



On the initial–boundary problem for fourth order wave equations with damping, strain and source terms



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ABSTRACT

In this paper we study the global existence and blow-up of solutions to the fourth order equations

$$u_{tt} + u_t + \Delta^2 u - \alpha \Delta u - \sum_{i=1}^n \frac{\partial}{\partial x_i} (\theta_i(u_{x_i})) = f(u), \quad x \in \Omega, t > 0,$$

where $\alpha \geq 0$.

Under appropriate assumptions on the initial data and parameters in the above equation we establish two results on blow-up of solutions with arbitrary initial energy, $-\infty < E(0) < +\infty$. Also, by using a potential well we show the global existence of solutions for the fourth order wave equation with some $\theta_i(s)$ and $f(s)$. Especially, it is proved that the energy decays exponentially as $t \rightarrow \infty$.

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1. Introduction

Consider the initial–boundary value problem of the fourth order wave equation with the nonlinear strain and source terms as follows

$$\begin{cases} u_{tt} + u_t + \Delta^2 u - \alpha \Delta u - \sum_{i=1}^n \frac{\partial}{\partial x_i} (\theta_i(u_{x_i})) = f(u), & x \in \Omega, t > 0, \\ u(0, t) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \Omega, \\ u = \frac{\partial u}{\partial n} = 0 \quad \text{or} \quad u = \Delta u = 0, & x \in \partial\Omega, t \geq 0, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $\alpha \geq 0$.

The one-dimension case of the fourth order wave equation is written as

$$u_{tt} + u_{xxxx} - a(u_x^2)_x = f(x), \quad x \in \Omega \subset \mathbb{R}, t > 0, \quad (2)$$

which was first introduced in [2] to describe the elasto-plastic-microstructure models for the longitudinal motion of an elasto-plastic bar. Chen and Yang [3] studied the Cauchy problem for the more general equation (2).

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For the following equation

$$u_{tt} + u_t + \Delta^2 u + \sum_{i=1}^n \frac{\partial}{\partial x_i} (\theta_i(u_{x_i})) = 0, \quad x \in \Omega, \quad t > 0, \quad (3)$$

Yang [19] studied the initial-boundary value problem in the high dimension space and established the blow-up result with negative potential energy. Very recently Liu and Xu [15] have established the threshold result of the blow-up and global existence for Eq. (3) by the potential well method.

We next recall some existing results of the global existence and blow-up of solutions for Eq. (1) without linear damping term u_t . Esquivel-Avila [5] studied the initial-boundary value problem for the fourth order wave equation

$$u_{tt} + \Delta^2 u - \alpha \Delta u \pm \beta \sum_{i=1}^n \frac{\partial}{\partial x_i} (|u_{x_i}|^{m-2} u_{x_i}) = \mu |u|^{r-2} u, \quad x \in \Omega, \quad t > 0, \quad (4)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and r, m satisfy

$$2 < r \leq \frac{2(n-2)}{n-4} \quad \text{if } n \geq 5,$$

$$2 < m \leq \frac{2(n-1)}{n-2} \quad \text{if } n \geq 3.$$

The following invariant sets were defined in [5]

$$V_1 = \{u : I(u) > 0, J(u) < d\} \cup \{0\},$$

$$V_2 = \{u : I(u) < 0, J(u) < d\},$$

where

$$J(u) = \frac{1}{2} \|\Delta u\|^2 + \frac{\alpha}{2} \|\nabla u\|^2 \mp \frac{\beta}{m} \|\nabla u\|_m^m - \frac{\mu}{r} \|u\|_r^r,$$

$$I(u) = \|\Delta u\|^2 + \alpha \|\nabla u\|^2 \mp \beta \|\nabla u\|_m^m - \mu \|u\|_r^r,$$

$$d = \inf_{u \in M} J(u),$$

$$M = \{u : I(u) = 0, u \neq 0\}.$$

He then studied the qualitative behavior of a solution for Eq. (4). Recently, by using a family of potential wells, Liu and Xu [14] obtained the threshold result of the global existence and blow-up of the solution of the following equation as

$$u_{tt} + \Delta^2 u - \alpha \Delta u + \beta \sum_{i=1}^n \frac{\partial}{\partial x_i} \theta_i(u_{x_i}) = \mu |u|^{r-2} u, \quad x \in \Omega, \quad t > 0. \quad (5)$$

Up until now there are still some open problems for Eq. (1):

- (i) Whether there is some initial data such that the solution of Eq. (1) blows up in finite time with the arbitrarily positive initial energy?
- (ii) How about the energy of Eq. (1) as $t \rightarrow \infty$?

Up until now there are some blow-up results for the second order nonlinear evolution equations with arbitrarily positive initial energy (see, e.g., [7,13,16,18]). But there is no blow-up result for the higher order nonlinear evolution equations with the arbitrary initial energy; for example, the fourth order wave equation. Thus, one of the main objective of the present paper is to establish a blow-up result with arbitrarily positive initial energy. In other words, in the present paper a complete blow-up result for Eq. (1) is established in the sense of the initial energy, $-\infty < E(0) < +\infty$.

The technique used in the proof of the blow-up results is essentially the concavity method, developed by Levine [10,11]. Many researchers have used this method to study blow-up of the solution of some hyperbolic equations. We restrict ourselves to cite papers [16,4,6,12,17] referring to their bibliographies for a broader list of work, although still not exhaustive. But in this paper, in order to prove the blow-up result with arbitrarily positive initial energy, we need first to study an invariant set. Since we will break through the potential well, our invariant set is different from the invariant set used in the method of the potential well. Thus we can use the concavity argument to prove our desired result.

Concerning problem (ii), we investigate the exponential energy decay by an integral inequality for Eq. (1) with some $\theta_i(s)$ and $f(s)$. Indeed, the result of the exponential energy decay can be established for Eq. (1) with some more general $\theta_i(s)$ and $f(s)$, but for the simplicity of the proof in the paper we just consider some special cases of $\theta_i(s)$ and $f(s)$. Our result extend the corresponding result in [19]. The proof is based on a known integral inequality, and it is different from the proof used in [19]. By a similar way we can investigate the energy decay of Eq. (1) with the nonlinear damping term, $|u_t|^{m-2} u_t$.

The paper is organized as follows. In Section 2, some known results are introduced, and our main results of blow-up, global existence and exponential energy decay are stated. Section 3 is devoted to prove two blow-up theorems of Eq. (1) with arbitrary initial energy. Finally the global existence and exponential energy decay are studied for the fourth order wave equation (1) in Section 4.

2. Preliminaries and main results

In order to state our main results, we first introduce some notations. We use $\|\cdot\|_p$ to denote the norm in $L^p(\Omega)$. For simplicity, we always use $\|\cdot\|$ to denote $\|\cdot\|_2$, and let $H_0^1(\Omega)$ and $H_0^2(\Omega)$ be Sobolev spaces with $\|\cdot\|_{H_0^1(\Omega)}$ and $\|\cdot\|_{H_0^2(\Omega)}$, respectively. In addition, let

$$B = \left\{ u \in H_0^2(\Omega) : \frac{\partial u(x)}{\partial n} \equiv 0 \text{ or } \Delta u(x) \equiv 0 \text{ for every } x \in \partial\Omega \right\}$$

with the norm

$$\|u\|_B^2 = \|\Delta u\|^2 + \alpha \|\nabla u\|^2.$$

It is well-known that, for $\forall u \in H_0^2(\Omega)$, $\|\Delta u\|$ is equivalent to $\|u\|_{H_0^2(\Omega)}$ (for the proof, see [1,9]). Naturally we see that for any $u \in B$, $\|u\|_B$ is equivalent to $\|u\|_{H_0^2}$.

Let κ_p be the optimal constant of the Sobolev embedding $B \hookrightarrow L^p(\Omega)$ given by

$$\kappa_p^{-1} = \inf\{\|v\|_B : v \in B, \|v\|_p = 1\},$$

where $2 \leq p < \frac{2n}{n-4}$ as $n > 4$ and $2 \leq p < \infty$ as $n \leq 4$.

The following integral inequality plays an important role in our study of energy decay for Eq. (1).

Lemma 2.1. Assume that the function $\phi : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}^+ \cup \{0\}$ is a non-increasing function and there exists a constant $c > 0$ such that

$$\int_t^{+\infty} \phi(s) ds \leq c\phi(t)$$

for every $t \in [0, \infty)$. Then

$$\phi(t) \leq \phi(0) \exp(1 - t/c)$$

for every $t > c$.

For the proof, see [8].

In order to introduce our main results, we first state the local existence of a solution of the fourth order wave equation (1).

Theorem 2.2. Assume that $f(s) \in C(\mathbb{R})$ and $\theta_i \in C(\mathbb{R})$ satisfy the local Lipschitz condition with $f(0) = 0$ and $\theta_i(0) = 0$. For an initial data $(u_0, u_1) \in B \times L^2(\Omega)$, there exists a unique (local) weak solution $u(t)$ of Eq. (1), i.e.

$$\frac{d^2}{dt^2}(u(t), w) + (\Delta u(t), \Delta w) + \alpha(\nabla u(t), \nabla w) + \sum_{i=1}^n (\theta_i(u_{x_i}), w_{x_i}) = (f(u(t)), w)$$

a.e. in $[0, T_{\max})$ and for every $w \in B$, such that $u \in C([0, T_{\max}); B)$, where T_{\max} is the maximal time of the existence of solution.

If $T_{\max} < \infty$, then the solution $u(t)$ of Eq. (1) blows up in finite time, that is to say,

$$\lim_{t \rightarrow T_{\max}} \|u(t)\|_2 \rightarrow \infty.$$

By a simple calculation, we have

$$E(t) - E(0) = - \int_0^t \|u_\tau(\tau)\|^2 d\tau \leq 0 \tag{6}$$

for every $t \in [0, T_{\max})$, where

$$E(t) = \frac{1}{2} \|u_t(t)\|^2 + \frac{1}{2} \|\Delta u(t)\|^2 + \frac{\alpha}{2} \|\nabla u(t)\|^2 + \sum_{i=1}^n \int_\Omega \Theta_i(u_{x_i}(t, x)) dx - \int_\Omega F(u(t, x)) dx; \tag{7}$$

here

$$F(s) = \int_0^s f(\tau) d\tau \tag{8}$$

and

$$\Theta_i(s) = \int_0^s \theta_i(\tau) d\tau. \tag{9}$$

As in [5], the proof of Theorem 2.2 follows from the standard fixed point argument, we omit it here.

Before stating our blow-up results, we make the following assumptions on $f(s)$ and $\theta_i(s)$.

Assumption 2.1. f satisfies the Lipschitz condition with $f(0) = 0$. Moreover, $\exists \epsilon > 0$ such that

$$sf(s) \geq (2 + \epsilon)F(s), \quad \forall s \in \mathbb{R}, \quad (10)$$

where $F(s)$ is given in (8).

Assumption 2.2. $\theta_i(s)$ is a local Lipschitz function, and satisfies

$$(2 + \epsilon)\Theta_i(s) \geq s\theta_i(s) > 0$$

for every $s \in \mathbb{R}$, $\Theta_i(s)$ is defined by (9).

We next state our first blow-up result for Eq. (1).

Theorem 2.3. Let f and θ_i satisfy Assumptions 2.1 and 2.2. If a non-zero initial data $(u_0, u_1) \in B \times L^2(\Omega)$ satisfies

$$E(0) < 0,$$

or

$$\int_{\Omega} u_0(x)u_1(x)dx \geq 0 \quad \text{if } E(0) = 0,$$

then the corresponding solution $u(t)$ of Eq. (1) blows up in finite time.

To state our second blow-up result, we need to define a function

$$I(u(t)) = \|\Delta u\|^2 + \alpha \|\nabla u\|^2 + \sum_{i=1}^n \int_{\Omega} u_{x_i} \theta_i(u_{x_i}) dx - \int_{\Omega} u f(u) dx. \quad (11)$$

We are now in a position to introduce our blow-up result for Eq. (1) with arbitrarily positive initial energy.

Theorem 2.4. Let f and θ_i satisfy Assumptions 2.1 and 2.2. If a non-zero initial data $(u_0, u_1) \in B \times L^2(\Omega)$ satisfies

$$E(0) > 0, \quad (12)$$

$$\int_{\Omega} u_0 u_1 dx \geq 0, \quad (13)$$

$$I(u_0) < 0, \quad (14)$$

$$\|u_0\|_2^2 \geq \frac{2\kappa_2^2(2 + \epsilon)}{\epsilon} E(0), \quad (15)$$

then the corresponding solution $u(t)$ of Eq. (1) blows up in finite time.

For the special case, $\theta_i(s) = |s|^{m-2}s$ and $f(s) = |s|^{p-2}s$, Theorem 2.3 generalizes the result in [5]. Especially, Theorem 2.4 is the complementary of the result in [5]. Under the sense of the initial energy, $-\infty < E(0) < +\infty$, Theorems 2.3 and 2.4 imply the complete blow-up result for Eq. (1).

We next consider the global existence and energy decay of the solution of Eq. (1) with

$$\theta_i(s) = |s|^{m-2}s \quad \text{and} \quad f(s) = |s|^{p-2}s \quad (16)$$

for every $s \in \mathbb{R}$ and $i = 1, 2, \dots, n$, where $2 < m < p$, $2 < p < \infty$ when $n \leq 4$ and $2 < p < \frac{2n}{n-4}$ when $n > 4$. Obviously, the special case (16) satisfies Assumptions 2.1 and 2.2, and the energy function (7) can be rewritten as

$$E(t) = \frac{1}{2} \|u_t(t)\|^2 + \frac{1}{2} \|\Delta u(t)\|^2 + \frac{\alpha}{2} \|\nabla u(t)\|^2 + \frac{1}{m} \|\nabla u(t)\|_m^m - \frac{1}{p} \|u(t)\|_p^p \quad (17)$$

for every $t \in [0, T_{\max})$.

Here we note that the potential well theory is also used for Eq. (1) without the damping term in [5].

Theorem 2.5. Let f and θ_i be defined as in (16). If an initial data $(u_0, u_1) \in B \times L^2(\Omega)$ satisfies the following conditions:

$$\|u_0\|_B < s_0,$$

$$E(0) < E_0,$$

where

$$s_0 = \kappa_p^{-\frac{2p}{p-2}}, \quad (18)$$

$$E_0 = \frac{p-2}{2p} \kappa_p^{-\frac{2p}{p-2}}, \quad (19)$$

then the corresponding solution $u(t)$ of Eq. (1) exists globally on $[0, +\infty)$.

Furthermore, the solution $u(t)$ of Eq. (1) satisfies the following exponential energy decay

$$E(t) \leq E(0) \exp(1 - \delta t)$$

for $t \geq \delta^{-1}$, where

$$\delta = \frac{1 - \kappa_p^p \left(\frac{2p}{p-2} E(0) \right)^{\frac{p-2}{2}}}{1 + \frac{2p}{p-2} (2 + \kappa_2^2)}.$$

Remark 2.6. For Eq. (1) with (16) we can establish a blow-up result under the following condition

$$\|u_0\| > s_0,$$

$$E(0) < E_0,$$

which implies that we obtain a threshold result of the global existence and blow-up of Eq. (1) with $E(0) < E_0$.

Remark 2.7. Consider the following fourth order wave equation

$$u_{tt} + u_t + \Delta^2 u - \alpha \Delta u + \sum_{i=1}^n \frac{\partial}{\partial x_i} (\theta_i(u_{x_i})) = f(u), \quad x \in \Omega, \quad t > 0. \quad (20)$$

By a similar way in the following proofs of Theorems 2.3–2.5, we can also obtain similar results of blow-up and global existence for Eq. (20), respectively.

3. Proofs of Theorems 2.3 and 2.4

Before we prove these two theorems, the following lemmas are first introduced.

Lemma 3.1. Suppose that $h(t)$ is twice continuously differential satisfying

$$\begin{cases} h''(t) + h'(t) > 0, & t > 0 \\ h(0) > 0, & h'(0) \geq 0, \end{cases} \quad (21)$$

then $h(t)$ is strictly increasing for $t \geq 0$.

The proof of this lemma is very simple and omitted here.

Lemma 3.2. Assume that $(u_0, u_1) \in B \times L^2(\Omega)$ satisfies

$$\int_{\Omega} u_0 u_1 dx \geq 0. \quad (22)$$

If the corresponding solution $u(t)$ of Eq. (1) satisfies

$$I(u(t)) < 0 \quad (23)$$

for every $t \in [0, T_{\max})$, then the function $\|u(t)\|^2$ is strictly increasing on $[0, T_{\max})$.

Proof. Since $u(t)$ is a solution of Eq. (1), we then have

$$\begin{aligned} \frac{1}{2} \frac{d^2}{dt^2} \|u(t)\|^2 &= \|u_t(t)\|^2 - \int_{\Omega} u(t) u_t(t) dx + \int_{\Omega} u(t) f(u(t)) dx - \left(\|\Delta u\|^2 + \alpha \|\nabla u\|^2 + \sum_{i=1}^n \int_{\Omega} u_{x_i} \theta_i(u_{x_i}) dx \right) \\ &= \|u_t(t)\|^2 - \int_{\Omega} u(t) u_t(t) dx - I(u(t)) \end{aligned}$$

for every $t \in [0, T_{\max})$.

As a result, it follows from (23) that

$$\frac{d^2}{dt^2} \|u(t)\|^2 + \frac{d}{dt} \|u(t)\|^2 = 2\|u_t(t)\|^2 - 2I(u(t)) > 0$$

for every $t \in [0, T_{\max})$.

Thus, by Lemma 3.1, our desired result is obtained. \square

Proof of Theorem 2.3. First, we assume that the solution $u(t)$ globally exists, that is, $T_{\max} = \infty$. We next define a function

$$\phi(t) = \int_{\Omega} |u(t, x)|^2 dx + \int_0^t \|u(\tau, \cdot)\|^2 d\tau + (T_0 - t)\|u_0\|^2 + \omega(T_1 + t)^2 \quad (24)$$

where the positive constants T_0, T_1 and ω will be determined later.

After direct calculation, we have

$$\begin{aligned} \phi'(t) &= 2 \int_{\Omega} u(t)u_t(t)dx + \|u(t)\|^2 - \|u_0\|^2 + 2\omega(T_1 + t) \\ &= 2 \int_{\Omega} u(t)u_t(t)dx + 2 \int_0^t (u(\tau), u_{\tau}(\tau))d\tau + 2\omega(T_1 + t) \end{aligned} \quad (25)$$

and

$$\begin{aligned} \phi''(t) &= 2\|u_t(t)\|^2 + 2 \int_{\Omega} u(t)u_{tt}(t)dx + 2 \int_{\Omega} u(t)u_t(t)dx + 2\omega \\ &= 2\|u_t(t)\|^2 + 2 \int_{\Omega} u(t)f(u(t))dx - 2 \left(\|\Delta u\|^2 + \alpha \|\nabla u\|^2 + \sum_{i=1}^n \int_{\Omega} u_{x_i} \theta_i(u_{x_i})dx \right) + 2\omega. \end{aligned} \quad (26)$$

Second, if $E(0) < 0$ holds, then by (6), (7) and Assumption 2.1 we have

$$\begin{aligned} \int_{\Omega} F(u(t, x))dx &= \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2 + \frac{\alpha}{2}\|\nabla u(t)\|^2 + \sum_{i=1}^n \int_{\Omega} \Theta_i(u_{x_i}(t, x))dx - E(t) \\ &= \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2 + \frac{\alpha}{2}\|\nabla u(t)\|^2 + \sum_{i=1}^n \int_{\Omega} \Theta_i(u_{x_i}(t, x))dx - E(0) + \int_0^t \|u_{\tau}(\tau)\|^2 d\tau \\ &\leq \frac{1}{2+\epsilon} \int_{\Omega} u(t)f(u(t))dx. \end{aligned} \quad (27)$$

Hence it follows from (26) and (27) that

$$\begin{aligned} \phi''(t) &\geq (4 + \epsilon)\|u_t(t)\|^2 + 2(2 + \epsilon) \int_0^t \|u_{\tau}(\tau)\|^2 d\tau + \epsilon (\|\Delta u\|^2 + \alpha \|\nabla u\|^2) \\ &\quad - 2(2 + \epsilon)E(0) - 2 \sum_{i=1}^n \int_{\Omega} u_{x_i} \theta_i(u_{x_i})dx + 2(2 + \epsilon) \int_{\Omega} \Theta_i(u_{x_i}(t))dx + 2\omega \\ &\geq (4 + \epsilon)\|u_t(t)\|^2 + 2(2 + \epsilon) \int_0^t \|u_{\tau}(\tau)\|^2 d\tau + \epsilon (\|\Delta u(t)\|^2 + \alpha \|\nabla u(t)\|^2) - 2(2 + \epsilon)E(0) + 2\omega \end{aligned} \quad (28)$$

for every $t \in [0, +\infty)$, where the last inequality follows from Assumption 2.2.

Taking $\omega = -2E(0)$, we then obtain the inequality

$$\phi''(t) \geq (4 + \epsilon)\|u_t(t)\|^2 + 2(2 + \epsilon) \int_0^t \|u_{\tau}(\tau)\|^2 d\tau + (4 + \epsilon)\omega \geq 0$$

for every $t \in [0, +\infty)$.

Now we take T_1 sufficiently large such that

$$\phi'(0) > 0$$

and

$$\frac{\epsilon}{2} \left(\int_{\Omega} u_0(x)u_1(x)dx + \omega T_1 \right) > \int_{\Omega} |u_0(x)|^2 dx.$$

Setting

$$a = \int_{\Omega} |u(t, x)|^2 dx + \int_0^t \|u(\tau)\|^2 d\tau + \omega(T_1 + t)^2,$$

$$b = \frac{1}{2}\phi'(t),$$

$$c = \|u_t(t)\|^2 + \int_0^t \|u_{\tau}(\tau)\|^2 d\tau + \omega,$$

we have $\phi(t) \geq a$ for every $t \in [0, T_0]$, where T_0 may be chosen large enough such that

$$T_0 \geq \frac{4\phi(0)}{\epsilon\phi'(0)}.$$

Since $T_{\max} = \infty$, and $\phi''(t) \geq (4 + \epsilon)c$, we then have

$$\phi''(t)\phi(t) - \frac{4 + \epsilon}{4}(\phi'(t))^2 \geq (4 + \epsilon)(ac - b^2)$$

for every $t \in [0, T_0]$.

By simple calculation we see that

$$\begin{aligned} as^2 - 2bs + c &= \int_{\Omega} (su(t) + u_t(t))^2 dx + \int_0^t \|su(\tau) + u_\tau(\tau)\|^2 d\tau + \omega(s(T_1 + t) + 1)^2 \\ &\geq 0, \end{aligned}$$

for every $s \in \mathbb{R}$ and $t \in [0, T_0]$, from which, it follows that $b^2 - ac \leq 0$. Therefore, one has

$$\phi''(t)\phi(t) - \frac{4 + \epsilon}{4}(\phi'(t))^2 \geq 0$$

for every $t \in [0, T_0]$.

Put $\alpha = \frac{\epsilon}{4}$, and a direct calculation gives

$$\frac{d}{dt}\phi^{-\alpha}(t) = -\alpha\phi^{-\alpha-1}(t)\phi'(t) < 0, \quad (29)$$

$$\begin{aligned} \frac{d^2}{dt^2}\phi^{-\alpha}(t) &= -\alpha\phi^{-\alpha-2}(t)\left(\phi''(t)\phi(t) - \frac{4 + \epsilon}{4}(\phi'(t))^2\right) \\ &\leq 0 \end{aligned} \quad (30)$$

for every $t \in [0, T_0]$, which implies that $\phi^{-\alpha}(t)$ is concave.

Since $\phi(0) > 0$, by (29) and (30) we see that there exists T with $0 < T < \frac{4\phi(0)}{\epsilon\phi'(0)} \leq T_0$ such that $\phi^{-\alpha}(t) \rightarrow 0$ as $t \rightarrow T$ and $t < T$. Noting that $\phi(t)$ exists on $[0, T_0]$, we obtain a contradiction. Thus we have proved that when $E(0) < 0$, the corresponding solution $u(t)$ of Eq. (1) blows up in finite time, that is,

$$\lim_{t \rightarrow T_{\max}} \|u(t)\|^2 = \infty.$$

Finally, if $E(0) = 0$ with $\int_{\Omega} u_0 u_1 dx \geq 0$, by (6) and (7) we have

$$\frac{1}{2}\|\Delta u(t)\|^2 + \frac{\alpha}{2}\|\nabla u(t)\|^2 + \sum_{i=1}^n \int_{\Omega} \Theta_i(u_{x_i}(t, x)) dx - \int_{\Omega} F(u(t, x)) dx \leq 0$$

for every $t \in [0, +\infty)$.

Using Assumptions 2.1 and 2.2, we get

$$I(u(t)) < 0$$

for every $t \in [0, +\infty)$.

Thus, by Lemma 3.2 we see that $\|u(t)\|^2$ is strictly increasing on $[0, T_{\max})$. Now we consider the function ϕ defined as (24) again and obtain

$$\begin{aligned} \frac{1}{2}\phi''(t) &= \int_{\Omega} |u_t(t)|^2 dx - I(u(t)) + \omega \\ &> 0 \end{aligned}$$

for every $t \in [0, +\infty)$.

As (28) we also obtain

$$\begin{aligned} \phi''(t) &\geq (4 + \epsilon) \int_{\Omega} |u_t(t)|^2 dx + 2(2 + \epsilon) \int_0^t \|u_\tau(\tau)\|^2 d\tau + \epsilon(\|\Delta u(t)\|^2 + \alpha\|\nabla u(t)\|^2) + 2\omega \\ &\geq (4 + \epsilon) \int_{\Omega} |u_t(t)|^2 dx + 2(2 + \epsilon) \int_0^t \|u_\tau(\tau)\|^2 d\tau + \epsilon\kappa_2^{-1}\|u_0\|^2 + 2\omega, \end{aligned}$$

where the last inequality uses the fact that $\|u(t)\|^2$ is strictly increasing on $[0, +\infty)$.

We now choose ω to satisfy $0 < \omega < \frac{\epsilon}{2+\epsilon} \|u_0\|^2$, and two constants T_0, T_1 be large enough such that

$$T_0 \geq \frac{4\phi(0)}{\epsilon\phi'(0)},$$

$$\frac{\epsilon}{2} \left(\int_{\Omega} u_0 u_1 dx + \omega T_1 \right) > \int_{\Omega} |u_0|^2 dx.$$

Following a similar argument as in the case $E(0) < 0$, we can also obtain the blow-up result for the case $E(0) = 0$ with $\int_{\Omega} u_0 u_1 dx \geq 0$. \square

In order to show Theorem 2.3, we need the following lemma.

Lemma 3.3. Under Assumptions 2.1 and 2.2, if the initial data (u_0, u_1) satisfies (12)–(15), then the corresponding solution $u(t)$ of Eq. (1) satisfies

$$I(u(t)) < 0, \quad (31)$$

$$\|u(t)\|^2 \geq \frac{2\kappa_2^2(2+\epsilon)}{\epsilon} E(0) \quad (32)$$

for every $t \in [0, T_{\max})$.

Proof. We use a contradiction argument to prove this lemma. Assume that (31) is false, that is to say, there exists a time $T > 0$ such that

$$T = \min\{t \in (0, T_{\max}) : I(u(t)) = 0\},$$

which implies $I(u(t)) < 0$ for every $t \in [0, T)$.

By Lemma 3.2 we see that $\|u(t)\|^2$ is strictly increasing on $[0, T)$. As a result, we have

$$\|u(t)\|^2 > \|u_0\|^2 > \frac{2\kappa_2^2(2+\epsilon)}{\epsilon} E(0)$$

for every $t \in [0, T)$.

And by the continuity of $\|u(t)\|^2$, we have

$$\|u(T)\|^2 > \frac{2\kappa_2^2(2+\epsilon)}{\epsilon} E(0). \quad (33)$$

On the other hand, it follows from (6) and (7) that

$$\|\Delta u(t)\|^2 + \alpha \|\nabla u(t)\|^2 + 2 \sum_{i=1}^n \int_{\Omega} \Theta_i(u_{x_i}(t, x)) dx - 2 \int_{\Omega} F(u(t, x)) dx \leq 2E(t) \leq 2E(0). \quad (34)$$

Since $I(u(T)) = 0$, by Assumption 2.1 we have

$$\|\Delta u(T)\|^2 + \alpha \|\nabla u(T)\|^2 + \sum_{i=1}^n \int_{\Omega} u_{x_i}(T) \theta_i(u_{x_i}(T)) dx \geq (2+\epsilon) \int_{\Omega} F(u(T)) dx. \quad (35)$$

Assumption 2.2, (34) and (35) imply that

$$\|\Delta u(T)\|^2 + \alpha \|\nabla u(T)\|^2 \leq \frac{2(2+\epsilon)}{\epsilon} E(0), \quad (36)$$

from which, by the embedding $B \hookrightarrow L^2(\Omega)$, it follows that

$$\|u(T)\|^2 \leq \frac{2\kappa_2^2(2+\epsilon)}{\epsilon} E(0). \quad (37)$$

Obviously there exists a contradiction between (33) and (37). Therefore, we have proved that

$$I(u(t)) < 0$$

for every $t \in [0, T_{\max})$.

In addition, by Lemma 3.2, we see that $\|u(t)\|^2$ is strictly increasing on $[0, T_{\max})$. Thus,

$$\|u(t)\|^2 \geq \frac{2\kappa_2^2(2+\epsilon)}{\epsilon} E(0)$$

for every $t \in [0, T_{\max})$. \square

Lemma 3.3 implies that the following set V is invariant under the flow of Eq. (1):

$$V = \{(u_0, u_1) \in B \times L^2(\Omega) : (u_0, u_1) \text{ satisfies the conditions (12)–(15)}\}.$$

Proof of Theorem 2.4. Assume that the solution $u(t)$ of Eq. (1) with the initial data satisfying (12)–(15) globally exists, that is to say, $T_{\max} = +\infty$. We let $\phi(t)$ be defined as (24).

Since

$$\phi''(t) = 2\|u_t(t)\|^2 - 2I(u(t)) + 2\omega,$$

by **Lemma 3.3** we see that $\phi''(t) > 0$ for every $t \in [0, +\infty)$.

And by (13) we have that $\phi'(0) \geq 0$. Thus, the functions, $\phi(t)$ and $\phi'(t)$, are strictly increasing on $[0, +\infty)$.

Using the following fact

$$\begin{aligned} & \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2 + \frac{\alpha}{2}\|\nabla u(t)\|^2 + \sum_{i=1}^n \int_{\Omega} \Theta_i(u_{x_i}(t, x)) dx - E(0) + \int_0^t \|u_{\tau}(\tau)\|^2 d\tau \\ & \leq \frac{1}{2+\epsilon} \int_{\Omega} u(t)f(u(t)) dx, \end{aligned}$$

we have

$$\begin{aligned} \phi''(t) & \geq (4+\epsilon)\|u_t(t)\|^2 + 2(2+\epsilon) \int_0^t \|u_{\tau}(\tau)\|^2 d\tau + \epsilon(\|\Delta u(t)\|^2 + \alpha\|\nabla u(t)\|^2) - 2(2+\epsilon)E(0) + 2\omega \\ & \geq (4+\epsilon)\|u_t(t)\|^2 + 2(2+\epsilon) \int_0^t \|u_{\tau}(\tau)\|^2 d\tau + \epsilon\kappa_2^{-2}\|u_0\|^2 - 2(2+\epsilon)E(0) + 2\omega \end{aligned}$$

where the second inequality comes from **Lemma 3.2**.

By (15),

$$\epsilon\kappa_2^{-2}\|u_0\|^2 - 2(2+\epsilon)E(0) + 2\omega > 0.$$

Thus, we choose ω such that

$$0 < \omega < \frac{\epsilon}{(2+\epsilon)\kappa_2^2} \|u_0\|^2 - 2E(0).$$

And we let two constants, T_0 and T_1 , be large enough such that

$$\begin{aligned} T_0 & \geq \frac{4\phi(0)}{\epsilon\phi'(0)}, \\ \frac{\epsilon}{2} \left(\int_{\Omega} u_0 u_1 dx + \omega T_1 \right) & > \int_{\Omega} |u_0|^2 dx. \end{aligned}$$

Let a , b and c denote the same functions as in the proof of **Theorem 2.3**. Then we have $\phi(t) \geq a$ and $\phi''(t) \geq (4+\epsilon)c$ for every $t \in [0, T_0]$. Thus, by a similar way as in the proof of **Theorem 2.3** we have that

$$\phi(t)\phi''(t) - \frac{4+\epsilon}{4}(\phi'(t))^2 \geq (4+\epsilon)(ac - b^2) \geq 0.$$

As the proof of **Theorem 2.2**, by the standard concavity argument we obtain our desired result. \square

4. Proof of Theorem 2.5

We begin this section with the next lemma.

Lemma 4.1. Suppose that θ_i and f satisfy (16). If the initial data (u_0, u_1) satisfies

$$\begin{aligned} \|u_0\|_B & < s_0, \\ E(0) & < E_0, \end{aligned}$$

where s_0 and E_0 are defined as (18) and (19), respectively, then the corresponding solution $u(t)$ of Eq. (1) satisfies

$$\begin{aligned} \|u(t)\|_B & < s_0, \\ E(t) & < E_0 \end{aligned}$$

for every $t \in [0, T_{\max})$.

Proof. From (6) and (17), it follows that

$$\begin{aligned} E(t) &= \frac{1}{2} \|u_t(t)\|^2 + \frac{1}{2} \|\Delta u(t)\|^2 + \frac{\alpha}{2} \|\nabla u(t)\|^2 + \frac{1}{m} \|\nabla u(t)\|_m^m - \frac{1}{p} \|u(t)\|_p^p \\ &\geq \frac{1}{2} \|u(t)\|_B^2 - \frac{1}{p} \|u(t)\|_p^p. \end{aligned}$$

By the continuous embedding $B \hookrightarrow L^p(\Omega)$, we have

$$E(t) \geq \Phi(\|u(t)\|_B)$$

where $\Phi(s)$ is defined as

$$\Phi(s) = \frac{1}{2} s^2 - \frac{\kappa_p^p}{p} s^p. \quad (38)$$

Obviously $\Phi(s)$ ($s \in \mathbb{R}^+ \cup \{0\}$) has the following property:

$$\begin{cases} \Phi(s) \text{ is strictly increasing on } [0, s_0), \\ \Phi(s) \text{ takes its maximum value } E_0 \text{ at } s_0, \\ \Phi(s) \text{ is strictly decreasing on } (s_0, \infty), \end{cases}$$

where s_0 and E_0 are defined as (18) and (19), respectively.

By (6) we see that $E(t) < E_0$. We next prove $\|u(t)\|_B \in [0, s_0)$ by a contradiction argument.

Suppose not, using the continuity of $u(t, x)$ as a function of t , then there exists a time t_0 such that

$$\|u(t_0)\|_B = s_0,$$

which implies that $E(t_0) \geq \Phi(s_0) = E_0$; this contradicts $E(t) < E_0$.

Thus we see that $\|u(t)\|_B \in [0, s_0)$. \square

Proof of Theorem 2.5. By Lemma 4.1,

$$\begin{aligned} \|u(t)\|^2 &\leq c_1 \|u(t)\|_B^2 \\ &\leq c_1 s_0^2, \end{aligned}$$

where $c_1 > 0$ is a constant. Thus, using the standard continuation argument, we see that $T_{\max} = +\infty$.

We next prove exponential energy decay of Eq. (1). Since $p > 2$, we then have

$$\begin{aligned} I(u(t)) &= \|\Delta u(t)\|^2 + \alpha \|\nabla u(t)\|^2 + \|\nabla u(t)\|_m^m - \|u(t)\|_p^p \\ &\geq 2\Phi(\|u(t)\|_B), \end{aligned}$$

from which, we deduce that, if $\|u(t)\|_B \in (0, s_0)$, then $I(u(t)) \geq 0$, where Φ is defined by (38). By Lemma 4.1 we see that $\|u(t)\|_B < s_0$ for every $t \in [0, T_{\max})$. Thus $I(u(t)) \geq 0$ for every time $t \in [0, \infty)$.

Using $I(u(t)) \geq 0$ and $2 < m < p$, we then have

$$\begin{aligned} E(0) &\geq E(t) \\ &\geq \left(\frac{1}{2} \|\Delta u(t)\|^2 + \frac{\alpha}{2} \|\nabla u(t)\|^2 + \frac{1}{m} \|\nabla u(t)\|_m^m \right) - \frac{1}{p} (\|\Delta u(t)\|^2 + \alpha \|\nabla u(t)\|^2 + \|\nabla u(t)\|_m^m) \\ &\geq \left(\frac{1}{2} - \frac{1}{p} \right) \|u(t)\|_B^2. \end{aligned}$$

From the above inequality, we deduce that

$$\|u(t)\|_B^2 \leq \frac{2p}{p-2} E(t) \leq \frac{2p}{p-2} E(0), \quad (39)$$

which leads to

$$\begin{aligned} \left(1 - \frac{2}{p}\right) \|u(t)\|_p^p &\leq \left(1 - \frac{2}{p}\right) \kappa_p^p \|u(t)\|_B^p \\ &\leq 2\kappa_p^p \left(\frac{2p}{p-2} E(0)\right)^{\frac{p-2}{2}} E(t) \\ &\leq 2(1 - \varsigma) E(t), \end{aligned} \quad (40)$$

where $\varsigma = 1 - \kappa_p^p \left(\frac{2p}{p-2} E(0)\right)^{\frac{p-2}{2}}$. Noting that $E(0) < E_0 = \frac{p-2}{2p} \kappa_p^{-\frac{2p}{p-2}}$, we have $\varsigma > 0$.

Multiplying Eq. (1) by $u(t)$ and integrating over $\Omega \times [t_1, t_2]$, we have

$$\begin{aligned}
 0 &= \int_{\Omega} \int_{t_1}^{t_2} u(t) \left(u_{tt} + u_t + \Delta^2 u - \alpha \Delta u - \sum_{i=1}^n \frac{\partial}{\partial x_i} (|u_{x_i}(t)|^{m-2} u_{x_i}(t)) - |u(t)|^{p-2} u(t) \right) dt dx \\
 &= \left[\int_{\Omega} u(t) u_t(t) dx \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \|u_t(t)\|^2 dt + \int_{t_1}^{t_2} \int_{\Omega} u(t) u_t(t) dx dt \\
 &\quad + \int_{t_1}^{t_2} (\|\Delta u(t)\|^2 + \alpha \|\nabla u(t)\|^2 + \|\nabla u(t)\|_m^m - \|u(t)\|_p^p) dt \\
 &= \left[\int_{\Omega} u(t) u_t(t) dx \right]_{t_1}^{t_2} - 2 \int_{t_1}^{t_2} \|u_t(t)\|^2 dt + \int_{t_1}^{t_2} \int_{\Omega} u(t) u_t(t) dx dt \\
 &\quad + \int_{t_1}^{t_2} 2E(t) dt + \int_{t_1}^{t_2} \int_{\Omega} \left[\left(\frac{2}{p} - 1 \right) |u(t)|^p + \left(1 - \frac{2}{m} \right) |\nabla u|^m \right] dx dt. \tag{41}
 \end{aligned}$$

Thus, by (40) and (41) we have

$$2\zeta \int_{t_1}^{t_2} E(t) dt \leq \int_{\Omega} |u(t_1) u_t(t_1)| dx + \int_{\Omega} |u(t_2) u_t(t_2)| dx + 2 \int_{t_1}^{t_2} \|u_t(t)\|^2 dt + \left| \int_{t_1}^{t_2} \int_{\Omega} u(t) u_t(t) dx dt \right|. \tag{42}$$

Next we consider the integral term $\int_{\Omega} u(t) u_t(t) dt$. First, by Hölder's inequality and the continuous embedding $B \hookrightarrow L^2(\Omega)$ we have

$$\begin{aligned}
 \int_{\Omega} |u(t) u_t(t)| dx &\leq \int_{\Omega} |u(t)|^2 dx + \int_{\Omega} |u_t(t)|^2 dx \\
 &\leq \lambda \|u(t)\|_B^2 + \|u_t(t)\|^2,
 \end{aligned}$$

where $\lambda > 0$ is a constant.

Further, from (39) it follows that

$$\int_{\Omega} |u(t) u_t(t)| dx \leq c_2 E(t),$$

where $c_2 > 0$ is a constant.

This implies that

$$\int_{\Omega} |u(t_1) u_t(t_1)| dx + \int_{\Omega} |u(t_2) u_t(t_2)| dx \leq 2c_2 E(t_1).$$

We also have the following estimate

$$\begin{aligned}
 2 \int_{t_1}^{t_2} \int_{\Omega} u(t) u_t(t) dx dt &= \int_{t_1}^{t_2} \int_{\Omega} \frac{d}{dt} \|u(t)\|^2 dx dt \\
 &= \|u(t_2)\|^2 - \|u(t_1)\|^2 \\
 &\leq \frac{4c_1 p}{p-2} E(t_1),
 \end{aligned}$$

where the last inequality follows from the fact that $E(t)$ is non-increasing on $[0, \infty)$.

In addition, from (6) we deduce that

$$\int_{t_1}^{t_2} \|u_t(t)\|^2 dt \leq E(t_1).$$

By (42) we then obtain

$$\int_{t_1}^{t_2} E(t) dt \leq \frac{c_3}{5} E(t_1)$$

where $c_3 > 0$ is a constant, which implies that

$$\int_t^\infty E(\tau) d\tau \leq \delta^{-1} E(t)$$

where $\delta = \left(1 - \kappa_p^p \left(\frac{2p}{p-2} E(0)\right)^{\frac{p-2}{2}}\right) c_3^{-1}$.

By Lemma 2.1 we have that

$$E(t) \leq E(0) \exp(1 - \delta t)$$

for $t \geq \delta$.

The proof is completed. \square

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