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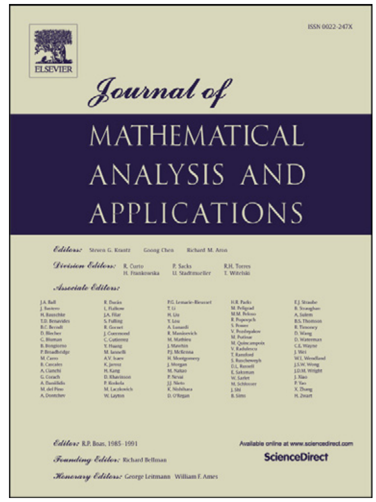

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The local well-posedness and global solution for a modified periodic two-component Camassa-Holm system

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Abstract

The local well-posedness and global solution for a modified periodic two-component Camassa-Holm system on the circle $\mathbb{S}$ are established in the Soblev space $H^s \times H^{s-2}$ with $s > \frac{7}{2}$, which are different from that of the two-component Camassa-Holm system.

MSC: 35D05, 35G25, 35L05, 35Q35.

Key words: A modified periodic two-component Camassa-Holm system; Local well-posedness; Global solution.

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1 Introduction

In this article, we will consider a modified periodic two-component Camassa-Holm system on the circle $\mathbb{S}$ with $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ (the circle of unit lengh).

$$\begin{cases} m_t + um_x + 2u_x m + \rho\rho_x = 0, \quad t > 0, x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, \quad t > 0, x \in \mathbb{R}, \\ m(0,x) = m_0(x), \quad x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), \quad x \in \mathbb{R}, \\ m(t, x+1) = m(t,x), \quad t > 0, x \in \mathbb{R}, \\ \rho(t, x+1) = \rho(t,x), \quad t > 0, x \in \mathbb{R} \end{cases} \tag{1}$$

where $m = (1 - 2\partial_x^2 + \partial_x^4)u = (1 - \partial_x^2)^2 u$ and $\mathbb{R}$ is real number. In fact, system (1) is a two-component generalization of the following equation (If $\rho = 0$ in system (1))

$$m_t + um_x + 2u_x m = 0, \quad m = (1 - \partial_x^2)^2 u. \tag{2}$$

Eq.(2) is firstly derived as the Euler-Poincare differential equation on the Bott-Virasoro group with respect to the H^2 metric [1], and it is known as a modified Camassa-Holm equation and also viewed as a geodesic equation on some diffeomorphism group [1]. It is shown in [1] that the well-posedness and dynamics of equation (2) on the unit circle $\mathbb{S}$ are significant different from that of the Camassa-Holm equation. For example, Eq.(2) does not conform with blow-up solution in finite time.

If $m = (1 - \partial_x^2)u$ in system (1), system (1) becomes the two-component Camassa-Holm system.

$$\begin{cases} (1 - \partial_x^2)u_t + u(1 - \partial_x^2)u_x + 2u_x(1 - \partial_x^2)u + \rho\rho_x = 0, \quad t > 0, x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, \quad t > 0, x \in \mathbb{R}, \end{cases} \tag{3}$$

where the variable $u(t,x)$ represents the horizontal velocity of the fluid, and $\rho(t,x)$ is related to the free surface elevation from equilibrium with the boundary assumptions, $u \to 0$ and $\rho \to 1$ as $\mid x \mid \to \infty$. System (3) was found originally in [2], but it was firstly derived rigorously by Constantin and Ivanov [3]. The system has bi-Hamiltonian structure and is complete integrability. Since the birth of the system, a lot of literature was devoted to the study of the two-component Camassa-Holm system. Some mathematical and physical properties of the system have been obtained. Chen et al.[4] established a reciprocal transformation between the two-component Camassa-Holm system and the first negative flow of the AKNS hierarchy. Escher et al.[5] used Kato's theory to establish local well-posedness for the two-component system and presented some precise blow-up scenarios for strong solutions of the system. In [3], Constantin and Ivanov described the sufficient conditions for wave-breaking and global solution to the system. It is worthwhile to mention that the wave-breaking criterions of strong solutions are established in the Soblev space $H^s \times H^{s-1}$ with $s > \frac{5}{2}$ and some examples are given to illustrate the application of the results [6]. The other results related to the system can be found in [7–17].

The main goal of present paper is to study the local well-posedness and global existence for the modified periodic two-component Camassa-Holm system (1). We use the Kato's theory [18] to prove the local well-posedness theorem in in the Soblev space $H^s \times H^{s-2}$ with $s > \frac{7}{2}$. On the other hand, we derive a sufficient condition for global solution in the Soblev space $H^s \times H^{s-2}$ with $s > \frac{7}{2}$, which can be done because $\| u_{xxx} \|_{L^\infty}$ and $\| \rho_x \|_{L^\infty}$ can be controlled by $\| u \|_{H^s}$ and $\| \rho \|_{H^{s-2}}$ separately if $s > \frac{7}{2}$.

2 The main results

We denote by $*$ the convolution and let $[A, B] = AB - BA$ denote the commutator between A and B. Note that if $g(x) := 1 + 2\sum_{n=1}^{\infty} \frac{1}{1+2n^2+n^4} \cos(nx)$, then $(1 - \partial_x^2)^{-2} f = g * f$ for all $f \in L^2(R)$ and $g * m = u$. We let C denote all of different positive constants which depend on initial data. To investigate dynamics of system(1), we can rewrite system (1) in the form

$$\begin{cases} u_t + uu_x + \partial_x g * [u^2 + u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x u_{xxx} + \frac{1}{2}\rho^2] = 0, t > 0, x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, \quad t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), \quad x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), \quad x \in \mathbb{R}, \\ u(t, x + 1) = u(t, x), \quad t > 0, x \in \mathbb{R}, \\ \rho(t, x + 1) = \rho(t, x), \quad t > 0, x \in \mathbb{R}. \end{cases} \tag{4}$$

The main results of present paper are listed as follows.

Theorem 2.1. *Given $z_0 = (u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-2}(\mathbb{S})(s > \frac{7}{2})$, there exist a maximal $T = T(\| z_0 \|_{H^s(\mathbb{S})\times H^{s-2}(\mathbb{S})})$ and a unique solution $z = (u, \rho)$ to problem (4), such that*

$$z = z(\cdot, z_0) \in C\Big([0, T); H^s \times H^{s-2}\Big) \cap C^1\Big([0, T); H^{s-1} \times H^{s-3}\Big).$$

Moreover, the solution depends continously on the initial data, the mapping

$$\begin{aligned} z_0 \to z(\cdot, z_0) : H^s \times H^{s-2} \\ \to C\Big([0, T); H^s \times H^{s-2}\Big) \cap C^1\Big([0, T); H^{s-1} \times H^{s-3}\Big) \end{aligned}$$

is continuous.

A sufficient condition of global existence is given in the following.

Theorem 2.2. *Let $z_0 = (u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-2}(\mathbb{S})$, $s > \frac{7}{2}$. Then system (4) admits a unique solution satisfying*

$$z = (u, \rho) \in C\Big([0, \infty); H^s \times H^{s-2}\Big) \cap C^1\Big([0, \infty); H^{s-1} \times H^{s-3}\Big).$$

3 Local well-posedness

In this section, we establish the local well-posedness for system (1) by using the Kato's theory [18].

Set $Y = H^s(\mathbb{S}) \times H^{s-2}(\mathbb{S})$, $X = H^{s-1}(\mathbb{S}) \times H^{s-3}(\mathbb{S})$, $\Lambda = (1 - \partial_x^2)^{\frac{1}{2}}$, $Q = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$ and $f(z) = \begin{pmatrix} -\partial_x(1-\partial_x^2)^{-2}(u^2 + u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x u_{xxx} + \frac{1}{2}\rho^2) \\ -u_x\rho \end{pmatrix}$.

In order to verify Theorem 2.1, we need the following Lemmas in which μ_1, μ_2, μ_3 and μ_4 are constants depending only on $\max\{\| z \|_Y, \| y \|_Y\}$.

Lemma 3.1. ([19]) If X_1 and X_2 are Banach spaces and $A_i \in G(X_i, 1, \beta)$, $i = 1, 2$. Then the operator $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ with $D(A) = D(A_1) \times D(A_2)$.

Let T(t) be a C_0-semigroup on X with generator -A and assume that Y is continuously embedded in X. We say that Y is A-admissible if $T(t)Y \subset Y$ for all $t \geq 0$ and the restriction of $T(t)$ to Y is a C_0-semigroup on Y.

For later purpose we need the following result.

Lemma 3.2. ([20]) The operator $A(u) = u\partial_x$ with $u \in H^s$, $s > \frac{3}{2}$ belongs to $G(H^s, 1, \beta)$.

From Lemmas 3.1-3.2, we have the following lemma.

Lemma 3.3. The operator $A(z) = \begin{pmatrix} u\partial_x & 0 \\ 0 & u\partial_x \end{pmatrix}$ belongs to $G(H^{s-1}(\mathbb{S}) \times H^{s-3}(\mathbb{S}), 1, \beta)$.

Lemma 3.4. Let $A(z) = \begin{pmatrix} u\partial_x & 0 \\ 0 & u\partial_x \end{pmatrix}$. Then $A(z) \in L(H^s(\mathbb{S}) \times H^{s-2}(\mathbb{S}), H^{s-1}(\mathbb{S}) \times H^{s-3}(\mathbb{S}))$. Moreover, for all $z, y, w \in H^s(\mathbb{S}) \times H^{s-2}(\mathbb{S})$,

$$\| (A(z) - A(y))w \|_{H^{s-1}\times H^{s-3}} \leq \mu_1 \| z - y \|_{H^{s-1}\times H^{s-3}} \| w \|_{H^s \times H^{s-2}} .$$

Proof. Let $z, y, w \in H^s \times H^{s-2}$, $s > \frac{7}{2}$. Then

$$\begin{aligned} (A(z) - A(y))w &= \begin{pmatrix} (u-v)\partial_x & 0 \\ 0 & (u-v)\partial_x \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &= \begin{pmatrix} (u-v)\partial_x w_1 \\ (u-v)\partial_x w_2 \end{pmatrix}. \end{aligned}$$

Thus, we have

$$\begin{aligned} &\| (A(z) - A(y))w \|_{H^{s-1}\times H^{s-3}} \\ &\leq \| (u-v)\partial_x w_1 \|_{H^{s-1}} + \| (u-v)\partial_x w_2 \|_{H^{s-3}} \\ &\leq \| u - v \|_{H^{s-1}} \| \partial_x w_1 \|_{H^{s-1}} + \| u - v \|_{H^{s-3}} \| \partial_x w_2 \|_{H^{s-3}} \\ &\leq C \| z - y \|_{H^{s-1}\times H^{s-3}} \| w \|_{H^s\times H^{s-2}} . \end{aligned}$$

Taking $y = 0$ in the above inequality, we obtain that $A(z) \in L(H^s(\mathbb{S}) \times H^{s-2}(\mathbb{S}), H^{s-1}(\mathbb{S}) \times H^{s-3}(\mathbb{S}))$. This completes the proof of Lemma 3.4.

Lemma 3.5. ([21]) Let $f \in H^s$, $s > \frac{3}{2}$. Then

$$\| \Lambda^{-r}[\Lambda^{r+t+1}, M_f]\Lambda^{-t} \|_{L^2} \leq C \| f \|_{H^s}, |r|, |t| \leq s - 1,$$

where M_f is the operator of multiplication by f and C is a positive constant depending only on r, t.

Lemma 3.6. ([18]) Let r, t be real numbers such that $-r < t \leq r$. Then

$$\| fg \|_{H^t} \leq C \| f \|_{H^r} \| g \|_{H^t}, \quad if \quad r > \frac{1}{2},$$
$$\| fg \|_{H^{t+r-\frac{1}{2}}} \leq C \| f \|_{H^r} \| g \|_{H^t}, \quad if \quad r < \frac{1}{2},$$

where C is a positive constant depending on r, t.

Lemma 3.7. For $s > \frac{7}{2}$, $z, y \in H^s(\mathbb{S}) \times H^{s-2}(\mathbb{S})$ and $w \in H^{s-1}(\mathbb{S}) \times H^{s-3}(\mathbb{S})$, it holds that $B(z) = QA(z)Q^{-1} - A(z) \in L(H^{s-1} \times H^{s-2})$ and

$$\| (B(z) - B(y))w \|_{H^{s-1} \times H^{s-3}} \leq \mu_2 \| z - y \|_{H^s \times H^{s-2}} \| w \|_{H^{s-1} \times H^{s-3}} .$$

Proof. Let $z, y \in H^s \times H^{s-2}$, $w \in H^{s-1} \times H^{s-3}$, $s > \frac{7}{2}$. Then

$$(B(z) - B(y))w = \begin{pmatrix} \Lambda(u-v)\partial_x \Lambda^{-1} w_1 - (u-v)\partial_x w_1 \\ \Lambda(u-v)\partial_x \Lambda^{-1} w_2 - (u-v)\partial_x w_2 \end{pmatrix}.$$

Thus, we have

$$\begin{aligned}
&\| (B(z) - B(y))w \|_{H^{s-1} \times H^{s-3}} \\
&\leq \| \Lambda(u-v)\partial_x \Lambda^{-1} w_1 - (u-v)\partial_x w_1 \|_{H^{s-1}} \\
&\qquad + \| \Lambda(u-v)\partial_x \Lambda^{-1} w_2 - (u-v)\partial_x w_2 \|_{H^{s-3}} \\
&\leq \| [\Lambda, (u-v)\partial_x]\Lambda^{-1} w_1 \|_{H^{s-1}} + \| [\Lambda, (u-v)\partial_x]\Lambda^{-1} w_2 \|_{H^{s-3}} \\
&\leq \| \Lambda^{s-1}[\Lambda, (u-v)\partial_x]\Lambda^{-1} w_1 \|_{L^2} + \| \Lambda^{s-3}[\Lambda, (u-v)\partial_x]\Lambda^{-1} w_2 \|_{L^2} \\
&\leq \| \Lambda^{s-1}[\Lambda, (u-v)]\Lambda^{1-s} \|_{L^2} \| \Lambda^{s-2}\partial_x w_1 \|_{L^2} \\
&\qquad + \| \Lambda^{s-3}[\Lambda, (u-v)]\Lambda^{3-s} \|_{L^2} \| \Lambda^{s-4}\partial_x w_2 \|_{L^2} \\
&\leq \| u - v \|_{H^{s-1}} \| w_1 \|_{H^{s-1}} + \| u - v \|_{H^{s-3}} \| w_2 \|_{H^{s-3}} \\
&\leq \mu_2 \| z - y \|_{H^s \times H^{s-2}} \| w \|_{H^{s-1} \times H^{s-3}} .
\end{aligned}$$

where we applied Lemma 3.5 with $r = 1 - s$ and $t = s - 1$ (with $r = 3 - s$ and $t = s - 3$). Taking $y = 0$ in the above inequality, we obtain $B(z) \in L(H^{s-1} \times H^{s-3})$. This completes the proof of Lemma 3.7.

Lemma 3.8. Let

$$f(z) = \begin{pmatrix} -\partial_x(1-\partial_x^2)^{-2}(u^2 + u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x u_{xxx} + \frac{1}{2}\rho^2) \\ -u_x \rho \end{pmatrix}.$$

Then $f(z)$ is bounded on bounded sets in $H^s(\mathbb{S}) \times H^{s-2}(\mathbb{S})$ with $s > \frac{7}{2}$ and satisfies

(a) $\| f(z) - f(y) \|_{H^s \times H^{s-2}} \leq \mu_3 \| z - y \|_{H^s \times H^{s-2}}, \quad z, y \in H^s \times H^{s-2}$.

(b) $\| (f(z) - f(y)) \|_{H^{s-1} \times H^{s-3}} \leq \mu_4 \| z - y \|_{H^{s-1} \times H^{s-3}}, \quad z, y \in H^{s-1} \times H^{s-3}$.

Proof. (a) Let $y = (v, \sigma)$, we have

$$\begin{aligned}
&\| f(z) - f(y) \|_{H^s \times H^{s-2}} \\
&\leq \| \partial_x (1 - \partial_x^2)^{-2} (u^2 + u_x^2 - \frac{7}{2} u_{xx}^2 - 3 u_x u_{xxx} - v^2 - v_x^2 + \frac{7}{2} v_{xx}^2 \\
&\quad + 3 v_x v_{xxx}) \|_{H^s} + \| \partial_x (1 - \partial_x^2)^{-2} (\frac{1}{2} \rho^2 - \frac{1}{2} \sigma^2) \|_{H^s} \\
&\quad + \| u_x \rho - v_x \sigma \|_{H^{s-2}} \\
&\leq C (\| u^2 - v^2 \|_{H^{s-3}} + \| u_x^2 - v_x^2 \|_{H^{s-3}} + \| u_{xx}^2 - v_{xx}^2 \|_{H^{s-3}} \\
&\quad + \| u_x u_{xxx} - v_x v_{xxx} \|_{H^{s-3}} + \| \rho^2 - \sigma^2 \|_{H^{s-3}} + \| u_x \rho - v_x \sigma \|_{H^{s-2}}). \qquad (5)
\end{aligned}$$

Using Lemma 3.6 with $t = r$, one has

$$\| u^2 - v^2 \|_{H^{s-3}} \leq C (\| u + v \|_{H^{s-3}} \| u - v \|_{H^{s-3}}) \leq C \| u - v \|_{H^s}, \qquad (6)$$

$$\| u_x^2 - v_x^2 \|_{H^{s-3}} \leq C (\| u + v \|_{H^{s-2}} \| u - v \|_{H^{s-2}}) \leq C \| u - v \|_{H^s}, \qquad (7)$$

$$\| u_{xx}^2 - v_{xx}^2 \|_{H^{s-3}} \leq C \| u - v \|_{H^s}, \qquad (8)$$

$$\| \rho^2 - \sigma^2 \|_{H^{s-3}} \leq C \| \rho - \sigma \|_{H^{s-2}} \qquad (9)$$

and

$$\begin{aligned}
&\| u_x \rho - v_x \sigma \|_{H^{s-2}} \\
&\leq \| u_x \rho - u_x \sigma \|_{H^{s-2}} + \| u_x \sigma - v_x \sigma \|_{H^{s-2}} \\
&\leq C \| u \|_{H^s} \| \rho - \sigma \|_{H^{s-2}} + C \| \sigma \|_{H^{s-2}} \| u - v \|_{H^s} . \qquad (10)
\end{aligned}$$

For the forth term in (5), we get from Lemma 3.6 with $t = r$ that

$$
\begin{aligned}
&\| u_x u_{xxx} - v_x v_{xxx} \|_{H^{s-3}} \\
&\leq \| u_x u_{xxx} - u_x v_{xxx} + u_x v_{xxx} - v_x v_{xxx} \|_{H^{s-3}} \\
&\leq \| u_x u_{xxx} - u_x v_{xxx} \|_{H^{s-3}} + \| u_x v_{xxx} - v_x v_{xxx} \|_{H^{s-3}} \\
&\leq C \| u - v \|_{H^s}, \qquad (11)
\end{aligned}
$$

Therefore, from (6)-(11), we obtain

$$
\begin{aligned}
\| f(z) - f(y) \|_{H^s \times H^{s-2}} &\leq C \| u - v \|_{H^s} + C \| \rho - \sigma \|_{H^{s-2}} \\
&= \mu_3 \| z - y \|_{H^s \times H^{s-2}}, \qquad (12)
\end{aligned}
$$

from which we know (a) holds.

Now, we prove (b).

$$
\begin{aligned}
&\| f(z) - f(y) \|_{H^{s-1} \times H^{s-3}} \\
&\leq \| \partial_x (1 - \partial_x^2)^{-2} (u^2 + u_x^2 - \frac{7}{2} u_{xx}^2 - 3 u_x u_{xxx} - v^2 - v_x^2 + \frac{7}{2} v_{xx}^2 \\
&\quad + 3 v_x v_{xxx}) \|_{H^{s-1}} + \| \partial_x (1 - \partial_x^2)^{-2} (\frac{1}{2} \rho^2 - \frac{1}{2} \sigma^2) \|_{H^{s-1}} \\
&\quad + \| u_x \rho - v_x \sigma \|_{H^{s-3}} \\
&\leq \| u^2 - v^2 \|_{H^{s-4}} + \| u_x^2 - v_x^2 \|_{H^{s-4}} + \| u_{xx}^2 - v_{xx}^2 \|_{H^{s-4}} \\
&\quad + \| u_x u_{xxx} - v_x v_{xxx} \|_{H^{s-4}} + c \| \rho^2 - \sigma^2 \|_{H^{s-4}} + \| u_x \rho - v_x \sigma \|_{H^{s-3}} \qquad (13)
\end{aligned}
$$

We will estimate each of the terms on the right-hand side of (13). For the first term , we get from Lemma 3.6 that

$$
\begin{aligned}
\| u^2 - v^2 \|_{H^{s-4}} &\leq \| (u + v)(u - v) \|_{H^{s-4}} \\
&\leq \| u + v \|_{H^{s-4}} \| u - v \|_{H^{s-3}} \leq C \| u - v \|_{H^{s-1}} . \qquad (14)
\end{aligned}
$$

In an analogous way to the first term, we have

$$
\| u_x^2 - v_x^2 \|_{H^{s-4}} \leq C \| u - v \|_{H^{s-1}}, \qquad (15)
$$

$$
\| u_{xx}^2 - v_{xx}^2 \|_{H^{s-4}} \leq C \| u - v \|_{H^{s-1}} \qquad (16)
$$

and

$$
\| \rho^2 - \sigma^2 \|_{H^{s-4}} \leq C \| \rho - \sigma \|_{H^{s-3}} \qquad (17)
$$

For the forth term in (13), one has

$$\begin{aligned}
&\| u_x u_{xxx} - v_x v_{xxx} \|_{H^{s-4}} \\
&\leq \| u_x u_{xxx} - u_x v_{xxx} + u_x v_{xxx} - v_x v_{xxx} \|_{H^{s-4}} \\
&\leq \| u_x u_{xxx} - u_x v_{xxx} \|_{H^{s-4}} + \| u_x v_{xxx} - v_x v_{xxx} \|_{H^{s-4}} \\
&\leq \| u_x \|_{H^{s-3}} \| u_{xxx} - v_{xxx} \|_{H^{s-4}} + \| v_{xxx} \|_{H^{s-4}} \| u_x - v_x \|_{H^{s-3}} \\
&\leq C \| u - v \|_{H^{s-1}} . \qquad (18)
\end{aligned}$$

For the last term, note that $s > \frac{7}{2}$, it yields

$$\begin{aligned}
&\| u_x \rho - v_x \sigma \|_{H^{s-3}} \\
&\leq \| u_x \rho - u_x \sigma \|_{H^{s-3}} + \| u_x \sigma - v_x \sigma \|_{H^{s-3}} \\
&\leq C \| \rho - \sigma \|_{H^{s-3}} + C \| u - v \|_{H^{s-1}} . \qquad (19)
\end{aligned}$$

Therefore, from (13)-(19), we deduce

$$\begin{aligned}
\| f(z) - f(y) \|_{H^{s-1} \times H^{s-3}} &\leq C \| u - v \|_{H^{s-1}} + C \| \rho - \sigma \|_{H^{s-3}} \\
&= \mu_4 \| z - y \|_{H^{s-1} \times H^{s-3}} . \qquad (20)
\end{aligned}$$

This completes the proof of Lemma 3.8.

Proof of Theorem 2.1. Applying the Kato Theorem for abstract quasi-linear evolution equations of hyperbolic type [18], Lemmas 3.3-3.4, 3.7 and 3.8, we obtain the local well-posedness of system (4) in $H^s \times H^{s-2}$, $s > \frac{7}{2}$, and

$$z = z(\cdot, z_0) \in C\Big([0, T); H^s \times H^{s-2}\Big) \cap C^1\Big([0, T); H^{s-1} \times H^{s-3}\Big).$$

4 Global solution

To prove Theorem 2.2, we need the following lemmas.

Lemma 4.1.([22]) *The following estimates hold*

(i) For $s \geq 0$,

$$\| fg \|_{H^s} \leq C(\| f \|_{H^s} \| g \|_{L^\infty} + \| f \|_{L^\infty} \| g \|_{H^s}). \tag{21}$$

(ii) For $s > 0$,

$$\| f\partial_x g \|_{H^s} \leq C(\| f \|_{H^{s+1}} \| g \|_{L^\infty} + \| f \|_{L^\infty} \| \partial_x g \|_{H^s}). \tag{22}$$

Lemma 4.2. ([24]) *Let $r > 0$. If $u \in H^r \cap W^{1,\infty}$ and $v \in H^{r-1} \cup L^\infty$, then*

$$\| [\Lambda^r, u]v \|_{L^2} \leq C(\| u_x \|_{L^\infty} \| \Lambda^{r-1} \|_{L^2} + \| \Lambda^r u \|_{L^2} \| v \|_{L^\infty}).$$

Lemma 4.3. *Let $z_0 = (u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-2}(\mathbb{S})$, $s > \frac{7}{2}$. Then $\| z \|_{H^s \times H^{s-2}} = \| (u, \rho) \|_{H^s \times H^{s-2}}$ is finite for $0 < t < \infty$.*

Proof. Applying Λ^s to $u_t = -uu_x - f(u)$, where $f(u) = \partial_x \Lambda^{-4}(u^2 + u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x u_{xxx} + \frac{1}{2}\rho^2)$, and multiplying by $\Lambda^s u$ and the integrating over $\mathbb{S}$, we have

$$\frac{d}{dt}\int_S (\Lambda^s u)^2 dx = -2\int_S \Lambda^s u \Lambda^s u u_x dx - 2\int_S \Lambda^s u \Lambda^s f(u) dx. \tag{23}$$

From lemma 4.2 and the Cauchy inequality, we reach

$$\begin{aligned}\int_S \Lambda^s u \Lambda^s u u_x dx &\leq \int_S \Lambda^s u(\Lambda^s u u_x - u\Lambda^s u_x) dx + \int_S (\Lambda^s u) u \Lambda^s u_x dx \\ &\leq C \| u_x \|_{L^\infty} \| u \|_{H^s}^2 .\end{aligned} \tag{24}$$

The Cauchy inequality ensures

$$\int_S \Lambda^s u \Lambda^s f(u) dx \leq \| u \|_{H^s} \| f(u) \|_{H^s} \tag{25}$$

and

$$\| f(u) \|_{H^s} \leq C \| u^2 + u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x u_{xxx} + \frac{1}{2}\rho^2 \|_{H^{s-3}}$$

$$\begin{aligned}
&\leq C(\| u^2 \|_{H^{s-3}} + \| u_x^2 \|_{H^{s-3}} + \| u_{xx}^2 \|_{H^{s-3}} + \| u_x u_{xxx} \|_{H^{s-3}} \\
&\quad + \| \rho^2 \|_{H^{s-3}}) \\
&\leq C(\| u \|_{L^\infty} \| u \|_{H^{s-3}} + \| u_x \|_{L^\infty} \| u_x \|_{H^{s-3}} \\
&\quad + \| u_{xx} \|_{L^\infty} \| u_{xx} \|_{H^{s-3}} + \| u_x \|_{H^{s-2}} \| u_{xx} \|_{L^\infty} \\
&\quad + \| u_x \|_{L^\infty} \| u_{xxx} \|_{H^{s-3}} + \| \rho \|_{L^\infty} \| \rho \|_{H^{s-3}}),
\end{aligned} \tag{26}$$

where we have used Lemma 4.1.

Hence,

$$\frac{d}{dt} \| u \|_{H^s}^2 \leq C_1(\| u \|_{H^s}^2 + \| u \|_{H^s}^3 + \| u \|_{H^s} \| \rho \|_{H^{s-2}}^2), \tag{27}$$

where $C_1 = C_1(\| z_0 \|_{H^s \times H^{s-2}})$.

Applying Λ^{s-2} to $\rho_t = -u_x \rho - u \rho_x$, and multiplying by $\Lambda^{s-2}\rho$ and the integrating over $\mathbb{S}$, we have

$$\frac{d}{dt} \int_S (\Lambda^{s-2}\rho)^2 dx = -2 \int_S \Lambda^{s-2}\rho \Lambda^{s-2}(u_x \rho) dx - 2 \int_S \Lambda^{s-2}\rho \Lambda^{s-2}(u \rho_x) dx. \tag{28}$$

From Lemma 4.2 and the Cauchy inequality, we have

$$\frac{d}{dt} \| \rho \|_{H^{s-2}}^2 \leq C(\| u \|_{H^s} \| \rho \|_{H^{s-2}}^2 + \| \rho \|_{H^{s-2}}^2), \tag{29}$$

which together to (27) yields

$$\begin{aligned}
&\frac{d}{dt}(\| u \|_{H^s}^2 + \| \rho \|_{H^{s-2}}^2) \\
&\leq C(\| u \|_{H^s}^2 + \| u \|_{H^s}^3 + \| u \|_{H^s} \| \rho \|_{H^{s-2}}^2 + \| \rho \|_{H^{s-2}}^2) \\
&\leq C(\| u \|_{H^s}^2 + \| \rho \|_{H^{s-2}}^2)(\| u \|_{H^s}^2 + \| \rho \|_{H^{s-2}}^2 + 1),
\end{aligned} \tag{30}$$

which implies

$$\frac{\| u \|_{H^s}^2 + \| \rho \|_{H^{s-2}}^2}{\| u \|_{H^s}^2 + \| \rho \|_{H^{s-2}}^2 + 1} \leq \frac{\| u_0 \|_{H^s}^2 + \| \rho_0 \|_{H^{s-2}}^2}{\| u_0 \|_{H^s}^2 + \| \rho_0 \|_{H^{s-2}}^2 + 1} e^{Ct}. \tag{31}$$

Note that $0 \leq t < \infty$, we get from (31) that

$$\frac{\| u \|_{H^s}^2 + \| \rho \|_{H^{s-2}}^2}{\| u \|_{H^s}^2 + \| \rho \|_{H^{s-2}}^2 + 1} \leq \frac{\| u_0 \|_{H^s}^2 + \| \rho_0 \|_{H^{s-2}}^2}{\| u_0 \|_{H^s}^2 + \| \rho_0 \|_{H^{s-2}}^2 + 1},$$

which results in

$$\| u \|_{H^s}^2 + \| \rho \|_{H^{s-2}}^2 \leq \| u_0 \|_{H^s}^2 + \| \rho_0 \|_{H^{s-2}}^2 . \tag{32}$$

This completes the proof of Lemma 4.3.

Proof of Theorem 2.2. Theorem 2.2 is a direct consequence of Theorem 2.1 and Lemma 4.3.

Remark. We have investigated the local well-posedness and global existence of system (1) on the periodic case. In fact, the above results hold true with $m = (1 - \partial_x^2)^k u$, $k \geq 2$ on the periodic case.

$$\begin{cases} m_t + u m_x + 2u_x m + \rho\rho_x = 0, \quad t > 0, x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, \quad t > 0, x \in \mathbb{R}. \end{cases} \tag{33}$$

More precisely, the local well-posedness Theorem 2.1 and global existence result Theorem 2.2 hold true in the Soblev space $H^s(\mathbb{S}) \times H^{s-k}(\mathbb{S})$ with $s > 2k - \frac{1}{2}$. More dynamics related to system (1) will be discussed in forthcoming paper.

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