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Existence of positive solutions to the Schrödinger–Poisson system without compactness conditions[☆]

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ABSTRACT

In this paper, we discuss the existence of a positive radial solution to a generalized Schrödinger–Poisson system without compactness conditions. By the method of the combination of a cut-off function, a monotonicity trick and a Pohozaev type identity, we obtain the boundedness of a Palais–Smale sequence.

1. Introduction

In this paper, we consider the existence of positive solutions to the following nonlinear generalized Schrödinger–Poisson system

$$\begin{cases} -\Delta u + u + q\phi f(u) = g(u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = 2qF(u), & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $q \geqslant 0$ is a parameter, f and g satisfy the following conditions:

(f) $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and there exists $c > 0$ such that $|f(t)| \leqslant c(|t| + |t|^\alpha)$ for all $t \in \mathbb{R}_+ = [0, \infty)$, where $\alpha \in (2, 4)$;
(g_1) $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ and there exists $c_1 > 0$ such that $|g(t)| \leqslant c_1(1 + |t|^{p-1})$ for all $t \in \mathbb{R}_+$ and some $p \in (2, 6)$;
(g_2) $\lim_{t\to 0^+} g(t)/t = 0$;
(g_3) $\lim_{t\to\infty} g(t)/t = \infty$.

When the function $f(t) = t$, this system represents the well known Schrödinger–Poisson system

$$\begin{cases} -\Delta u + u + q\phi u = g(u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = qu^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

Such a system, also known as the nonlinear Schrödinger–Maxwell equations, arises in many mathematical physics context. Indeed, according to a classical model, the interaction of a charged particle with an electromagnetic field can be described by coupling the nonlinear Schrödinger and the Poisson equation (we refer the reader to [8] for details on the physical aspects). Recently, the problem (1.2) has been studied widely by using the modern variational method and the critical point

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theory under various assumptions; see [1–3,6,7,18,10,11,17,20,21,25,27] and the references therein. The greatest part of the literature focuses on the study of existence, nonexistence of solutions, multiplicity of solutions, ground states, radial and non-radial solutions of such a system with very special nonlinearity $g(t) = |t|^{p-1}t$ (see [2,7,18,10,11,17,21]). In [18], the author proved the existence of a nontrivial radial solution of (1.2) when $p \in (3, 5)$. The same result was obtained in [11] for $p \in [3, 5)$. By using a Pohozaev type identity, in [10], the authors proved that the problem (1.2) has no nontrivial solutions for $p \notin (1, 5)$. This result was completed in [21], where by discussing the behavior of the energy functional and the scopes of the parameters, the author showed that if $p \leqslant 2$, the problem (1.2) does not admit any nontrivial solution, and if $p \in (2, 5)$, there exists a nontrivial radial solution of (1.2). The author also pointed that the case $p = 2$ turns out to be the critical. On the basis of the result in [21], in [2], the authors discussed the existence of multiplicity solutions to the problem (1.2) by using the method of critical point theory. In [12], the existence of non-radially symmetric solutions was obtained for $p \in (3, 5)$.

In [3,6] the authors discussed the problem (1.2) with the nonlinearity g satisfying the general hypotheses introduced by Berestycki and Lions [9]. In [3], by using a concentration and compactness argument, the author proved the existence of a nontrivial non-radial solution to the problem (1.2). In [6], the authors discussed the existence of a nontrivial radial solution by using the method of a cut-off function.

There are also many references which investigated the well-known Schrödinger–Poisson system in a bounded domain; see [4,5,8,19,22,23]. In [4], the authors considered the following problem involving the critical growing nonlinearity

$$\begin{cases} -\Delta u = \lambda u + q|u|^3 u\phi, & \text{in } B_R, \\ -\Delta \phi = q|u|^5, & \text{in } B_R, \\ u = \phi = 0, & \text{on } \partial B_R, \end{cases}$$

where B_R is a ball in $\mathbb{R}^3$ centered at the origin and with radius R. They proved the existence and nonexistence results by discussing the scope of the parameter λ.

By using the method of a cut-off function and the variational arguments, the authors in [5] studied the following Schrödinger–Poisson system in a bounded domain:

$$\begin{cases} -\Delta u + \varepsilon q\phi f(u) = \eta|u|^{p-1}u, & \text{in } \Omega, \\ -\Delta \phi = 2qF(u), & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $1 < p < 5$, $q > 0$, $\varepsilon, \eta = \pm 1$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $F(t) = \int_0^t f(s)ds$. They prove the existence and multiplicity results assuming on f a subcritical growth condition and also they consider the existence and nonexistence results under the critical case.

To our knowledge, there were few papers considering the more general Schrödinger–Poisson system (1.1) in $\mathbb{R}^3$, where f is subcritical and $g(s) \neq |s|^{p-1}s$ is also general. We will use the variational method and the trick of a cut-off to discuss the problem (1.1). Our main idea is somehow similar to that of [5]. But our nonlinearity $g(s)$ is not equal to $|s|^{p-1}s$ and also does not satisfy the following global Ambrosetti–Rabinowitz growth hypothesis

(g_4) there exists $\mu > 2$ such that $0 < \mu G(t) \leqslant tg(t)$ for all $t \in \mathbb{R}$.

Although, we can verify that the corresponding functional has the mountain pass geometry under our weaker conditions (g_1)–(g_3). But the boundedness of Palais–Smale cannot be obtained only by the standard argument. So, motivated by the method in [6,5], we combine the method of a monotonicity trick [24] and a cut-off function to get the bounded sequence which is the main step for the problem (1.1).

Our main result is as follows.

Theorem 1.1. *If f satisfies (f) and g satisfies (g_1), (g_2) and (g_3), then there exists $q_0 > 0$ such that, for any $q \in [0, q_0)$, the problem (1.1) has at least a positive radial solution $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$.*

The paper is organized as follows. In Section 2, we give preliminaries and the variational framework to our problem. In order to obtain the boundedness of Palais–Smale sequences, a Pohozaev type identity is also given in this section. In Section 3, we give the proof of Theorem 1.1.

Throughout the paper, we denote by C_i various positive constants which may vary from line to line.

2. Preliminaries

Let $H^1(\mathbb{R}^3)$ be the usual Sobolev space equipped with the inner product and norm

$$(u, v) = \int_{\mathbb{R}^3} [\nabla u \cdot \nabla v + uv], \quad \|u\| = (u, u)^{1/2}.$$

We denote by $|\cdot|_s$ the usual $L^s(\mathbb{R}^3)$ norm. Then we have that $H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ continuously for $s \in [2, 2^*]$. Hence there exists γ_s such that

$$|u|_s \leqslant \gamma_s \|u\|, \quad u \in H^1(\mathbb{R}^3).$$

Let $H = H_r^1(\mathbb{R}^3)$ be the subspace of $H^1(\mathbb{R}^N)$ containing only the radial functions. Then $H \hookrightarrow L^s(\mathbb{R}^3)$ compactly for $s \in (2, 6)$ [26, Corollary 1.26, p.18]. Let $\mathcal{D}^{1,2}(\mathbb{R}^3)$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_{\mathcal{D}^{1,2}} = (\int_{\mathbb{R}^3} |\nabla u|^2)^{1/2}$. It is well known that $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ continuously. Let $S > 0$ be the embedding constant, i.e,

$$|u|_6^2 \leqslant S^{-1}\|u\|^2_{\mathcal{D}^{1,2}}, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^3).$$

In this paper, since we concern the existence of positive solutions to (1.1), we assume that $f(t) = g(t) = 0$ for $t < 0$.

By standard arguments, we can prove that the problem (1.1) is variational and that the associated C^1 functional $\varepsilon_q : H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ is given by

$$\varepsilon_q(u, \phi) = \frac{1}{2}\int_{\mathbb{R}^3} [|\nabla u|^2 + u^2] - \frac{1}{4}\int_{\mathbb{R}^3} |\nabla \phi|^2 + q\int_{\mathbb{R}^3} F(u)\phi - \int_{\mathbb{R}^3} G(u), \tag{2.1}$$

where $G(t) = \int_0^t g(s)ds$. Similar to the problem (1.2), by using the Lax–Milgram theorem, we can obtain that the second equation has a unique solution ϕ_u, substituting ϕ_u to the first equation of the problem (1.1), then the problem can be transformed to a one variable equation. In fact, we first have the following lemma.

Lemma 2.1. *By condition (f), for any $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ solution of*

$$-\Delta\phi = 2qF(u), \quad \text{in } \mathbb{R}^3. \tag{2.2}$$

Moreover

(i) $\|\phi_u\|^2_{\mathcal{D}^{1,2}} = 2q\int_{\mathbb{R}^3} F(u)\phi_u$;
(ii) $\phi_u \geqslant 0$;
(iii) $\|\phi_u\|_{\mathcal{D}^{1,2}} \leqslant qC(\|u\|^2 + \|u\|^{1+\alpha})$;
(iv) $\int_{\mathbb{R}^3} F(u)\phi_u \leqslant q\widetilde{C}(\|u\|^4 + \|u\|^{2(1+\alpha)})$, *where $\widetilde{C}$ is only dependent on $c, S, \gamma_{12/5}$ and $\gamma_{6(1+\alpha)/5}$;*
(v) *if u is a radial function then ϕ_u is radial, too.*

Proof. By condition (f), we can get that there exists $C_1 > 0$ such that

$$F(t) \leqslant C_1(|t|^2 + |t|^{1+\alpha}), \quad t \in \mathbb{R}.$$

Then, for any $u \in H^1(\mathbb{R}^3)$, we have

$$|F(u)|_{6/5} \leqslant C_1(|u|^2_{12/5} + |u|^{1+\alpha}_{6(1+\alpha)/5}) \leqslant C_2(\|u\|^2 + \|u\|^{1+\alpha}). \tag{2.3}$$

Now, for given $u \in H^1(\mathbb{R}^3)$, a linear functional $\Phi : \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ is defined as

$$\Phi(v) = \int_{\mathbb{R}^3} 2qF(u)v, \quad v \in \mathcal{D}^{1,2}(\mathbb{R}^3).$$

By (2.3), we have

$$|\Phi(v)| \leqslant 2q|F(u)|_{6/5}|v|_6 \leqslant 2qS^{-1/2}C_2(\|u\|^2 + \|u\|^{1+\alpha})\|v\|_{\mathcal{D}^{1,2}}. \tag{2.4}$$

Hence, $\Phi : \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ is continuous. Then, by the Lax–Milgram theorem, there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \nabla\phi_u \cdot \nabla v = 2q\int_{\mathbb{R}^3} F(u)v, \quad v \in \mathcal{D}^{1,2}(\mathbb{R}^3).$$

Therefore, ϕ_u is a weak solution of (2.2), and by [13, Theorem 9.9, p.230] or [16, Lemma 2.3] the integral expression of ϕ_u is in the form

$$\phi_u(x) = 2q\int_{\mathbb{R}^3} \frac{F(u(y))}{|x-y|}dy, \quad x \in \mathbb{R}^3, \forall u \in H^1(\mathbb{R}^3).$$

Moreover, $\phi_u > 0$ when $u \neq 0$. By (2.2) and (2.4), the relations

$$\|\phi_u\|_{\mathcal{D}^{1,2}} \leqslant qC_3(\|u\|^2 + \|u\|^{1+\alpha}),$$

and

$$\int_{\mathbb{R}^3} F(u)\phi_u = \frac{1}{2q}\|\phi_u\|^2_{\mathcal{D}^{1,2}} \leqslant q\widetilde{C}(\|u\|^4 + \|u\|^{2(1+\alpha)})$$

hold, where $\widetilde{C}$ is only dependent on $c, S, \gamma_{12/5}$ and $\gamma_{6(1+\alpha)/5}$. The proof is completed. □

Now by Lemma 2.1 and (2.1), we can prove that $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of (1.1) if and only if $u \in H^1(\mathbb{R}^3)$ is a critical point of the one variable functional defined as

$$J_q(u) = \frac{1}{2}\|u\|^2 + \frac{q}{2}\int_{\mathbb{R}^3} F(u)\phi_u - \int_{\mathbb{R}^3} G(u).$$

It follows from (g_1) and (g_2) that J_q is well defined on H and is of C^1 for all $q \geqslant 0$, and

$$(J_q'(u), v) = (u, v) + q\int_{\mathbb{R}^3} f(u)\phi_u v - \int_{\mathbb{R}^3} g(u)v, \quad u, v \in H.$$

By the conditions (f), (g_1)–(g_3) and Lemma 2.1, it is easy to see that the functional J_q has the mountain pass geometry, but the standard arguments used to prove the boundedness of Palais–Smale sequences do not work. Under our general assumptions, we need a different approach. Following [15], we introduce a cut-off function $\chi \in C^\infty(\mathbb{R}_+, [0, 1])$ satisfying

$$\chi(t) = \begin{cases} 1, & t \in [0, 1/2], \\ 0, & t \geqslant 1, \end{cases} \qquad |\chi'|_\infty \leqslant 4,$$

and study the following modified functional $J_q^T : H \to \mathbb{R}$ defined as

$$J_q^T(u) = \frac{1}{2}\|u\|^2 + \frac{q}{2}h_T(u)\int_{\mathbb{R}^3} F(u)\phi_u - \int_{\mathbb{R}^3} G(u), \quad u \in H,$$

where, for every $T > 0$, $h_T(u) = \chi(T^{-2}\|u\|^2)$.

In the following, we will discuss the existence of a critical point of J_q^T. In fact, for $T > 0$ sufficiently large and q sufficiently small, we can find a critical point of J_q^T such that $\|u\| \leqslant T/\sqrt{2}$, so u is also a critical point of J_q. We recall the following result. The "monotonicity trick" at the core of this theorem was invented by Struwe (see [24]).

Theorem 2.2 ([14]). *Let $(X, \|\cdot\|)$ be a Banach space and $I \subset \mathbb{R}_+$ an interval. Consider the family of C^1 functionals on X*

$$J_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in I,$$

with B nonnegative and either $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$ and such that $J_\lambda(0) = 0$.

For any $\lambda \in I$ we set

$$\Gamma_\lambda = \{\gamma \in C([0, 1], X) : \gamma(0) = 0, J_\lambda(\gamma(1)) < 0\}.$$

If for every $\lambda \in I$ the set Γ_λ is nonempty and

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda}\max_{t\in[0,1]} J_\lambda(\gamma(t)) > 0,$$

then for almost every $\lambda \in I$ there is a sequence $\{u_n\} \subset X$ such that

(i) *$\{u_n\}$ is bounded;*
(ii) *$J_\lambda(u_n) \to c_\lambda$;*
(iii) *$J_\lambda'(u_n) \to 0$ in the dual X^{-1} of X.*

As our case, $X = H$, $I = [\delta, 1]$, where $\delta \in (0, 1)$ is a positive constant,

$$A(u) = \frac{1}{2}\|u\|^2 + \frac{q}{2}h_T(u)\int_{\mathbb{R}^3} F(u)\phi_u, \qquad B(u) = \int_{\mathbb{R}^3} G(u). \tag{2.5}$$

So that the perturbed functional which we discuss is

$$J_{q,\lambda}^T(u) = \frac{1}{2}\|u\|^2 + \frac{q}{2}h_T(u)\int_{\mathbb{R}^3} F(u)\phi_u - \lambda\int_{\mathbb{R}^3} G(u).$$

Actually, this functional is a restriction to the radial functions of a C^1 functional defined on the whole space $H^1(\mathbb{R}^3)$ and for any $u, v \in H$,

$$((J_{q,\lambda}^T)'(u), v) = \left[1 + a_q^T(u)\right](u, v) + qh_T(u)\int_{\mathbb{R}^3} f(u)\phi_u v - \lambda\int_{\mathbb{R}^3} g(u)v, \tag{2.6}$$

where

$$a_q^T(u) = qT^{-2}\chi'(T^{-2}\|u\|^2)\int_{\mathbb{R}^3} F(u)\phi_u. \tag{2.7}$$

In the next section, we will verify that the functional $J_{q,\lambda}^T$ satisfies the conditions of Theorem 2.2. In order to obtain our result, we need the following Pohozaev type identity.

Lemma 2.3. *If $u \in H$ is a weak solution of*

$$\begin{cases} \left[1 + a_q^T(u)\right](-\Delta u + u) + qh_T(u)\phi f(u) = g(u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = 2qF(u), & \text{in } \mathbb{R}^3, \end{cases} \tag{2.8}$$

then the following Pohozaev type identity holds

$$\left[1 + a_q^T(u)\right] \int_{\mathbb{R}^3} \left[\frac{1}{2}|\nabla u|^2 + \frac{3}{2}u^2\right] + \frac{5}{2}qh_T(u) \int_{\mathbb{R}^3} F(u)\phi = 3\int_{\mathbb{R}^3} G(u).$$

Proof. Since $u \in H$ is a weak solution of (2.8), by the standard regularity results, $u \in H^2_{\mathrm{loc}}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$. By the divergence theorem, for any $R > 0$, we have

$$\int_{B_R} -\Delta u x \cdot \nabla u = -\frac{1}{2}\int_{B_R} |\nabla u|^2 - \frac{1}{R}\int_{\partial B_R} |x \cdot \nabla u|^2 + \frac{R}{2}\int_{\partial B_R} |\nabla u|^2,$$
$$\int_{B_R} ux \cdot \nabla u = -\frac{3}{2}\int_{B_R} u^2 + \frac{R}{2}\int_{\partial B_R} u^2,$$
$$\int_{B_R} f(u)\phi x \cdot \nabla u = -3\int_{B_R} F(u)\phi - \int_{B_R} F(u)x \cdot \nabla\phi + R\int_{\partial B_R} F(u)\phi,$$
$$\int_{B_R} g(u)x \cdot \nabla u = -3\int_{B_R} G(u) + R\int_{\partial B_R} G(u).$$

Multiplying the first equation of (2.8) by $x \cdot \nabla u$, the second equation by $x \cdot \nabla\phi$ and integrating on B_R, by above, we have

$$\begin{aligned} &\left[1 + a_q^T(u)\right]\left(-\frac{1}{2}\int_{B_R} |\nabla u|^2 - \frac{1}{R}\int_{\partial B_R} |x \cdot \nabla u|^2 + \frac{R}{2}\int_{\partial B_R} |\nabla u|^2 - \frac{3}{2}\int_{B_R} u^2 + \frac{R}{2}\int_{\partial B_R} u^2\right) \\ &\quad + qh_T(u)\left(-3\int_{B_R} F(u)\phi - \int_{B_R} F(u)x \cdot \nabla\phi + R\int_{\partial B_R} F(u)\phi\right) \\ &= -3\int_{B_R} G(u) + R\int_{\partial B_R} G(u), \end{aligned} \tag{2.9}$$

and

$$2q\int_{B_R} F(u)x \cdot \nabla\phi = \int_{B_R} -\Delta\phi x \cdot \nabla\phi = -\frac{1}{2}\int_{B_R} |\nabla\phi|^2 - \frac{1}{R}\int_{\partial B_R} |x \cdot \nabla\phi|^2 + \frac{R}{2}\int_{\partial B_R} |\nabla\phi|^2. \tag{2.10}$$

Substituting (2.10) into (2.9), we obtain that

$$\begin{aligned} &\left[1 + a_q^T(u)\right]\left(-\frac{1}{2}\int_{B_R} |\nabla u|^2 - \frac{3}{2}\int_{B_R} u^2\right) - 3qh_T(u)\int_{B_R} F(u)\phi + \frac{1}{4}h_T(u)\int_{B_R} |\nabla\phi|^2 + 3\int_{B_R} G(u) \\ &= R\int_{\partial B_R} G(u) - \left[1 + a_q^T(u)\right]\left(-\frac{1}{R}\int_{\partial B_R} |x \cdot \nabla u|^2 + \frac{R}{2}\int_{\partial B_R} |\nabla u|^2 + \frac{R}{2}\int_{\partial B_R} u^2\right) \\ &\quad + h_T(u)\left(-\frac{1}{2R}\int_{\partial B_R} |x \cdot \nabla\phi|^2 + \frac{R}{4}\int_{\partial B_R} |\nabla\phi|^2\right) - qh_T(u)R\int_{\partial B_R} F(u)\phi. \end{aligned} \tag{2.11}$$

We can choose a suitable sequence $\{R_n\}$ with $R_n \to \infty$ such that the right hand of (2.11) converges to 0. Since

$$\int_{B_{R_n}} |\nabla u|^2 \to \int_{\mathbb{R}^3} |\nabla u|^2, \qquad \int_{B_{R_n}} u^2 \to \int_{\mathbb{R}^3} u^2, \qquad \int_{B_{R_n}} G(u) \to \int_{\mathbb{R}^3} G(u),$$

and $\int_{\mathbb{R}^3} |\nabla\phi|^2 = 2q\int_{\mathbb{R}^3} F(u)\phi$, we get

$$\left[1 + a_q^T(u)\right]\left(\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3}{2}\int_{\mathbb{R}^3} u^2\right) + \frac{5}{2}qh_T(u)\int_{\mathbb{R}^3} F(u)\phi = 3\int_{\mathbb{R}^3} G(u).$$

The proof is completed. □

3. Proof of the main result

In this section, we first verify that the perturbed functional $J^T_{q,\lambda}$ satisfies the conditions of Theorem 2.2; see Lemmas 3.1 and 3.2. So there is a bounded Palais–Smale sequence $\{u_n^\lambda\}$ to $J^T_{q,\lambda}$. Then, for almost every $\lambda \in I$ and under proper assumptions of q, T, we obtain that there is a subsequence of $\{u_n^\lambda\}$ converging to u^λ which is a nontrivial critical point of the functional $J_{q,\lambda}$; see Lemmas 3.3 and 3.4. As a consequence, when $\lambda_n \to 1$, there exists a sequence $\{u^{\lambda_n}\}$ (denoted by $\{u_n\}$) of critical point of J^T_{q,λ_n}. In Lemma 3.5, we give the proof of $\|u_n\| \leqslant T$ for sufficiently large $T > 0$ by using the Pohozaev type identity. At last, we give the proof of our main result.

Lemma 3.1. *$\Gamma_\lambda \neq \emptyset$ for all $\lambda \in I$.*

Proof. We choose a radial function $\psi \in C_0^\infty(\mathbb{R}^3)$ with $\psi \geqslant 0$, $\|\psi\| = 1$ and $\mathrm{supp}(\psi) \subset B_R$ for some $R > 0$, by (g_3), we have that for any C_1 with $2C_1\delta \int_{B_R} \psi^2 > 1$, there exists $C_2 > 0$ such that

$$G(t) \geqslant C_1|t|^2 - C_2, \quad t \in \mathbb{R}_+. \tag{3.1}$$

Then, for $t > T$,

$$\begin{aligned} J^T_{q,\lambda}(t\psi) &= \frac{1}{2}t^2 + \frac{q}{2}\chi(T^{-2}t^2)\int_{\mathbb{R}^3} F(t\psi)\phi_{t\psi} - \lambda\int_{\mathbb{R}^3} G(t\psi) \\ &= \frac{1}{2}t^2 - \lambda\int_{B_R} G(t\psi) \\ &\leqslant \frac{1}{2}t^2 - \delta C_1 t^2 \int_{B_R} \psi^2 + C_3. \end{aligned}$$

Then we can choose $t > 0$ large such that $J^T_{q,\lambda}(t\psi) < 0$. The proof is completed. □

Lemma 3.2. *There exists a constant $c > 0$ such that $c_\lambda \geqslant c$ for all $\lambda \in I$.*

Proof. By the conditions (g_1) and (g_2), for $\varepsilon \in (0, \gamma_2^{-2}/2)$, there exists $C_\varepsilon > 0$ such that

$$g(t) \leqslant \varepsilon|t| + C_\varepsilon|t|^{p-1}, \quad t \in \mathbb{R}, \tag{3.2}$$

$$g(t) \leqslant \varepsilon|t| + C_\varepsilon|t|^5, \quad t \in \mathbb{R}, \tag{3.3}$$

and

$$G(t) \leqslant \frac{\varepsilon}{2}|t|^2 + \frac{C_\varepsilon}{p}|t|^p, \quad t \in \mathbb{R}. \tag{3.4}$$

Hence, for any $u \in H$ and $\lambda \in I$, we have

$$\begin{aligned} J^T_{q,\lambda}(u) &\geqslant \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3}\left(\frac{1}{2}\varepsilon u^2 + \frac{C_\varepsilon}{p}|u|^p\right) \\ &\geqslant \frac{1}{4}\|u\|^2 - \frac{C_\varepsilon}{p}\gamma_p^p\|u\|^p. \end{aligned}$$

Since $p > 2$, we conclude that there exists $\rho > 0$ such that $J^T_{q,\lambda}(u) > 0$ for any $\lambda \in I$ and $u \in H$ with $\|u\| \in (0, \rho]$. In particular, for $\|u\| = \rho$, we have $J^T_{q,\lambda}(u) \geqslant c > 0$. Fix $\lambda \in I$ and $\gamma \in \Gamma_\lambda$, by the definition of Γ_λ, we have $\|\gamma(1)\| > \rho$. By the continuity of γ, there exists $t_\gamma \in (0, 1)$ such that $\|\gamma(t_\gamma)\| = \rho$. Therefore, for any $\lambda \in I$, we have

$$c_\lambda \geqslant \inf_{\gamma\in\Gamma_\lambda} J^T_{q,\lambda}(\gamma(t_\gamma)) \geqslant c > 0.$$

The proof is completed. □

Lemma 3.3. *For any $\lambda \in I$ and $4q^2\widetilde{T} < 1$, each bounded Palais–Smale sequence of the functional $J^T_{q,\lambda}$ admits a convergent subsequence, where $\widetilde{T} = \widetilde{C}(T^2 + T^{2\alpha})$.*

Proof. Let $\lambda \in I$ and $\{u_n\}$ be a bounded (PS) sequence of $J^T_{q,\lambda}$, that is, $\{u_n\}$ and $\{J^T_{q,\lambda}(u_n)\}$ are bounded, $(J^T_{q,\lambda})'(u_n) \to 0$ in H', where H' is the dual space of H. We may assume that, up to a subsequence,

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } H, \\ &u_n \to u \quad \text{in } L^p(\mathbb{R}^3), \\ &u_n \to u \quad \text{a.e. in } \mathbb{R}^3. \end{aligned}$$

By (3.2), we obtain that

$$\begin{aligned}\left|\int_{\mathbb{R}^3} g(u_n)(u_n-u)\right| &\leqslant \int_{\mathbb{R}^3}[\varepsilon|u_n|+C_\varepsilon|u_n|^{p-1}]|u_n-u|\\ &\leqslant \varepsilon|u_n|_2|u_n-u|_2+C_\varepsilon|u_n|_p^{p-1}|u_n-u|_p\\ &\leqslant \varepsilon\gamma_2^2\|u_n\|\|u_n-u\|+C_\varepsilon\gamma_p^{p-1}\|u_n\|^{p-1}|u_n-u|_p.\end{aligned}$$

It follows that

$$\int_{\mathbb{R}^3} g(u_n)(u_n-u)\to 0.$$

By the condition (f) and Hölder's inequality, we have that

$$\begin{aligned}\left|\int_{\mathbb{R}^3} f(u_n)\phi_{u_n}(u_n-u)\right| &\leqslant c\int_{\mathbb{R}^3}[|u_n|+|u_n|^\alpha]\phi_{u_n}|u_n-u|\\ &\leqslant c\left[|\phi_{u_n}|_6|u_n|_{12/5}|u_n-u|_{12/5}+|\phi_{u_n}|_6|u_n|_6^\alpha|u_n-u|_\beta\right],\end{aligned}$$

where $\beta=6/(5-\alpha)\in(2,6)$. Then, by Lemma 2.1 (iii) and Sobolev's embedding theorem, we have

$$\int_{\mathbb{R}^3} f(u_n)\phi_{u_n}(u_n-u)\to 0.$$

Thus, by (2.6), we have

$$\begin{aligned}((J_{q,\lambda}^T)'(u_n),u_n-u) &= \left[1+a_q^T(u_n)\right](u_n,u_n-u)+qh_T(u_n)\int_{\mathbb{R}^3} f(u_n)\phi_{u_n}(u_n-u)-\lambda\int_{\mathbb{R}^3} g(u_n)(u_n-u)\\ &= \left[1+a_q^T(u_n)\right](u_n,u_n-u)+o(1),\end{aligned}$$

and then

$$\left[1+a_q^T(u_n)\right](u_n,u_n-u)\to 0.$$

When $\|u_n\|\leqslant T$, by Lemma 2.1 (iv), we obtain that

$$\left|\int_{\mathbb{R}^3} F(u_n)\phi_{u_n}\right|\leqslant q\widetilde{C}\left(\|u_n\|^4+\|u_n\|^{2(1+\alpha)}\right)\leqslant q\widetilde{C}\left(T^4+T^{2(1+\alpha)}\right)=qT^2\widetilde{T}. \tag{3.5}$$

By (2.7)

$$\left|a_q^T(u_n)\right|\leqslant qT^{-2}\left|\chi'\left(T^{-2}\|u_n\|^2\right)\right|\left|\int_{\mathbb{R}^3} F(u_n)\phi_{u_n}\right|\leqslant 4q^2\widetilde{T}. \tag{3.6}$$

It follows from the condition $4q^2\widetilde{T}<1$ that $1+a_q^T(u_n)\geqslant 1-4q^2\widetilde{T}>0$ and $\|u_n\|\to\|u\|$. This together with $u_n\rightharpoonup u$ shows that $u_n\to u$ in H. The proof is completed. □

Lemma 3.4. *Let $4q^2\widetilde{T}<1$. Then for almost every $\lambda\in I$, there exists $u^\lambda\in H\setminus\{0\}$ such that $(J_{q,\lambda}^T)'(u^\lambda)=0$ and $J_{q,\lambda}^T(u^\lambda)=c_\lambda$.*

Proof. First, by (2.5), it is easy to see that B is nonnegative, $A(u)\to\infty$ as $\|u\|\to\infty$ and $J_{q,\lambda}^T(0)=0$. Then by Lemmas 3.1 and 3.2 and Theorem 2.2, for almost every $\lambda\in I$, there exists a bounded sequence $\{u_n^\lambda\}\subset H$ such that

$$J_{q,\lambda}^T(u_n^\lambda)\to c_\lambda,$$
$$(J_{q,\lambda}^T)'(u_n^\lambda)\to 0.$$

By Lemma 3.3, we can obtain that there exists $u^\lambda\in H$ such that $u_n^\lambda\to u^\lambda$ in H. Therefore, $(J_{q,\lambda}^T)'(u^\lambda)=0$ and $J_{q,\lambda}^T(u^\lambda)=c_\lambda$. It follows from Lemma 3.2 that $u^\lambda\in H\setminus\{0\}$. □

According to Lemma 3.4, when $\lambda_n\to 1^-$ with $\{\lambda_n\}\subset I$, we can find a sequence $\{u^{\lambda_n}\}$ (denoted by $\{u_n\}$ for simplicity) satisfying

$$J_{q,\lambda_n}^T(u_n)=c_{\lambda_n},\quad (J_{q,\lambda_n}^T)'(u_n)=0.$$

The following lemma shows that $\|u_n\|\leqslant T$ for all $n\in\mathbb{N}=\{1,2,\ldots\}$ which is the key for this paper.

Lemma 3.5. *Let u_n be a critical point of J_{q,λ_n}^T at level c_{λ_n}. Then for $T>1$ sufficiently large, there exists $q_0>0$ with $8q_0^2T^2\widetilde{T}<1$ such that for any $q\in[0,q_0)$, $\|u_n\|\leqslant T/\sqrt{2}$ for all $n\in\mathbb{N}$.*

Proof. First, since $(J^T_{q,\lambda_n})'(u_n) = 0$, by (2.6), u_n is a weak solution of (2.8) with g replaced by $\lambda_n g$. So by Lemma 2.3, u_n satisfies the following Pohozaev identity

$$\left[1 + a^T_q(u_n)\right]\left(\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2 + \frac{3}{2}\int_{\mathbb{R}^3}u_n^2\right) + \frac{5}{2}qh_T(u_n)\int_{\mathbb{R}^3}F(u_n)\phi_{u_n} = 3\lambda_n\int_{\mathbb{R}^3}G(u_n). \tag{3.7}$$

On the other hand, by $J^T_{q,\lambda_n}(u_n) = c_{\lambda_n}$, we have that

$$\frac{3}{2}\int_{\mathbb{R}^3}[|\nabla u_n|^2 + u_n^2] + \frac{3}{2}qh_T(u_n)\int_{\mathbb{R}^3}F(u_n)\phi_{u_n} - 3\lambda_n\int_{\mathbb{R}^3}G(u_n) = 3c_{\lambda_n}. \tag{3.8}$$

Hence, by (3.7), (3.8) and $8q^2\widetilde{T} < 1$, we can obtain that

$$\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2 \leqslant \left[1 + a^T_q(u_n)\right]\int_{\mathbb{R}^3}|\nabla u_n|^2 = 3c_{\lambda_n} + \frac{3}{2}a^T_q(u_n)\|u_n\|^2 + qh_T(u_n)\int_{\mathbb{R}^3}F(u_n)\phi_{u_n}. \tag{3.9}$$

We now estimate the right hand side of (3.9). By the min–max definition of the mountain pass level, Lemma 3.1 and (3.1), we have

$$\begin{aligned} c_{\lambda_n} &\leqslant \max_t J^T_{q,\lambda_n}(t\psi) \\ &\leqslant \max_t\left\{\frac{1}{2}t^2 - \lambda_n\int_{\mathbb{R}^3}G(t\psi)\right\} + \max_t\frac{q}{2}\chi(t^2T^{-2})\int_{\mathbb{R}^3}F(t\psi)\phi_{t\psi} \\ &\leqslant \max_t\left\{\frac{1}{2}t^2 - \delta C_1 t^2\int_{B_R}\psi^2 + C_3\right\} + A_q(T) \\ &= C_3 + A_q(T). \end{aligned}$$

If $t \geqslant T$, then $\chi(t^2T^{-2}) = 0$. Thus, by (3.5), we have that

$$A_q(T) \leqslant \frac{q}{2}\max_{t\in[0,T]}\left|\int_{\mathbb{R}^3}F(t\psi)\phi_{t\psi}\right| \leqslant \frac{1}{2}q^2T^2\widetilde{T}.$$

By (3.5) and (3.6), we have also that

$$qh_T(u_n)\int_{\mathbb{R}^3}F(u_n)\phi_{u_n} \leqslant q^2T^2\widetilde{T},$$

and

$$|a^T_q(u_n)|\|u_n\|^2 \leqslant 4q^2T^2\widetilde{T}.$$

Then, by (3.9), we have

$$\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2 \leqslant 3\left(C_3 + \frac{1}{2}q^2T^2\widetilde{T}\right) + 6q^2T^2\widetilde{T} + q^2T^2\widetilde{T} = 3C_3 + \frac{17}{2}q^2T^2\widetilde{T}. \tag{3.10}$$

On the other hand, since $((J^T_{q,\lambda_n})'(u_n), u_n) = 0$, by (2.6) and (3.3), we have that

$$[1 + a^T_q(u_n)]\|u_n\|^2 + qh_T(u_n)\int_{\mathbb{R}^3}f(u_n)\phi_{u_n}u_n = \lambda_n\int_{\mathbb{R}^3}g(u_n)u_n \leqslant \varepsilon|u_n|_2^2 + C_\varepsilon|u_n|_6^6. \tag{3.11}$$

Thus, by (3.6), (3.10) and (3.11), we obtain that

$$(1/2 - \varepsilon\gamma_2^2)\|u_n\|^2 \leqslant C_\varepsilon|u_n|_6^6 \leqslant S^{-3}C_\varepsilon|\nabla u_n|_2^6 \leqslant S^{-3}C_\varepsilon\left(6C_3 + 17q^2T^2\widetilde{T}\right)^3,$$

and then

$$\|u_n\|^2 \leqslant C_4\left(6C_3 + 17q^2T^2\widetilde{T}\right)^3. \tag{3.12}$$

One chooses T sufficiently large such that $T^2 \geqslant 2C_4(6C_3 + 17/8)^3$. We then choose $q_0 > 0$ such that $8q_0^2T^2\widetilde{T} < 1$. Hence, for all $q \in [0, q_0)$, we have from (3.12) that

$$\|u_n\|^2 \leqslant C_4(6C_3 + 17/8)^3 \leqslant T^2/2,$$

that is $\|u_n\| \leqslant T/\sqrt{2}$. Thus we obtain the conclusion. $\square$

4. Proof of Theorem 1.1

Proof of Theorem 1.1. Let T, q_0 be defined as in Lemma 3.5, and let u_n be a critical point for J^T_{q,λ_n} at level c_{λ_n}. Then from Lemma 3.5 we have that $\|u_n\| \leqslant T/\sqrt{2}$ and $\{c_{\lambda_n}\}$ is bounded. Hence

$$J^T_{q,\lambda_n}(u_n) = \frac{1}{2}\|u_n\|^2 + \frac{q}{2}\int_{\mathbb{R}^3} F(u_n)\phi_{u_n} - \lambda_n \int_{\mathbb{R}^3} G(u_n).$$

In the following, we show that $\{u_n\}$ is a (PS) sequence of J_q. Indeed, since

$$J_q(u_n) = J_{q,\lambda_n}(u_n) + (\lambda_n - 1)\int_{\mathbb{R}^3} G(u_n),$$

$$(J'_q(u_n), v) = ((J^T_{q,\lambda_n})'(u_n), v) + (\lambda_n - 1)\int_{\mathbb{R}^3} g(u_n)v, \quad v \in H.$$

From (3.2) and (3.4), the boundedness of $\{u_n\}$ implies $\int_{\mathbb{R}^3} G(u_n)$ is bounded and $\left|\int_{\mathbb{R}^3} g(u_n)v\right| \leqslant C\|v\|$. Thus, when $\lambda_n \to 1$, we have that $\{J_q(u_n)\}$ is bounded and $J'_q(u_n) \to 0$. Therefore $\{u_n\}$ is a bounded (PS) sequence of J_q.

By Lemma 3.3, $\{u_n\}$ has a convergent subsequence. We may assume that $u_n \to u$. Consequently, $J'_q(u) = 0$. According to Lemma 3.2, we have that $J_q(u) = \lim_{n\to\infty} J_q(u_n) = \lim_{n\to\infty} J^T_{q,\lambda_n}(u_n) \geqslant c > 0$ and u is a positive solution by the conditions (f) and (g_1). The proof is completed. □

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