mathcal{C} \to K(\mathcal{C})$ is upper semicontinuous. Now let $\Omega \subset \mathcal{C}$ be a bounded subset. Condition $(Q2)$ implies that the set $Q(\Omega)$ is bounded in $\mathcal{L}^2$. Therefore, the set $j \circ Q(\Omega)$ is a relatively compact in $\mathcal{C}$. Thus, $j \circ Q$ is a completely continuous multioperator.

Assume that there exists $u_* \in \mathcal{C}$, such that $u_* \in j \circ Q(u_*)$. Notice that then $u_*(0) = u_*(1) = 0$. Then there is $f_* \in Q(u_*)$ such that $u_*''(t) = f_*(t)$ for a.e. $t \in [0,1]$, and hence

$$\langle f_*, u_*\rangle = \left\langle u_*'', u_*\right\rangle = -\left\langle u_*', u_*'\right\rangle \leqslant 0.$$

Therefore, $\|u_*\|_2 \leqslant N$.

For every $t \in [0,1]$, we have

$$\left|u_*(t)\right| \leqslant \int_0^1 \left|G(t,s)\right|\left|f_*(s)\right| ds \leqslant \int_0^1 \left|f_*(s)\right| ds \leqslant \|f_*\|_2.$$

From $(Q2)$ it follows that for every $t \in [0,1]$

$$\left|u_*(t)\right| \leqslant \|f_*\|_2 \leqslant \left\|Q(u_*)\right\|_2 \leqslant q\big(1 + N^p\big),$$

hence, $\|u_*\|_{\mathcal{C}} \leqslant q(1 + N^p)$.

Now set $R = qN^p + q + 1$. Consider the multimap $\Psi : B_{\mathcal{C}}(0,R) \times [0,1] \to K(\mathcal{C})$,

$$\Psi(u,\lambda) = j \circ \big((1-\lambda)\delta u + \lambda Q(u)\big),$$

where $0 < \delta < \frac{1}{N}$ is an arbitrary number.

Let us show that Ψ is a completely continuous CJ-multimap. In fact, from condition $(Q1)$ it follows that we may represent the multimap $j \circ Q$ as $\varphi \circ F \in CJ(\mathcal{C},\mathcal{C})$, where $F : \mathcal{C} \to K(Y)$ is a J-multimap from $\mathcal{C}$ to some metric space Y and $\varphi : Y \to \mathcal{C}$ is a continuous map. Define the multimap

$$\widetilde{F}: B_{\mathcal{C}}(0,R)\times[0,1]\to K(\mathcal{C}\times Y\times\mathbb{R}),$$
$$\widetilde{F}(u,\lambda)=\{u\}\times F(u)\times\{\lambda\},$$

and a map

$$\tilde{\varphi}:\mathcal{C}\times Y\times\mathbb{R}\to K(\mathcal{C}),$$
$$\tilde{\varphi}(u,v,\lambda)=\delta(1-\lambda)ju+\lambda\varphi(v).$$

It is clear that $\widetilde{F}$ is a J-multimap, $\tilde{\varphi}$ is a continuous map and for every $(u,\lambda)\in B_{\mathcal{C}}(0,R)\times[0,1]$ we have $\Psi(u,\lambda)=\tilde{\varphi}\circ\widetilde{F}(u,\lambda)$. So, Ψ is a CJ-multimap. Further, for every bounded subset $\Omega\subset\mathcal{C}$ the sets $j\circ Q(\Omega)$ and $j(\Omega)$ are relatively compact in $\mathcal{C}$, therefore $\Psi(\Omega\times[0,1])$ is a relatively compact set in $\mathcal{C}$, too. So, the multimap Ψ is completely continuous.

Now, we will show that Ψ has no fixed points on $\partial B_{\mathcal{C}}(0,R)\times[0,1]$. To the contrary, assume that there exists $(u_*,\lambda_*)\in\partial B_{\mathcal{C}}(0,R)\times[0,1]$ such that $u_*\in\Psi(u_*,\lambda_*)$. Then there is a function $f_*\in Q(u_*)$ such that

$$u_*(t)=\int_0^1 G(t,s)\big((1-\lambda_*)\delta u_*(s)+\lambda_* f_*(s)\big)\,ds,\quad \forall t\in[0,1], \tag{3.2}$$

or equivalently,

$$\begin{cases} u_*''(t)=(1-\lambda_*)\delta u_*(t)+\lambda_* f_*(t) & \text{for a.e. } t\in[0,1],\\ u_*(0)=u_*(1)=0. \end{cases} \tag{3.3}$$

Assume that $\|u_*\|_2\leqslant N$. Then from (3.2) we have

$$\big|u_*(t)\big|\leqslant\delta(1-\lambda_*)\|u_*\|_2+\lambda_*\|f_*\|_2\leqslant\delta(1-\lambda_*)N+\lambda_* q\big(1+N^p\big)<R,$$

for all $t\in[0,1]$. Hence $u_*\notin\partial B_{\mathcal{C}}(0,R)$, that is a contradiction. Therefore, $\|u_*\|_2>N$. From (3.3) it follows that

$$\big\langle u_*'',u_*\big\rangle=\delta(1-\lambda_*)\langle u_*,u_*\rangle+\lambda_*\langle u_*,f_*\rangle>0,$$

giving a contradiction.

Thus, Ψ is a homotopy joining $\Psi(\cdot,0)=\delta j\circ i$ and $\Psi(\cdot,1)=j\circ Q$, where i denotes the inclusion map. By virtue of the homotopic invariance property of the topological degree we have

$$\deg\big(i-j\circ Q,B_{\mathcal{C}}(0,R)\big)=\deg\big(i-\delta j\circ i,B_{\mathcal{C}}(0,R)\big).$$

For a sufficiently small $\delta>0$ we have

$$\big\|u-(u-\delta ju)\big\|_{\mathcal{C}}=\delta\|ju\|_{\mathcal{C}}<\|u\|_{\mathcal{C}}$$

for all $u\in\partial B_{\mathcal{C}}(0,R)$.

Then the vector fields i and $i-\delta j\circ i$ are homotopic on $\partial B_{\mathcal{C}}(0,R)$, so

$$\deg\big(i-j\circ Q,B_{\mathcal{C}}(0,R)\big)=\deg\big(i-\delta j\circ i,B_{\mathcal{C}}(0,R)\big)=\deg\big(i,B_{\mathcal{C}}(0,R)\big)=1.$$

Hence problem (3.1) has a solution $u\in B_{\mathcal{C}}(0,R)$. $\square$

Now we can formulate the above result in terms of the guiding functions. Notice that for every continuous function $V:\mathbb{R}\to\mathbb{R}$ the following map

$$V^\sharp:\mathcal{C}\to\mathcal{L}^2,\qquad V^\sharp(u)(t)=V\big(u(t)\big),$$

is continuous.

Definition 5. (See [13,18].) A continuous function $V:\mathbb{R}\to\mathbb{R}$ is said to be an integral guiding function for problem (3.1), if:

$(V1)$ there are $\alpha\geqslant 0$ and $\beta>0$ such that

$$\big|V(t)\big|\leqslant\alpha+\beta|t|,\quad \forall t\in\mathbb{R};$$

$(V2)$ there exists $N>0$ such that for every $u\in\mathcal{C}$, $\|u\|_2>N$, the following relation holds:

$$\big\langle V^\sharp(u),f\big\rangle>0\quad\text{for all } f\in Q(u);$$

$(V3)$ for every $u\in\mathring{W}_2^2[0,1]$, from $\|u\|_2>N$ it follows that

$$\big\langle u'',V^\sharp(u)\big\rangle\leqslant 0,$$

where N is the same constant as in $(V2)$.

Theorem 2. *Let conditions* $(Q1)$–$(Q2)$ *hold. Assume that there exists an integral guiding function* V *for problem* (3.1)*. Then problem* (3.1) *has a solution.*

Proof. Set $R = qN^p + q + 1$ and consider the multimap

$$\Psi : B_{\mathcal{C}}(0, R) \times [0, 1] \to K(\mathcal{C}),$$
$$\Psi(u, \lambda) = j \circ \big((1-\lambda)\delta V^{\sharp}(u) + \lambda Q(u)\big),$$

where δ, $0 < \delta < \frac{1}{\alpha+\beta N}$ is an arbitrary number, with N being the number in $(V2)$.

Similarly as in the proof of Theorem 1, one can verify that Ψ is a CJ-multimap. Let $\Omega \subset \mathcal{C}$ be a bounded subset. From $(Q2)$ and $(V1)$ it follows that the sets $Q(\Omega)$ and $V^{\sharp}(\Omega)$ are bounded in $\mathcal{L}^2$. Since the operator j is completely continuous, the sets $j \circ Q(\Omega)$ and $j \circ V^{\sharp}(\Omega)$ are relatively compact in $\mathcal{C}$. Hence the set $\Psi(\Omega \times [0, 1])$ is relatively compact in $\mathcal{C}$. Thus, Ψ is a completely continuous CJ-multimap.

Let us show that Ψ has no fixed points on $\partial B_{\mathcal{C}}(0, R) \times [0, 1]$. To the contrary, assume that there is $(u_*, \lambda_*) \in \partial B_{\mathcal{C}}(0, R) \times [0, 1]$ such that $u_* \in \Psi(u_*, \lambda_*)$. Then there is a function $f_* \in Q(u_*)$ such that

$$u_*(t) = \int_0^1 G(t,s)\big((1-\lambda_*)\delta V\big(u_*(s)\big) + \lambda_* f_*(s)\big)\, ds, \quad \forall t \in [0, 1], \tag{3.4}$$

or equivalently,

$$\begin{cases} u_*''(t) = (1-\lambda_*)\delta V\big(u_*(t)\big) + \lambda_* f_*(t) & \text{for a.e. } t \in [0, 1], \\ u_*(0) = u_*(1) = 0. \end{cases} \tag{3.5}$$

Assume that $\|u_*\|_2 \leqslant N$. Then from $(Q2)$, $(V1)$ and (3.4) we have

$$\begin{aligned} \big|u_*(t)\big| &\leqslant \delta(1-\lambda_*)\int_0^1 \big|V\big(u_*(t)\big)\big|\, dt + \lambda_* \int_0^1 \big|f_*(t)\big|\, dt \\ &\leqslant \delta(1-\lambda_*)\int_0^1 \big(\alpha + \beta\big|u_*(t)\big|\big)\, dt + \lambda_* \|f_*\|_2 \\ &\leqslant \delta(1-\lambda_*)\big(\alpha + \beta\|u_*\|_2\big) + \lambda_* q\big(1+N^p\big) \leqslant \delta(\alpha+\beta N) + q\big(1+N^p\big) < R, \end{aligned}$$

for all $t \in [0, 1]$. Hence $u_* \notin \partial B_{\mathcal{C}}(0, R)$, that is a contradiction. Therefore, $\|u_*\|_2 > N$. Notice that $u_* \in \mathring{W}_2^2[0, 1] \subset \mathcal{C}$, then from $(V2)$–$(V3)$ and (3.5) it follows that

$$\big\langle u_*'', V^{\sharp}(u_*)\big\rangle = \delta(1-\lambda_*)\big\langle V^{\sharp}(u_*), V^{\sharp}(u_*)\big\rangle + \lambda_*\big\langle V^{\sharp}(u_*), f_*\big\rangle > 0,$$

giving a contradiction.

So, we again obtain that Ψ is a homotopy and for sufficiently small $\delta > 0$ we have

$$\deg\big(i - j \circ Q, B_{\mathcal{C}}(0, R)\big) = \deg\big(i - \delta\, j \circ V^{\sharp}, B_{\mathcal{C}}(0, R)\big) = \deg\big(i, B_{\mathcal{C}}(0, R)\big) = 1.$$

Thus problem (3.1) has a solution. □

4. Applications

4.1. Feedback control system

We consider a feedback control system of the form

$$\begin{cases} u''(t) - \lambda u(t) = f\big(t, u(t), v(t)\big) & \text{for a.e. } t \in [0, 1], \\ v'(t) \in G\big(t, v(t), u(t)\big) & \text{for a.e. } t \in [0, 1], \\ u(0) = u(1) = 0, \qquad v(0) = v_0, \end{cases} \tag{4.1}$$

where $v_0 \in \mathbb{R}$, $\lambda > 0$, the map $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ and the multimap $G : [0, 1] \times \mathbb{R}^2 \to Kv(\mathbb{R})$ satisfy the following conditions:

$(f1)$ for every $(x, y) \in \mathbb{R}^2$ the function $f(\cdot, x, y) : [0, 1] \to \mathbb{R}$ is measurable;
$(f2)$ for a.e. $t \in [0, 1]$ the map $f(t, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ is continuous;

$(f3)$ there is $\alpha > 0$ such that

$$\left|f(t,x,y)\right| \leqslant \alpha\bigl(1+|x|+|y|\bigr)$$

for all $(x, y) \in \mathbb{R}^2$ and a.e. $t \in [0, 1]$;

$(G1)$ for every $(x, y) \in \mathbb{R}^2$ the multifunction $G(\cdot, x, y) : [0, 1] \to Kv(\mathbb{R})$ has a measurable selection;

$(G2)$ for a.e. $t \in [0, 1]$ the multimap $G(t, \cdot, \cdot) : \mathbb{R}^2 \to Kv(\mathbb{R})$ is u.s.c.;

$(G3)$ the multimap G is uniformly continuous with respect to the third argument in the sense: for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$G(t,x,\bar{y}) \subset O_\varepsilon\bigl(G(t,x,y)\bigr) \quad \forall (t,x) \in [0,1]\times\mathbb{R}$$

whenever $|\bar{y} - y| < \delta$;

$(G4)$ there is $\beta > 0$ such that

$$\bigl\|G(t,x,y)\bigr\| \leqslant \beta\bigl(1+|x|+|y|\bigr)$$

for all $(x, y) \in \mathbb{R}^2$ and a.e. $t \in [0, 1]$, where

$$\bigl\|G(t,x,y)\bigr\| = \max\bigl\{|z|:\ z \in G(t,x,y)\bigr\}.$$

For a given function $u \in \mathcal{C}$ define the multimap

$$G_u : [0,1]\times\mathbb{R} \to Kv(\mathbb{R}), \qquad G_u(t,x) = G\bigl(t,x,u(t)\bigr).$$

By virtue of Theorem 1.3.5 [19] for each $x \in \mathbb{R}$ the multifunction $G_u(\cdot, x)$ has a measurable selection. Further, from conditions $(G2)$ and $(G3)$ it follows that for a.e. $t \in [0, 1]$ the multimap $G_u(t, x)$ upper semicontinuously depends on (u, x). We have the following assertion (see, e.g., [14,19]): for each $u \in \mathcal{C}$ the set Π_u of all solutions of the following problem

$$\begin{cases} v'(t) \in G\bigl(t, v(t), u(t)\bigr) & \text{for a.e. } t \in [0,1],\\ v(0) = v_0 \end{cases}$$

is an R_δ-set in $\mathcal{C}$ and the multimap

$$\Pi : \mathcal{C} \to K(\mathcal{C}), \qquad \Pi(u) = \Pi_u,$$

is upper semicontinuous.

By a solution to problem (4.1) we mean a function $u \in \overset{\circ}{W}{}^2_2[0,1]$ such that there is an absolutely function $v \in \Pi(u)$ satisfying

$$u''(t) - \lambda u(t) = f\bigl(t, u(t), v(t)\bigr) \quad \text{for a.e. } t \in [0,1].$$

In what follows, we need the following assertion.

Lemma 1 *(Gronwall's Lemma). (See, e.g., [16].) Let $u, v : [a, b] \to \mathbb{R}$ be continuous nonnegative functions and $C \geqslant 0$ be a constant and*

$$v(t) \leqslant C + \int_a^t u(s)v(s)\,ds, \quad a \leqslant t \leqslant b.$$

Then

$$v(t) \leqslant Ce^{\int_a^t u(s)\,ds}, \quad a \leqslant t \leqslant b.$$

Theorem 3. *Let conditions $(f1)$–$(f3)$ and $(G1)$–$(G4)$ hold. Then for each*

$$\lambda > \alpha\bigl(1 + \beta e^\beta\bigr)$$

the feedback control system (4.1) has a solution.

Proof. Let $\widetilde{\Pi} : \mathcal{C} \to K(\mathcal{C}\times\mathcal{C})$

$$\widetilde{\Pi}(u) = \{u\} \times \Pi(u),$$

and $\tilde{f} : \mathcal{C}\times\mathcal{C} \to \mathcal{L}^2$,

$$\tilde{f}(u,v)(t) = \lambda u(t) + f\bigl(t, u(t), v(t)\bigr), \quad t \in [0,1].$$

Then we can substitute the feedback control system (4.1) by the following problem

$$Lu \in Q(u), \tag{4.2}$$

where $L : \mathring{W}_2^2[0,1] \to \mathcal{L}^2$, $Lu = u''$ and $Q : \mathcal{C} \to K(\mathcal{L}^2)$,

$$Q(u) = \tilde{f} \circ \widetilde{\Pi}(u).$$

From the continuity of the operator $\tilde{f}$ and $\widetilde{\Pi} \in J(\mathcal{C}, \mathcal{C} \times \mathcal{C})$ it follows that $j \circ Q \in CJ(\mathcal{C}, \mathcal{C})$.

Now letting $g \in Q(u)$ there exists $v \in \Pi(u)$ such that

$$g(s) = \tilde{f}(u, v)(s) = \lambda u(s) + f\bigl(s, u(s), v(s)\bigr), \quad \forall s \in [0,1].$$

From $v \in \Pi(u)$ it follows that there is a function $h \in L_1[0,1]$ such that

$$h(t) \in G\bigl(t, v(t), u(t)\bigr) \quad \text{for a.e. } t \in [0,1],$$

and

$$v(t) = v_0 + \int_0^t h(s)\,ds, \quad 0 \leqslant t \leqslant 1.$$

From $(G4)$ it follows that for every $t \in [0,1]$ the following relations hold

$$\begin{aligned}
\bigl|v(t)\bigr| &\leqslant |v_0| + \int_0^t \bigl|h(s)\bigr|\,ds \leqslant |v_0| + \int_0^t \beta\bigl(1 + \bigl|v(s)\bigr| + \bigl|u(s)\bigr|\bigr)\,ds \\
&\leqslant |v_0| + \beta + \beta\|u\|_2 + \int_0^t \beta\bigl|v(s)\bigr|\,ds.
\end{aligned}$$

By virtue of Lemma 1 we obtain

$$\bigl|v(t)\bigr| \leqslant \bigl(|v_0| + \beta + \beta\|u\|_2\bigr)e^{\beta}, \quad \forall t \in [0,1].$$

Applying $(f3)$ we have

$$\begin{aligned}
\|g\|_2^2 &= \int_0^1 g^2(s)\,ds = \int_0^1 \bigl(\lambda u(s) + f\bigl(s, u(s), v(s)\bigr)\bigr)^2\,ds \\
&\leqslant \int_0^1 \bigl(\lambda^2 + 1\bigr)\bigl(u^2(s) + f^2\bigl(s, u(s), v(s)\bigr)\bigr)\,ds \\
&\leqslant \bigl(\lambda^2 + 1\bigr)\left(\|u\|_2^2 + \int_0^1 \alpha^2\bigl(1 + \bigl|u(s)\bigr| + \bigl|v(s)\bigr|\bigr)^2\,ds\right) \\
&\leqslant \bigl(\lambda^2 + 1\bigr)\left(\|u\|_2^2 + \int_0^1 3\alpha^2\bigl(1 + u^2(s) + v^2(s)\bigr)\,ds\right) \\
&\leqslant \bigl(\lambda^2 + 1\bigr)\bigl(\bigl(1 + 3\alpha^2\bigr)\|u\|_2^2 + 3\alpha^2 + 3\alpha^2\bigl(|v_0| + \beta + \beta\|u\|_2\bigr)^2 e^{2\beta}\bigr).
\end{aligned}$$

Therefore, the multimap Q satisfies condition $(Q2)$.

Now for every $u \in \mathcal{C}$, choosing an arbitrary $g \in Q(u)$, we have that

$$\begin{aligned}
\langle g, u\rangle &= \int_0^1 u(s)\bigl(\lambda u(s) + f\bigl(s, u(s), v(s)\bigr)\bigr)\,ds \\
&\geqslant \lambda\|u\|_2^2 - \int_0^1 \bigl|f\bigl(s, u(s), v(s)\bigr)\bigr|\bigl|u(s)\bigr|\,ds \\
&\geqslant \lambda\|u\|_2^2 - \alpha\int_0^1 \bigl|u(s)\bigr|\bigl(1 + \bigl|u(s)\bigr| + \bigl|v(s)\bigr|\bigr)\,ds
\end{aligned}$$

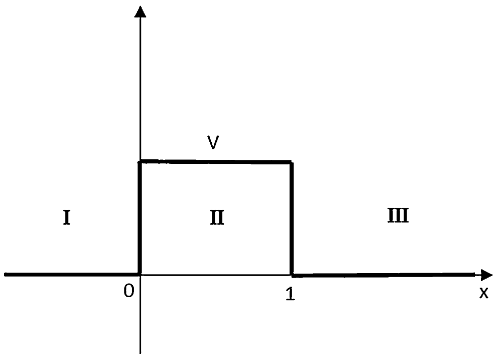

Fig. 1. The potential energy function $V(x)$.

$$\geqslant (\lambda-\alpha)\|u\|_2^2 - \alpha\int_0^1 |u(s)|\,ds - \alpha e^{\beta}\big(|v_0|+\beta+\beta\|u\|_2\big)\int_0^1 |u(s)|\,ds$$

$$\geqslant \big(\lambda-\alpha-\alpha\beta e^{\beta}\big)\|u\|_2^2 - \alpha\big(1+e^{\beta}\big(|v_0|+\beta\big)\big)\|u\|_2 > 0$$

provided

$$\|u\|_2 > \frac{\alpha(1+e^{\beta}(|v_0|+\beta))}{\lambda-\alpha(1+\beta e^{\beta})}.$$

By virtue of Theorem 1 problem (4.2) has a solution, and hence the feedback control system (4.1) has a solution. □

4.2. A model of a motion of a particle in a 1-dimensional potential

It is well known that for a single particle in a 1-dimensional potential energy V, the time-independent Schrödinger equation takes the form:

$$\frac{-\hbar^2}{2m}\Psi''(x) + V(x)\Psi(x) = E\Psi(x), \tag{4.3}$$

where m is the particle's mass, $\hbar$ – the reduced Planck constant, E – the total energy of the particle, $V(x)$ is the potential energy at the position x and $\Psi(x)$ is the wave function.

Here we consider the case when the potential energy $V(x)$ has a form:

$$V(x) = \begin{cases} V, & 0 \leqslant x \leqslant 1, \\ 0, & \text{otherwise}, \end{cases} \tag{4.4}$$

where V is a constant (see Fig. 1).

In this case, Eq. (4.3) must be solved in three regions:

$$I\ (x<0), \qquad II\ (0 \leqslant x \leqslant 1) \quad \text{and} \quad III\ (x>1).$$

The corresponding solutions of Eq. (4.3) in the first and third regions are

$$\Psi_I(x) = A\sin kx + B\cos kx \quad \text{and} \quad \Psi_{III}(x) = C\sin kx + D\cos kx,$$

where $k = \frac{\sqrt{2mE}}{\hbar}$ and A, B, C, D are constants.

So now we focus our attention on solving Eq. (4.3) in the second region. In this region the Schrödinger equation has the form:

$$\Psi_{II}''(x) = \frac{2m}{\hbar^2}V\Psi_{II}(x) - \frac{2m}{\hbar^2}E\Psi_{II}(x). \tag{4.5}$$

We assume that the potential V is connected with the wave function Ψ_{II} by the following relation:

$$V \in F(\Psi_{II}), \tag{4.6}$$

where $F : \mathcal{L}^2 \to K(\mathbb{R}_+)$ is a J-multimap, $\mathbb{R}_+ = [0,+\infty)$.

From the continuity of Ψ it follows that in region II the boundary conditions for Eq. (4.5) are:

$$\Psi_{II}(0) = \Psi_I(0) = B \quad \text{and} \quad \Psi_{II}(1) = \Psi_{III}(1) = C\sin k + D\cos k. \tag{4.7}$$

By a solution of problem (4.5)–(4.7) we mean a function $\Psi_{II} \in W_2^2[0,1]$ for which there exists $V \in F(\Psi_{II})$ such that Eq. (4.5) and condition (4.7) hold true.

Theorem 4. *Assume that there exist $a, b > 0$ such that*

$$F(u) \subseteq \left[a\|u\|_2, b\left(1+\|u\|_2\right)\right] \quad \textit{for all } u \in \mathcal{L}^2.$$

Then problem (4.5)–(4.7) *has a solution.*

Proof. Set $\alpha = B$, $\beta = C\sin k + D\cos k$ and $g(x) = \beta x + \alpha(1-x)$. For every $x \in [0,1]$ let $\varphi(x) = \Psi_{II}(x) - g(x)$. Then problem (4.5)–(4.7) can be replaced with the following system

$$\begin{cases} \varphi''(x) = \dfrac{2m}{\hbar^2} V\left(\varphi(x)+g(x)\right) - \dfrac{2m}{\hbar^2} E\left(\varphi(x)+g(x)\right), \\ V \in F(\varphi+g), \\ \varphi(0) = \varphi(1) = 0, \end{cases}$$

or equivalently,

$$\varphi \in j \circ Q(\varphi), \tag{4.8}$$

where

$$Q(\varphi) = \frac{2m}{\hbar^2}(\varphi+g)F(\varphi+g) - \frac{2m}{\hbar^2}E(\varphi+g),$$

and the operator j is defined as in Section 3.

It is easy to see that the multimap Q satisfies conditions $(Q1)$–$(Q2)$. For every $w \in Q(\varphi)$ there is $V \in F(\varphi+g)$ such that

$$w = \frac{2m}{\hbar^2}(V-E)\varphi + \frac{2m}{\hbar^2}(V-E)g.$$

We have

$$\begin{aligned} \langle \varphi, w\rangle &= \frac{2m}{\hbar^2}(V-E)\|\varphi\|_2^2 + \frac{2m}{\hbar^2}(V-E)\langle g, \varphi\rangle \\ &\geqslant \frac{2m}{\hbar^2}\left(a\|\varphi+g\|_2 - E\right)\|\varphi\|_2^2 - \frac{2m}{\hbar^2}V\|g\|_2\|\varphi\|_2 - \frac{2m}{\hbar^2}E\|g\|_2\|\varphi\|_2 \\ &\geqslant \frac{2m}{\hbar^2}\left(a\|\varphi\|_2 - a\|g\|_2 - E\right)\|\varphi\|_2^2 - \frac{2m}{\hbar^2}\left(b + b\|\varphi\|_2 + b\|g\|_2\right)\|g\|_2\|\varphi\|_2 - \frac{2m}{\hbar^2}E\|g\|_2\|\varphi\|_2. \end{aligned}$$

Therefore

$$\langle \varphi, w\rangle \geqslant \frac{2m}{\hbar^2}a\|\varphi\|_2^3 - \frac{2m}{\hbar^2}\left(a\|g\|_2 + E + b\|g\|_2\right)\|\varphi\|_2^2 - \frac{2m}{\hbar^2}\|g\|_2\left(b + b\|g\|_2 + E\right)\|\varphi\|_2 > 0$$

for sufficiently large $\|\varphi\|_2$.

From Theorem 1 it follows that inclusion (4.8) is solvable, and hence problem (4.5)–(4.7) has a solution. $\square$

5. Generalization to the case of an infinite-dimensional Hilbert space

Let H be a real infinite-dimensional Hilbert space with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. For every $n \in \mathbb{N}$, let H_n be an n-dimensional subspace H with basis $\{e_k\}_{k=1}^{n}$ and P_n be the projection onto H_n. By $\langle x, y\rangle_H$ we denote the inner product of elements $x, y \in H$. Set $I = [0,1]$ and denote by $C(I,H)$ $[L_2(I,H)]$ the collection of all continuous [resp., square summable] functions $u : I \to H$. The symbol $\langle f, g\rangle_L$ denotes the inner product of elements $f, g \in L_2(I,H)$. A ball of radius r centered at 0 in $C(I,H)$ is denoted by $B_C(0,r)$. As in Section 3, we will consider the Sobolev space $W_2^k(I,H)$ and its subspace $\mathring{W}_2^k(I,H)$. Notice that for every $k \geqslant 1$ the embedding $W_2^k(I,H) \hookrightarrow C(I,H)$ is continuous (but not compact). The weak convergence in $W_2^k(I,H)$ $[L_2(I,H)]$ is denoted by $u_n \xrightarrow{W_2^k} u_0$ [resp., $f_n \xrightarrow{L} f_0$].

For every $n \in \mathbb{N}$ let $\mathcal{J}_n : L_2(I,H) \to C(I,H_n)$ be the operator defined by

$$\mathcal{J}_n(f)(t) = \sum_{k=1}^{n}\left(\int_0^1 G(t,s) f_{(k)}(s)\, ds\right)e_k,$$

where the function $G(t,s)$ is defined as in Section 3 and

$$f(t) = \sum_{k=1}^{\infty} f_{(k)}(t)e_k, \quad \text{for all } t \in I.$$

It is clear that the operator $\mathcal{J}_n$ is completely continuous and for each $t \in I$ we have

$$\begin{aligned} \|\mathcal{J}_n(f)(t)\|_{H_n} &= \left(\sum_{k=1}^{n}\left(\int_0^1 G(t,s) f_{(k)}(s)\,ds\right)^2\right)^{\frac{1}{2}} \\ &\leqslant \left(\sum_{k=1}^{n}\int_0^1 G^2(t,s)\,ds \int_0^1 f^2_{(k)}(s)\,ds\right)^{\frac{1}{2}} \\ &\leqslant \left(\int_0^1 \sum_{k=1}^{n} f^2_{(k)}(s)\,ds\right)^{\frac{1}{2}} \leqslant \left(\int_0^1 \|f(s)\|_H^2\,ds\right)^{\frac{1}{2}} = \|f\|_2. \end{aligned} \tag{5.1}$$

For $n \in \mathbb{N}$, define the projection $\mathbb{P}_n : L_2(I,H) \to L_2(I,H_n)$ generated by P_n as

$$(\mathbb{P}_n f)(t) = P_n f(t) \quad \text{for a.e. } t \in I.$$

Consider now the following operator-differential inclusion

$$\begin{aligned} &u'' \in Q(u), \\ &u(0) = u(1) = 0, \end{aligned} \tag{5.2}$$

where the multimap $Q : C(I,H) \to C(L_2(I,H))$ satisfies the following conditions:

$(Q1)'$ for each $m \in \mathbb{N}$ the restriction $(\mathcal{J}_m \circ Q)_{|C(I,H_m)}$ belongs to $CJ(C(I,H_m); C(I,H_m))$;
$(Q2)'$ there are constants $p_1, q_1 > 0$ such that

$$\|Q(u)\|_2 \leqslant q_1\left(1 + \|u\|_2^{p_1}\right)$$

for all $u \in C(I,H)$;
$(Q3)$ for every bounded closed subset $M \subset \overset{\circ}{W}{}^2_2(I,H)$, if there exist the sequences $\{n_k\}$ and $\{u_k\}$, $u_k \in M \cap \overset{\circ}{W}{}^2_2(I,H_{n_k})$ such that

$$u_k'' \in \mathbb{P}_{n_k} Q(u_k),$$

then there is $u_* \in M$ such that $u_*'' \in Q(u_*)$.

By a solution to problem (5.2) we mean a function $u \in \overset{\circ}{W}{}^2_2(I,H)$ such that there is a function $f \in Q(u)$ and

$$u''(t) = f(t) \quad \text{for a.e. } t \in [0,1].$$

Theorem 5. *Let conditions $(Q1)'$–$(Q2)'$ and $(Q3)$ hold. Assume that there exists $N > 0$ such that for every $u \in C(I,H)$, $\|u\|_2 > N$, the following relation holds*:

$$\langle u, f\rangle_L > 0, \quad \text{for all } f \in Q(u). \tag{5.3}$$

Then problem (5.2) *has a solution.*

Proof. For each $n \in \mathbb{N}$, consider the auxiliary problem

$$\begin{aligned} &u'' \in \mathbb{P}_n Q(u), \\ &u(0) = u(1) = 0. \end{aligned}$$

It is clear that this problem is equivalent to the following fixed point problem

$$u \in \Sigma_n(u), \tag{5.4}$$

where $\Sigma_n : C(I,H_n) \to K(C(I,H_n))$, $\Sigma_n(u) = \mathcal{J}_n \circ Q(u)$.

From $(Q1)'$–$(Q2)'$ if follows that Σ_n is a completely continuous CJ-multimap.

Assume that $u \in C(I,H_n)$ is a solution of inclusion (5.4). Then there is $f \in Q(u)$ such that

$$u''(t) = P_n f(t) \quad \text{for a.e. } t \in I.$$

We have

$$\langle u, f\rangle_L = \langle u, \mathbb{P}_n f\rangle_L = \langle u, u''\rangle_L \leqslant 0.$$

Therefore, $\|u\|_2 \leqslant N$. From $(Q2)'$ and (5.1) it follows that

$$\|u\|_C = \max_{t\in I} \|u(t)\|_H \leqslant q\big(1+N^p\big).$$

Now set $R = qN^p + q + 1$ and define a multimap

$$\Psi_n : B_C^{(n)}(0,R) \times [0,1] \to K\big(C(I,H_n)\big),$$
$$\Psi_n(u,\lambda) = \mathcal{J}_n \circ \big(\delta(1-\lambda)u + \lambda Q(u)\big),$$

where $0 < \delta < \frac{1}{N}$ and $B_C^{(n)}(0,R) = B_C(0,R) \cap C(I,H_n)$.

By following the method given in the proof of Theorem 1, one can obtain that Ψ_n is a completely continuous CJ-multimap which has no fixed points on $\partial B_C^{(n)}(0,R) \times [0,1]$. Then for sufficiently small δ we have

$$\deg\big(i - \mathcal{J}_n \circ Q, B_C^{(n)}(0,R)\big) = \deg\big(i - \delta\mathcal{J}_n \circ i, B_C^{(n)}(0,R)\big) = \deg\big(i, B_C^{(n)}(0,R)\big) = 1.$$

Therefore, there is $u_n \in B_C(0,R) \cap \mathring{W}_2^2(I,H_n)$ such that $u_n'' = \mathbb{P}_n Q(u_n)$. By virtue of $(Q3)$ there exists $u_* \in B_C(0,R) \cap \mathring{W}_2^2(I,H)$ such that $u_*'' \in Q(u_*)$. The function u_* is a solution of problem (5.2). □

As an example, setting $Y = C[0,h]$ and $H = W_2^1[0,h]$ $(h>0)$, we can consider the following feedback control system

$$\begin{cases} w''(t) = f\big(t, w(t), \varphi(t)\big) & \text{for a.e. } t \in [0,1],\\ \varphi'(t) \in G\big(t, \varphi(t), w(t)\big) & \text{for a.e. } t \in [0,1],\\ w(0) = w(1) = 0, \qquad \varphi(0) = 0, \end{cases} \tag{5.5}$$

where $f : I \times Y \times Y \to Y$ is a continuous map and $G : I \times Y \times Y \to Cv(H)$ is a multimap.

Assume that the map f and the multimap G satisfy the following conditions:

$(f1)'$ the restriction $f_{|I\times H\times Y}$ takes values in H;
$(f2)'$ for every $(y,z) \in Y \times Y$ the function $f(\cdot,y,z) : I \to Y$ is measurable;
$(f3)'$ for a.e. $t \in I$ the map $f(t,\cdot,\cdot) : Y \times Y \to Y$ is continuous;
$(f4)'$ there is $c > 0$ such that

$$\big\|f(t,y,z)\big\|_H \leqslant c\big(1 + \|y\|_H + \|z\|_Y\big),$$

for all $(y,z) \in H \times Y$ and a.e. $t \in I$;
$(G1)'$ for each $(y,z) \in Y \times Y$ the multifunction $G(\cdot,y,z) : I \to Cv(H)$ has a measurable selection;
$(G2)'$ for every $t \in I$ the multimap $G(t,\cdot,\cdot) : Y \times Y \to Cv(H)$ is u.s.c.;
$(G3)'$ the multimap G is uniformly continuous with respect to the third argument in the following sense: for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$G(t,y,\bar z) \subset O_\varepsilon\big(G(t,y,z)\big), \quad \forall (t,y) \in I \times Y$$

provided $\|\bar z - z\| < \delta$;
$(G4)'$ there is $d > 0$ such that

$$\big\|G(t,y,z)\big\|_H \leqslant d\big(1 + \|y\|_Y + \|z\|_Y\big)$$

for all $(t,y,z) \in I \times Y \times Y$.

By a solution to problem (5.5) we mean a function $w \in \mathring{W}_2^2(I,H)$ such that there exists $\varphi \in W_2^1(I,H)$ with

$$\varphi'(t) \in G\big(t, \varphi(t), w(t)\big) \quad \text{for a.e. } t \in I,$$
$$\varphi(0) = 0,$$

and

$$w''(t) = f\big(t, w(t), \varphi(t)\big) \quad \text{for a.e. } t \in I.$$

We will need the following assertion.

Lemma 2. *(See Theorem 70.12 [14].) Let E be a separable Banach space and $\Phi : I \times E \to Kv(E)$ be a multimap satisfying the following conditions*:

$(\Phi1)$ *for each $y \in E$ the multifunction $\Phi(\cdot,y)$ has a measurable selection*;

($\Phi 2$) *for every* $t \in I$ *the multimap* $\Phi(t, \cdot)$ *is completely continuous*;
($\Phi 3$) *the set* $\Phi(A)$ *is compact for every compact subset* $A \subset I \times E$;
($\Phi 4$) *there is* $\omega \in L_2^+[0,1]$ *such that*

$$\left\|\Phi(t, y)\right\|_E \leqslant \omega(t)\big(1 + \|y\|_E\big),$$

for all $(t, y) \in I \times E$.

Then the set of all solutions to the following problem

$$\begin{cases} g'(t) \in \Phi\big(t, g(t)\big), & t \in I, \\ g(0) = g_0 \in E, \end{cases}$$

is an R_δ*-set in* $C(I, E)$.

Theorem 6. *Let conditions* $(f1)'$–$(f4)'$ *and* $(G1)'$–$(G4)'$ *hold. Then problem* (5.5) *can be represented as problem* (5.2) *with conditions* $(Q1)'$–$(Q2)'$ *and* $(Q3)$.

Proof. Let us mention that Y is a separable space and the embedding $H \hookrightarrow Y$ is compact. From $(G4)'$ it follows that for every $(t, y, z) \in I \times Y \times Y$ the set $G(t, y, z)$ is bounded in H, and hence it is a compact set in Y.

For a given function $w \in C(I, H)$ consider the following multimap

$$G_w : I \times Y \to Kv(Y), \qquad G_w(t, y) = G\big(t, y, w(t)\big).$$

It is easy to verify that G_w satisfies conditions $(\Phi 1) - (\Phi 4)$. Notice that $(\Phi 4)$ follows from $(G4)'$ and the fact that for every $y \in H$ the following relation holds:

$$\|y\|_Y \leqslant \max\left\{\sqrt{h}, \frac{1}{\sqrt{h}}\right\}\|y\|_H.$$

So we obtain that for every $w \in C(I, H)$ the set Π_w of all solutions to the following problem

$$\begin{cases} \varphi'(t) \in G\big(t, \varphi(t), w(t)\big), & t \in I, \\ \varphi(0) = 0 \end{cases}$$

is an R_δ-set in $C(I, Y)$. Define the multimap

$$\Pi : C(I, H) \to K\big(C(I, Y)\big), \qquad \Pi(w) = \Pi_w.$$

From Theorem 5.2.5 [19] it follows that the multimap Π is upper semicontinuous.

Now set $\widetilde{\Pi} : C(I, H) \to K(C(I, H) \times C(I, Y))$

$$\widetilde{\Pi}(w) = \{w\} \times \Pi(w),$$

and $\tilde{f} : C(I, H) \times C(I, Y) \to L_2(I, H)$,

$$\tilde{f}(w, \varphi)(t) = f\big(t, w(t), \varphi(t)\big).$$

Then problem (5.5) can be written in the form

$$w'' \in Q(w),$$

$$w(0) = w(1) = 0,$$

where $Q : C(I, H) \to K(L_2(I, H))$, $Q(w) = \tilde{f} \circ \widetilde{\Pi}(w)$.

Now we will show that the multimap Q satisfies conditions $(Q1)'$–$(Q2)'$ and $(Q3)$.

In fact, from the continuity of $\tilde{f}$ and the fact that $\widetilde{\Pi}$ belongs to $J(C(I, H); C(I, H) \times C(I, Y))$ it follows that $Q \in CJ(C(I, H); L_2(I, H))$. So for every $n \in \mathbb{N}$ the restriction $(\mathcal{J}_n \circ Q)_{|C(I,H_n)}$ is in $CJ(C(I, H_n); C(I, H_n))$. Hence, condition $(Q1)'$ holds. Notice that condition $(Q2)'$ immediately follows from $(f4)'$, $(G4)'$ and Lemma 1.

Let us verify now the condition $(Q3)$. Let $M \subset \overset{\circ}{W}{}_2^2(I, H)$ be a bounded closed subset and assume that there are $\{n_k\}$ and $\{w_k\}$, $w_k \in M \cap \overset{\circ}{W}{}_2^2(I, H_{n_k})$, such that

$$w_k'' \in \mathbb{P}_{n_k} Q(w_k).$$

The set $\{w_k\}_{k=1}^{\infty}$ is bounded, so it is weakly compact. W.l.o.g. assume that

$$w_k \xrightarrow{W_2^2} w_0 \in M.$$

Therefore, $w_k'' \xrightarrow{L} w_0''$ and $w_k(t) \xrightarrow{H} w_0(t)$ for every $t \in I$. From the compactness of the embedding $H \hookrightarrow Y$ it follows that

$$w_k(t) \xrightarrow{Y} w_0(t), \tag{5.6}$$

for every $t \in I$.

Set $h_k \in Q(w_k)$, such that

$$w_k'' = \mathbb{P}_{n_k} h_k.$$

From $(Q2)'$ it follows that the set $\{Q(w_k)\}_{k=1}^{\infty}$ is bounded, hence the set $\{h_k\}_{k=1}^{\infty}$ is bounded in $L_2(I, H)$, and therefore it is weakly compact. W.l.o.g. assume that

$$h_k \xrightarrow{L} h_0 \in L_2(I, H).$$

Now we will show that $\mathbb{P}_{n_k} h_k \xrightarrow{L} h_0$. At first we claim that

$$\lim_{n\to\infty} \mathbb{P}_n h_0 = h_0.$$

Indeed, since

$$L_2(I, H) = \overline{\bigcup_{n=1}^{\infty} L_2(I, H_n)},$$

there are sequences $\{\tilde{n}_m\}_{m=1}^{\infty} \subset \mathbb{N}$ and $\{\tilde{h}_m\}_{m=1}^{\infty}$, $\tilde{h}_m \in L_2(I, H_{\tilde{n}_m})$ such that $\tilde{h}_m \to h_0$. We have

$$\begin{aligned} \|\mathbb{P}_{\tilde{n}_m} h_0 - h_0\|_2 &\leqslant \|\mathbb{P}_{\tilde{n}_m} h_0 - \mathbb{P}_{\tilde{n}_m} \tilde{h}_m\|_2 + \|\mathbb{P}_{\tilde{n}_m} \tilde{h}_m - h_0\|_2 \\ &\leqslant 2\|\tilde{h}_m - h_0\|_2 \to 0 \quad \text{provided } m \to \infty. \end{aligned}$$

Further, for $n > \tilde{n}_m$

$$\|\mathbb{P}_n h_0 - \mathbb{P}_{\tilde{n}_m} h_0\|_2 = \|\mathbb{P}_n h_0 - \mathbb{P}_n \mathbb{P}_{\tilde{n}_m} h_0\|_2 \leqslant \|h_0 - \mathbb{P}_{\tilde{n}_m} h_0\|_2,$$

and hence

$$\begin{aligned} \|\mathbb{P}_n h_0 - h_0\|_2 &\leqslant \|\mathbb{P}_n h_0 - P_{\tilde{n}_m} h_0\|_2 + \|\mathbb{P}_{\tilde{n}_m} h_0 - h_0\|_2 \\ &\leqslant 2\|h_0 - \mathbb{P}_{\tilde{n}_m} h_0\|_2. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} \mathbb{P}_n h_0 = h_0.$$

For all $g \in L_2(I, H)$ we have

$$\begin{aligned} \langle \mathbb{P}_{n_k} h_k - h_0, g\rangle_L &= \langle \mathbb{P}_{n_k} h_k - \mathbb{P}_{n_k} h_0, g\rangle_L + \langle \mathbb{P}_{n_k} h_0 - h_0, g\rangle_L \\ &= \langle h_k - h_0, \mathbb{P}_{n_k} g\rangle_L + \langle \mathbb{P}_{n_k} h_0 - h_0, g\rangle_L \\ &= \langle h_k - h_0, g\rangle_L + \langle h_k - h_0, \mathbb{P}_{n_k} g - g\rangle_L + \langle \mathbb{P}_{n_k} h_0 - h_0, g\rangle_L. \end{aligned}$$

Thus

$$\lim_{k\to\infty} \langle \mathbb{P}_{n_k} h_k - h_0, g\rangle_L = 0.$$

On the other hand

$$\mathbb{P}_{n_k} h_k = w_k'' \xrightarrow{L} w_0''.$$

So we obtain that $w_0'' = h_0$, i.e., $h_k \xrightarrow{L} w_0''$.

From $h_k \in Q(w_k)$ it follows that there is a sequence $\{\varphi_k\}_{k=1}^{\infty}$, $\varphi_k \in \Pi(w_k)$, such that

$$h_k(t) = f\big(t, w_k(t), \varphi_k(t)\big) \quad \text{for a.e. } t \in I. \tag{5.7}$$

Set $\widehat{W}_2^1(I, H) = \{u \in W_2^1(I, H)\colon\ u(0) = 0\}$. It is clear that $\widehat{W}_2^1(I, H)$ is a subspace of $W_2^1(I, H)$. The set $\{\varphi_k\}_{k=1}^{\infty}$ is bounded in $\widehat{W}_2^1(I, H)$ and so it is weakly compact. W.l.o.g. assume that

$$\varphi_k \xrightarrow{W_2^1} \varphi_0 \in \widehat{W}_2^1(I, H).$$

Therefore

$$\varphi_k' \xrightarrow{L} \varphi_0', \quad \text{and} \quad \varphi_k(t) \xrightarrow{Y} \varphi_0(t), \quad \text{for every } t \in I. \tag{5.8}$$

From $\varphi_k \in \Pi(w_k)$ it follows that there is $\{g_k\}_{k=1}^{\infty} \subset L_2(I, H)$ such that

$$g_k(t) \in G\bigl(t, \varphi_k(t), w_k(t)\bigr) \quad \text{for a.e. } t \in I,$$

and

$$\varphi_k'(t) = g_k(t) \quad \text{for a.e. } t \in I.$$

So $g_k \xrightarrow{L} \varphi_0'$. By virtue of Mazur's Lemma (see, e.g., [10]) there are sequences of convex combinations $\{\hat{g}_m\}$ and $\{\hat{h}_m\}$

$$\hat{g}_m = \sum_{k=m}^{\infty} \lambda_{mk} g_k, \quad \lambda_{mk} \geqslant 0 \quad \text{and} \quad \sum_{k=m}^{\infty} \lambda_{mk} = 1,$$
$$\hat{h}_m = \sum_{k=m}^{\infty} \tilde{\lambda}_{mk} h_k, \quad \tilde{\lambda}_{mk} \geqslant 0 \quad \text{and} \quad \sum_{k=m}^{\infty} \tilde{\lambda}_{mk} = 1,$$

which converge in $L_2(I, H)$ to φ_0' and w_0'', respectively. Applying Theorem 38 [25] we again can assume w.l.o.g. that

$$\hat{g}_m(t) \xrightarrow{H} \varphi_0'(t) \quad \text{and} \quad \hat{h}_m(t) \xrightarrow{H} w_0''(t) \tag{5.9}$$

for a.e. $t \in I$.

From (5.6), (5.8) and $(G2)'$ it follows that for every $t \in I$ and $\varepsilon > 0$ there is $i_0 = i_0(\varepsilon, t)$ such that

$$G\bigl(t, \varphi_i(t), w_i(t)\bigr) \subset O_\varepsilon^H\bigl(G\bigl(t, \varphi_0(t), w_0(t)\bigr)\bigr), \quad \text{for all } i \geqslant i_0.$$

Then $g_i(t) \in O_\varepsilon^H(G(t, \varphi_0(t), w_0(y)))$ for all $i \geqslant i_0$, and hence, from the convexity of the set $O_\varepsilon^H(G(t, \varphi_0(t), w_0(t)))$ we have

$$\hat{g}_m(t) \in O_\varepsilon^H\bigl(G\bigl(t, \varphi_0(t), w_0(t)\bigr)\bigr), \quad \text{for all } m \geqslant i_0.$$

Thus, $\varphi_0'(t) \in G(t, \varphi_0(t), w_0(t))$ for a.e. $t \in I$, i.e., $\varphi_0 \in \Pi(w_0)$.

Now from (5.6), (5.8) and $(f3)'$ we have

$$\lim_{k \to \infty} f\bigl(t, w_k(t), \varphi_k(t)\bigr) = f\bigl(t, w_0(t), \varphi_0(t)\bigr)$$

for a.e. $t \in I$.

So for a.e. $t \in I$ and $\varepsilon > 0$ there is $\tau_0 = \tau_0(\varepsilon, t)$ such that

$$f\bigl(t, w_\tau(t), \varphi_\tau(t)\bigr) \in O_\varepsilon^Y\bigl(f\bigl(t, w_0(t), \varphi_0(t)\bigr)\bigr), \quad \text{for all } \tau \geqslant \tau_0.$$

From (5.7) and (5.9) we obtain

$$w_0''(t) = f\bigl(t, w_0(t), \varphi_0(t)\bigr) \quad \text{for a.e. } t \in I.$$

So condition $(Q3)$ holds. $\square$

Now let the map f in (5.5) have the following form:

$$f\bigl(t, w(t), \varphi(t)\bigr) = b + aw(t) + \hat{f}\bigl(t, w(t), \varphi(t)\bigr),$$

where $a > 0$, $b \in \mathbb{R}$.

Theorem 7. *Let conditions $(G1)'$–$(G4)'$ hold. Assume that the map*

$$\hat{f} : I \times Y \times Y \to H$$

satisfies conditions $(f2)'$–$(f4)'$ and

$(\hat{f})$ *$a > c(1 + dr^2 e^{rd})$, where $r = \max\{\sqrt{h}, \frac{1}{\sqrt{h}}\}$ and c, d are constants from $(f4)'$ and $(G4)'$, respectively.*

Then problem (5.5) has a solution.

Proof. It is easy to see that the map f satisfies conditions $(f1)'$–$(f4)'$. And hence, from Theorem 6 it follows that the multimap Q satisfies conditions $(Q1)'$ and $(Q2)$–$(Q3)$.

Now for every $w \in \overset{\circ}{W}{}^2_2(I,H)$, choose an arbitrary $\gamma \in Q(w)$. Then there is a function $\varphi \in \Pi(w)$ such that

$$\gamma = b + aw + f^*(w,\varphi),$$

where $f^*(w,\varphi)(t) = \hat{f}(t, w(t), \varphi(t))$ for $t \in I$.

From $\varphi \in \Pi(w)$ it follows that there exists $g \in L_2(I,H)$ such that

$$g(t) \in G\big(t, \varphi(t), w(t)\big) \quad \text{for a.e. } t \in I,$$

and

$$\varphi(t) = \int_0^t g(s)\,ds \quad t \in I.$$

Therefore

$$\begin{aligned}\big\|\varphi(t)\big\|_H &\leqslant \int_0^t \big\|g(s)\big\|_H\,ds \leqslant d\int_0^t \big(1 + \big\|\varphi(s)\big\|_Y + \big\|w(s)\big\|_Y\big)\,ds \\ &\leqslant d + dr\int_0^1 \big\|w(s)\big\|_H\,ds + \int_0^t rd\big\|\varphi(s)\big\|_H\,ds.\end{aligned}$$

By virtue of Lemma 1 we obtain

$$\big\|\varphi(t)\big\|_H \leqslant \left(d + dr\int_0^1 \big\|w(s)\big\|_H\,ds\right)e^{rd} \quad \text{for all } t \in I.$$

Notice that for every $s \in I$: $u = w(s)$ and $v = \hat{f}(s, w(s), \varphi(s))$ are elements of H. We have

$$\begin{aligned}\big\langle w(s), b + aw(s) + \hat{f}\big(s, w(s), \varphi(s)\big)\big\rangle_H &= \langle u, b + au + v\rangle_H \\ &= \int_0^h a\big(u^2(\tau) + u'^2(\tau)\big)\,d\tau + b\int_0^h u(\tau)\,d\tau + \int_0^h \big(u(\tau)v(\tau) + u'(\tau)v'(\tau)\big)\,d\tau \\ &\geqslant a\|u\|_H^2 - b\sqrt{h}\|u\|_H - \|u\|_H\|v\|_H.\end{aligned}$$

So we obtain the following estimation

$$\begin{aligned}\langle w, \gamma\rangle_L &= \int_0^1 \big\langle w(s), b + aw(s) + \hat{f}\big(s, w(s), \varphi(s)\big)\big\rangle_H\,ds \\ &\geqslant \int_0^1 \big(a\big\|w(s)\big\|_H^2 - |b|\sqrt{h}\,\big\|w(s)\big\|_H - \big\|\hat{f}\big(s, w(s), \varphi(s)\big)\big\|_H\big\|w(s)\big\|_H\big)\,ds \\ &\geqslant a\|w\|_2^2 - |b|\sqrt{h}\|w\|_2 - \int_0^1 \big\|w(s)\big\|_H c\big(1 + \big\|w(s)\big\|_H + \big\|\varphi(s)\big\|_Y\big)\,ds \\ &\geqslant (a-c)\|w\|_2^2 - (b\sqrt{h} + c)\|w\|_2 - cr\int_0^1 \big\|w(s)\big\|_H\big\|\varphi(s)\big\|_H\,ds \\ &\geqslant (a-c)\|w\|_2^2 - (b\sqrt{h} + c)\|w\|_2 - cr\left(d + dr\int_0^1 \big\|w(s)\big\|_H\,ds\right)e^{rd}\int_0^1 \big\|w(s)\big\|_H\,ds \\ &\geqslant \big(a - c - cdr^2e^{rd}\big)\|w\|_2^2 - \big(b\sqrt{h} + c + cdre^{rd}\big)\|w\|_2 > 0\end{aligned}$$

provided

$$\|w\|_2 > \frac{b\sqrt{h} + c + cdre^{rd}}{a - c - cdr^2e^{rd}}.$$

From Theorem 5 it follows that problem (5.5) has a solution. $\square$

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References

[1] D. Affane, D. Azzam-Laouir, A control problem governed by a second order differential inclusion, Appl. Anal. 88 (12) (2009) 1677–1690.
[2] R.P. Agarwal, S.R. Grace, D. O'Regan, Oscillation theorems for second order differential inclusions, Int. J. Dyn. Syst. Differ. Equ. 1 (2) (2007) 85–88.
[3] J. Andres, L. Malaguti, M. Pavlačkova, Strictly localized bounding functions for vector second-order boundary value problems, Nonlinear Anal. 71 (12) (2009) 6019–6028.
[4] E.P. Avgerinos, N.S. Papageorgiou, N. Yannakakis, Periodic solutions for second order differential inclusions with nonconvex and unbounded multifunction, Acta Math. Hungar. 83 (4) (1999) 303–314.
[5] M. Benchohra, S.K. Ntouyas, Controllability of second-order differential inclusions in Banach spaces with nonlocal conditions, J. Optim. Theory Appl. 107 (3) (2000) 559–571.
[6] Y.G. Borisovich, B.D. Gelman, A.D. Myshkis, V.V. Obukhovskii, Introduction to the Theory of Multivalued Maps and Differential Inclusions, second edition, Librokom, Moscow, 2011 (in Russian).
[7] K. Borsuk, Theory of Retracts, Monografie Mat., vol. 44, PWN, Warszawa, 1967.
[8] Z. Denkowski, S. Migórski, N.S. Papageorgiou, An Introduction to Nonlinear Analysis: Theory, Kluwer Academic Publishers, Boston, MA, 2003.
[9] S. Domachowski, J. Gulgowski, A global bifurcation theorem for convex-valued differential inclusions, Z. Anal. Anwend. 23 (2) (2004) 275–292.
[10] I. Ekeland, R. Temam, Convex Analysis and Variation Problems, North-Holland, Amsterdam, 1979.
[11] L. Erbe, W. Krawcewicz, Existence of solutions to boundary value problems for impulsive second order differential inclusions, Rocky Mountain J. Math. 22 (2) (1992) 519–539.
[12] L. Erbe, W. Krawcewicz, Boundary value problems for second order nonlinear differential inclusions, in: Qualitative Theory of Differential Equations, Szeged, 1988, in: Colloq. Math. Soc. Janos Bolyai, vol. 53, North-Holland, Amsterdam, 1990, pp. 163–171.
[13] A. Fonda, Guiding functions and periodic solutions to functional differential equations, Proc. Amer. Math. Soc. 99 (1) (1987) 79–85.
[14] L. Górniewicz, Topological Fixed Point Theory of Multivalued Mappings, second edition, Springer, Dordrecht, 2006.
[15] S.R. Grace, R.P. Agarwal, D. O'Regan, A selection of oscillation criteria for second-order differential inclusions, Appl. Math. Lett. 22 (2) (2009) 153–158.
[16] Ph. Hartman, Ordinary Differential Equations, Classics Appl. Math., vol. 38, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. Corrected reprint of the second (1982) edition, Birkhäuser, Boston, MA.
[17] D.M. Hyman, On decreasing sequences of compact absolute retracts, Fund. Math. 64 (1969) 91–97.
[18] S.V. Kornev, V.V. Obukhovskii, On some developments of the method of integral guiding functions, Funct. Differ. Equ. 12 (3–4) (2005) 303–310.
[19] M. Kamenskii, V. Obukhovskii, P. Zecca, Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces, de Gruyter Ser. Nonlinear Anal. Appl., vol. 7, Walter de Gruyter, Berlin, New York, 2001.
[20] M.A. Krasnosel'skii, The Operator of Translation Along the Trajectories of Differential Equations, Nauka, Moscow, 1966 (in Russian); English translation: Transl. Math. Monogr., vol. 19, Amer. Math. Soc., Providence, RI, 1968.
[21] M.A. Krasnosel'skii, A.I. Perov, On a principle of the existence of bounded, periodic and almost periodic solutions for the systems of ordinary differential equations, Dokl. Akad. Nauk 123 (2) (1958) 235–238 (in Russian).
[22] M.A. Krasnosel'skii, P.P. Zabreiko, Geometrical Methods of Nonlinear Analysis, Nauka, Moscow, 1975 (in Russian); English translation: Series of Comprehensive Studies in Math., vol. 263, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.
[23] S. Kyritsi, N. Matzakos, N.S. Papageorgiou, Periodic problems for strongly nonlinear second-order differential inclusions, J. Differential Equations 183 (2) (2002) 279–302.
[24] A.D. Myshkis, Generalizations of the theorem on a fixed point of a dynamical system inside of a closed trajectory, Mat. Sb. 34 (3) (1954) 525–540 (in Russian).
[25] L. Schwartz, Cours d'Analyse. 1, second edition, Hermann, Paris, 1981.