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Global weak solution for a periodic generalized Hunter–Saxton equation

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ABSTRACT

In this paper, we consider a periodic generalized Hunter–Saxton equation. We obtain the existence of global weak solutions to the equation. First, we give the well-posedness result of the viscous approximate equations and establish the basic energy estimates. Then, we show that the limit of the viscous approximation solutions is a global weak solution to the equation.

1. Introduction

In this paper we consider the Cauchy problem of the following periodic generalized Hunter–Saxton equation:

$$\begin{cases} \mu(u_t) - u_{txx} = -2\mu(u)u_x + 2u_x u_{xx} + uu_{xxx}, & t > 0,\ x \in \mathbb{R}, \\ u(t, x+1) = u(t, x), & t \geqslant 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $u = u(t, x)$ is a time-dependent function on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ and $\mu(u) = \int_{\mathbb{S}} u\,dx$ denotes its mean. The equation lies 'mid-way' between the periodic Hunter–Saxton and Camassa–Holm equations, and describes evolution of rotators in liquid crystals with external magnetic and self-interaction [14].

In [14], they proved that Eq. (1.1) is the Euler equation on the diffeomorphism group of the circle corresponding to a natural right-invariant Sobolev metric. They showed that Eq. (1.1) is bi-Hamiltonian and admits both cusped and smooth traveling-wave solutions which are natural candidates for solitons. They also proved that Eq. (1.1) is locally well-posed and has blowing-up solutions and global solutions with non-negative angular momentum density. Fu et al. investigated the blow-up phenomena and blow-up rate in [8]. According to [8,14], the periodic generalized Hunter–Saxton equation is the periodic generalized Camassa–Holm equation.

The Camassa–Holm equation can be regarded as a shallow water wave equation [3]. It has a bi-Hamiltonian structure [7] and is completely integrable [5]. Obviously, if $\mu(u) = 0$, which implies $\mu(u_t) = 0$, then this equation reduces to the Hunter–Saxton equation describing the director field of a nematic liquid crystal [10], which is a short wave limit of the Camassa–Holm equation. The Hunter–Saxton equation has also a bi-Hamiltonian structure [10,17] and is completely integrable [1,11]. Yin studied the periodic Hunter–Saxton in [20]. He proved the local existence of strong solutions of the periodic Hunter–Saxton equation and showed that all strong solutions except space-independent solutions blow up in finite time. Recently, Wei and Yin also studied the periodic Hunter–Saxton equation with weak dissipation [18].

Recently, global dissipative and conservative weak solutions for the initial boundary value problem of the classical Hunter–Saxton equation on the half-line were investigated extensively, cf. [2,11,12,15,21–23]. The authors in [12,15,23] constructed the viscous approximate solution sequence by zero-viscosity method.

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The aim of this paper is to prove the existence of global weak solutions to generalized Hunter–Saxton equation (1.1). Thus, we have to use the viscous approximation method [4,12,15,19,23] and the theory L^p Young measure [15,19] to prove the existence of global weak solutions to Eq. (1.1).

By reformulating Eq. (1.1), we write Eq. (1.1) as follows:

$$\begin{cases} u_t + uu_x + \partial_x A^{-1}\left(2\mu(u)u + \frac{1}{2}u_x^2\right) = 0, & t > 0,\ x \in \mathbb{R}, \\ u(t, x+1) = u(t, x), & t \geqslant 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $A = \mu - \partial_x^2$ is an isomorphism between $H^s(\mathbb{S})$ and $H^{s-2}(\mathbb{S})$ with the inverse $v = A^{-1}(w)$ given explicitly by

$$v(x) = \left(\frac{x^2}{2} - \frac{x}{2} + \frac{13}{12}\right)\mu(w) + \left(x - \frac{1}{2}\right)\int_0^1\int_0^y w(s)\,ds\,dy - \int_0^x\int_0^y w(s)\,ds\,dy + \int_0^1\int_0^y\int_0^s w(r)\,dr\,ds\,dy. \tag{1.3}$$

Since A^{-1} and ∂_x commute, the following identities hold

$$A^{-1}\partial_x w(x) = \left(x - \frac{1}{2}\right)\int_0^1 w(x)\,dx - \int_0^x w(y)\,dy + \int_0^1\int_0^x w(y)\,dy\,dx, \tag{1.4}$$

and

$$A^{-1}\partial_x^2 w(x) = -w(x) + \int_0^1 w(x)\,dx. \tag{1.5}$$

If we write the inverse of the operator $A = \mu - \partial_x^2$ in terms of a Green's function, we find $(A^{-1}m)(x) = \int_0^1 g(x - x')m(x')\,dx' = (g * m)(x)$. Eq. (1.2) is equivalent to

$$\begin{cases} u_t + uu_x + \partial_x g * \left(2\mu(u)u + \frac{1}{2}u_x^2\right) = 0, & t > 0,\ x \in \mathbb{R}, \\ u(t, x+1) = u(t, x), & t \geqslant 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.6}$$

where the Green's function $g(x)$ [8] is given by

$$g(x) = \frac{1}{2}x(x-1) + \frac{13}{12}, \quad \text{for } x \in [0, 1) \simeq S^1, \tag{1.7}$$

and is extended periodically to the real line. In other words,

$$g(x - x') = \frac{(x - x')^2}{2} - \frac{|x - x'|}{2} + \frac{13}{12}, \quad \text{for } x, x' \in [0, 1) \simeq S^1. \tag{1.8}$$

In particular, $\mu(g) = 1$.

In the current paper, the existence of global weak solutions to generalized Hunter–Saxton equation (1.1) was investigated. Firstly, Eq. (1.1) has been added the terms $\mu(u)$ and $\mu(u_t)$, this leads to the essential difficulty comparing for the classical Hunter–Saxton equation. Secondly, [15] discussed the case of half-space; nevertheless we consider the case of circle. Thirdly, the existence of global weak solutions to Eq. (1.1) has not been discussed, the result is new.

Motivated by this, we first introduce the definition of a weak solution to the Cauchy problem (1.2).

Definition 1.1. u is a dissipative weak solution to the Cauchy problem (1.2) if

$$u(t, x) \in L^\infty_{loc}\big((0, \infty); H^1(\mathbb{S})\big)$$

satisfies Eq. (1.2) and $u(t, \cdot) \to u_0$ as $t \to 0^+$ in the sense of distributions on $\mathbb{R}_+ \times \mathbb{R}$. Moreover,

$$\int_{\mathbb{S}} u_x^2(t, x)\,dx \leqslant \int_{\mathbb{S}} u_{0,x}^2(x)\,dx. \tag{1.9}$$

Therefore in this paper, the main result is to give the existence of a globe-in-time weak solution u to the Cauchy problem (1.2) with the initial $u_0 \in H^1(\mathbb{S})$. The main result is as follows:

Theorem 1.1. *Let $u_0 \in H^1(\mathbb{S})$. Then Eq.* (1.2) *has a dissipative weak solution in the sense of Definition* 1.1.

Remark 1.1. There are global strong solutions of the classical Hunter–Saxton equation in [20], but the strong solutions are not unique. Therefore, one cannot prove the uniqueness of dissipative weak solutions.

The organization of the paper is as follows. In Section 2, we give the well-posedness result of the viscous approximate to Eq. (1.2) and establish the basic energy estimate on u_ϵ. In Section 3, the uniform a priori one-sided super-norm estimate and local space–time higher integrability estimate for $\partial_x u_\epsilon$ are established. In Section 4, the strong convergence of $\partial_x u_\epsilon$ in $L^2_{loc}(\mathbb{R}_+ \times \mathbb{S})$ is carried out and we complete the proof of the main result.

2. Viscous approximate solutions

In this section, we construct the approximate solution sequence $u_\epsilon = u_\epsilon(t,x)$. Hence, we consider the viscous problem of Eq. (1.3) as follows:

$$\begin{cases} (u_\epsilon)_t + u_\epsilon (u_\epsilon)_x + \partial_x g * \left(2\mu(u_\epsilon)u_\epsilon + \dfrac{1}{2}(u_\epsilon)_x^2\right) = \epsilon (u_\epsilon)_{xx}, & t>0,\ x\in\mathbb{R}, \\ u_\epsilon(t,x+1) = u_\epsilon(t,x), & t \geqslant 0,\ x\in\mathbb{R}, \\ u_\epsilon(0,x) = u_{\epsilon,0}(x), & x\in\mathbb{R}, \end{cases} \tag{2.1}$$

or the equivalent form:

$$\begin{cases} (u_\epsilon)_t + u_\epsilon (u_\epsilon)_x = -\partial_x A^{-1} \left(2\mu(u_\epsilon)u_\epsilon + \dfrac{1}{2}(u_\epsilon)_x^2\right) = \epsilon (u_\epsilon)_{xx}, & t>0,\ x\in\mathbb{R}, \\ u_\epsilon(t,x+1) = u_\epsilon(t,x), & t \geqslant 0,\ x\in\mathbb{R}, \\ u_\epsilon(0,x) = u_{\epsilon,0}(x), & x\in\mathbb{R}, \end{cases} \tag{2.2}$$

where $u_{\epsilon,0}(x) = (\phi_\epsilon * u_0)(x)$, and

$$\phi_\epsilon(x) := \left(\int_{\mathbb{R}} \phi(\xi)\,d\xi\right)^{-1} \frac{1}{\epsilon}\phi\left(\frac{x}{\epsilon}\right), \quad x\in\mathbb{R},\ \epsilon>0,$$

where $\phi \in C_c^\infty(\mathbb{R})$ is defined by

$$\phi(x) = \begin{cases} e^{1/(x^2-1)}, & |x|<1, \\ 0, & |x| \geqslant 1. \end{cases}$$

Then, we have

$$\|u_{\epsilon,0}\|_{L^2(\mathbb{S})} \leqslant \|u_0\|_{L^2(\mathbb{S})}, \qquad \left\|(u_\epsilon)_{0,x}\right\|_{L^2(\mathbb{S})} \leqslant \|u_{0,x}\|_{L^2(\mathbb{S})}$$

and

$$u_{\epsilon,0} \to u_0, \quad \text{in } H^1(\mathbb{S}).$$

Integrating both sides of Eq. (2.1) over the circle and using periodicity imply that $\mu(u_\epsilon)_t = \mu(u_{\epsilon t}) = 0$. Moreover, for the sake of convenience, let

$$\mu(u_\epsilon) = \int_{\mathbb{S}} u_\epsilon\,dx = \mu(u_{\epsilon,0}) = \int_{\mathbb{S}} u_{\epsilon,0}\,dx,$$

and

$$a_\epsilon(t) = \frac{1}{2}\int_{\mathbb{S}} (u_\epsilon)_x^2\,dx, \qquad a_{\epsilon,0} = \frac{1}{2}\int_{\mathbb{S}} (u_\epsilon)_{0,x}^2\,dx.$$

Differentiating Eq. (2.2) with respect to x yields

$$(u_\epsilon)_{tx} + (u_\epsilon)_x^2 + u_\epsilon(u_\epsilon)_{xx} = -A^{-1}\partial_x^2\left(2\mu(u_{\epsilon,0})u_\epsilon + \frac{1}{2}(u_\epsilon)_{\epsilon x}^2\right) + \epsilon(u_\epsilon)_{xxx},$$

in view of (1.5), we have

$$(u_\epsilon)_{tx} = -\frac{1}{2}(u_\epsilon)_x^2 - u_\epsilon(u_\epsilon)_{xx} + 2\mu(u_{\epsilon,0})u_\epsilon - 2\mu^2(u_{\epsilon,0}) - a_\epsilon(t) + \epsilon(u_\epsilon)_{xxx}. \tag{2.3}$$

The existence, uniqueness, and basic energy estimate on this approximate solution of (2.1) are given in the following theorem. We first recall the following three lemmas.

Lemma 2.1. *(See [13].) If $r > 0$, then $H^r(\mathbb{S}) \cap L^\infty(\mathbb{S})$ is an algebra. Moreover*

$$\|fg\|_{H^r(\mathbb{S})} \leqslant c\big(\|f\|_{L^\infty(\mathbb{S})}\|g\|_{H^r(\mathbb{S})} + \|f\|_{H^r(\mathbb{S})}\|g\|_{L^\infty(\mathbb{S})}\big),$$

where c is a constant depending only on r.

Lemma 2.2. *(See [13].) If $r > 0$, then*

$$\big\|[\Lambda^r, f]g\big\|_{L^2(\mathbb{S})} \leqslant c\big(\|\partial_x f\|_{L^\infty(\mathbb{S})}\big\|\Lambda^{r-1}g\big\|_{L^2(\mathbb{S})} + \big\|\Lambda^r f\big\|_{L^2(\mathbb{S})}\|g\|_{L^\infty(\mathbb{S})}\big),$$

where c is a constant depending only on r.

Lemma 2.3. *(See Appendix C of [16].) Let X be a separable reflexive Banach space and let f^n be bounded in $L^\infty(0,T;X)$ for some $T \in (0,\infty)$. We assume that $f^n \in C([0,T];Y)$ where Y is a Banach space such that $X \hookrightarrow Y$, Y' is separable and dense in X'. Furthermore, $(\phi, f^n(t))_{Y'\times Y}$ is uniformly continuous in $t \in [0,T]$ and uniformly in $n \geqslant 1$. Then f^n is relatively compact in $C^w([0,T];X)$, the space of continuous functions from $[0,T]$ with values in X when the latter space is equipped with its weak topology.*

Remark 2.1. If the conditions which f^n satisfies in Lemma 2.3 are replaced by the following conditions:

$$f^n \in L^\infty(0,T;X), \qquad \partial_t f^n \in L^p(0,T;Y) \quad \text{for some } p \in (1,\infty),$$

and

$$\big\|f^n\big\|_{L^\infty(0,T;X)}, \big\|\partial_t f^n\big\|_{L^p(0,T;Y)} \leqslant C, \quad \forall n \geqslant 1,$$

then the conclusion of Lemma 2.3 holds true.

Theorem 2.1. *Let $\epsilon > 0$ and $u_{\epsilon,0} \in H^s(\mathbb{S})$, $s \geqslant 2$. Then there exists a unique $u_\epsilon \in C(\mathbb{R}_+; H^s(\mathbb{S})) \cap C^1(\mathbb{R}_+; H^{s-1}(\mathbb{S}))$, $s \geqslant 2$, to Eq. (2.1). Moreover, for each $t \geqslant 0$,*

$$\int_{\mathbb{S}} \left(\frac{\partial u_\epsilon}{\partial x}\right)^2 + 2\epsilon \int_0^t \int_{\mathbb{S}} \left(\frac{\partial^2 u_\epsilon}{\partial x^2}\right)^2 (s,x)\,dx\,ds = \big\|(u_\epsilon)_{0,x}\big\|_{L^2(\mathbb{S})}^2 \leqslant \|u_{0,x}\|_{L^2(\mathbb{S})}^2. \tag{2.4}$$

For the convenience of presentation, we will omit the subscripts in u_ϵ in the following proof.

Proof of Theorem 2.1. First, following the standard argument for a nonlinear parabolic equation, we can obtain that for $u_0 \in H^s(\mathbb{S})$, $s \geqslant 2$, there exists a positive constant $T > 0$ such that Eq. (2.2) has a unique solution $u = u(t,x) \in C([0,T]; H^s(\mathbb{S})) \cap C^1([0,T]; H^{s-1}(\mathbb{S}))$.

Second, we show that if T is the maximal existence time of the corresponding solution $u(t,x)$ of Eq. (2.2) with the initial data u_0, then the $H^s(\mathbb{S})$-norm of $u(t,\cdot)$ blows up if and only if

$$\limsup_{t\to T}\big\|u_x(t,\cdot)\big\|_{L^\infty(\mathbb{S})} = \infty.$$

Multiplying Eq. (2.3) by u_x and integrating over $\mathbb{S}$, we obtain

$$\frac{d}{dt}\int_{\mathbb{S}} u_x^2(t,x)\,dx = -2\epsilon \int_{\mathbb{S}} u_{xx}^2\,dx.$$

Integrating the above inequality over $(0,t)$, we get

$$\big\|u_x(t,\cdot)\big\|_{L^2(\mathbb{S})}^2 + 2\epsilon \int_0^t \int_{\mathbb{S}} u_{xx}^2\,dx\,dt = \|u_{0,x}\|_{L^2(\mathbb{S})}^2 \equiv 2a_{\epsilon,0}. \tag{2.5}$$

Applying the operator Λ^s to Eq. (2.2), multiplying by $\Lambda^s u$, and integrating over $\mathbb{S}$, we observe that

$$\frac{d}{dt}\|u\|^2_{H^s(\mathbb{S})} = -2(uu_x, u)_s + 2\left(u, -\partial_x A^{-1}\left(2\mu(u_0)u + \frac{1}{2}u_x^2\right)\right)_s + 2\epsilon(u_{xx}, u)_s.$$

By a direct calculation and Lemma 2.2 with $r = s$, we get

$$\begin{aligned}
\left|(uu_x, u)_s\right| &= \left|\left(\Lambda^s(u\partial_x u), \Lambda^s u\right)_0\right| \\
&= \left|\left(\left[\Lambda^s, u\right]\partial_x u, \Lambda^s u\right)_0 + \left(u\Lambda^s\partial_x u, \Lambda^s u\right)_0\right| \\
&\leqslant \left\|\left[\Lambda^s, u\right]\partial_x u\right\|_{L^2}\left\|\Lambda^s u\right\|_{L^2} + \frac{1}{2}\left|\left(u_x\Lambda^s u, \Lambda^s u\right)_0\right| \\
&\leqslant \left(c\|u_x\|_{L^\infty} + \frac{1}{2}\|u_x\|_{L^\infty}\right)\|u\|^2_{H^s} \\
&\leqslant c\|u_x\|_{L^\infty}\|u\|^2_{H^s}.
\end{aligned}$$

By (1.5) and Lemma 2.2 with $r = s-1$, we have

$$\begin{aligned}
\left\|A^{-1}\partial_x u\right\|_{H^s(\mathbb{S})} &= \left(\left\|A^{-1}\partial_x u\right\|_{L^2(\mathbb{S})} + \left\|\partial_x A^{-1}\partial_x u\right\|_{H^{s-1}(\mathbb{S})}\right) \\
&\leqslant 3\|u\|_{L^2(\mathbb{S})} + \left\|-u + \int_{\mathbb{S}} u\,dx\right\|_{H^{s-1}(\mathbb{S})} \\
&\leqslant 3\|u\|_{L^2(\mathbb{S})} + 2\|u\|_{H^{s-1}(\mathbb{S})} = 5\|u\|_{H^s(\mathbb{S})},
\end{aligned}$$

and

$$\left\|A^{-1}\partial_x u_x^2\right\|_{H^s(\mathbb{S})} \leqslant 5\left\|u_x^2\right\|_{H^s(\mathbb{S})} \leqslant 5\|u_x\|_{L^\infty(\mathbb{S})}\|u\|_{H^s(\mathbb{S})}.$$

It follows that

$$\left|\left(u, A^{-1}\partial_x u\right)_{H^s(\mathbb{S})}\right| \leqslant c\|u\|_{H^s(\mathbb{S})}\left\|A^{-1}\partial_x u\right\|_{H^s(\mathbb{S})} \leqslant c\|u\|^2_{H^s(\mathbb{S})},$$

and

$$\left|\left(u, A^{-1}\partial_x u_x^2\right)_{H^s(\mathbb{S})}\right| \leqslant c\|u\|_{H^s(\mathbb{S})}\left\|A^{-1}\partial_x u_x^2\right\|_{H^s(\mathbb{S})} \leqslant c\|u_x\|_{L^\infty(\mathbb{S})}\|u\|^2_{H^s(\mathbb{S})}.$$

In view of the above estimates, we obtain

$$2\left(u, -\partial_x A^{-1}\left(2\mu(u_0)u + \frac{1}{2}u_x^2\right)\right)_s \leqslant c\left(1 + \|u_x\|_{L^\infty(\mathbb{S})}\right)\|u\|^2_{H^s(\mathbb{S})}.$$

By $2\epsilon(u_{xx}, u)_s = -2\epsilon\|u_x\|_{H^s(\mathbb{S})} \leqslant 0$, we get

$$\frac{d}{dt}\|u\|^2_{H^s(\mathbb{S})} \leqslant c\left(1 + \|u_x\|_{L^\infty(\mathbb{S})}\right)\|u\|^2_{H^s(\mathbb{S})}.$$

Gronwall's inequality and the assumption of the theorem yield

$$\|u\|^2_{H^s(\mathbb{S})} \leqslant \exp\left(c\int_0^t \left(1 + \|u_x\|_{L^\infty(\mathbb{S})}\right) ds\right)\|u_0\|^2_{H^s(\mathbb{S})}.$$

Then, we have the second conclusion.

Next, we derive the a priori bound on $\|u_x(t,\cdot)\|_{L^\infty(\mathbb{S})}$. In view of (2.4) and Eq. (2.2), we deduce that

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{S}} u^2(t,x)\,dx &= -2\int_{\mathbb{S}} u\partial_x A^{-1}\left(2\mu(u_0)u + \frac{1}{2}u_x^2\right)dx + \epsilon\int_{\mathbb{S}} uu_{xx}\,dx \\
&\leqslant \int_{\mathbb{S}} u^2\,dx + \int_{\mathbb{S}}\left(\partial_x A^{-1}\left(2\mu(u_0)u + \frac{1}{2}u_x^2\right)\right)^2 dx \\
&= \int_{\mathbb{S}} u^2\,dx + \int_{\mathbb{S}}\left(g_x * \left(2\mu(u_0)u + \frac{1}{2}u_x^2\right)\right)^2 dx
\end{aligned}$$

$$\leqslant \int_{\mathbb{S}} u^2\,dx + \|g_x\|_{L^2(\mathbb{S})}^2 \left\| 2\mu(u_0)u + \frac{1}{2}u_x^2 \right\|_{L^1(\mathbb{S})}^2$$
$$\leqslant \left(8\mu^2(u_0)+1\right) \int_{\mathbb{S}} u^2(t,x)\,dx + a_{\epsilon,0}^2,$$

where we used $\|g_x\|_{L^2(\mathbb{S})} \leqslant 1$. By Gronwall's inequality we have

$$\int_{\mathbb{S}} u^2(t,x)\,dx \leqslant e^{(8\mu^2(u_0)+1)t}\left(\int_{\mathbb{S}} u_0^2\,dx + a_{\epsilon,0}^2\right). \tag{2.6}$$

In view of (2.5)–(2.6), we have

$$\left\|u(t,\cdot)\right\|_{H^1(\mathbb{S})} \leqslant \sqrt{C(t)}, \tag{2.7}$$

where $C(t) = e^{(8\mu^2(u_0)+1)t}(\int_{\mathbb{S}} u_0^2\,dx + a_{\epsilon,0}^2) + 2a_{\epsilon,0}$. Sobolev's imbedding theorem yields

$$\left\|u(t,\cdot)\right\|_{L^\infty(\mathbb{S})} \leqslant \frac{13}{12}\sqrt{C(t)} \equiv C_1(t). \tag{2.8}$$

Due to (2.8) and the Sobolev inequality, we only need to derive an a priori estimate on $\|u_{xx}(t,\cdot)\|_{L^2(\mathbb{S})}$. In view of Eq. (2.3), we get

$$\frac{d}{dt}\left\|u_{xx}(t,\cdot)\right\|_{L^2(\mathbb{S})}^2 + 2\epsilon\left\|u_{xxx}(t,\cdot)\right\|_{L^2(\mathbb{S})}^2 = 6\int_{\mathbb{S}} u u_{xx} u_{xxx}\,dx.$$

Integrating over $(0,t)$, we have

$$\begin{aligned}\left\|u_{xx}(t,\cdot)\right\|_{L^2(\mathbb{S})}^2 + 2\epsilon\int_0^t \left\|u_{xxx}(s,\cdot)\right\|_{L^2(\mathbb{S})}^2\,ds &= 6\int_0^t\int_{\mathbb{S}} u u_{xx} u_{xxx}\,dx\,ds + \left\|u_{0,xx}(t,\cdot)\right\|_{L^2(\mathbb{S})}^2 \\ &\leqslant \epsilon\int_0^t\int_{\mathbb{S}} u_{xxx}^2\,dx\,ds + C(\epsilon)C_1^2(t)\int_0^t\int_{\mathbb{S}} u_{xx}^2\,dx\,ds + \left\|u_{0,xx}(t,\cdot)\right\|_{L^2(\mathbb{S})}^2.\end{aligned}$$

Then, from (2.5) we obtain

$$\left\|u_{xx}(t,\cdot)\right\|_{L^2(\mathbb{S})}^2 + \epsilon\int_0^t \left\|u_{xxx}(s,\cdot)\right\|_{L^2(\mathbb{S})}^2\,ds \leqslant \frac{C(\epsilon)}{\epsilon}C_1^2(t)a_{\epsilon,0} + \left\|u_{0,xx}(t,\cdot)\right\|_{L^2(\mathbb{S})}^2.$$

Combing this with (2.7), we show that there exists a positive constant $C_2(\epsilon, t, \|u_{0,xx}(t,\cdot)\|_{L^2(\mathbb{S})})$ such that

$$\|u_x\|_{L^\infty(\mathbb{S})} \leqslant C\|u\|_{H^2(\mathbb{S})} \leqslant C_2(t) < +\infty, \quad \forall t > 0.$$

Finally, the global existence solution follows from these a priori estimates and the standard continuation argument. Furthermore, (2.4) holds on $[0,\infty)$. This completes the proof of Theorem 2.1. □

Remark 2.2. For given $\epsilon > 0$, we set $a_\epsilon(t) = \frac{1}{2}\int_{\mathbb{S}}(u_\epsilon)_x^2\,dx$, $a_{\epsilon,0} = \frac{1}{2}\int_{\mathbb{S}}(u_\epsilon)_{0,x}^2\,dx$ and $a_0 = \frac{1}{2}\|u_{0,x}\|_{L^2(\mathbb{S})}^2$. Then from the proof of Theorem 2.1, we see that

$$a_\epsilon(t) + \epsilon\int_0^t\int_{\mathbb{S}} (u_\epsilon)_{xx}^2\,dx\,dt \leqslant a_{\varepsilon,0} = \frac{1}{2}\left\|(u_\epsilon)_{0,x}\right\|_{L^2(\mathbb{S})}^2 \leqslant \frac{1}{2}\|u_{0,x}\|_{L^2(\mathbb{S})}^2 \equiv a_0.$$

3. Uniform a priori estimates

In this section, we derive the uniform one-sided super-norm estimate and the space–time higher integrability estimates on $\partial_x u_\epsilon(t,x)$, which are essential for our compactness argument. We denote $q_\epsilon(t,x) = \partial_x u_\epsilon(t,x)$ in the following text.

Lemma 3.1. *For fixed $T > 0$, $\forall (t,x) \in (0,T] \times \mathbb{S}$, we have*

$$\partial_x u_\epsilon(t,x) \leqslant \frac{2}{t} + \sqrt{2K(T)}. \tag{3.1}$$

Proof. From Eqs. (2.3) and (2.7), we get

$$\begin{aligned}\partial_t q_\epsilon + u_\epsilon \partial_x q_\epsilon + \frac{1}{2} q_\epsilon^2 - \epsilon \partial_x^2 q_\epsilon &= 2\mu(u_{\epsilon,0}) u_\epsilon - 2\mu^2(u_{\epsilon,0}) - a_\epsilon(t) \\ &\leqslant C_1(T) + 2\mu^2(u_{\epsilon,0}) + a_0 \equiv K(T).\end{aligned} \tag{3.2}$$

Let $f_\epsilon = f_\epsilon(t)$ be the solution of

$$\partial_t f_\epsilon + \frac{1}{2} f_\epsilon^2 = K(T), \qquad f_\epsilon(0) = \big\| (u_\epsilon)_{0,x} \big\|_{L^\infty(\mathbb{S})}^2. \tag{3.3}$$

The comparison principle for parabolic equations yields

$$q_\epsilon(t,x) \leqslant f_\epsilon(t), \quad \forall (t,x) \in (0,T] \times \mathbb{S}.$$

Consider the map $F(t) := \frac{2}{t} + \sqrt{2K(T)}$, $t \in [0,T]$. One observes that $\partial_t F + \frac{1}{2} F^2 - K(T) = \frac{2\sqrt{2K(T)}}{t} > 0$, so that $F(t)$ is a super-solution of (3.3). Therefore, the estimate (3.1) holds. □

Lemma 3.2. *Let $0 < \alpha < 1$, $T > 0$. Then there exists a positive constant C depending only on $\|u_0\|_{H^1(\mathbb{S})}$ and T, but independent of ϵ, such that*

$$\int_0^T \int_{\mathbb{S}} \big| \partial_x u_\epsilon(t,x) \big|^{2+\alpha} \leqslant C. \tag{3.4}$$

Proof. Consider the map $\theta(\xi) := \xi(|\xi|+1)^\alpha$, $\xi \in \mathbb{R}$, which was introduced in [4]. Obviously,

$$\begin{aligned}&\theta'(\xi) = \big((\alpha+1)|\xi|+1\big)\big(|\xi|+1\big)^{\alpha-1}, \\ &\big|\theta(\xi)\big| \leqslant |\xi|^{\alpha+1} + |\xi|, \qquad 0 < \theta'(\xi) \leqslant (\alpha+1)|\xi|+1, \qquad \big|\theta''(\xi)\big| \leqslant 2\alpha,\end{aligned} \tag{3.5}$$

and

$$\xi\theta(\xi) - \frac{1}{2}\xi^2\theta'(\xi) \geqslant \frac{1-\alpha}{2}\xi^2\big(|\xi|+1\big)^\alpha. \tag{3.6}$$

Multiplying (3.2) by $\theta'(q_\epsilon)$ and integrating over $\prod_T := [0,T] \times \mathbb{S}$, we obtain

$$\begin{aligned}\int_{\prod_T} q_\epsilon \theta(q_\epsilon)\,dx\,dt - \frac{1}{2}\int_{\prod_T} q_\epsilon^2 \theta'(q_\epsilon)\,dx\,dt &= \int_{\mathbb{S}} \big(\theta\big(q_\epsilon(T,x)\big) - \theta\big(q_\epsilon(0,x)\big)\big)\,dx + \int_{\prod_T} \big(a_\epsilon(t) + 2\mu^2(u_{\epsilon,0})\big)\theta'(q_\epsilon)\,dx\,dt \\ &\quad - 2\int_{\prod_T} \mu(u_{\epsilon,0}) u_\epsilon \theta'(q_\epsilon)\,dx\,dt + \epsilon \int_{\prod_T} \left(\frac{\partial q_\epsilon}{\partial x}\right)^2 \theta''(q_\epsilon)\,dx\,dt.\end{aligned} \tag{3.7}$$

By (3.6), we observe that

$$\begin{aligned}\int_{\prod_T} q_\epsilon \theta(q_\epsilon)\,dx\,dt - \frac{1}{2}\int_{\prod_T} q_\epsilon^2 \theta'(q_\epsilon)\,dx\,dt &= \int_{\prod_T} \left(q_\epsilon \theta(q_\epsilon) - \frac{1}{2} q_\epsilon^2 \theta'(q_\epsilon)\right) dx\,dt \\ &\geqslant \frac{1-\alpha}{2} \int_{\prod_T} q_\epsilon^2 \big(|q_\epsilon|+1\big)^\alpha\,dx\,dt.\end{aligned} \tag{3.8}$$

In view of $0 < \alpha < 1$, the first part of (3.5) and (2.5), using Hölder's inequality, we get

$$\begin{aligned}\int_{\mathbb{S}} \theta(q_\epsilon)\,dx &\leqslant \int_{\mathbb{S}} \big(|q_\epsilon|^{\alpha+1} + |q_\epsilon|\big)\,dx \\ &\leqslant \big\|q_\epsilon(t,\cdot)\big\|_{L^2(\mathbb{S})}^{\alpha+1} + \big\|q_\epsilon(t,\cdot)\big\|_{L^2(\mathbb{S})} \leqslant \|u_0\|_{H^1(\mathbb{S})}^{\alpha+1} + \|u_0\|_{H^1(\mathbb{S})}.\end{aligned} \tag{3.9}$$

By the second part of (3.5) and Remark 2.1, we obtain

$$\int\limits_{\prod_T} \big(a_\epsilon(t) + 2\mu^2(u_{\epsilon,0})\big)\theta'(q_\epsilon)\,dx\,dt \leqslant \big(2a_{\epsilon,0} + 2\mu^2(u_{\epsilon,0})\big)\int\limits_{\prod_T} \big(|q_\epsilon| + 1\big)\,dx\,dt$$
$$\leqslant \big(2a_{\epsilon,0} + 2\mu^2(u_{\epsilon,0})\big)\big(\|u_0\|_{H^1(\mathbb{S})} + 1\big)T. \tag{3.10}$$

From (2.7) and the second part of (3.5), we have

$$2\int\limits_{\prod_T} \mu(u_{\epsilon,0})u_\epsilon\theta'(q_\epsilon)\,dx\,dt \leqslant 2\mu(u_{\epsilon,0})C_1(t)\big(\|u_0\|_{H^1(\mathbb{S})} + 1\big)T. \tag{3.11}$$

Using (2.4) and the third part of (3.5), we deduce that

$$\epsilon\left|\int\limits_{\prod_T} \left(\frac{\partial q_\epsilon}{\partial x}\right)^2 \theta''(q_\epsilon)\,dx\,dt\right| \leqslant 2\alpha\epsilon\left|\int\limits_{\prod_T} \left(\frac{\partial q_\epsilon}{\partial x}\right)^2 dx\,dt\right| \leqslant \alpha\|u_0\|^2_{H^1(\mathbb{S})}. \tag{3.12}$$

From (3.7)–(3.12), we see that there exists a constant $c > 0$ depending only on $\|u_0\|_{H^1(\mathbb{S})}$, α and $T > 0$, but independent of ϵ, such that

$$\frac{1-\alpha}{2}\int\limits_{\prod_T} q_\epsilon^2\big(|q_\epsilon| + 1\big)^\alpha\,dx\,dt \leqslant c.$$

Then

$$\int_0^T\int_{\mathbb{S}} \left|\frac{\partial u_\epsilon}{\partial x}(t,x)\right|^{2+\alpha} dx\,dt \leqslant \int\limits_{\prod_T} q_\epsilon^2\big(|q_\epsilon| + 1\big)^\alpha\,dx\,dt \leqslant \frac{2c}{1-\alpha}.$$

This completes the proof of Lemma 3.2. □

4. Precompactness

In this section, we are now ready to obtain the necessary compactness of the viscous approximate solution $u_\epsilon(t,x)$. We start with the weak compactness in $L^\infty_{loc}(\mathbb{R}_+, H^1(\mathbb{S}))$. For convenience, we denote $q_\epsilon = \partial_x u_\epsilon$ and $\mu_\epsilon = \frac{1}{2}\int_{\mathbb{S}} q_\epsilon^2\,dx$.

Lemma 4.1. *Under the assumption of Theorem* 1.1, *there exist a subsequence $\{u_{\epsilon_k}(t,x), a_{\epsilon_k}(t)\}$ of the sequence $\{u_\epsilon(t,x), a_\epsilon(t)\}$ and functions $u(t,x)$, $\bar a(t)$ with $u \in L^\infty_{loc}(\mathbb{R}_+, H^1(\mathbb{S}))$ and $0 \leqslant \bar a(t) \leqslant a_0$ such that*

$$u_{\epsilon_k} \to u \quad \text{as } k \to \infty,$$

uniformly on any compact subset of $\mathbb{R}_+ \times \mathbb{R}$, and

$$a_\epsilon \to \bar a \quad \text{in } L^p(0,T),\ \text{as } k \to \infty,$$

for any $1 \leqslant p < \infty$.

Proof. For any fixed $T > 0$, we claim that $\{u_\epsilon(t,x)\}$ is uniformly bounded in $L^\infty((0,T), H^1(\mathbb{S}))$. From Theorem 2.1, we have

$$\|u_\varepsilon u_{\varepsilon x}\|_{L^2((0,T)\times\mathbb{S})} \leqslant C_1(t)\|u_{\varepsilon x}\|_{L^2((0,T)\times\mathbb{S})} \leqslant 2C_1(T)a_{\epsilon,0},$$

and

$$\left\|g_x * \left(2\mu(u_{\epsilon,0})u_\varepsilon + \frac{1}{2}u_{\varepsilon x}^2\right)\right\|^2_{L^2((0,T)\times\mathbb{S})} = \int_0^T \left\|g_x * \left(2\mu(u_{\epsilon,0})u_\varepsilon + \frac{1}{2}u_{\varepsilon x}^2\right)(t,\cdot)\right\|^2_{L^2(\mathbb{S})} dt$$
$$\leqslant \int_0^T \|g_x\|^2_{L^2(\mathbb{S})}\left\|2\mu(u_{\epsilon,0})u_\varepsilon + \frac{1}{2}u_{\varepsilon x}^2\right\|^2_{L^1(\mathbb{S})} dt$$
$$\leqslant \int_0^T \left(8\mu^2(u_{\epsilon,0})\int_{\mathbb{S}} u_\varepsilon^2\,dx + \frac{1}{2}\left(\int_{\mathbb{S}} u_{\varepsilon x}^2\,dx\right)\right)^2 dt$$
$$\leqslant T\big(8\mu^2(u_{\epsilon,0})C_1(T) + a^2_{\epsilon,0}\big),$$

and by Eq. (2.1) we have $\{\partial_t u_\epsilon(t,x)\}$ is uniformly bounded in $L^2((0,T)\times\mathbb{S})$. Thus, by Lemma 2.3, there exist $u \in L^\infty_{loc}(\mathbb{R}_+, H^1(\mathbb{S}))$ and a subsequence $\{u_{\epsilon_k}(t,x)\}$ such that $\{u_{\epsilon_k}(t,x)\}$ is weakly compact in $L^\infty([0,T],H^1(\mathbb{S}))$ and $\{u_\epsilon(t,x)\}$ converges to $u(t,x)$ in the space $H^1((0,T)\times\mathbb{S})$. Moreover, $u(t,x)$ is a continuous function.

Next, we turn to the compactness of $\{a_\epsilon(t)\}$. Since $\{a_\epsilon(t)\}$ are uniformly bounded on $\mathbb{R}_+$, so they are uniformly bounded in $L^1(0,T)$. In view of Theorem 2.1, we have $|\frac{d}{dt}a_\epsilon| = \epsilon\int_{\mathbb{S}} u^2_{\epsilon,xx}\,dx$ are uniformly bounded in $L^1(0,T)$. Thus, we deduce that $\{a_\epsilon\}\in W^{1,1}(0,T)$. Furthermore, by Sobolev's compact imbedding theorem $W^{1,1}(0,T)\hookrightarrow\hookrightarrow L^P(0,T)$, $1\leqslant p<+\infty$, there exist $\overline{a(t)}\in L^p(0,T)$ and a subsequence $\{a_{\epsilon_k}(t)\}$ such that $a_{\epsilon_k}\to\overline{a(t)}$ in $L^p(0,T)$, for any $1<p<\infty$. This completes the proof of Lemma 4.1. □

Remark 4.1. Let $\mu(u_\epsilon)=\int_{\mathbb{S}} u_\epsilon dx$. From Lemma 4.1, we see that

$$\mu(u_\epsilon)=\mu(u_{\epsilon,0})\to\mu_0=\int_{\mathbb{S}} u_0\,dx.$$

Remark 4.2. From Lemmas 3.2 and 4.1, we can deduce that there exist two functions $q\in L^p_{loc}(\mathbb{R}_+\times\mathbb{R})$ and $\overline{q^2}\in L^r_{loc}(\mathbb{R}_+\times\mathbb{R})$ such that

$$q_{\epsilon_k}\rightharpoonup q \quad \text{in } L^p_{loc}(\mathbb{R}_+\times\mathbb{R}),\qquad q^2_{\epsilon_k}\rightharpoonup\overline{q^2}\quad\text{in } L^r_{loc}(\mathbb{R}_+\times\mathbb{R}),\tag{4.1}$$

for every $1<p<3$, $1<r<\frac{3}{2}$. Moreover,

$$q^2(t,x)\leqslant\overline{q^2}(t,x),\quad \text{a.e. } (t,x)\in\mathbb{R}_+\times\mathbb{R}.\tag{4.2}$$

In view of (4.1), we conclude that for any $\eta\in C^1(\mathbb{R})$ with η' bounded, Lipschitz continuous on $\mathbb{R}$, $\eta(0)=0$ and any $1<p<3$, we have

$$\eta(q_{\epsilon_k})\rightharpoonup\overline{\eta(q)}\quad\text{in } L^p_{loc}(\mathbb{R}_+\times\mathbb{R}).\tag{4.3}$$

Lemma 4.2. *There exist a subsequence of the solution sequence $q_\varepsilon\equiv\partial_x u_\varepsilon$, and a family Young measure $\mu_{t,x}(\lambda)$, such that for any continuous functions $f(t,x,\lambda)=O(|\lambda|^r)$, and $\partial_\lambda f(t,x,\lambda)=O(|\lambda|^{r-1})$ as $|\lambda|\to\infty$ for $r<2$, and for any $\psi(x)\in L^s_c(\mathbb{R})$ with $\frac{1}{s}+\frac{r}{2}=1$, we have*

$$\lim_{\varepsilon\to0^+}\int_{\mathbb{R}} f\big(t,x,q_\varepsilon(t,x)\big)\psi(x)\,dx=\int_{\mathbb{R}}\overline{f(t,x,q)}\psi(x)\,dx,\tag{4.4}$$

uniformly on each compact subset of $[0,\infty)$, *where*

$$\overline{f(t,x,q)}:=\int_{\mathbb{R}} f(t,x,\lambda)\,d\mu_{t,x}(\lambda)\in C\big([0,\infty),L_{loc}^{\frac{r'}{r}(\mathbb{R})}\big),\tag{4.5}$$

for any $r'\in(r,2)$. Moreover, for any $T>0$, there hold

$$\lim_{\varepsilon\to0^+}\int_0^T\int_{\mathbb{R}} g(t,x,q_\varepsilon)\varphi\,dx\,dt=\int_0^T\int_{\mathbb{R}}\overline{g(t,x,q)}\varphi\,dx\,dt,\tag{4.6}$$

where the continuous function $g(t,x,\lambda)=O(|\lambda|^\iota)$, as $|\lambda|\to\infty$ for some $\iota<3$ and $\varphi(t,x)\in L^m(Q_T)$ with $\frac{\iota}{3}+\frac{1}{m}<1$. In addition,

$$\lambda\in L^\iota_{loc}\Big(\mathbb{R}^+\times\mathbb{R}\times\mathbb{R},dt\otimes dx\otimes d\mu_{t,x}(\lambda)\Big).\tag{4.7}$$

Proof. The proof is similar to that of Lemma 3 in [19]. □

Theorem 4.1. *Let $\mu_{t,x}(\lambda)$ be the Young measure given in Lemma 4.2. Then*

$$\mu_{t,x}(\lambda)=\delta_{\overline{q(t,x)}}(\lambda)\quad\text{for a.e. } (t,x)\in\mathbb{R}^+\times\mathbb{R}.$$

Proof. Step 1: Multiplying Eq. (2.3) by $\eta'(q_\epsilon)$, we get

$$\frac{\partial}{\partial t}\eta(q_\epsilon)+\frac{\partial}{\partial x}\big(u_\epsilon\eta(q_\epsilon)\big)=q_\epsilon\eta(q_\epsilon)-\frac{1}{2}q_\epsilon^2\eta'(q_\epsilon)+2\mu(u_{\epsilon,0})u_\epsilon\eta'(q_\epsilon)\\ -\big(a_\epsilon+2\mu^2(u_{\epsilon,0})\big)\eta'(q_\epsilon)+\epsilon\partial_x\big(\eta'(q_\epsilon)\partial_x q_\epsilon\big)-\epsilon\eta''(q_\epsilon)(\partial_x q_\epsilon)^2.$$

Note that $\{\sqrt{\epsilon}\partial_x q_\epsilon\}$ is uniformly bounded in $L^2(\mathbb{R}_+\times\mathbb{S})$ due to Theorem 2.1. Taking $\epsilon\to 0$, by Lemma 4.1, Remark 4.1 and (4.3) we obtain

$$\frac{\partial\overline{\eta(q)}}{\partial t}+\frac{\partial}{\partial x}\big(u\overline{\eta(q)}\big)\leqslant\overline{q\eta(q)}-\frac{1}{2}\overline{q^2\eta'(q)}+2\mu_0 u\overline{\eta'(q)}-\big(\bar a+2\mu_0^2\big)\cdot\overline{\eta'(q)},\tag{4.8}$$

in the sense of distributions on $\mathbb{R}_+\times\mathbb{S}$, here $\overline{f}$ is the limit of f_{ϵ_k} in the sense of distributions on $\mathbb{R}_+\times\mathbb{S}$.

Step 2: Using Eq. (2.3), Lemma 4.1 and Remarks 4.1–4.2, letting $\epsilon\to 0$, we have

$$\frac{\partial q}{\partial t}+u\frac{\partial}{\partial x}q=\frac{1}{2}\overline{q^2}-q^2+2\mu_0 u-2\mu_0^2-\bar a,\tag{4.9}$$

in the sense of distributions on $\mathbb{R}_+\times\mathbb{R}$.

Denote $q^\epsilon(t,x):=(q(t,\cdot)*\phi_\epsilon)(x)$. According to Lemma II.1 of [6], it follows from (4.9) that q^ϵ solves

$$\frac{\partial q^\epsilon}{\partial t}+u\frac{\partial q^\epsilon}{\partial x}=\left(\frac{1}{2}\overline{q^2}-q^2+2\mu_0 u-2\mu_0^2-\bar a\right)*\phi_\epsilon+\tau_\epsilon,\tag{4.10}$$

where the error τ_ϵ tends to zero in $L^1_{loc}(\mathbb{R}_+\times\mathbb{R})$. Multiplying (4.10) by $\eta'(q^\epsilon)$, we get

$$\frac{\partial\eta(q^\epsilon)}{\partial t}+\frac{\partial}{\partial x}\big(u\eta(q^\epsilon)\big)=\left(\left(\frac{1}{2}\overline{q^2}-q^2+2\mu_0 u-2\mu_0^2-\bar a\right)*\phi_\epsilon\right)\eta'(q^\epsilon)+q\eta(q^\epsilon)+\tau_\epsilon\eta'(q^\epsilon).\tag{4.11}$$

Using the boundedness of η, η' and sending $\epsilon\to 0$ in (4.11), we obtain

$$\frac{\partial\eta(q)}{\partial t}+\frac{\partial}{\partial x}\big(u\eta(q)\big)=q\eta(q)+\left(-q^2+\frac{1}{2}\overline{q^2}+2\mu_0 u-2\mu_0^2-\bar a\right)\eta'(q).\tag{4.12}$$

Subtracting (4.12) from (4.8) yields

$$\frac{\partial}{\partial t}\big(\overline{\eta(q)}-\eta(q)\big)+\frac{\partial}{\partial x}\big(u\big(\overline{\eta(q)}-\eta(q)\big)\big)\\ \leqslant\int_{\mathbb{R}}\left\{\lambda\eta(\lambda)-\frac{1}{2}\eta'(\lambda)\lambda^2\right\}d\mu_{t,x}(\lambda)\\ +\frac{1}{2}\eta'(q)q^2-q\eta(q)-\frac{1}{2}\eta'(q)\big(\overline{q^2}-q^2\big)+\big(2\mu_0 u-2\mu_0^2-\bar a\big)\eta'(q).\tag{4.13}$$

Next we will apply (4.13) to a family of suitably chosen E's to deduce that the Young measure $\mu_{t,x}(\lambda)$ must be a Dirac measure.

Step 3: Define

$$h_+=\max(h,0),\qquad h_-=\min(h,0),$$
$$\eta_R(\lambda)=\begin{cases}\frac{1}{2}\lambda^2, & |\lambda|\leqslant R,\\ R|\lambda|-\frac{1}{2}R^2, & |\lambda|>R,\end{cases}$$
$$\eta_R^+(\lambda):=\eta_R(\lambda)\chi_{[0,\infty)}(\lambda),\qquad \eta_R^-(\lambda):=\eta_R(\lambda)\chi_{(-\infty,0]}(\lambda),$$

then

$$\eta_R^+(\lambda)=\frac{1}{2}(\lambda_+)^2-\frac{1}{2}(R-\lambda)^2\chi_{(R,\infty)}(\lambda),\qquad \big(\eta_R^+\big)'(\lambda)=\lambda_++(R-\lambda)\chi_{(R,\infty)}(\lambda),\tag{4.14}$$

$$\eta_R^-(\lambda)=\frac{1}{2}(\lambda_-)^2-\frac{1}{2}(R+\lambda)^2\chi_{(-\infty,-R)}(\lambda),\qquad \big(\eta_R^-\big)'(\lambda)=\lambda_--(R+\lambda)\chi_{(-\infty,-R)}(\lambda).\tag{4.15}$$

By Lemma 2.4 and results in [4] or [19], for each $R>0$ we have

$$\lim_{t\to 0^+}\int_{\mathbb{S}}\big(\overline{\eta_R^\pm(q)}(t,x)-\eta_R^\pm\big(q(t,x)\big)\big)\,dx=0.$$

Similar to the arguments in [9], there holds

$$\lim_{t\to 0^+}\int_{\mathbb{S}} q^2(t,x)\,dx=\lim_{t\to 0^+}\int_{\mathbb{S}} \overline{q^2}(t,x)\,dx=\int_{\mathbb{S}} u_{0,x}^2(x)\,dx. \tag{4.16}$$

Step 4: We now apply (4.13) to the entropy η_R^+ to obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\big(\overline{\eta_R^+(q)}-\eta_R^+(q)\big)+\frac{\partial}{\partial x}\big(u\big[\overline{\eta_R^+(q)}-\eta_R^+(q)\big]\big)\\
&\quad\leqslant \frac{R}{2}\left\{\int_{\mathbb{R}}\lambda(\lambda-R)\chi_{(R,+\infty)}\,d\mu_{t,x}(\lambda)-q(q-R)\chi_{(R,+\infty)}\right\}\\
&\qquad-\frac{1}{2}\big(\overline{q^2}-q^2\big)\big(\eta_R^+\big)'(q)-\big(\bar a+2\mu_0^2-2\mu_0 u\big)\big(\overline{\big(\eta_R^+\big)'(q)}-\big(\eta_R^+\big)'(q)\big).
\end{aligned}\tag{4.17}$$

Since η_R^+ is increasing, by (4.2), we get

$$-\frac{1}{2}\big(\overline{q_+^2}-q_+^2\big)\big(\eta_R^+\big)'(q)\leqslant 0. \tag{4.18}$$

Note that both $q(t,x)$ and $q_\varepsilon(t,x)$ are bounded above by $\frac{2}{t}+\sqrt{2K(T)}$. Thus $\sup\mu_{t,x}(\cdot)\subset(-\infty,\frac{2}{t}+\sqrt{2K(T)})$. Hence for $R>\frac{2}{t}+\sqrt{2K(T)}$, i.e. $R-\sqrt{2K(T)}<t<T$, (4.17) becomes

$$\begin{aligned}
&\frac{\partial}{\partial t}\big(\overline{\eta_R^+(q)}-\eta_R^+(q)\big)+\frac{\partial}{\partial x}\big(u\big[\overline{\eta_R^+(q)}-\eta_R^+(q)\big]\big)\\
&\quad\leqslant -\frac{1}{2}\big(\overline{q^2}-q^2\big)\big(\eta_R^+\big)'(q)-\big(\bar a+2\mu_0^2-2\mu_0 u\big)\big(\overline{\big(\eta_R^+\big)'(q)}-\big(\eta_R^+\big)'(q)\big).
\end{aligned}\tag{4.19}$$

Let $\Omega_R:=(\frac{2}{R-\sqrt{2K(T)}},T)\times\mathbb{R}$. In view of (4.14), we obtain

$$\begin{aligned}
\overline{\eta_R^+(q)}-\eta_R^+(q)&=\frac{1}{2}\overline{(q_+)^2}-\frac{1}{2}(q_+)^2-\frac{1}{2}\left\{\int_{\mathbb{R}}(\lambda-R)^2\chi_{(R,+\infty)}\,d\mu_{t,x}(\lambda)-(q-R)^2\chi_{(R,+\infty)}\right\}\\
&=\frac{1}{2}\overline{(q_+)^2}-\frac{1}{2}(q_+)^2,\\
\overline{\big(\eta_R^+\big)'(q)}-\big(\eta_R^+\big)'(q)&=\overline{q_+}-q_++\frac{1}{2}\left\{\int_{\mathbb{R}}(R-\lambda)\chi_{(R,+\infty)}\,d\mu_{t,x}(\lambda)-(R-q)\chi_{(R,+\infty)}\right\}.
\end{aligned}$$

From (4.19), and integrating over $(\frac{2}{R-\sqrt{2K(T)}},T)\times\mathbb{S}$, then we deduce

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{S}}\big(\overline{(q_+)^2}(t,x)-(q_+)^2(t,x)\big)\,dx\\
&\quad\leqslant\int_{\mathbb{S}}\big(\overline{(q_+)^2}-(q_+)^2\big)\left(\frac{2}{R-\sqrt{2K(T)}},x\right)dx\\
&\qquad-\int_{\frac{2}{R-\sqrt{2K(T)}}}^{t}\int_{\mathbb{S}}\big(\bar a+2\mu_0^2-2\mu_0 u\big)\big(\overline{q_+}-q_+\big)\,dx\,ds\\
&\qquad+\int_{\frac{2}{R-\sqrt{2K(T)}}}^{t}\int_{\mathbb{S}}\big(\bar a+2\mu_0^2-2\mu_0 u\big)\left\{\int_{\mathbb{R}}(R-\lambda)\chi_{(R,+\infty)}\,d\mu_{t,x}(\lambda)-(R-q)\chi_{(R,+\infty)}\right\}dx\,ds.
\end{aligned}\tag{4.20}$$

Using (4.16) and sending $R\to\infty$ in (4.20), we get

$$\frac{1}{2}\int_{\mathbb{S}}\big(\overline{(q_+)^2}-(q_+)^2\big)(t,x)\,dx\leqslant-\int_0^t\int_{\mathbb{S}}\big(\bar a+2\mu_0^2\big)\big(\overline{q_+}(s,x)-q_+\big)\,dx\,ds+2\mu_0\int_0^t\int_{\mathbb{S}}u\big(\overline{q_+}(s,x)-q_+\big)\,dx\,ds. \tag{4.21}$$

Step 5: We now apply (4.13) and use the entropy η_R^- to get

$$\begin{aligned}&\frac{\partial}{\partial t}\big(\overline{\eta_R^-(q)}-\eta_R^-(q)\big)+\frac{\partial}{\partial x}\big(u\big[\overline{\eta_R^-(q)}-\eta_R^-(q)\big]\big)\\&\quad\leqslant -\frac{R}{2}\bigg\{\int_{\mathbb{R}}\lambda(\lambda+R)\chi_{(-\infty,-R)}\,d\mu_{t,x}(\lambda)-q(q+R)\chi_{(-\infty,-R)}\bigg\}\\&\qquad-\frac{1}{2}\big(\overline{q^2}-q^2\big)\big(\eta_R^-\big)'(q)-\big(\bar a+2\mu_0^2-2\mu_0 u\big)\big(\overline{(\eta_R^-)'(q)}-\big(\eta_R^-\big)'(q)\big).\end{aligned}\tag{4.22}$$

Since $-R\leqslant(\eta_R^-)'\leqslant 0$, by (4.2) we have

$$-\frac{1}{2}\big(\overline{q^2}-q^2\big)\big(\eta_R^-\big)'(q)\leqslant\frac{R}{2}\big(\overline{q^2}-q^2\big).\tag{4.23}$$

Substituting (4.23) into (4.22) and integrating over $(0,t)\times\mathbb{S}$ yield

$$\begin{aligned}&\int_{\mathbb{S}}\big(\overline{\eta_R^-(q)}(t,x)-\eta_R^-\big(q(t,x)\big)\big)\,dx\\&\quad\leqslant-\frac{R}{2}\int_0^t\int_{\mathbb{S}}\big[\lambda(\lambda+R)\chi_{(-\infty,-R)}\,d\mu_{t,x}(\lambda)-q(q+R)\chi_{(-\infty,-R)}\big]\,dx\,ds\\&\qquad+\frac{R}{2}\int_0^t\int_{\mathbb{S}}\big(\overline{q^2}-q^2\big)\,dx\,ds-\int_0^t\int_{\mathbb{S}}\big(\bar a+2\mu_0^2\big)\big(\overline{(\eta_R^-)'(q)}-\big(\eta_R^-\big)'(q)\big)\,dx\,ds\\&\qquad+2\mu_0\int_0^t\int_{\mathbb{S}}u\big(\overline{(\eta_R^-)'(q)}-\big(\eta_R^-\big)'(q)\big)\,dx\,ds.\end{aligned}\tag{4.24}$$

Applying the identity $\frac{R}{2}(R+q)^2-\frac{R}{2}q(R+q)=\frac{R^2}{2}(R+q)$ we deduce

$$\begin{aligned}&\int_{\mathbb{S}}\big(\overline{\eta_R^-(q)}(t,x)-\eta_R^-\big(q(t,x)\big)\big)\,dx\\&\quad\leqslant\frac{R^2}{2}\int_0^t\int_{\mathbb{S}}\big[(\lambda+R)\chi_{(-\infty,-R)}\,d\mu_{t,x}(\lambda)-(q+R)\chi_{(-\infty,-R)}\big]\,dx\,ds\\&\qquad+\frac{R}{2}\int_0^t\int_{\mathbb{S}}\big(\overline{q^2}-q^2\big)\,dx\,ds-\int_0^t\int_{\mathbb{S}}\big(\bar a+2\mu_0^2\big)\big(\overline{(\eta_R^-)'(q)}-\big(\eta_R^-\big)'(q)\big)\,dx\,ds\\&\qquad+2\mu_0\int_0^t\int_{\mathbb{S}}u\big(\overline{(\eta_R^-)'(q)}-\big(\eta_R^-\big)'(q)\big)\,dx\,ds.\end{aligned}\tag{4.25}$$

Step 6: Using (4.15), we have the trivial identity

$$\overline{\eta_R^-(q)}-\eta_R^-(q)=\frac{1}{2}\big(\overline{(q_-)^2}-(q_-)^2\big)-\frac{1}{2}\bigg(\int(\lambda+R)^2\chi_{(-\infty,-R)}d\mu_{t,x}(\lambda)-(q+R)^2\chi_{(-\infty,-R)}\bigg).$$

Note that $\overline{q^2}-q^2=\overline{q_+^2}-q_+^2+\overline{q_-^2}-q_-^2$. Adding (4.21) and (4.25) yields

$$\begin{aligned}&\int_{\mathbb{S}}\bigg(\frac{1}{2}\big[\overline{(q_+)^2}-(q_+)^2\big]+\overline{\eta_R^-(q)}(t,x)-\eta_R^-(q)(t,x)\bigg)\,dx\\&\quad\leqslant\frac{R^2}{2}\int_0^t\int_{\mathbb{S}}\big[(\lambda+R)\chi_{(-\infty,-R)}\,d\mu_{t,x}(\lambda)-(q+R)\chi_{(-\infty,-R)}\big]\,dx\,ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{R}{2}\int_0^t\int_{\mathbb{S}}\big[\overline{(q_+)^2}-(q_+)^2+\overline{\eta_R^-(q)}-\eta_R^-(q)\big]\,dx\,ds\\
&-\big(\bar a+2\mu_0^2\big)\int_0^t\int_{\mathbb{S}}\big(\overline{q_+}-q_++\overline{\big(\eta_R^-\big)'(q)}-\big(\eta_R^-\big)'(q)\big)\,dx\,ds\\
&+2\mu_0\int_0^t\int_{\mathbb{S}}u(s,x)\big(\overline{q_+}-q_++\overline{\big(\eta_R^-\big)'(q)}-\big(\eta_R^-\big)'(q)\big)\,dx\,ds. \qquad (4.26)
\end{aligned}
$$

Note that

$$
\begin{aligned}
0&\leqslant\big(\overline{q_+}-q_++\overline{\big(\eta_R^-\big)'(q)}-\big(\eta_R^-\big)'(q)\big)\\
&=-\bigg[\int_{\mathbb{S}}(\lambda+R)\chi_{(-\infty,-R)}\,d\mu_{t,x}(\lambda)-(q+R)\chi_{(-\infty,-R)}\bigg].
\end{aligned}
$$

By the proof of Theorem 2.1, we see that

$$
\|u\|_{L^\infty(\mathbb{S})}\leqslant C(T).
$$

Since $\lambda\to(R+\lambda)\chi_{(-\infty,-R)}(\lambda)$ is concave and choose R large enough, it yields that

$$
\begin{aligned}
&\frac{R^2}{2}\int_0^t\int_{\mathbb{S}}\big[(\lambda+R)\chi_{(-\infty,-R)}\,d\mu_{t,x}(\lambda)-(q+R)\chi_{(-\infty,-R)}\big]\,dx\,ds\\
&\quad+2\mu_0\int_0^t\int_{\mathbb{S}}u(s,x)\big(\overline{q_+}-q_++\overline{\big(\eta_R^-\big)'(q)}-\big(\eta_R^-\big)'(q)\big)\,dx\,ds\\
&\leqslant\bigg(\frac{R^2}{2}-C(T)\bigg)\int_0^t\int_{\mathbb{S}}\bigg[\int_{\mathbb{S}}(\lambda+R)\chi_{(-\infty,-R)}\,d\mu_{t,x}(\lambda)-(q+R)\chi_{(-\infty,-R)}\bigg]\,dx\,ds\\
&\leqslant 0. \qquad (4.27)
\end{aligned}
$$

Then, from (4.26)–(4.27), we obtain

$$
\begin{aligned}
0&\leqslant\int_{\mathbb{S}}\bigg(\frac12\big[\overline{(q_+)^2}-(q_+)^2\big]+\big[\overline{\eta_R^-(q)}-\eta_R^-(q)\big]\bigg)(t,x)\,dx\\
&\leqslant R\int_0^t\int_{\mathbb{R}}\bigg(\frac12\big[\overline{(q_+)^2}-(q_+)^2\big]+\big[\overline{\eta_R^-(q)}-\eta_R^-(q)\big]\bigg)\,dx\,ds.
\end{aligned}
$$

Using the Gronwall inequality and (4.16), we conclude that

$$
\int_{\mathbb{S}}\bigg(\frac12\big[\overline{(q_+)^2}-(q_+)^2\big]+\big[\overline{\eta_R^-(q)}-\eta_R^-(q)\big]\bigg)(t,x)\,dx=0.
$$

By the Fatou lemma and (4.2), sending $R\to\infty$ yields

$$
0\leqslant\int_{\mathbb{S}}\big(\overline{q^2}-q^2\big)(t,x)\,dx\leqslant 0,\quad t>0.
$$

Consequently,

$$
\mu_{t,x}(\lambda)=\delta_{\overline{q(t,x)}}(\lambda)\quad\text{for a.e. }(t,x)\in\mathbb{R}^+\times\mathbb{R}.
$$

This completes the proof of Theorem 4.1. □

Theorem 4.2. *Under the assumption of Theorem* 1.1, *there holds*

$$q_\epsilon(t,x) \to q(t,x) = u_x(t,x) \quad \text{a.e. on } \mathbb{R}_+ \times \mathbb{R}. \tag{4.28}$$

Proof. In view of Lemma 3.2 and Theorem 4.1, we deduce $q_\epsilon(t,x) \to q(t,x) = u_x(t,x)$ a.e. on $\mathbb{R}_+ \times \mathbb{S}$, we get the conclusion (4.28). □

And $q_\epsilon(t,\cdot)$ and $q(t,\cdot)$ being in $L^2(\mathbb{S})$, we obtain that

$$(u_\epsilon)_x \to u_x, \quad \text{in } L^2_{loc}(\mathbb{R}_+ \times \mathbb{S}). \tag{4.29}$$

With all the preparations given in the previous, it is easy to see that the proof of Theorem 1.1 is completed. Let u be the limit of the viscous approximate solutions u_ϵ as $\epsilon \to 0^+$. From Theorem 2.1 and Lemma 4.1, it follows that $u \in C([0,+\infty) \times \mathbb{R}) \cap L^\infty_{loc}(\mathbb{R}_+, H^1(\mathbb{S}))$ and (1.9)–(1.10) hold. From Lemma 4.1, Theorem 4.2 and (4.29), we deduce that $u_\epsilon \to u$, $\mu(u_\epsilon) \to \mu(u)$, $((u_\epsilon)_x)^2 \to (u_x)^2$, $u_\epsilon(u_\epsilon)_{xx} \to uu_x$ and $a_\epsilon(t) = \frac{1}{2}\int_{\mathbb{S}} (u_\epsilon)_x^2(t,x)\,dx \to a(t) = \frac{1}{2}\int_{\mathbb{S}} u_x^2(t,x)\,dx$, in the sense of distributions on $\mathbb{R}_+ \times \mathbb{R}$. Then we obtain that

$$\partial_x g * \left(2\mu(u_\epsilon)u_\epsilon + \frac{1}{2}(u_\epsilon)_x^2\right) \to \partial_x g * \left(2\mu(u)u + \frac{1}{2}u_x^2\right).$$

Thus we see that u is a dissipative weak solution to (1.1). This completes the proof of Theorem 1.1.

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