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Positivity of Toeplitz operators via Berezin transform ☆

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ABSTRACT

In this paper, we study positive Toeplitz operators on the Bergman space via their Berezin transforms. Surprisingly we show that the positivity of a Toeplitz operator on the Bergman space is not completely determined by the positivity of the Berezin transform of its symbol. In fact, we show that even if the minimal value of the Berezin transform of a quadratic polynomial of $|z|$ on the unit disk is positive, the Toeplitz operator with the function as the symbol may not be positive.

1. Introduction

Let dA denote Lebesgue area measure on the unit disk $\mathbb{D}$ in the complex plane, normalized so that the measure of the disk $\mathbb{D}$ is 1. The Bergman space L_a^2 is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are square integrable with respect to the measure dA. For $\varphi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_φ with symbol φ, is the operator defined on L_a^2 by

$$T_\varphi f = P(\varphi f),$$

where $P : L^2(\mathbb{D}, dA) \to L_a^2$ is the orthogonal projection. Using the reproducing kernel $K_z(w) = \frac{1}{(1-\bar{z}w)^2}$, we express the Toeplitz operator to be an integral operator:

$$\begin{aligned} T_\varphi f(z) &= \int_{\mathbb{D}} \varphi(w) f(w) \overline{K_z(w)}\, dA(w) \\ &= \int_{\mathbb{D}} \frac{\varphi(w) f(w)}{(1-\overline{w}z)^2}\, dA(w). \end{aligned}$$

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This immediately gives

$$\langle T_\varphi f, g\rangle = \int_{\mathbb{D}} \varphi(w) f(w)\overline{g(w)}\, dA(w)$$

for $f, g \in L_a^2$, and hence T_φ is self-adjoint if and only if φ is real-valued. If $\varphi(z) \geqslant 0$ for $z \in \mathbb{D}$, then T_φ is positive.

Like pseudodifferential operators, there are deep connections between properties of Toeplitz operators and properties of their symbols [1,5,13]. In this paper we study the simple but fundamental problem when T_φ is positive. We will show that the positivity of T_φ is not completely determined by the positivity of φ.

First we observe that if T_φ is positive, then

$$\langle T_\varphi k_z, k_z\rangle \geqslant 0,$$

where k_z called the *normalized Bergman reproducing kernel* of L_a^2 given by

$$k_z(w) = \frac{K_z(w)}{\|K_z\|} = \frac{1-|z|^2}{(1-\bar{z}w)^2}.$$

For A a bounded operator on L_a^2, the *Berezin transform* of A is the function $\widetilde{A}$ on $\mathbb{D}$ defined by

$$\widetilde{A}(z) = \langle A k_z, k_z\rangle.$$

For $\varphi \in L^\infty(\mathbb{D})$, $\widetilde{\varphi}$ is called the Berezin transform of φ given by

$$\widetilde{\varphi}(z) = \widetilde{T_\varphi}(z).$$

The Berezin transform is very useful in studying Toeplitz operators on the Bergman space and enjoys many nice properties:

(1) $T_\varphi = 0$ iff $\widetilde{\varphi}(z) = 0$ for $z \in \mathbb{D}$; moreover, $A = 0$ iff $\widetilde{A}(z) = 0$ for all $z \in \mathbb{D}$ (see [12]);
(2) T_φ is self-adjoint on L_a^2 iff $\widetilde{\varphi}$ is real-valued;
(3) T_φ is compact on the Bergman space iff $\widetilde{\varphi}(z) \to 0$ as $|z| \to 1^-$ (see [2]);
(4) There is no constant $M > 0$ such that

$$\|T_f\| \leqslant M\|\widetilde{f}\|_\infty$$

for all $f \in L^\infty(\mathbb{D})$ [10]. In general Coburn [3] showed that there is no constant $M > 0$ such that

$$\|A\| \leqslant M\|\widetilde{A}\|_\infty$$

for all bounded operators A on the Bergman space.

These lead to the following natural question:

Is T_φ positive if the Berezin transform $\widetilde{\varphi}(z)$ is nonnegative on $\mathbb{D}$?

The answer to the analogous question for Toeplitz operators on Hardy space H^2 is positive by means of well known result of the spectral theorem for the self-adjoint Hardy Toeplitz operators or the harmonic extension. On the Bergman space the answer is also affirmative if the symbol of the Toeplitz operator is harmonic on the unit disk.

To study the above question, we note that the Berezin transform $\widetilde{\varphi}$ has a deep connection with the Mellin transform of a function g integrable on $[0,1]$, which is defined by

$$\widehat{g}(z) = \int_0^1 g(r) r^{z-1}\, dr.$$

Every function φ in $L^2(\mathbb{D}, dA)$ can be written in the polar form:

$$\varphi(re^{i\theta}) = \sum_{k=-\infty}^{\infty} \varphi_k(r) e^{ik\theta}.$$

Cuckovic obtained in [4] the connection between the Berezin transform and the Mellin transform:

$$\widetilde{\varphi}(re^{i\theta}) = 2(1-r^2)^2 \sum_{k=-\infty}^{\infty} r^{|k|} \left[\sum_{n=1}^{\infty} n(n+|k|)\widehat{\varphi_k}(2n+|k|) r^{2(n-1)}\right] e^{ik\theta}.$$

For a radial function $\varphi(z)$, i.e. $\varphi(re^{i\theta}) = \varphi(r)$ for $z \in \mathbb{D}$, the affirmative answer to the above question is equivalent to

$$\widetilde{\varphi}(z) \geqslant 0 \quad \text{on } \mathbb{D} \quad \text{iff} \quad \widehat{\varphi}(2n) \geqslant 0 \quad \text{for } n \geqslant 1.$$

But using (4) we will show that the answer is no for a bounded function φ on the unit disk. In this paper, we will show that the positivity of T_φ is not completely determined by the positivity of the Berezin transform of φ. However, it is more difficult to characterize the positivity of a Bergman Toeplitz operator even if the symbol φ is a continuous function on the closure of the unit disk. We consider the special case of the radial function $\varphi(z) = |z|^2 + a|z| + b$, where a and b are both real numbers. For this type of φ, we will prove that $T_\varphi \geqslant 0$ and $\widetilde{\varphi} \geqslant 0$ are "almost" equivalent, but in the last section we will show an example that $\widetilde{\varphi}$ is strictly positive on the unit disk and T_φ is not positive.

2. Main results

First let us consider the Toeplitz operators on Hardy spaces. The following theorem is a consequence of Hartman–Wintner's theorem or the harmonic extension. We will present two proofs which both work on the Bergman space for harmonic symbols.

Theorem 2.1. *Let $\varphi \in L^\infty(\partial\mathbb{D})$, then the Toeplitz operator T_φ is positive on the Hardy space if and only if $\varphi \geqslant 0$ a.e. on the unit circle $\partial\mathbb{D}$.*

Proof. As we pointed out in the introduction, by the integral representation of Toeplitz operators, we see that if the symbol φ is nonnegative, then T_φ is positive. So we need only to show that if T_φ is positive, then φ is nonnegative. The first proof follows from Hartman–Wintner's theorem in [5] or [7]

$$\sigma(T_\varphi) = \left[\operatorname{ess\,inf}(\varphi), \operatorname{ess\,sup}(\varphi)\right]$$

if φ is a real-valued function in $L^\infty(\partial\mathbb{D})$. If T_φ is positive, then the spectrum $\sigma(T_\varphi)$ must be nonnegative. Thus $\varphi(z)$ is nonnegative (almost everywhere on the unit circle).

The second method uses the harmonic extension. Let k_z denote the reproducing kernel of the Hardy space at $z \in \mathbb{D}$. If T_φ is positive, then

$$\langle T_\varphi k_z, k_z \rangle \geqslant 0.$$

On the other hand,

$$\langle T_\varphi k_z, k_z \rangle = \int_{\partial \mathbb{D}} \frac{1-|z|^2}{|1-\bar{z}e^{i\theta}|^2} \varphi\left(e^{i\theta}\right) d\theta,$$

which is the harmonic extension of φ at z. The radial limit of the above functions converges to $\varphi(e^{i\theta})$ almost everywhere if $z = re^{i\theta} \to e^{i\theta}$ [5]. Thus $\varphi(e^{i\theta}) \geqslant 0$ almost everywhere on the unit circle. □

Now we consider the Toeplitz operator T_φ acting on L_a^2. If φ is harmonic on $\mathbb{D}$, then we have the same result as one in Theorem 2.1. The following theorem is also a consequence of the spectrum of T_φ for a real valued harmonic function φ in [9]. But we present the proof by using the Berezin transform to follow the second method in the proof of the above theorem.

Theorem 2.2. *Let $\varphi \in L^\infty(\mathbb{D})$ be a harmonic function. $T_\varphi \geqslant 0$ on the Bergman space iff $\varphi(z) \geqslant 0$ for all $z \in \mathbb{D}$.*

Proof. If T_φ is positive, then

$$\langle T_\varphi k_z, k_z \rangle \geqslant 0$$

for $z \in \mathbb{D}$. Thus this implies

$$\widetilde{\varphi}(z) \geqslant 0$$

for $z \in \mathbb{D}$. So we obtain

$$\varphi(z) = \widetilde{\varphi}(z) \geqslant 0.$$

The first equality follows from the fact that

$$\widetilde{\varphi}(z) = \varphi(z)$$

for all $z \in \mathbb{D}$ (see Proposition 6.13 in [13]) since φ is harmonic.

Conversely, in the first section we have pointed out that T_φ is positive if $\varphi(z) \geqslant 0$ for $z \in \mathbb{D}$. □

For a harmonic function φ, the above theorem implies that T_φ is positive on the Bergman space iff the Berezin transform $\widetilde{\varphi}(z) \geqslant 0$ on the unit disk. For a general function φ in $L^\infty(\mathbb{D})$, the following theorem shows that this is not true.

Theorem 2.3. *The positivity of Toeplitz operators on the Bergman space is not completely determined by the positivity of the Berezin transform of their symbols.*

Proof. Suppose that for any real-valued functions f in L^∞, if $\widetilde{f}(z) \geqslant 0$ on $\mathbb{D}$, then T_f is positive. We will show that this implies the following inequality

$$\|T_f\| \leqslant 2\|\widetilde{f}\|_\infty$$

for all f in $L^\infty(\mathbb{D})$, which is a contraction to the following fact in [10] that there is no constant $M > 0$ such that

$$\|T_f\| \leqslant M\|\widetilde{f}\|_\infty$$

for all $f \in L^\infty(\mathbb{D})$.

To do this, we consider that f is a real-valued function in $L^\infty(\mathbb{D})$. Then

$$\widetilde{\|\widetilde{f}\|_\infty \mp f}(z) \geqslant 0$$

for $z \in \mathbb{D}$. Thus we have

$$T_{\|\widetilde{f}\|_\infty \mp f} \geqslant 0,$$

to get

$$\|\widetilde{f}\|_\infty \geqslant \pm T_f.$$

So this gives that for any h in L_a^2,

$$\|\widetilde{f}\|_\infty \|h\|_2^2 \geqslant \pm\langle T_f h, h\rangle.$$

Hence

$$\|\widetilde{f}\|_\infty \|h\|_2^2 \geqslant \bigl|\langle T_f h, h\rangle\bigr|.$$

Since for a self-adjoint operator T_f,

$$\|T_f\| = \sup_{\|h\|\leqslant 1,\, h\in L_a^2} \bigl|\langle T_f h, h\rangle\bigr|,$$

we obtain

$$\|T_f\| \leqslant \|\widetilde{f}\|_\infty.$$

Next for $f \in L^\infty(\mathbb{D})$, we also have

$$\|\widetilde{\Re f}\|_\infty \leqslant \|\widetilde{f}\|_\infty,$$

and

$$\|\widetilde{\Im f}\|_\infty \leqslant \|\widetilde{f}\|_\infty,$$

to get

$$\begin{aligned}
\|T_f\| &= \|T_{\Re f} + iT_{\Im f}\| \\
&\leqslant \|T_{\Re f}\| + \|T_{\Im f}\| \\
&\leqslant \|\widetilde{\Re f}\|_\infty + \|\widetilde{\Im f}\|_\infty \\
&\leqslant 2\|\widetilde{f}\|_\infty.
\end{aligned}$$

This completes the proof. $\square$

Little is known concerning the positivity of the Toeplitz operator with symbol φ in $C(\overline{\mathbb{D}})$, but for the special case of $\varphi(z) = |z|^2 + a|z| + b$ we have the following result.

Theorem 2.4. *Let $\varphi(z) = |z|^2 + a|z| + b$ $(a, b \in \mathbb{R})$. Suppose $a \in \mathbb{R} \setminus (-2, -\frac{5}{4})$, then T_φ is positive if and only if $\widetilde{\varphi}(z)$ is a nonnegative function on $\mathbb{D}$.*

On the other hand, we have the following counter examples of the above theorem even for quadratic polynomials of $|z|$.

Theorem 2.5. *For each $a \in (-\frac{14}{9}, -\frac{5}{4}) \subset (-2, -\frac{5}{4})$, there exist some $b \in \mathbb{R}$ and $\delta > 0$ such that the Berezin transform*

$$\widetilde{\varphi}(z) \geqslant \delta$$

of $\varphi(z) = |z|^2 + a|z| + b$ for all z in $\mathbb{D}$, but T_φ is not positive.

The details of Theorem 2.5 will be contained in Section 5 and the proof of Theorem 2.4 will be presented in Section 4.

3. Berezin transform $\widetilde{\varphi}$ and the matrix representation of T_φ

Let $e_n(z) = \sqrt{n+1}z^n$, then $\{e_n\}_{n=0}^\infty$ form an orthonormal basis of the Bergman space L_a^2. For the special case of $\varphi(z) = |z|^2 + a|z| + b$ we can find the relationship between the positivity of T_φ and the Berezin transform $\widetilde{\varphi}$ by its matrix. The matrix of T_φ with respect to this basis is a diagonal matrix. More precisely, we have the following lemma.

Lemma 3.1. *Let $\varphi(z) = |z|^2 + a|z| + b$ $(a, b \in \mathbb{R})$ and $\{e_n\}_{n=0}^\infty$ be as above, then the matrix of the Toeplitz operator T_φ under the basis is*

$$\operatorname{diag}\left(\left\{\frac{2n+2}{2n+4} + a\frac{2n+2}{2n+3} + b\right\}_{n=0}^\infty\right).$$

Proof. For each $k \in \mathbb{N}$, we have

$$\begin{aligned}
T_{|z|^k} e_n(z) &= \langle T_{|z|^k} e_n, K_z\rangle \\
&= \langle |w|^k e_n, K_z\rangle \\
&= \sqrt{n+1} \int_{\mathbb{D}} |w|^k \frac{w^n}{(1-\overline{w}z)^2}\, dA(w) \\
&= \sqrt{n+1} \int_{\mathbb{D}} |w|^k w^n \sum_{j=0}^\infty (j+1)\overline{w}^j z^j\, dA(w) \\
&= \sqrt{n+1}\frac{1}{\pi} \int_0^{2\pi}\int_0^1 r^k \cdot r^n \cdot e^{in\theta} \sum_{j=0}^\infty (j+1) r^j e^{-j(i\theta)} z^j r\, dr\, d\theta \\
&= \frac{2\sqrt{n+1}(n+1)}{2n+k+2} z^n \\
&= \frac{2n+2}{2n+k+2} e_n(z).
\end{aligned}$$

Thus the matrix representation of $T_{|z|^k}$ is a diagonal matrix under the basis $\{e_n\}_{n=0}^{\infty}$. So is the matrix representation of $T_\varphi = T_{|z|^2+a|z|+b} = T_{|z|^2} + aT_{|z|} + bI$ since it is a linear combination of $T_{|z|^k}$. In fact we have

$$T_\varphi e_n(z) = \left[\frac{2n+2}{2n+4} + a\frac{2n+2}{2n+3} + b\right] e_n(z).$$

This gives the matrix representation as desired in the theorem. □

The above matrix representation of T_φ immediately gives the following criterion on the positivity of T_φ for $\varphi = |z|^2 + a|z| + b$.

Lemma 3.2. *Let $\varphi(z) = |z|^2 + a|z| + b$ $(a, b \in \mathbb{R})$. T_φ is positive if and only if*

$$1 + a + b \geqslant \frac{a}{2n+3} + \frac{2}{2n+4}$$

for $n \geqslant 0$.

On the other hand, for $\varphi(z) = |z|^2 + a|z| + b$ $(a, b \in \mathbb{R})$, we are going to compute the Berezin transform of φ to get the following precise formula.

Lemma 3.3. *Let $\varphi(z) = |z|^2 + a|z| + b$ and let $\widetilde{\varphi}(z)$ be the Berezin transform of $\varphi(z)$, then*

$$\widetilde{\varphi}(z) = \left[2 - \frac{1}{|z|^2} - \frac{(1-|z|^2)^2}{|z|^4}\log\bigl(1-|z|^2\bigr)\right] + \frac{a}{2}\left[3 - \frac{1}{|z|^2} + \frac{(1-|z|^2)^2}{2|z|^3}\log\frac{1+|z|}{1-|z|}\right] + b$$

for all $z \in \mathbb{D}$.

Proof. First, for each $l \in \mathbb{N}$, we compute $\widetilde{|z|^l}$. By the definition of the Berezin transform, we have

$$\begin{aligned}
\widetilde{|z|^l} &= \langle T_{|z|^l} k_z, k_z\rangle \\
&= \int_{\mathbb{D}} |w|^l \bigl|k_z(w)\bigr|^2\, dA(w) \\
&= \int_{\mathbb{D}} |w|^l \frac{(1-|z|^2)^2}{|1-\bar{z}w|^4}\, dA(w) \\
&= \bigl(1-|z|^2\bigr)^2 \frac{1}{\pi}\int_0^{2\pi}\int_0^1 r^l \left(\sum_{n=0}^{\infty}(n+1)\bar{z}^n r^n e^{in\theta}\right)\left(\sum_{m=0}^{\infty}(m+1)z^m r^m e^{-im\theta}\right) r\, dr\, d\theta \\
&= \bigl(1-|z|^2\bigr)^2 \frac{1}{\pi}\int_0^{2\pi}\int_0^1 r^l \left(\sum_{n=0}^{\infty}(n+1)^2|z|^{2n} r^{2n}\right) r\, dr\, d\theta \\
&= 2\bigl(1-|z|^2\bigr)^2 \sum_{n=0}^{\infty}\frac{(n+1)^2}{2n+l+2}|z|^{2n}.
\end{aligned}$$

Using the above formula for $l = 1, 2$, we get

$$\widetilde{\varphi}(z) = \int_{\mathbb{D}} \varphi(w)\bigl|k_z(w)\bigr|^2\, dA(w) = 2\bigl(1-|z|^2\bigr)^2\left[\sum_{n=0}^{\infty}\frac{(n+1)^2}{2n+4}|z|^{2n} + a\sum_{n=0}^{\infty}\frac{(n+1)^2}{2n+3}|z|^{2n}\right] + b.$$

Simple calculations give that for $z \in \mathbb{D}$,

$$\sum_{n=0}^{\infty} \frac{(n+1)^2}{2n+4}|z|^{2n} = \frac{1}{2}\left[\frac{|z|^2}{(1-|z|^2)^2} - \frac{1}{|z|^2} - \frac{\log(1-|z|^2)}{|z|^4}\right],$$

and

$$\sum_{n=0}^{\infty} \frac{(n+1)^2}{2n+3}|z|^{2n} = \frac{1}{4}\left[\frac{1+|z|^2}{(1-|z|^2)^2} - \frac{1}{|z|^2} + \frac{1}{2|z|^3}\log\frac{1+|z|}{1-|z|}\right].$$

Combining above three formulae, we obtain

$$\widetilde{\varphi}(z) = \left[2 - \frac{1}{|z|^2} - \frac{(1-|z|^2)^2}{|z|^4}\log\bigl(1-|z|^2\bigr)\right] + \frac{a}{2}\left[3 - \frac{1}{|z|^2} + \frac{(1-|z|^2)^2}{2|z|^3}\log\frac{1+|z|}{1-|z|}\right] + b$$

for all $z \in \mathbb{D}$, to complete the proof. □

The following proposition gives values of $\widetilde{\varphi}$ at 0 and 1, which is useful to get necessary conditions for $\widetilde{\varphi}(z) \geqslant 0$ on $\mathbb{D}$.

Proposition 3.4. *Let $\varphi(z) = |z|^2 + a|z| + b$ $(a, b \in \mathbb{R})$, then $\widetilde{\varphi}(0) = b + \frac{2a}{3} + \frac{1}{2}$ and $\widetilde{\varphi}(1) = b + a + 1$.*

Proof. Using

$$\widetilde{\varphi}(z) = 2\bigl(1-|z|^2\bigr)^2\left[\sum_{n=0}^{\infty} \frac{(n+1)^2}{2n+4}|z|^{2n} + a\sum_{n=0}^{\infty} \frac{(n+1)^2}{2n+3}|z|^{2n}\right] + b$$

and letting $|z| = 0$ in the above power series, we obtain

$$\widetilde{\varphi}(0) = 2\left(\frac{1}{4} + \frac{a}{3}\right) + b.$$

This gives $\widetilde{\varphi}(0) = b + \frac{2a}{3} + \frac{1}{2}$.

Since $\varphi(z) = |z|^2 + a|z| + b$ is continuous on the closure of the unit disk, we obtain that $\widetilde{\varphi}$ is also continuous on the closure of the unit disk $\mathbb{D}$ and $\widetilde{\varphi} = \varphi$ on $\partial\mathbb{D}$ (see Proposition 6.14 in [13]). Thus we get

$$\widetilde{\varphi}(1) = \varphi(1) = 1 + a + b,$$

to complete the proof. □

If $\psi(z) = |z| - a$, the following theorem says that the positivity of T_ψ is completely determined by the positivity of the Berezin transform of ψ but not the positivity of ψ.

Theorem 3.5. *Let $\psi(z) = |z| - a$, where $a \in \mathbb{R}$. Then the following are equivalent:*

(i) $T_\psi \geqslant 0$;
(ii) $a \leqslant \frac{2}{3}$;
(iii) $\widetilde{\psi}(z) \geqslant 0$ *for all* $z \in \mathbb{D}$.

Proof. First we show that (i) $\Longleftrightarrow$ (ii). The proof of Lemma 3.1 gives that the matrix of T_ψ is a diagonal operator with diagonal entries

$$\left\{\frac{2n+2}{2n+3}-a\right\}_{n=0}^{\infty}.$$

Thus

$$T_\psi \geqslant 0 \quad \Longleftrightarrow \quad a \leqslant \frac{2n+2}{2n+3} \quad (\forall n \geqslant 0).$$

Since $\{\frac{2n+2}{2n+3}\}_{n=0}^{\infty}$ is an increasing sequence, we see that

$$a \leqslant \frac{2n+2}{2n+3} \quad (\forall n \geqslant 0) \quad \Longleftrightarrow \quad a \leqslant \frac{2}{3},$$

to get that (i) $\Longleftrightarrow$ (ii).

In the first section we have pointed out that if T_ψ is positive, then the Berezin transform $\widetilde{\psi}(z)$ is nonnegative on the unit disk. Thus (i) $\Longrightarrow$ (iii). To complete the proof we need only to verify that (iii) $\Longrightarrow$ (ii). To do this, we note that

$$\widetilde{\psi}(z) = 2\big(1-|z|^2\big)^2\left[\sum_{n=0}^{\infty}\frac{(n+1)^2}{2n+3}|z|^{2n}\right] - a \quad \text{(by the proof of Proposition 3.4)}.$$

Letting $z=0$ in the above equality gives

$$\widetilde{\psi}(0) = \frac{2}{3} - a.$$

If $\widetilde{\psi}(z) \geqslant 0$ on the unit disk, we get $a \leqslant \frac{2}{3}$, which proves the theorem. $\square$

Example. Let $\psi(z) = |z| - a$ and $0 < a \leqslant \frac{2}{3}$. The above theorem gives that T_ψ is positive but $\psi(0) = -a < 0$ and hence $\psi(z)$ is not a nonnegative function on the unit disk.

4. Proof of Theorem 2.4

In this section we give the proof of Theorem 2.4. Let $\varphi(z) = |z|^2 + a|z| + b$ $(a, b \in \mathbb{R})$. Assume

$$\widetilde{\varphi}(z) \geqslant 0$$

for all $z \in \mathbb{D}$. By Proposition 3.4, we have

$$\widetilde{\varphi}(0) = b + \frac{2a}{3} + \frac{1}{2}$$

and

$$\widetilde{\varphi}(1) = b + a + 1.$$

Thus

$$b + \frac{2a}{3} + \frac{1}{2} \geqslant 0$$

and

$$b + a + 1 \geqslant 0.$$

By Lemma 3.2, we have

$$T_\varphi \geqslant 0 \quad \Longleftrightarrow \quad 1 + a + b \geqslant \frac{a}{2n+3} + \frac{2}{2n+4} \quad \text{for all } n \geqslant 0.$$

We need only to show

$$1 + a + b \geqslant \frac{a}{2n+3} + \frac{2}{2n+4} \quad (n \geqslant 0).$$

To do this, we consider the following three cases.

Case I. Suppose $-\infty < a \leqslant -2$. In this case, we have

$$a \leqslant -2 + \frac{2}{2n+4}$$

for all $n \geqslant 0$. Thus

$$\frac{a}{2n+3} + \frac{2}{2n+4} \leqslant 0 \quad (\forall n \geqslant 0).$$

Recall that

$$1 + a + b \geqslant 0,$$

so we obtain

$$1 + a + b \geqslant \frac{a}{2n+3} + \frac{2}{2n+4} \quad (\forall n \geqslant 0).$$

Case II. Suppose $-\frac{9}{8} \leqslant a < +\infty$. In this case, we have

$$a \geqslant -\frac{3}{2} + \frac{3}{4(n+2)}$$

for all $n \geqslant 0$. This implies

$$\frac{a}{2n+3} + \frac{2}{2n+4} - \frac{a}{3} - \frac{1}{2} \leqslant 0 \quad (\forall n \geqslant 0).$$

Thus we have

$$b + \frac{2a}{3} + \frac{1}{2} \geqslant 0 \geqslant \frac{a}{2n+3} + \frac{2}{2n+4} - \frac{a}{3} - \frac{1}{2}$$

for all $n \geqslant 0$, to get

$$1 + a + b \geqslant \frac{a}{2n+3} + \frac{2}{2n+4}$$

for all $n \geqslant 0$, as desired.

Case III. Suppose $-\frac{5}{4} \leqslant a \leqslant -\frac{9}{8}$.

First we observe that

$$\frac{\sqrt{\frac{-a}{2}}}{1-\sqrt{\frac{-a}{2}}} \in [3,5].$$

Next we want to find the maximal term of the sequence $\{\frac{a}{2n+3} + \frac{2}{2n+4}\}_{n=0}^{\infty}$. To do this, let

$$F(x) = \frac{a}{x} + \frac{2}{x+1},$$

where $x = 2n+3 \geqslant 3$. A simple calculation gives that $F(x)$ is increasing if $x < \frac{\sqrt{\frac{-a}{2}}}{1-\sqrt{\frac{-a}{2}}}$ and $F(x)$ is decreasing if $x \geqslant \frac{\sqrt{\frac{-a}{2}}}{1-\sqrt{\frac{-a}{2}}}$. This implies that the maximal term of the above sequence is

$$\begin{aligned} \max\left\{\frac{a}{2n+3} + \frac{2}{2n+4}\colon\ n \geqslant 0\right\} &= F(3) \quad \left(\text{since } a \geqslant -\frac{5}{4} \text{ and } F(3) \geqslant F(5)\right) \\ &= \frac{a}{3} + \frac{1}{2}. \end{aligned}$$

Since $\widetilde{\varphi}$ is nonnegative, Proposition 3.4 gives

$$b + \frac{2a}{3} + \frac{1}{2} \geqslant 0.$$

Thus we obtain

$$\begin{aligned} 1+a+b &\geqslant \frac{a}{3} + \frac{1}{2} \\ &= \max\left\{\frac{a}{2n+3} + \frac{2}{2n+4}\colon\ n \geqslant 0\right\}, \end{aligned}$$

to complete the proof. □

5. Proof of Theorem 2.5

Theorem 2.4 tells us that there are many real numbers a and b such that T_φ is positive if and only if the Berezin transform $\widetilde{\varphi}$ of $\varphi(z) = |z|^2 + a|z| + b$ is nonnegative. In this section we will show that the positivity of a Toeplitz operator with the symbol $|z|^2 + a|z| + b$ is not completely determined by the positivity of the Berezin transform of its symbol. The following lemma will be used in the proof of Theorem 2.5.

Lemma 5.1. *For each $r \in (-\frac{2}{15}, -\frac{1}{18})$, the polynomial*

$$K(x) = x^3 + \frac{r-1}{2}x^2 + \left(r + \frac{1}{6}\right)x + \left(6r + \frac{1}{3}\right)$$

has exactly one real root x_0 in $(0,1)$.

In order to prove the above lemma, we need the following Sturm theorem [11]:

Theorem 5.2 *(Sturm). Let $p_0 = p, p_1, \dots, p_m$ be a Sturm chain, where p is a square-free polynomial, and let $\sigma(\xi)$ denote the number of sign changes (ignoring zeroes) in the sequence*

$$p_0(\xi), p_1(\xi), p_2(\xi), \dots, p_m(\xi).$$

For two real numbers $a < b$, the number of distinct roots of p in the half-open interval $(a, b]$ is $\sigma(a) - \sigma(b)$.

To obtain a Sturm chain, apply Euclid's algorithm to p and its derivative p':

$$\begin{aligned}
p_0(x) &:= p(x),\\
p_1(x) &:= p'(x),\\
p_2(x) &:= -\mathrm{rem}(p_0, p_1) = p_1(x)q_0(x) - p_0(x),\\
&\ \ \vdots\\
p_{i+1}(x) &:= -\mathrm{rem}(p_{i-1}, p_i) = p_i(x)q_{i-1}(x) - p_{i-1}(x) \quad (1 \leqslant i \leqslant m-1),\\
&\ \ \vdots\\
0 &= -\mathrm{rem}(p_{m-1}, p_m).
\end{aligned}$$

Note that each $q_{i-1}(x)$ is the quotient of the polynomial long division of p_{i-1} by p_i, $-p_{i+1}(x) = \mathrm{rem}(p_{i-1}, p_i)$ is the remainder. It can be seen that $\{p_i(x)\}$ is a sequence of polynomials of decreasing degree, which must eventually terminate in a polynomial $p_m(x)$, where $m \leqslant \deg(p)$ is the minimal number of polynomial divisions needed to obtain a zero remainder, $p_m(x)$ is the greatest common divisor of $p_0(x)$ and $p_1(x)$ and hence of every $p_i(x)$. That is, the final polynomial p_m is the greatest common divisor of p and its derivative.

Remarks. Usually, we use the canonical Sturm sequence to determine the number of zeros of a square-free polynomial p in some open interval (a, b). However, even if p is not square-free, the difference $\sigma(a) - \sigma(b)$ is the number of distinct roots of p in (a, b) whenever $a < b$ are real numbers such that $p(a) \neq 0$ and $p(b) \neq 0$, see Sturm's theorem in [6].

Proof of Lemma 5.1. If $7 - 5\sqrt{2} \leqslant r < -\frac{1}{18}$, we use the criteria on cubic equations [8] to get that $K(x)$ has only one root in $(0, 1)$ since

$$\begin{aligned}
9 \times 1 \times \left(r + \frac{1}{6}\right) - 3\left(\frac{r-1}{2}\right)^2 &= -\frac{3}{4}\left(r^2 - 14r - 1\right)\\
&\geqslant 0 \quad \left(\text{since } 7 - 5\sqrt{2} \leqslant r < -\frac{1}{18}\right)
\end{aligned}$$

and

$$K(0) = 6r + \frac{1}{3} < 0, \qquad K(1) = \frac{15r}{2} + 1 > 0.$$

If $-\frac{2}{15} \leqslant r < 7 - 5\sqrt{2}$, we will use the above Sturm theorem to find the number of roots in $(0, 1)$ for the polynomial $K(x)$. First we get the Sturm sequence by long division. Let

$$p_0(x) = K(x) = x^3 + \frac{r-1}{2}x^2 + \left(r + \frac{1}{6}\right)x + \left(6r + \frac{1}{3}\right).$$

Applying polynomial long division to the pair $p_0(x)$ and

$$p_1(x) = p_0'(x) = 3x^2 + (r-1)x + \left(r + \frac{1}{6}\right)$$

gives the remainder $r_1(x)$. Multiplying $r_1(x)$ by -1 we obtain

$$p_2(x) = \left(\frac{r^2}{18} - \frac{7}{9}r - \frac{1}{18}\right)x + \left(\frac{r^2}{18} - \frac{653}{108}r - \frac{37}{108}\right).$$

Next dividing $p_1(x)$ by $p_2(x)$ and then multiplying the remainder by -1, we obtain a constant $p_3(x)$.

Then evaluating $p_0(x)$, $p_1(x)$, $p_2(x)$ and $p_3(x)$ at 0 gives

$$\begin{aligned}
&p_0(0) = \frac{1}{3} + 6r < 0;\\
&p_1(0) = r + \frac{1}{6} > 0;\\
&p_2(0) = \frac{1}{108}\left(6r^2 - 653r - 37\right) > 0 \quad (\text{since } r < 7 - 5\sqrt{2});\\
&p_3(0) \text{ is a constant.}
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
&p_0(1) = \frac{15}{2}r + 1 > 0;\\
&p_1(1) = 2r + \frac{13}{6} > 0;\\
&p_2(1) = \frac{1}{108}\left(12r^2 - 737r - 43\right) > 0 \quad (\text{since } r < 7 - 5\sqrt{2});\\
&p_3(1) = p_3(0) \text{ is a constant.}
\end{aligned}$$

Thus we obtain that $K(x)$ has a root $x_0 \in (0,1)$ as $p_0(0)$ is negative and $p_0(1)$ is positive. To finish the proof we need only to show that x_0 is the unique root of $K(x)$ in $(0,1)$. To do so, we consider the following three cases:

(1) If $p_3(0)$ is negative, then the sequence of signs of $p_0(0)$, $p_1(0)$, $p_2(0)$, $p_3(0)$ is $\{-,+,+,-\}$ and the sequence of signs of $p_0(1)$, $p_1(1)$, $p_2(1)$, $p_3(1)$ is $\{+,+,+,-\}$, thus $\sigma(0) = 2$ and $\sigma(1) = 1$. So the Sturm theorem gives that the number of roots of $K(x)$ in $(0,1)$ equals

$$\sigma(0) - \sigma(1) = 2 - 1 = 1;$$

(2) If $p_3(0)$ is positive, the sequences of signs are $\{-,+,+,+\}$ and $\{+,+,+,+\}$, thus $\sigma(0) = 1$ and $\sigma(1) = 0$. So the Sturm theorem gives that the number of roots of $K(x)$ in $(0,1)$ equals

$$\sigma(0) - \sigma(1) = 1 - 0 = 1;$$

(3) If $p_3(0)$ equals 0, note that neither 0 nor 1 is a root of $K(x)$, it is easy to see that the number of roots is $1 - 0 = 1$. This completes the proof. □

We are now ready to prove Theorem 2.5.

Proof of Theorem 2.5. The main idea of this proof is to estimate the minimal value of $\widetilde{\varphi}(z)$ on the unit disk. To do so, let $x = |z|^2$. Then x is in $(0,1)$ if z is in the unit disk. Simple calculations give

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(n+1)^2}{2n+4}x^n &= \frac{1}{4}+\sum_{n=1}^{\infty}\frac{n^2+2n+1}{2n+4}x^n\\ &= \frac{1}{4}+\frac{1}{2}\sum_{n=1}^{\infty}\left(n+\frac{1}{n+2}\right)x^n\\ &= \frac{1}{4}+\frac{1}{2}\sum_{n=1}^{\infty}nx^n+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n+2}x^n\\ &= \frac{1}{4}+\frac{x}{2(1-x)^2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n+2}x^n\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(n+1)^2}{2n+3}x^n &= \frac{1}{3}+\sum_{n=1}^{\infty}\frac{n^2+2n+1}{2n+3}x^n\\ &= \frac{1}{3}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(n^2+\frac{3}{2}n)+(\frac{1}{2}n+1)}{n+\frac{3}{2}}x^n\\ &= \frac{1}{3}+\frac{1}{2}\sum_{n=1}^{\infty}nx^n+\frac{1}{4}\sum_{n=1}^{\infty}x^n+\sum_{n=1}^{\infty}\frac{1}{8n+12}x^n\\ &= \frac{1}{3}+\frac{x}{2(1-x)^2}+\frac{x}{4(1-x)}+\sum_{n=1}^{\infty}\frac{1}{8n+12}x^n.\end{aligned}$$

Combining the above two series with the proof of Lemma 3.3 gives

$$\begin{aligned}\widetilde{\varphi}(z) &= 2(1-x)^2\left[\sum_{n=0}^{\infty}\frac{(n+1)^2}{2n+4}x^n+a\sum_{n=0}^{\infty}\frac{(n+1)^2}{2n+3}x^n\right]+b\\ &= \left(\frac{1}{2}+\frac{a}{6}\right)x^2+\frac{a}{6}x+\left(\frac{1}{2}+\frac{2}{3}a+b\right)+\frac{(1-x)^2}{2}\sum_{n=1}^{\infty}\frac{(4+a)n+(2a+6)}{(n+2)(2n+3)}x^n\\ &\geqslant \left(\frac{1}{2}+\frac{a}{6}\right)x^2+\frac{a}{6}x+\left(\frac{1}{2}+\frac{2}{3}a+b\right)+\frac{(1-x)^2}{2}\sum_{n=1}^{2}\frac{(4+a)n+(2a+6)}{(n+2)(2n+3)}x^n \quad (\text{since } a>-2)\\ &= \frac{1}{420}\left[(30a+105)x^4-(18a+70)x^3+(16a+35)x^2+(112a+140)x\right]+\left(\frac{1}{2}+\frac{2}{3}a+b\right)\\ &= \frac{30a+105}{420}\left[x^4-\frac{18a+70}{30a+105}x^3+\frac{16a+35}{30a+105}x^2+\frac{112a+140}{30a+105}x\right]+\left(\frac{2a}{3}+b+\frac{1}{2}\right).\end{aligned}$$

Letting

$$r=\frac{1}{10+\frac{35}{a}},$$

we have that r is in $(-\frac{2}{15},-\frac{1}{18})$ (since $a\in(-2,-\frac{5}{4})$) and

$$\begin{aligned}G(x) &:= x^4-\frac{18a+70}{30a+105}x^3+\frac{16a+35}{30a+105}x^2+\frac{112a+140}{30a+105}x\\ &= x^4+\left(\frac{2r}{3}-\frac{2}{3}\right)x^3+\left(\frac{1}{3}+2r\right)x^2+\left(\frac{4}{3}+24r\right)x.\end{aligned}$$

Thus

$$\begin{aligned}\widetilde{\varphi}(z) &\geqslant \frac{30a+105}{420}G(x)+\left(\frac{2a}{3}+b+\frac{1}{2}\right)\\ &\geqslant \frac{30a+105}{420}\inf_{x\in[0,1]}G(x)+\left(\frac{2a}{3}+b+\frac{1}{2}\right).\end{aligned}$$

To finish the proof, it suffices to show that for each $a\in(-\frac{14}{9},-\frac{5}{4})$, there exists a real constant b such that

$$\inf_{z\in\mathbb{D}}\widetilde{\varphi}(z)>0$$

and T_φ is not positive. As we show above, we need

$$\delta:=\frac{30a+105}{420}\inf_{x\in[0,1]}G(x)+\left(\frac{2a}{3}+b+\frac{1}{2}\right)>0.$$

This gives

$$-\frac{2}{3}a-\frac{1}{2}-\frac{30a+105}{420}\inf_{x\in[0,1]}G(x)<b.$$

By Lemma 3.2 we have that T_φ is positive if and only if

$$1+a+b\geqslant\max\left\{\frac{a}{2n+3}+\frac{2}{2n+4}\colon\ n\geqslant 0\right\}.$$

In order that T_φ is not positive, the above inequality gives

$$1+a+b<\max\left\{\frac{a}{2n+3}+\frac{2}{2n+4}\colon\ n\geqslant 0\right\}.$$

These are equivalent that there exists a real constant b such that

$$-\frac{2}{3}a-\frac{1}{2}-\frac{30a+105}{420}\inf_{x\in[0,1]}G(x)<b<\max\left\{\frac{a}{2n+3}+\frac{2}{2n+4}\colon\ n\geqslant 0\right\}-a-1.$$

Indeed, we will show that

$$\max\left\{\frac{a}{2n+3}+\frac{2}{2n+4}\colon\ n\geqslant 0\right\}+\frac{30a+105}{420}\inf_{x\in[0,1]}G(x)>\frac{a}{3}+\frac{1}{2}$$

for each $a\in(-\frac{14}{9},-\frac{5}{4})$ i.e. for each $r\in(-\frac{1}{12.5},-\frac{1}{18})$. To do this, the proof will be divided into four steps.

Let $K(x)=\frac{1}{4}G'(x)$ $(x\in(0,1))$ and recall that $F(x)=\frac{a}{x}+\frac{2}{x+1}$ $(x=2n+3\geqslant 3)$.

Step 1. Suppose r is in $(-\frac{1}{15.2},-\frac{1}{18})$. Then

$$\frac{\sqrt{\frac{-a}{2}}}{1-\sqrt{\frac{-a}{2}}}\in[3,5]\quad\text{and}\quad F(5)\geqslant F(3),$$

and so

$$\begin{aligned}\max\left\{\frac{a}{2n+3}+\frac{2}{2n+4}\colon\ n\geqslant 0\right\}&=F(5)\quad\text{(by the proof of Theorem 2.4)}\\&=\frac{a}{5}+\frac{1}{3}.\end{aligned}$$

Thus we are going to show that

$$\inf_{x\in[0,1]}G(x)>\frac{2}{3}+12r$$

for each $r\in(-\frac{1}{15.2},-\frac{1}{18})$. We consider the following three cases:

(I) If r is in $(-\frac{1}{15.9},-\frac{1}{17.59})$, then a simple computation gives

$$\begin{aligned}K(0)&=\frac{1}{3}+6r\\&<\frac{1}{3}-\frac{6}{18}\quad\left(\text{since } r<-\frac{1}{18}\right)\\&=0\end{aligned}$$

and

$$\begin{aligned}K\left(\frac{1}{2}\right)&=\frac{53r}{8}+\frac{5}{12}\\&>-\frac{53}{8}\times\frac{1}{15.9}+\frac{5}{12}\quad\left(\text{since } r>-\frac{1}{15.9}\right)\\&=-\frac{53}{8}\times\frac{10}{159}+\frac{5}{12}\\&=0.\end{aligned}$$

Thus there exists a point $x_0\in(0,\frac{1}{2})$ such that $K(x_0)=0$. By Lemma 5.1 above, we see that x_0 is the unique point where $G(x)$ reaches its minimal value. Moreover, x_0 satisfies the following equation:

$$x_0^3+\frac{r-1}{2}x_0^2+\left(r+\frac{1}{6}\right)x_0+\left(6r+\frac{1}{3}\right)=0,$$

and hence

$$x_0^4+\frac{r-1}{2}x_0^3+\left(r+\frac{1}{6}\right)x_0^2+\left(6r+\frac{1}{3}\right)x_0=0.$$

So we have

$$\inf_{x\in[0,1]}G(x)=\frac{r-1}{6}x_0^3+\frac{1+6r}{6}x_0^2+(1+18r)x_0.$$

Let

$$L(t)=\frac{r-1}{6}t^3+\frac{1+6r}{6}t^2+(1+18r)t\quad\bigl(t\in(0,1)\bigr).$$

Taking derivative of $L(t)$ gives

$$L'(t) = \frac{r-1}{2}t^2 + \frac{1+6r}{3}t + (1+18r) < 0$$

for all $t \in (0,1)$ if $r < -\frac{1}{17.59}$. This yields

$$\begin{aligned}\inf_{x\in[0,1]} G(x) &= L(x_0)\\ &\geqslant L\left(\frac{1}{2}\right) \quad \left(\text{since } x_0 \in \left(0, \frac{1}{2}\right)\right)\\ &= \frac{445}{48}r + \frac{25}{48}\\ &> \frac{2}{3} + 12r \quad \left(\text{since } r < -\frac{1}{17.59}\right).\end{aligned}$$

(II) Suppose r is in $(-\frac{1}{15.2}, -\frac{1}{15.9}]$. Using the same method as one in (I), we get

$$K\left(\frac{1}{2}\right) < 0, \qquad K\left(\frac{3}{5}\right) > 0,$$

and hence $x_0 \in (\frac{1}{2}, \frac{3}{5})$. Thus we have

$$\begin{aligned}\inf_{x\in[0,1]} G(x) &= L(x_0)\\ &\geqslant L\left(\frac{3}{5}\right) \quad \left(\text{since } x_0 < \frac{3}{5}\right)\\ &> \frac{2}{3} + 12r \quad \left(\text{since } r \leqslant -\frac{1}{15.9}\right).\end{aligned}$$

(III) Suppose that r is in $(-\frac{1}{17.59}, -\frac{1}{18})$. In this case, we need to consider the following three subcases.

(1) Let r be in $(-\frac{1}{17.59}, -\frac{1}{17.6}]$. Simple computations give

$$K(0.131) < 0 \quad \text{and} \quad K(0.141) > 0.$$

This implies that x_0 is in $(0.131, 0.141)$. Noting

$$\begin{aligned}L''(t) &= (r-1)t + \frac{1}{3} + 2r\\ &> (r-1) \times 0.141 + \frac{1}{3} + 2r\\ &= 2.141r + \frac{1}{3} - 0.141\\ &> 2.141 \times \left(-\frac{1}{17.59}\right) + \frac{1}{3} - 0.141\\ &> 0,\end{aligned}$$

we see that $L'(t)$ is increasing in $(0.131, 0.141)$, and hence

$$\begin{aligned}L'(t) &\leqslant L'(0.141)\\ &< 18.29r + 1.038\end{aligned}$$

$$\begin{aligned}&\leqslant 18.29 \times \left(-\frac{1}{17.6}\right) + 1.038\\&< 0.\end{aligned}$$

This implies that $L(t)$ is decreasing in $(0.131, 0.141)$. Thus we obtain

$$\begin{aligned}L(x_0) - \left(\frac{2}{3} + 12r\right) &\geqslant L(0.141) - \left(\frac{2}{3} + 12r\right)\\&> 2.56r + 0.14 - \frac{2}{3} - 12r\\&> \frac{9}{1000} \quad \left(\text{since } r \leqslant -\frac{1}{17.6}\right).\end{aligned}$$

(2) Let r be in $(-\frac{1}{17.6}, -\frac{1}{17.7}]$. Using the same idea as one in (1) of (III) above, we have that

$$\inf_{x\in[0,1]} G(x) > \frac{2}{3} + 12r$$

for each $r \in (-\frac{1}{17.6}, -\frac{1}{17.7}]$.

(3) Let r be in $(-\frac{1}{17.7}, -\frac{1}{18})$. Observe that

$$\begin{aligned}\inf_{x\in[0,1]} G(x) &= \frac{r-1}{6}x_0^3 + \frac{1+6r}{6}x_0^2 + (1+18r)x_0\\&= -\frac{1}{36}\left[3\left(r^2 - 14r - 1\right)x_0^2 + \left(6r^2 - 653r - 37\right)x_0 + 2\left(18r^2 - 17r - 1\right)\right].\end{aligned}$$

It suffices to show that

$$-\frac{1}{36}\left[3\left(r^2 - 14r - 1\right)x_0^2 + \left(6r^2 - 653r - 37\right)x_0 + 2\left(18r^2 - 17r - 1\right)\right] > \frac{2}{3} + 12r.$$

To get the above inequality, we need to show

$$3\left(r^2 - 14r - 1\right)x_0^2 + \left(6r^2 - 653r - 37\right)x_0 + 2\left(18r^2 + 199r + 11\right) < 0.$$

To do so, let

$$R(t) = 3\left(r^2 - 14r - 1\right)t^2 + \left(6r^2 - 653r - 37\right)t + 2\left(18r^2 + 199r + 11\right).$$

Taking derivative of $R(t)$ gives

$$R'(t) = 6\left(r^2 - 14r - 1\right)t + \left(6r^2 - 653r - 37\right).$$

If r is in $(-\frac{1}{17.7}, -\frac{1}{18})$, then

$$r^2 - 14r - 1 < 0 \quad \text{and} \quad 6r^2 - 653r - 37 < 0.$$

This implies that $R(t)$ is decreasing in $(0, 1)$ and so

$$\begin{aligned}R(x_0) &\leqslant R(0) \quad \left(\text{since } x_0 \in (0,1)\right)\\&= 2\left(18r^2 + 199r + 11\right)\end{aligned}$$

$$= 36(r+11)\left(r+\frac{1}{18}\right)$$
$$< 0.$$

Step 2. Suppose that r is in $(-\frac{1}{14}, -\frac{1}{15.2}]$. Then

$$\frac{\sqrt{\frac{-a}{2}}}{1-\sqrt{\frac{-a}{2}}} \in [5,7] \quad \text{and} \quad F(5) \geqslant F(7).$$

Using the same idea as one in (I) of Step 1, we need only to show

$$\inf_{x\in[0,1]} G(x) > \frac{2}{3} + 12r$$

for each $r \in (-\frac{1}{14}, -\frac{1}{15.2}]$. Indeed,

$$K\left(\frac{1}{2}\right) < 0, \qquad K(0.63) > 0,$$

so $x_0 \in (\frac{1}{2}, 0.63)$ and $L(x_0) \geqslant L(0.63) > \frac{2}{3} + 12r$.

Step 3. Suppose r is in $(-\frac{7}{90}, -\frac{1}{14}]$. Then

$$\frac{\sqrt{\frac{-a}{2}}}{1-\sqrt{\frac{-a}{2}}} \in [5,7] \quad \text{and} \quad F(5) < F(7),$$

and thus

$$\max\left\{\frac{a}{2n+3} + \frac{2}{2n+4}\colon\ n \geqslant 0\right\} = F(7) \quad \text{(by the proof of Theorem 2.4)}$$
$$= \frac{a}{7} + \frac{1}{4}.$$

So we need to prove

$$\inf_{x\in[0,1]} G(x) > 1 + \frac{50}{3}r$$

for each $r \in (-\frac{7}{90}, -\frac{1}{14}]$. One can show the above inequality using the same method as one in (I) and the details are omitted here.

Step 4. Suppose r is in $(-\frac{1}{12.5}, -\frac{7}{90}]$. In this case, we have

$$\frac{\sqrt{\frac{-a}{2}}}{1-\sqrt{\frac{-a}{2}}} \in [7,9] \quad \text{and} \quad F(7) \geqslant F(9).$$

Thus we also need to prove

$$\inf_{x\in[0,1]} G(x) > 1 + \frac{50}{3}r$$

for each $r \in (-\frac{1}{12.5}, -\frac{7}{90}]$. This can be showed easily by the method used in (I) of Step 1. This completes the proof. □

Acknowledgments

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References

[1] S. Axler, Bergman spaces and their operators, in: Surveys of Some Recent Results in Operator Theory, vol. I, in: Pitman Res. Notes Math. Ser., vol. 171, Longman Sci. Tech., Harlow, 1988, pp. 1–50.
[2] S. Axler, D. Zheng, Compact operators via the Berezin transform, Indiana Univ. Math. J. 47 (1998) 387–400.
[3] L. Coburn, Berezin transform and Weyl-type unitary operators on the Bergman space, Proc. Amer. Math. Soc. 140 (2012) 3445–3451.
[4] Z. Cuckovic, Berezin versus Mellin, J. Math. Anal. Appl. 287 (2003) 234–243.
[5] R. Douglas, Banach Algebra Techniques in Operator Theory, second ed., Grad. Texts in Math., vol. 179, Springer, New York, 1998.
[6] H. Fine, A College Algebra, Ginn, 1904.
[7] P. Halmos, A Hilbert Space Problem Book, van Nostrand-Reinhold, Princeton, NJ, 1967.
[8] N. Jacobson, Basic Algebra I, second ed., W.H. Freeman, New York, 1985.
[9] G. McDonald, C. Sundberg, Toeplitz operators on the disc, Indiana Univ. Math. J. 28 (1979) 595–611.
[10] F. Nazarov, private communication.
[11] J. Stoer, R. Bulirsch, Introduction to Numerical Analysis, second ed., Springer, New York, 1993.
[12] K. Stroethoff, The Berezin transform and operators on spaces of analytic functions, in: Banach Center Publ., vol. 38, Polish Academy of Sciences, Warsaw, 1997, pp. 361–380.
[13] K. Zhu, Operator Theory in Function Spaces, second ed., Math. Surveys Monogr., vol. 138, American Mathematical Society, 2007.