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Large time behavior of solutions to the compressible Navier–Stokes equations with potential force

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A R T I C L E I N F O

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A B S T R A C T

The compressible Navier–Stokes equation with a potential external force is considered in $\mathbb{R}^3$ in the present paper. Under the smallness assumption on both the external force and the initial perturbation of the stationary solution in some Sobolev spaces, the existence theory of global solutions to the stationary profile is established. Furthermore, when the $\dot{H}^{-s}$ norm ($s \in (0, \frac{1}{2})$) of initial perturbation is finite, we obtain the optimal time decay rates of the solutions in L^2-norm. As a corollary, the L^p–L^q ($3/2 < p \leq 2$) type of the decay rates follows without requiring that the L^p norm of initial perturbation is small.

1. Introduction

In this paper, we consider the initial value problem of the compressible Navier–Stokes equations with a potential force as follows:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ u_t + (u \cdot \nabla)u + \dfrac{\nabla P(\rho)}{\rho} = \dfrac{\mu}{\rho}\Delta u + \dfrac{\mu+\lambda}{\rho}\nabla \operatorname{div} u - \nabla\phi(x), \\ (\rho, u)(0, x) = (\rho_0, u_0)(x) \to (\rho_\infty, 0), \quad \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $t > 0$. Here, $\rho = \rho(x,t) > 0$, $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $P = P(\rho)$ denote the density, velocity and the pressure function, respectively; $-\nabla\phi(x)$ is the time independent potential force; μ, λ are viscosity constants, satisfying $\mu > 0$, $2\mu + 3\lambda \geq 0$ which deduce $\mu + \lambda > 0$. In addition, $(\rho_\infty, 0)$ is the state of initial data at infinity, while ρ_∞ is a positive constant and $P(\rho)$ is smooth in a neighborhood of ρ_∞ with $P'(\rho_\infty) > 0$.

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For the Navier–Stokes equations (1.1)$_1$–(1.1)$_2$ with potential force, the stationary solution $(\rho_*, u_*)(x)$ satisfies, cf. [17]

$$\int_{\rho_\infty}^{\rho_*(x)} \frac{P'(z)}{z} \mathrm{d}z + \phi(x) = 0, \qquad u_*(x) = 0. \tag{1.2}$$

There are many works which were devoted to proving the global existence, unique and time decay rates of solutions to the compressible Navier–Stokes equations with or without external forces, cf. [2,5–9,11,12,14–18,22,24] and references therein. In the following, we mainly mention some studies on the time decay rates of the solutions.

When omitting the external force, the stationary solution is just a constant. Matsumura and Nishida in [16] obtained the first global existence of small solutions when the initial perturbation is small in $H^3(\mathbb{R}^3)$. They also studied the L^2-norm decay rates in $H^4(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, see [15]. Moreover, the optimal L^p-norm time decay rates were proved by Ponce in [18]. Furthermore, the pointwise estimates of solutions were shown in [6,7,14] when the small initial perturbation in $H^N(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $N \geq [\frac{n}{2}] + 3$. Under the framework of $H^2(\mathbb{R}^3)$, by some elaborate estimates, the global existence of a strong solution and its optimal decay estimates were obtained in [24] when the initial data is bounded in L^1-norm. Recently, by using a nonnegative and negative Sobolev space $H^N(\mathbb{R}^3) \cap \dot{H}^{-s}(\mathbb{R}^3)$ ($s \in [0, 3/2)$) to replace $H^N(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ with $N \geq 3$, Guo and Wang in [5] developed a general energy method and obtained the following optimal time-decay estimates of solutions, i.e.

$$\left\|\nabla^k(\rho - \rho_*, u)(t)\right\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{k+s}{2}}, \quad \text{for } -s < k \leq N.$$

When a general external force is involved, the stationary solution (ρ_*, u_*) may not be a constant. For this, when the initial disturbance belongs to $H^3(\mathbb{R}^3) \cap L^{6/5}(\mathbb{R}^3)$, the following convergence rate was obtained by Shibata and Tanaka in [20] for isentropic viscous fluid

$$\left\|\nabla(\rho - \rho_*, u - u_*)(t)\right\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{1}{2}+\kappa},$$

for any small constant $\kappa > 0$. The same decay rate for non-isentropic case was established by Qian and Yin in [19]. For the external potential force, based on the energy method and the spectral analysis on the linearized system (see (4.19)), Duan et al. studied the optimal time decay estimates

$$\left\|(\rho - \rho_*, u)(t)\right\|_{L^q(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})},$$

and

$$\left\|\nabla(\rho - \rho_*, u)(t)\right\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}},$$

when the initial perturbation belongs to $H^3(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, cf. [4] for $p = 1$ and [3] for $1 \leq p < 6/5$.

Motivated by [3,4], we prove L^p–L^q type time decay estimates of solutions for $3/2 < p \leq 2$ by employing a negative Sobolev space $\dot{H}^{-s}(\mathbb{R}^3)$ to replace $L^p(\mathbb{R}^3)$. To be specific, we study the global existence and optimal time decay estimate of solutions to the problem (1.1) in both H^2-framework and H^3-framework.

First of all, when the initial perturbation is small in $H^2(\mathbb{R}^3)$, for the existence part, the difficulty mainly comes from the appearance of non-trivial stationary solutions. We should avoid the terms $\int_0^t \|\nabla^3(\rho - \rho_*)(\tau)\|_{L^2} \mathrm{d}\tau$ and $\int_0^t \|\nabla^4 u(\tau)\|_{L^2} \mathrm{d}\tau$ during the process of deducing *a priori* estimates. To overcome the difficulties, we employ a refined energy method. Moreover, when the initial perturbation is bounded in $\dot{H}^{-s}(\mathbb{R}^3)$, we obtain the optimal time decay estimates by the general energy method introduced in [5].

Finally, when the initial perturbation belongs to $H^3(\mathbb{R}^3)$, based on the existence of the global solutions obtained in [3,16], we establish the optimal L^2-norm time decay estimates of the solution by employing the representation of the solution and the L^p–L^q type estimate on linearized system. As an immediate by-product, the optimal L^p–L^q $(3/2 < p \leq 2)$ time decay rate is shown directly by the Hardy–Littlewood–Sobolev theorem (see Lemma A.5 in Appendix A). It is also worth mentioning that we don't require that the L^p-norm of the initial perturbation is small.

The rest of the paper is organized as follows. After reformulating the problem and stating the main results in the next section, we give the global existence of solutions when the initial perturbation is in $H^2(\mathbb{R}^3)$ by using the energy method in Section 3. In Section 4, we obtain the decay estimates of solutions to the Cauchy problem (1.1) in both H^2-framework and H^3-framework. In Appendix A, we show some useful inequalities.

2. Preliminary and main results

In this section, we first introduce some notation for later use. Then, we give the reformulation of the problem (1.1) and show the main results in this paper.

Let C be denoted as a generic positive constant. The notation "$a \lesssim b$" means that "$a \leq Cb$" for a universal constant $C > 0$. For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ and $|\alpha| = \sum_{i=1}^3 \alpha_i$. We denote a set composed of all mth partial derivatives with respect to the variable x by ∇^m. $H^m(\mathbb{R}^3)$, $m \in \mathbb{Z}_+$, denotes the usual Sobolev space with its norm

$$\|f\|_{H^m(\mathbb{R}^3)} := \sum_{k=0}^{m} \big\|\nabla^k f\big\|_{L^2(\mathbb{R}^3)}.$$

In particular, we use $\|\cdot\|_m = \|\cdot\|_{H^m(\mathbb{R}^3)}$ and $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^3)}$. And $\langle\cdot,\cdot\rangle$ denotes the inner-product in $L^2(\mathbb{R}^3)$. In addition, let $\hat{f}(\xi)$ be the Fourier transform of $f(x)$ with respect to $x \in \mathbb{R}^3$, i.e. $\hat{f}(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} f(x)\mathrm{e}^{-ix\cdot\xi}\mathrm{d}x$. The operator Λ^s, $s \in \mathbb{R}$, is defined by

$$\Lambda^s f(x) = \int_{\mathbb{R}^3} |\xi|^s \hat{f}(\xi)\mathrm{e}^{ix\cdot\xi}\mathrm{d}\xi.$$

We define the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3)$ of all f for which $\|f\|_{\dot{H}^s}$ is finite, where

$$\|f\|_{\dot{H}^s} := \big\|\Lambda^s f\big\|_{L^2} = \big\||\xi|^s \hat{f}\big\|_{L^2}.$$

Throughout this paper, we will use the non-positive index s. For convenience, we will change the index to be "$-s$" with $s \geq 0$.

Now, we will reformulate the problem (1.1) as follows. Set

$$\mu_1 = \frac{\mu}{\rho_\infty}, \qquad \mu_2 = \frac{\mu+\lambda}{\rho_\infty}, \qquad \gamma = \sqrt{P'(\rho_\infty)}.$$

Taking change of variables by

$$\tilde{\rho}(x,t) = \rho(x,t) - \rho_*(x), \qquad \tilde{u}(x,t) = u(x,t),$$

and

$$\bar{\rho}(x) = \rho_*(x) - \rho_\infty,$$

the initial value problem (1.1) is reformulated as

$$\begin{cases} \tilde{\rho}_t + \rho_\infty \nabla \cdot \tilde{u} = \tilde{S}_1, \\ \tilde{u}_t - \mu_1 \Delta \tilde{u} - \mu_2 \nabla \operatorname{div} \tilde{u} + \dfrac{P'(\rho_\infty)}{\rho_\infty} \nabla \tilde{\rho} = \tilde{S}_2, \\ (\tilde{\rho}, \tilde{u})(x,t)|_{t=0} = (\rho_0 - \rho_*, u_0)(x) \to (0,0), \quad \text{as } |x| \to \infty, \end{cases} \tag{2.1}$$

where $\tilde{S}_1$ and $\tilde{S}_2$ are the source terms. Denote

$$\sigma(x,t) = \tilde{\rho}(x,t), \qquad \omega(x,t) = \frac{\rho_\infty}{\sqrt{P'(\rho_\infty)}} \tilde{u}(x,t),$$

by (1.2), then the problem (2.1) can be rewritten as

$$\begin{cases} \sigma_t + \gamma \nabla \cdot \omega = S_1, \\ \omega_t - \mu_1 \Delta \omega - \mu_2 \nabla \operatorname{div} \omega + \gamma \nabla \sigma = S_2, \\ (\sigma, \omega)(x,t)|_{t=0} = (\sigma_0, \omega_0)(x), \end{cases} \tag{2.2}$$

where

$$S_1 = -\frac{\mu_1 \gamma}{\mu} \operatorname{div}\big[(\sigma + \bar{\rho})\omega\big], \tag{2.3}$$

$$\begin{aligned} S_2 = &-\frac{\mu_1^2 \gamma^2}{\mu^2} (\omega \cdot \nabla)\omega - \mu_1 \frac{\sigma + \bar{\rho}}{\sigma + \rho_*} \Delta \omega - \mu_2 \frac{\sigma + \bar{\rho}}{\sigma + \rho_*} \nabla \operatorname{div} \omega \\ &- \frac{\mu_1 \gamma}{\mu} \left[\frac{P'(\sigma + \rho_*)}{\sigma + \rho_*} - \frac{P'(\rho_*)}{\rho_*} \right] \nabla \bar{\rho} - \frac{\mu_1 \gamma}{\mu} \left[\frac{P'(\sigma + \rho_*)}{\sigma + \rho_*} - \frac{P'(\rho_\infty)}{\rho_\infty} \right] \nabla \sigma, \end{aligned} \tag{2.4}$$

and

$$(\sigma_0, \omega_0)(x) = \left(\rho_0 - \rho_*, \frac{\rho_\infty}{\sqrt{P'(\rho_\infty)}} u_0 \right)(x) \to (0,0), \quad \text{as } |x| \to \infty.$$

We consider the global existence and time decay rates of the solution (ρ, u) to the steady state $(\rho_*, 0)$, that is, the existence and decay rates of the perturbed solution (σ, ω). In what follows, we begin to state our main result in H^2-framework as follows.

Theorem 2.1. *Let $(\sigma_0, \omega_0)(x) \in H^2(\mathbb{R}^3)$. If $\|(\sigma_0, \omega_0)\|_2 \le \epsilon$ and the potential function $\phi(x)$ satisfies*

$$\|\phi\|_{H^3 \cap L^\infty} + \big\|(1+|x|)\nabla\phi\big\|_{L^2 \cap L^3} + \big\|(1+|x|)\nabla^2\phi\big\|_{L^2 \cap L^3} + \big\|(1+|x|)\nabla^3\phi\big\|_{L^2} \le \epsilon, \tag{2.5}$$

for some small constant $\epsilon > 0$, then the Cauchy problem (2.2) admits a unique global solution (σ, ω) satisfying

$$\big\|(\sigma, \omega)(t)\big\|_2^2 + \int_0^t \big(\big\|\nabla\sigma(\tau)\big\|_1^2 + \big\|\nabla\omega(\tau)\big\|_2^2\big) \mathrm{d}\tau \le C \big\|(\sigma_0, \omega_0)\big\|_2^2, \quad t \ge 0. \tag{2.6}$$

In addition, for $s \in (0, 1/2)$, $(\sigma_0, \omega_0) \in \dot{H}^{-s}(\mathbb{R}^3)$ satisfies

$$\big\|(\sigma_0, \omega_0)\big\|_{\dot{H}^{-s}} < +\infty,$$

then there exists a constant C_0 such that

$$\left\|\Lambda^{-s}(\sigma,\omega)(t)\right\| \le C_0, \quad \textit{for } s \in (0,1/2), \tag{2.7}$$

and

$$\left\|\nabla^k(\sigma,\omega)(t)\right\| \le C_0(1+t)^{-\frac{s}{2}}, \quad k = 0,1,2. \tag{2.8}$$

The Hardy–Littlewood–Sobolev theorem (see Lemma A.5 in Appendix A) shows that for any $p \in (3/2, 2)$, $L^p \subset \dot{H}^{-s}$ with $s = 3(\frac{1}{p} - \frac{1}{2}) \in (0, 1/2)$. Then, from the decay estimates stated in Theorem 2.1, we have the following L^p–L^2 type time decay results.

Corollary 2.1. *Under the assumptions of Theorem 2.1 except that we replaced the $\dot{H}^{-s}$ assumption by that $(\sigma_0, \omega_0) \in L^p$ for $p \in (3/2, 2)$, the problem (2.2) admits a unique global solution (σ, ω) which enjoys the following time decay estimates:*

$$\left\|\nabla^k(\sigma,\omega)(t)\right\| \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})}, \quad \textit{for } k = 0,1,2.$$

Under the framework of $H^3(\mathbb{R}^3)$, the global existence and energy inequalities of solutions to the problem (2.2) near the steady state was proved by Matsumura–Nishida in [16] and Duan et al. in [3]. Precisely, the results can be stated as follows.

Theorem 2.2. *(See [3,16].) Let $(\sigma_0, \omega_0)(x) \in H^3(\mathbb{R}^3)$. If $\|(\sigma_0, \omega_0)\|_3 \le \epsilon_1$ and the potential function $\phi(x)$ satisfies*

$$\|\phi\|_{H^3 \cap L^\infty} + \left\|(1+|x|)\nabla\phi\right\|_{L^2 \cap L^\infty} + \sum_{2 \le k \le 4} \left\|(1+|x|)\nabla^k\phi\right\|_{L^\infty} \le \epsilon_1, \tag{2.9}$$

for some small constant $\epsilon_1 > 0$, then there exists a unique global solution (σ, ω) of the Cauchy problem (1.1) satisfying

$$\left\|(\sigma,\omega)(t)\right\|_3^2 + \int_0^t \left(\left\|\nabla\sigma(\tau)\right\|_2^2 + \left\|\nabla\omega(\tau)\right\|_3^2\right) \mathrm{d}\tau \le C\left\|(\sigma_0,\omega_0)\right\|_3^2, \quad t \ge 0. \tag{2.10}$$

Moreover, there is a Lyapunov-type energy inequality in the form of

$$\frac{\mathrm{d}\mathcal{L}(t)}{\mathrm{d}t} + \mathcal{L}(t) \le C\left\|\nabla(\sigma,\omega)(t)\right\|^2, \tag{2.11}$$

where $\mathcal{L}(t)$ is an energy functional which is equivalent to $\|\nabla(\sigma,\omega)(t)\|_2^2$.

The following is the optimal time decay rates of the solution when the initial perturbation is bounded in a negative Sobolev space $\dot{H}^{-s}(\mathbb{R}^3)$.

Theorem 2.3. *Assume all hypotheses of Theorem 2.2 hold. In addition, for $s \in (0, 1/2)$, (σ_0, ω_0) is bounded in $\dot{H}^{-s}(\mathbb{R}^3)$, then there exists a constant $\tilde{C}_0$ such that*

$$\left\|\Lambda^{-s}(\sigma,\omega)(t)\right\| \le \tilde{C}_0, \quad \textit{for } s \in (0,1/2),$$
$$\left\|(\sigma,\omega)(t)\right\| \le \tilde{C}_0(1+t)^{-\frac{s}{2}},$$

and

$$\left\|\nabla^k(\sigma,\omega)(t)\right\| \le \tilde{C}_0(1+t)^{-\frac{1+s}{2}}, \quad k = 1,2.$$

Similar to Corollary 2.1, we have the L^p–L^q type time decay estimates as follows.

Corollary 2.2. *Assume all hypotheses of Theorem 2.2 hold. If further,* $(\sigma_0, \omega_0) \in L^p$ *for* $p \in (3/2, 2)$*, the problem (2.2) admits a unique global solution* (σ, ω) *which enjoys the following time decay estimates:*

$$\big\|(\sigma,\omega)(t)\big\| \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})}, \tag{2.12}$$

$$\big\|\nabla^k(\sigma,\omega)(t)\big\| \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}, \quad \text{for } k=1,2. \tag{2.13}$$

Furthermore, for any $2 \le q \le 6$*, we have*

$$\big\|(\sigma,\omega)(t)\big\|_{L^q} \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}. \tag{2.14}$$

Remark 2.1. The decay estimate (2.14) is obtained directly by Lemmas A.1–A.2, i.e.

$$\begin{aligned}\big\|(\sigma,\omega)(t)\big\|_{L^q} &\le \big\|(\sigma,\omega)(t)\big\|_{L^6}^{\theta}\big\|(\sigma,\omega)(t)\big\|^{1-\theta}\\ &\le C\big\|\nabla(\sigma,\omega)(t)\big\|^{\theta}\big\|(\sigma,\omega)(t)\big\|^{1-\theta} \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})},\end{aligned} \tag{2.15}$$

where $\theta = \frac{3}{2} - \frac{3}{q}$. Compared with the decay result in [3], we expand the L^p–L^q type time decay estimate from $1 \le p < 6/5$ to $3/2 < p \le 2$.

3. Global existence

In this section, we are devoted to proving the existence part of Theorem 2.1, i.e. the global existence of solutions to the problem (2.2) when the initial perturbation is small in $H^2(\mathbb{R}^3)$.

3.1. Local existence

In this section, we will show the local existence of solution to the initial value problem (2.2). Before we proceed, we should remark that parts of our ideas come from [10,17]. First of all, (2.2) can be rewritten as

$$\begin{cases}\tilde\rho_t + \tilde u\cdot\nabla\tilde\rho = F_1[\tilde\rho,\tilde u],\\ \tilde u_t - \dfrac{\mu}{\tilde\rho+\rho_*}\Delta\tilde u - \dfrac{\mu+\lambda}{\tilde\rho+\rho_*}\nabla\operatorname{div}\tilde u = F_2[\tilde\rho,\tilde u],\\ (\tilde\rho,\tilde u)(x,t)|_{t=0} = (\tilde\rho_0,\tilde u_0)(x),\end{cases} \tag{3.1}$$

where

$$F_1[\tilde\rho,\tilde u] = -\tilde\rho\nabla\cdot\tilde u - \operatorname{div}(\rho_*\tilde u), \tag{3.2}$$

$$F_2[\tilde\rho,\tilde u] = -\frac{P'(\tilde\rho+\rho_*)}{\tilde\rho+\rho_*}\nabla\tilde\rho - \left[\frac{P'(\tilde\rho+\rho_*)}{\tilde\rho+\rho_*} - \frac{P'(\rho_*)}{\rho_*}\right]\nabla\rho_*. \tag{3.3}$$

Let's define the function set $X = X(0,T;E_0)$ as follows. $X(0,T;E_0)$ consists of functions $(\tilde\rho,\tilde u)$ satisfying the following properties, for any $0 \le T \le +\infty$,

$$\begin{aligned}&\tilde\rho \in C^0\big(0,T;H^2(\mathbb{R}^3)\big), \qquad \tilde\rho_t \in C^1\big(0,T;H^1(\mathbb{R}^3)\big),\\ &\tilde u \in C^0\big(0,T;H^2(\mathbb{R}^3)\big), \qquad \nabla\tilde u \in L^2\big(0,T;H^1(\mathbb{R}^3)\big),\\ &\tilde u_t \in C^0\big(0,T;L^2(\mathbb{R}^3)\big) \cap L^2\big(0,T;H^1(\mathbb{R}^3)\big),\end{aligned}$$

and

$$\sup_{0\le\tau\le t}\left\|(\tilde\rho,\tilde u)(\tau)\right\|_2^2+\int_0^t\left\|\nabla\tilde u(\tau)\right\|_2^2\mathrm{d}\tau\le 4E_0^2,$$
$$\left\|\partial_t\tilde\rho(t)\right\|_1+\left\|\partial_t\tilde u(t)\right\|\le C_1E_0,\quad \text{for } t\in[0,T],$$

where $E_0=\|(\tilde\rho_0,\tilde u_0)\|_2$ and C_1 is a suitable positive constant.

Theorem 3.1 *(Local existence). Suppose that $(\tilde\rho_0,\tilde u_0)(x)\in H^2(\mathbb{R}^3)$, (2.5) and E_0 is suitably small. Then there exists a positive constant $T_0>0$ depending on $\tilde\rho_0$ and $\tilde u_0$, such that the initial value problem (3.1) has a unique solution $(\tilde\rho,\tilde u)(x,t)\in X(0,T_0;E_0)$ which satisfies*

$$(\tilde\rho,\tilde u)\in C^0\big(0,T_0;H^2\big(\mathbb{R}^3\big)\big)\cap C^1\big(0,T_0;H^1\big(\mathbb{R}^3\big)\big).$$

Moreover, the solution verifies

$$\left\|(\tilde\rho,\tilde u)(t)\right\|_2^2+\int_0^t\left\|\nabla\tilde u(\tau)\right\|_2^2\mathrm{d}\tau\le C\left\|(\tilde\rho_0,\tilde u_0)\right\|_2^2,\quad \textit{for } t\in[0,T_0].$$

Proof. *Step 1.* We introduce the successive approximate sequence $\{(\tilde\rho^n,\tilde u^n)(x,t)\}$ for the initial value problem as follows $(\tilde\rho^0,\tilde u^0)(x,t)=(\tilde\rho_0,\tilde u_0)(x)$ and

$$\begin{cases}\tilde\rho_t^{n+1}+\tilde u^n\cdot\nabla\tilde\rho^{n+1}=F_1\big[\tilde\rho^n,\tilde u^n\big],\\ \tilde u_t^{n+1}-\dfrac{\mu}{\tilde\rho^n+\rho_*}\Delta\tilde u^{n+1}-\dfrac{\mu+\lambda}{\tilde\rho^n+\rho_*}\nabla\operatorname{div}\tilde u^{n+1}=F_2\big[\tilde\rho^n,\tilde u^n\big],\\ \big(\tilde\rho^{n+1},\tilde u^{n+1}\big)(x,t)\big|_{t=0}=(\tilde\rho_0,\tilde u_0)(x),\end{cases}\tag{3.4}$$

for $n=0,1,2,\ldots$. Now, we will show that if $(\tilde\rho^n,\tilde u^n)\in X(0,T;E_0)$ then $(\tilde\rho^{n+1},\tilde u^{n+1})\in X(0,T;E_0)$, provided that E_0 and T are chosen to be suitably small. This shows that $X(0,T;E_0)$ is an invariant set of the mapping $(\tilde\rho^n,\tilde u^n)\to(\tilde\rho^{n+1},\tilde u^{n+1})$. If $(\tilde\rho^n,\tilde u^n)\in X(0,T;E_0)$, by using Lemmas A.1 and A.3 and (2.5), it follows from (3.2) and (3.3) that

$$\left\|F_1\big[\tilde\rho^n,\tilde u^n\big]\right\|_2+\left\|F_2\big[\tilde\rho^n,\tilde u^n\big]\right\|_1\le C\left\|\nabla\big(\tilde\rho^n,\tilde u^n\big)\right\|_2.\tag{3.5}$$

By using the standard energy method and Lemmas A.1 and A.3, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\tilde\rho^{n+1}\right\|_2\le\left\|\nabla^2\tilde u^n\right\|\left\|\tilde\rho^{n+1}\right\|_2+\|F_1\|_2,\tag{3.6}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\tilde u^{n+1}\right\|_2^2+\left\|\nabla\tilde u^{n+1}\right\|_2^2+\left\|\operatorname{div}\tilde u^{n+1}\right\|_2^2\le C\left\|\nabla\tilde\rho^n\right\|_1\left\|\nabla\tilde u^{n+1}\right\|_2^2+\|F_2\|_1^2.\tag{3.7}$$

From (3.6) and (3.7), by noticing $(\tilde\rho^n,\tilde u^n)\in X(0,T;E_0)$ and using (3.5), the smallness of E_0 and the Gronwall inequality, we have

$$\left\|\big(\tilde\rho^{n+1},\tilde u^{n+1}\big)(t)\right\|_2^2+\int_0^t\left\|\nabla\tilde u^{n+1}\right\|_2^2\mathrm{d}\tau\le \mathrm{e}^{4t^{\frac12}E_0}\big\{2\left\|(\tilde\rho_0,\tilde u_0)\right\|_2^2+CE_0^2t\big\}.\tag{3.8}$$

Take T_0 so that

$$e^{4T_0^{\frac{1}{2}}E_0} \le \frac{4}{3}, \qquad CE_0^2 T_0 \le \|(\tilde{\rho}_0, \tilde{u}_0)\|_2^2.$$

Then (3.8) becomes

$$\|(\tilde{\rho}^{n+1}, \tilde{u}^{n+1})(t)\|_2^2 + \int_0^t \|\nabla \tilde{u}^{n+1}\|_2^2 d\tau \le 4E_0^2.$$

By using $(3.1)_1$–$(3.1)_2$ and (3.5), we have

$$\begin{aligned}\|\tilde{\rho}_t^{n+1}(t)\|_1 + \|\tilde{u}_t^{n+1}(t)\| &\le C_2(\|\nabla \tilde{\rho}^{n+1}\|_1 + \|\nabla^2 \tilde{u}^{n+1}\|) + \|F_1\|_1 + \|F_2\| \\ &\le (4C_2 + C)E_0 \le C_1 E_0,\end{aligned}$$

where C_2 is a positive constant.

Then, $\{(\tilde{\rho}^n, \tilde{u}^n)(x,t)\}$ is well defined and is uniformly bounded with respect to $n \ge 0$, i.e. $(\tilde{\rho}^n, \tilde{u}^n) \in X(0, T_0; E_0)$.

Step 2. Applying the standard energy estimate for the linear symmetric system satisfied by the difference $(\tilde{\rho}^{n+1} - \tilde{\rho}^n, \tilde{u}^{n+1} - \tilde{u}^n)$, we find that $\{(\tilde{\rho}^n, \tilde{u}^n)\}$ is a Cauchy sequence in $C^0(0, T_0; H^1(\mathbb{R}^3))$. So, there exists functions $(\tilde{\rho}, \tilde{u})(x,t)$ with $(\tilde{\rho}, \tilde{u}) \in C^0(0, T_0; H^1(\mathbb{R}^3))$ such that $(\tilde{\rho}^n, \tilde{u}^n) \to (\tilde{\rho}, \tilde{u})$ strongly in $C^0(0, T_0; H^1(\mathbb{R}^3))$ as $n \to +\infty$.

On the other hand, $\{(\tilde{\rho}^n, \tilde{u}^n)\}$ is uniformly bounded in $L^\infty(0, T_0; H^2(\mathbb{R}^3))$. Then, there is a subsequence (which is denoted by the same symbol) such that $(\tilde{\rho}^n, \tilde{u}^n) \to (\tilde{\rho}, \tilde{u})$ weakly star in $L^\infty(0, T_0; H^2(\mathbb{R}^3))$. Consequently, we have a solution $(\tilde{\rho}, \tilde{u})(x,t)$ of the problem (3.1) satisfying

$$\begin{gathered}\tilde{\rho} \in L^\infty(0, T_0; H^2(\mathbb{R}^3)), \\ \tilde{u} \in L^\infty(0, T_0; H^2(\mathbb{R}^3)) \cap L^2(0, T; H^3(\mathbb{R}^3)).\end{gathered}$$

Moreover, it follows from Eq. (3.1) and $(\tilde{\rho}, \tilde{u})(t) \in X(0, T_0; E_0)$ that

$$\begin{gathered}\tilde{\rho}_t \in L^\infty(0, T_0; H^1(\mathbb{R}^3)), \\ \tilde{u}_t \in L^\infty(0, T_0; L^2(\mathbb{R}^3)) \cap L^2(0, T_0; H^1(\mathbb{R}^3)).\end{gathered}$$

Similar to proving Lemma 2.6 in [10], we have $(\tilde{\rho}, \tilde{u}) \in C^0(0, T_0; H^2(\mathbb{R}^3))$ and $\tilde{\rho}_t \in C^0(0, T_0; H^1(\mathbb{R}^3))$ and $\tilde{u}_t \in C^0(0, T_0; L^2(\mathbb{R}^3))$. Thus, we finish the proof of Theorem 3.1. □

3.2. *Some* a priori *estimates*

In this section, we will establish some *a priori* estimates of solution to the problem (2.2). With the help of the local existence theory and those estimates, the global existence of solutions will be obtained by employing the stranded continuity argument. To begin with, we make *a priori* assumption

$$\sup_{0 \le \tau \le t} \|(\sigma, \omega)(\tau)\|_2 \le \delta, \tag{3.9}$$

where a constant $\delta > 0$ is sufficiently small. Now, we show the energy estimate for (σ, ω).

Lemma 3.1. *There exists a suitably large constant $D_1 > 0$, which is independent of δ, such that for $0 \le k \le 2$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|\nabla^k(\sigma,\omega)(t)\big\|^2 + D_1\big\|\nabla^{k+1}\omega(t)\big\|^2 \lesssim (\delta+\epsilon)\big(\big\|\nabla\sigma(t)\big\|_1^2 + \big\|\nabla\omega(t)\big\|_2^2\big). \tag{3.10}$$

Proof. Multiplying ∇^k (2.1)$_1$, ∇^k (2.1)$_2$ by $\nabla^k\sigma$ and $\nabla^k\omega$ respectively, and then integrating over $\mathbb{R}^3$, we have from the sum of the resultant equalities that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\big\|\nabla^k\sigma\big\|^2 + \big\|\nabla^k\omega\big\|^2\big) + \mu_1\big\|\nabla^{k+1}\omega\big\|^2 + \mu_2\big\|\nabla^k \operatorname{div}\omega\big\|^2\\ &\quad = \big\langle\nabla^k\sigma(t), \nabla^k S_1(t)\big\rangle + \big\langle\nabla^k\omega(t), \nabla^k S_2(t)\big\rangle\end{aligned} \tag{3.11}$$

Prior to estimating the terms on the right-hand side of (3.11), we notice that the source terms S_1 and S_2 have the following equivalent properties under the conditions (2.5) and (3.9):

$$S_1 \sim \nabla\sigma\cdot\omega + \sigma\nabla\cdot\omega + \nabla\bar{\rho}\cdot\omega + \bar{\rho}\nabla\cdot\omega, \tag{3.12}$$

$$S_2 \sim (\omega\cdot\nabla)\omega + \sigma\Delta\omega + \sigma\nabla\nabla\cdot\omega + \sigma\nabla\sigma + \bar{\rho}\Delta\omega + \bar{\rho}\nabla\nabla\cdot\omega + \nabla\bar{\rho}\sigma + \bar{\rho}\nabla\sigma. \tag{3.13}$$

When $k = 0$, by using the Hölder inequality, Lemma A.1, (1.2), (2.5) and the Young inequality, we obtain

$$\begin{aligned}\langle\sigma, S_1\rangle &\lesssim \|\sigma\|_{L^6}\|\nabla\sigma\|\|\omega\|_{L^3} + \|\sigma\|_{L^6}\|\sigma\|_{L^3}\|\nabla\omega\| + \|\sigma\|_{L^6}\big\|\big(1+|x|\big)\nabla\bar{\rho}\big\|_{L^3}\left\|\frac{\omega}{1+|x|}\right\| + \|\sigma\|_{L^6}\|\bar{\rho}\|_{L^3}\|\nabla\omega\|\\ &\lesssim (\delta+\epsilon)\big(\|\nabla\sigma\|^2 + \|\nabla\omega\|^2\big),\end{aligned} \tag{3.14}$$

where we have used the following Hardy inequality

$$\left\|\frac{\omega}{1+|x|}\right\| \le \|\nabla\omega\|.$$

From (3.13), similar to the proof of $\langle\sigma, S_1\rangle$, we get

$$\langle\omega, S_2\rangle \lesssim (\delta+\epsilon)\big(\|\nabla\sigma\|^2 + \|\nabla\omega\|_1^2\big). \tag{3.15}$$

Plugging (3.14)–(3.15) into (3.11) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|(\sigma,\omega)(t)\big\|^2 + \|\nabla\omega\|^2 \lesssim (\delta+\epsilon)\big(\|\nabla\sigma\|^2 + \|\nabla\omega\|_1^2\big).$$

When $1 \le k \le 2$, from (3.12), we have

$$\begin{aligned}\big\langle\nabla^k\sigma(t), \nabla^k S_1(t)\big\rangle &\lesssim \big|\big\langle\nabla^k\sigma(t), \nabla^k\big(\nabla\sigma(t)\cdot\omega(t)\big)\big\rangle\big| + \big|\big\langle\nabla^k\sigma(t), \nabla^k(\sigma\nabla\cdot\omega)\big\rangle\big|\\ &\quad + \big|\big\langle\nabla^k\sigma(t), \nabla^k(\nabla\bar{\rho}\cdot\omega)\big\rangle\big| + \big|\big\langle\nabla^k\sigma(t), \nabla^k(\bar{\rho}\nabla\omega)\big\rangle\big|\\ &:= I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{3.16}$$

For I_1, it holds from integration by parts, Lemmas A.1 and A.3, the assumption (3.9) and the Young inequality that

$$\begin{aligned}I_1 &\lesssim \big|\big\langle\nabla^k\sigma(t), \nabla^k\big(\nabla\sigma(t)\cdot\omega(t)\big)\big\rangle\big|\\ &\lesssim \|\nabla\omega\|_{L^\infty}\big\|\nabla^k\sigma\big\|^2 + \big\|\nabla^k\sigma\big\|\big\|\nabla^k(\nabla\sigma\cdot\omega) - \nabla^k\nabla\sigma\cdot\omega\big\|\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\nabla\omega\|_{L^\infty}\|\nabla^k\sigma\|^2 + \|\nabla^k\sigma\|\big(\|\nabla^k\sigma\|\|\nabla\omega\|_{L^\infty} + \|\nabla\sigma\|_{L^3}\|\nabla^k\omega\|_{L^6}\big)\\
&\lesssim \|\nabla^2\omega\|_1\|\nabla^k\sigma\|^2 + \|\nabla^k\sigma\|\big(\|\nabla^k\sigma\|\|\nabla^2\omega\|_1 + \|\nabla\sigma\|_1\|\nabla^{k+1}\omega\|\big)\\
&\lesssim \delta\big(\|\nabla^k\sigma\|^2 + \|\nabla^{k+1}\omega\|^2 + \|\nabla^2\omega\|_1^2\big).
\end{aligned}
\tag{3.17}
$$

Similarly, for I_2, we have

$$
\begin{aligned}
I_2 &\lesssim \|\nabla^k\sigma\|\big(\|\nabla^k\sigma\|\|\nabla\omega\|_{L^\infty} + \|\sigma\|_{L^\infty}\|\nabla^{k+1}\omega\|\big)\\
&\lesssim \delta\big(\|\nabla^k\sigma\|^2 + \|\nabla^{k+1}\omega\|^2 + \|\nabla^2\omega\|_1^2\big).
\end{aligned}
\tag{3.18}
$$

As to I_3 and I_4, by using Lemma A.1, (1.2), (2.5) and the Hölder inequality, we get

$$
\begin{aligned}
I_3 &\lesssim \big|\big\langle\nabla^k\sigma(t), \nabla^k(\nabla\bar\rho\cdot\omega)\big\rangle\big|\\
&\lesssim \|\nabla^k\sigma\|\bigg(\|\nabla^k\nabla\bar\rho\cdot\omega\| + \sum_{0\le l\le k-1}\|\nabla^l\nabla\bar\rho\cdot\nabla^{k-l}\omega\|\bigg)\\
&\lesssim \|\nabla^k\sigma\|\bigg(\|\nabla^{k+1}\bar\rho\|_{L^3}\|\omega\|_{L^6} + \sum_{0\le l\le k-1}\|\nabla^l\nabla\bar\rho\|_{L^3}\|\nabla^{k-l}\omega\|_{L^6}\bigg)\\
&\lesssim \|\nabla^k\sigma\|\bigg(\epsilon\|\nabla\omega\| + \epsilon\sum_{0\le l\le k-1}\|\nabla^{k-l+1}\omega\|\bigg)\\
&\lesssim \epsilon\|\nabla^k\sigma\|\sum_{1\le l\le k+1}\|\nabla^l\omega\|\\
&\lesssim \epsilon\|\nabla^k\sigma\|^2 + \epsilon\sum_{1\le l\le k+1}\|\nabla^l\omega\|^2,
\end{aligned}
\tag{3.19}
$$

and

$$
\begin{aligned}
I_4 &\lesssim \big|\big\langle\nabla^k\sigma(t), \nabla^k(\bar\rho\nabla\omega)\big\rangle\big|\\
&\lesssim \|\nabla^k\sigma\|\bigg(\|\bar\rho\nabla^k\nabla\omega\| + \sum_{1\le l\le k}\|\nabla^l\bar\rho\|_{L^3}\|\nabla^{k-l+1}\omega\|_{L^6}\bigg)\\
&\lesssim \|\nabla^k\sigma\|\bigg(\|\bar\rho\|_{L^\infty}\|\nabla^{k+1}\omega\| + \epsilon\sum_{1\le l\le k}\|\nabla^{k-l+2}\omega\|\bigg)\\
&\lesssim \epsilon\bigg(\|\nabla^k\sigma\|^2 + \sum_{2\le l\le k+1}\|\nabla^l\omega\|^2\bigg).
\end{aligned}
\tag{3.20}
$$

Putting (3.17)–(3.20) and (3.16) together yields

$$
\big\langle\nabla^k\sigma(t), \nabla^k S_1(t)\big\rangle \lesssim (\delta+\epsilon)\bigg(\|\nabla^k\sigma\|^2 + \sum_{1\le l\le 3}\|\nabla^l\omega\|^2\bigg).
\tag{3.21}
$$

By virtue of (3.12) and integration by parts over $\mathbb{R}^3$, the second term on the right-hand side of (3.11) can be estimated as follows:

$$
\begin{aligned}
\big\langle\nabla^k\omega(t), \nabla^k S_2(t)\big\rangle \lesssim{}& \big|\big\langle\nabla^{k+1}\omega(t), \nabla^{k-1}\big((\omega\cdot\nabla)\omega\big)\big\rangle\big| + \big|\big\langle\nabla^{k+1}\omega(t), \nabla^{k-1}(\sigma\Delta\omega)\big\rangle\big|\\
&+ \big|\big\langle\nabla^{k+1}\omega(t), \nabla^{k-1}(\sigma\nabla\nabla\cdot\omega)\big\rangle\big| + \big|\big\langle\nabla^{k+1}\omega(t), \nabla^{k-1}(\sigma\nabla\sigma)\big\rangle\big|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\left\langle\nabla^{k+1}\omega(t),\nabla^{k-1}(\bar\rho\Delta\omega)\right\rangle\right|+\left|\left\langle\nabla^{k+1}\omega(t),\nabla^{k-1}(\bar\rho\nabla\nabla\cdot\omega)\right\rangle\right|\\
&+\left|\left\langle\nabla^{k+1}\omega(t),\nabla^{k-1}(\nabla\bar\rho\sigma)\right\rangle\right|+\left|\left\langle\nabla^{k+1}\omega(t),\nabla^{k-1}(\bar\rho\nabla\sigma)\right\rangle\right|\\
:=\ &J_1+J_2+J_3+J_4+J_5+J_6+J_7+J_8. \qquad (3.22)
\end{aligned}
$$

Similar to the estimates on I_i $(i=1,2,3,4)$, for $1\le k\le 2$, we obtain

$$
J_1+J_4+J_7+J_8\lesssim(\delta+\epsilon)\left(\|\nabla\sigma\|_1^2+\sum_{1\le l\le k+1}\left\|\nabla^l\omega\right\|^2\right). \qquad (3.23)
$$

The estimate on J_2 follows from Lemma A.1, the assumption (3.9) and the Hölder inequality:

$$
\begin{aligned}
J_2&\lesssim\left|\left\langle\nabla^{k+1}\omega(t),\nabla^{k-1}(\sigma\Delta\omega)\right\rangle\right|\\
&\lesssim\left\|\nabla^{k+1}\omega\right\|\left\|\nabla^{k-1}(\sigma\Delta\omega)\right\|\\
&\lesssim\left\|\nabla^{k+1}\omega\right\|\left(\left\|\sigma\Delta\nabla^{k-1}\omega\right\|+\left\|\nabla\sigma\nabla^{k-2}\Delta\omega\right\|\right)\\
&\lesssim\|\sigma\|_{L^\infty}\left\|\nabla^{k+1}\omega\right\|^2+\left\|\nabla^{k+1}\omega\right\|\|\nabla\sigma\|_{L^3}\left\|\nabla^k\omega\right\|_{L^6}\\
&\lesssim\delta\left\|\nabla^{k+1}\omega\right\|^2, \qquad (3.24)
\end{aligned}
$$

where $\|\nabla\sigma\nabla^{k-2}\Delta\omega\|$ has vanished when $k=1$, and hereafter, etc.

For J_5, by using Lemma A.1 and (2.5), we have

$$
\begin{aligned}
J_5&\lesssim\left|\left\langle\nabla^{k+1}\omega(t),\nabla^{k-1}(\bar\rho\Delta\omega)\right\rangle\right|\\
&\lesssim\left\|\nabla^{k+1}\omega\right\|\left(\|\bar\rho\|_{L^\infty}\left\|\nabla^{k-1}\Delta\omega\right\|+\|\nabla\bar\rho\|_{L^3}\left\|\nabla^{k-2}\Delta\omega\right\|_{L^6}\right)\\
&\lesssim\epsilon\left\|\nabla^{k+1}\omega\right\|^2. \qquad (3.25)
\end{aligned}
$$

Similarly, J_3 and J_6 satisfy

$$
J_3+J_6\lesssim(\delta+\epsilon)\left\|\nabla^{k+1}\omega\right\|^2. \qquad (3.26)
$$

Then, (3.22)–(3.26) give

$$
\left\langle\nabla^k\omega(t),\nabla^kS_2(t)\right\rangle\lesssim(\delta+\epsilon)\left(\|\nabla\sigma\|_1^2+\sum_{1\le l\le k+1}\left\|\nabla^l\omega\right\|^2\right). \qquad (3.27)
$$

Substituting (3.21) and (3.27) into (3.11) and noticing the smallness of δ and ϵ, we get (3.10). Thus, we complete the proof of Lemma 3.1. □

Now, we show the dissipation estimate for σ by using Eq. (2.2).

Lemma 3.2. *There exists a suitably large constant $D_2>0$, which is independent of δ, such that for $0\le k\le 1$,*

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\nabla^k\nabla\sigma(t),\nabla^k\omega(t)\right\rangle+D_2\left\|\nabla^{k+1}\sigma(t)\right\|^2\lesssim\eta_0\left\|\nabla\sigma(t)\right\|_1^2+(\delta+\epsilon)\left\|\nabla\sigma(t)\right\|_1^2+\left\|\nabla\omega(t)\right\|_2^2, \qquad (3.28)
$$

where η_0 is a small positive constant.

Proof. From (2.2)$_2$, it is obvious that

$$\gamma\nabla\sigma = -\omega_t + \mu_1\Delta\omega + \mu_2\nabla \operatorname{div}\omega + S_2. \tag{3.29}$$

Multiplying (3.29) by $\nabla\sigma$ and integrating it over $\mathbb{R}^3$, and then using (2.2)$_1$ and the Young inequality, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\omega,\nabla\sigma\rangle + \gamma\|\nabla\sigma\|^2 &\lesssim \big|\langle\omega,\nabla\partial_t\sigma\rangle\big| + \big|\langle\Delta\omega,\nabla\sigma\rangle\big| + \big|\langle\nabla\operatorname{div}\omega,\nabla\sigma\rangle\big| + \big|\langle\nabla\sigma,S_2\rangle\big| \\ &\lesssim \big|\langle\nabla\cdot\omega,\nabla\cdot\omega\rangle\big| + \big|\langle\nabla\cdot\omega,S_1\rangle\big| + \big|\langle\Delta\omega,\nabla\sigma\rangle\big| + \big|\langle\nabla\operatorname{div}\omega,\nabla\sigma\rangle\big| + \big|\langle\nabla\sigma,S_2\rangle\big| \\ &\lesssim \eta_0\|\nabla\sigma\|^2 + \|\nabla\omega\|^2 + \big\|\nabla^2\omega\big\|^2 + \big|\langle\nabla\cdot\omega,S_1\rangle\big| + \big|\langle\nabla\sigma,S_2\rangle\big|. \end{aligned}\tag{3.30}$$

Then, a similar argument for obtaining (3.14) and (3.15) in the proof of Lemma 3.1 gives the following estimates

$$\big|\langle\nabla\cdot\omega,S_1\rangle\big| \lesssim (\delta+\epsilon)\big(\big\|\nabla^2\omega\big\|^2 + \|\nabla\omega\|^2\big), \tag{3.31}$$

and

$$\big|\langle\nabla\sigma,S_2\rangle\big| \lesssim (\delta+\epsilon)\big(\big\|\nabla^2\sigma\big\|^2 + \|\nabla\omega\|_1^2\big). \tag{3.32}$$

Then, by (3.31) and (3.32), (3.30) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\omega,\nabla\sigma\rangle + \gamma\|\nabla\sigma\|^2 \lesssim \eta_0\|\nabla\sigma\|^2 + (\delta+\epsilon)\big(\big\|\nabla^2\sigma\big\|^2 + \|\nabla\omega\|_1^2\big). \tag{3.33}$$

Applying ∂_x^α ($|\alpha| = 1$) to (3.29) and multiplying it by $\partial_x^\alpha\nabla\sigma$ and then integrating the resultant over $\mathbb{R}^3$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big\langle\partial_x^\alpha\omega,\partial_x^\alpha\nabla\sigma\big\rangle + \big\|\partial_x^\alpha\nabla\sigma\big\|^2 \\ &\quad\lesssim \big|\big\langle\partial_x^\alpha\omega,\partial_x^\alpha\nabla\partial_t\sigma\big\rangle\big| + \big|\big\langle\partial_x^\alpha\Delta\omega,\partial_x^\alpha\nabla\sigma\big\rangle\big| + \big|\big\langle\partial_x^\alpha\nabla\operatorname{div}\omega,\partial_x^\alpha\nabla\sigma\big\rangle\big| + \big|\big\langle\partial_x^\alpha S_2,\partial_x^\alpha\nabla\sigma\big\rangle\big| \\ &\quad:= K_1 + K_2 + K_3 + K_4. \end{aligned}\tag{3.34}$$

For K_1, it follows from (2.2)$_1$ and the Hölder inequality that

$$\begin{aligned}K_1 &\lesssim \big|\big\langle\partial_x^\alpha\nabla\cdot\omega,\partial_x^\alpha\nabla\cdot\omega\big\rangle\big| + \big|\big\langle\partial_x^\alpha\nabla\cdot\omega,\partial_x^\alpha S_1\big\rangle\big| \\ &\lesssim \big\|\nabla^2\omega\big\|^2 + \big\|\nabla^2\omega\big\|\big\|\partial_x^\alpha S_1\big\|. \end{aligned}\tag{3.35}$$

By using (3.12), the Hölder inequality, Lemma A.1 and the assumption (3.9), we have

$$\begin{aligned}\big\|\partial_x^\alpha S_1\big\| &\lesssim \big\|\partial_x^\alpha(\nabla\sigma\cdot\omega)\big\| + \big\|\partial_x^\alpha(\sigma\nabla\cdot\omega)\big\| + \big\|\partial_x^\alpha(\nabla\bar\rho\cdot\omega)\big\| + \big\|\partial_x^\alpha(\bar\rho\nabla\cdot\omega)\big\| \\ &\lesssim \big\|\partial_x^\alpha\nabla\sigma\big\|\|\omega\|_{L^\infty} + \|\nabla\sigma\|_{L^3}\big\|\partial_x^\alpha\omega\big\|_{L^6} + \big\|\partial_x^\alpha\sigma\big\|_{L^6}\|\nabla\cdot\omega\|_{L^3} \\ &\qquad + \|\sigma\|_{L^\infty}\big\|\partial_x^\alpha\nabla\cdot\omega\big\| + \big\|\partial_x^\alpha\nabla\bar\rho\cdot\omega\big\| + \big\|\nabla\bar\rho\cdot\partial_x^\alpha\omega\big\| + \big\|\bar\rho\partial_x^\alpha\nabla\cdot\omega\big\| + \big\|\partial_x^\alpha\bar\rho\nabla\cdot\omega\big\| \\ &\lesssim \delta\big\|\nabla^2(\sigma,\omega)\big\| + \big\|\nabla^2\bar\rho\big\|_{L^3}\|\omega\|_{L^6} + \|\bar\rho\|_{L^\infty}\big\|\nabla^2\omega\big\| + \|\nabla\bar\rho\|_{L^3}\|\nabla\omega\|_{L^6} \\ &\lesssim \delta\big\|\nabla^2\sigma\big\| + (\delta+\epsilon)\|\nabla\omega\|_1. \end{aligned}\tag{3.36}$$

Then, by the Young inequality, it is straightforward to show that

$$K_1 \lesssim \delta\left\|\nabla^2\sigma\right\|^2 + \|\nabla\omega\|_1^2. \tag{3.37}$$

Applying the Young inequality to K_2 and K_3, we have

$$K_2 + K_3 \lesssim \eta_0\left\|\nabla^2\sigma\right\|^2 + \left\|\nabla^3\omega\right\|^2. \tag{3.38}$$

It follows from (1.2), (2.5), (3.13) and Lemma A.1 that, for $|\alpha| = 1$,

$$\begin{aligned}
\left\|\partial_x^\alpha S_2\right\| &\lesssim \left\|\partial_x^\alpha\left[(\omega\cdot\nabla)\omega\right]\right\| + \left\|\partial_x^\alpha(\sigma\Delta\omega)\right\| + \left\|\partial_x^\alpha(\sigma\nabla\nabla\cdot\omega)\right\| + \left\|\partial_x^\alpha(\sigma\nabla\sigma)\right\| \\
&\quad + \left\|\partial_x^\alpha(\bar\rho\Delta\omega)\right\| + \left\|\partial_x^\alpha(\bar\rho\nabla\nabla\cdot\omega)\right\| + \left\|\partial_x^\alpha(\nabla\bar\rho\sigma)\right\| + \left\|\partial_x^\alpha(\bar\rho\nabla\sigma)\right\| \\
&\lesssim \|\nabla\omega\|_{L^6}\|\nabla\omega\|_{L^3} + \|\omega\|_{L^\infty}\left\|\partial_x^\alpha\nabla\omega\right\| + \left\|\sigma\partial_x^\alpha\Delta\omega\right\| + \left\|\partial_x^\alpha\sigma\Delta\omega\right\| + \left\|\sigma\partial_x^\alpha\nabla\nabla\cdot\omega\right\| \\
&\quad + \left\|\partial_x^\alpha\sigma\nabla\nabla\cdot\omega\right\| + \left\|\partial_x^\alpha\sigma\right\|_{L^6}\|\nabla\sigma\|_{L^3} + \|\sigma\|_{L^\infty}\left\|\partial_x^\alpha\nabla\sigma\right\| + \left\|\bar\rho\partial_x^\alpha\Delta\omega\right\| + \left\|\partial_x^\alpha\bar\rho\Delta\omega\right\| \\
&\quad + \left\|\partial_x^\alpha(\bar\rho\nabla\nabla\cdot\omega)\right\| + \left\|\partial_x^\alpha(\nabla\bar\rho\sigma)\right\| + \left\|\partial_x^\alpha(\bar\rho\nabla\sigma)\right\| \\
&\lesssim (\delta+\epsilon)\left(\|\nabla\sigma\|_1 + \|\nabla\omega\|_2\right).
\end{aligned}$$

Therefore, K_4 can be estimated by using the Hölder inequality as follows

$$K_4 \lesssim \left\|\partial_x^\alpha\nabla\sigma\right\|\left\|\partial_x^\alpha S_2\right\| \lesssim (\delta+\epsilon)\left(\|\nabla\sigma\|_1^2 + \|\nabla\omega\|_2^2\right). \tag{3.39}$$

Putting (3.37)–(3.39) into (3.34) and noticing that $|\alpha| = 1$ yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\nabla\omega, \nabla^2\sigma\right\rangle + \left\|\nabla^2\sigma\right\|^2 \lesssim \eta_0\left\|\nabla^2\sigma\right\|^2 + (\delta+\epsilon)\|\nabla\sigma\|_1^2 + \|\nabla\omega\|_2^2. \tag{3.40}$$

Combining (3.33) with (3.40) yields (3.28). Thus, we complete the proof of Lemma 3.2. □

By summing up (3.10) for from $k = 0$ to 2, since δ and ϵ are small, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{0\le k\le 2}\left\|\nabla^k(\sigma,\omega)(t)\right\|^2 + \sum_{0\le k\le 2}\left\|\nabla^{k+1}\omega(t)\right\|^2 \le C_1(\delta+\epsilon)\sum_{1\le k\le 2}\left\|\nabla^k\sigma(t)\right\|^2. \tag{3.41}$$

Summing up (3.28) for from $k = 0$ to 1, since η_0, δ and ϵ are small, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{0\le k\le 1}\left\langle\nabla^k\omega(t), \nabla^k\nabla\sigma(t)\right\rangle + C_2\sum_{0\le k\le 1}\left\|\nabla^k\nabla\sigma(t)\right\|^2 \le C\sum_{1\le k\le 3}\left\|\nabla^k\omega(t)\right\|^2. \tag{3.42}$$

Now, multiplying (3.42) by $2C_1(\delta+\epsilon)/C_2$, adding it with (3.41) and using the smallness of δ and ϵ, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sum_{0\le k\le 2}\left\|\nabla^k(\sigma,\omega)(t)\right\|^2 + \frac{2C_1(\delta+\epsilon)}{C_2}\sum_{0\le k\le 1}\left\langle\nabla^k\omega(t), \nabla^k\nabla\sigma(t)\right\rangle\right\} \\
&\quad + C_5\left\{\sum_{0\le k\le 1}\left\|\nabla^{k+1}\sigma(t)\right\|^2 + \sum_{0\le k\le 2}\left\|\nabla^{k+1}\omega(t)\right\|^2\right\} \le 0.
\end{aligned} \tag{3.43}$$

From (3.43) and the smallness of δ and ϵ, we obtain the a priori estimate (2.6). Thus, the global existence of solutions to the problem (2.2) stated in Theorem 2.1 is obtained.

4. Convergence rate

In this section, we first show negative Sobolev norm estimates of the solution (σ, ω) in both H^2-framework and H^3-framework. Then, we obtain time decay rates of the solution to the problem (2.2) under the two frameworks, respectively.

4.1. Negative Sobolev estimates

Lemma 4.1. *For $s \in (0, 1/2)$, there exists a constant $C_3 > 0$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\Lambda^{-s}\sigma(t)\|^2 + \|\Lambda^{-s}\omega(t)\|^2\big) + \|\nabla\Lambda^{-s}\omega(t)\|^2 \le C_3\eta_0\|\Lambda^{-s}\nabla\sigma(t)\| + C\epsilon\|\nabla\omega(t)\|_2^2. \tag{4.1}$$

Proof. Applying Λ^{-s} to $(2.1)_1$, $(2.1)_2$ and multiplying the resultant equalities by $\Lambda^{-s}\sigma, \Lambda^{-s}\omega$ respectively, combining them and then integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\big(|\Lambda^{-s}\sigma|^2 + |\Lambda^{-s}\omega|^2\big)\mathrm{d}x + \mu_1\|\nabla\Lambda^{-s}\omega\|^2 + \mu_2\|\operatorname{div}\Lambda^{-s}\omega\|^2 \\
&\quad\lesssim \big|\langle\Lambda^{-s}\sigma, \Lambda^{-s}S_1\rangle\big| + \big|\langle\Lambda^{-s}\omega, \Lambda^{-s}S_2\rangle\big|.
\end{aligned} \tag{4.2}$$

By using the Plancherel theorem and (3.12), we have

$$\begin{aligned}
\big|\langle\Lambda^{-s}\nabla\sigma, \Lambda^{-s-1}S_1\rangle\big| &= \big|\langle\Lambda^{1-s}\sigma, \Lambda^{-s-1}S_1\rangle\big| \\
&\lesssim \big|\langle\Lambda^{1-s}\sigma, \Lambda^{-s-1}(\nabla\sigma\cdot\omega)\rangle\big| + \big|\langle\Lambda^{1-s}\sigma, \Lambda^{-s-1}(\sigma\nabla\cdot\omega)\rangle\big| \\
&\quad + \big|\langle\Lambda^{1-s}\sigma, \Lambda^{-s-1}(\nabla\bar\rho\cdot\omega)\rangle\big| + \big|\langle\Lambda^{1-s}\sigma, \Lambda^{-s-1}(\bar\rho\nabla\omega)\rangle\big| \\
&:= U_1 + U_2 + U_3 + U_4.
\end{aligned} \tag{4.3}$$

For $s \in (0, 1/2)$, we have that $1/2 + (s+1)/3 < 1$ and $2 < 3/(s+1) < 3$. Then, using the Hölder inequality, Lemma A.2 and the Young inequality, for U_1, we have

$$\begin{aligned}
U_1 &\lesssim \|\nabla\sigma\|\|\omega\|_{L^{3/(s+1)}}\|\Lambda^{-s}\nabla\sigma\| \\
&\lesssim \|\nabla\sigma\|\|\omega\|^{s+1/2}\|\nabla\omega\|^{1/2-s}\|\Lambda^{-s}\nabla\sigma\| \\
&\lesssim \|\nabla\sigma\|^2\big(\|\omega\|^2 + \|\nabla\omega\|^2\big) + \eta_0\|\Lambda^{1-s}\sigma\|^2.
\end{aligned} \tag{4.4}$$

A similar argument leads to

$$U_2 \lesssim \|\nabla\omega\|^2\big(\|\sigma\|^2 + \|\nabla\sigma\|^2\big) + \eta_0\|\Lambda^{1-s}\sigma\|^2, \tag{4.5}$$

and

$$\begin{aligned}
U_4 &\lesssim \big|\langle\Lambda^{1-s}\sigma, \Lambda^{-s-1}(\bar\rho\nabla\omega)\rangle\big| \\
&\lesssim \|\bar\rho\|\|\nabla\omega\|_{L^{3/(s+1)}}\|\Lambda^{1-s}\sigma\| \\
&\lesssim \|\bar\rho\|\|\nabla\omega\|^{s+1/2}\|\nabla^2\omega\|^{1/2-s}\|\Lambda^{1-s}\sigma\| \\
&\lesssim \epsilon\big(\|\nabla\omega\|^2 + \|\nabla^2\omega\|^2\big) + \eta_0\|\Lambda^{1-s}\sigma\|^2.
\end{aligned} \tag{4.6}$$

For U_3, thank to the Hardy inequality, Lemmas A.1–A.2 and the Young inequality, we have

$$\begin{aligned}
U_3 &\lesssim \big\|(1+|x|)\nabla\bar{\rho}\big\|\left\|\frac{\omega}{1+|x|}\right\|_{L^{3/(s+1)}}\big\|\Lambda^{1-s}\sigma\big\| \\
&\lesssim \big\|(1+|x|)\nabla\bar{\rho}\big\|\left\|\frac{\omega}{1+|x|}\right\|^{s+1/2}\left\|\nabla\left(\frac{\omega}{1+|x|}\right)\right\|^{1/2-s}\big\|\Lambda^{1-s}\sigma\big\| \\
&\lesssim \big\|(1+|x|)\nabla\bar{\rho}\big\|\|\nabla\omega\|^{s+1/2}\|\nabla\omega\|_1^{1/2-s}\big\|\Lambda^{1-s}\sigma\big\|^2 \\
&\lesssim \epsilon\big(\|\nabla\omega\|^2+\|\nabla\omega\|_1^2\big)+\eta_0\big\|\Lambda^{1-s}\sigma\big\|^2. \qquad (4.7)
\end{aligned}$$

Then, putting (4.4)–(4.7) into (4.3) leads to

$$\big|\big\langle\Lambda^{-s}\nabla\sigma,\Lambda^{-s-1}S_1\big\rangle\big| \lesssim \eta_0\big\|\Lambda^{1-s}\sigma\big\|^2+\epsilon\|\nabla\omega\|_1^2. \qquad (4.8)$$

Similarly, we get

$$\big|\big\langle\Lambda^{-s}\omega,\Lambda^{-s}S_2\big\rangle\big| = \big|\big\langle\Lambda^{1-s}\omega,\Lambda^{-s-1}S_2\big\rangle\big| \lesssim \eta_0\big\|\Lambda^{1-s}\omega\big\|^2+\epsilon\|\nabla\omega\|_2^2. \qquad (4.9)$$

By using (4.8), (4.9) and (4.2) and noticing the smallness of constants η_0 and ϵ yield (4.1). Thus, the proof of Lemma 4.1 is finished. □

Lemma 4.2. *Let $s\in(0,1/2)$, then we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\langle\Lambda^{-s}\omega(t),\Lambda^{-s}\nabla\sigma(t)\big\rangle+\big\|\Lambda^{-s}\nabla\sigma(t)\big\|^2 \le C\big\|\Lambda^{-s}\nabla\omega(t)\big\|^2+C\big\|\nabla\sigma(t)\big\|_1^2+C\big\|\nabla\omega(t)\big\|_2^2. \qquad (4.10)$$

Proof. Applying Λ^{-s} to (3.29) and multiplying it by $\Lambda^{-s}\nabla\sigma$ and then integrating the resultant over $\mathbb{R}^3$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\big\langle\Lambda^{-s}\omega,\Lambda^{-s}\nabla\sigma\big\rangle+\gamma\big\|\Lambda^{-s}\nabla\sigma\big\|^2 &\lesssim \big|\big\langle\Lambda^{-s}\omega,\Lambda^{-s}\nabla\partial_t\sigma\big\rangle\big|+\big|\big\langle\Lambda^{-s}\Delta\omega,\Lambda^{-s}\nabla\sigma\big\rangle\big| \\
&\quad+\big|\big\langle\Lambda^{-s}\nabla\operatorname{div}\omega,\Lambda^{-s}\nabla\sigma\big\rangle\big|+\big|\big\langle\Lambda^{-s}S_2,\Lambda^{-s}\nabla\sigma\big\rangle\big| \\
&:= V_1+V_2+V_3+V_4. \qquad (4.11)
\end{aligned}$$

For $s\in(0,1/2)$, we have that $5/6+s/3<1$ and $2<3/(s+1)<3$. Then, as to V_1, by using $(2.2)_1$, integration by parts, the Hölder inequality, (3.12) and Lemma A.1, we have

$$\begin{aligned}
V_1 &\lesssim \big|\big\langle\Lambda^{-s}\omega,\Lambda^{-s}\nabla\operatorname{div}\omega\big\rangle\big|+\big|\big\langle\Lambda^{-s}\omega,\Lambda^{-s}\nabla S_1\big\rangle\big| \\
&\lesssim \big\|\Lambda^{1-s}\omega\big\|^2+\big\|\Lambda^{-s}\omega\big\|_{L^6}\big\|\Lambda^{-s}\nabla S_1\big\|_{L^{\frac{6}{5}}} \\
&\lesssim \big\|\Lambda^{1-s}\omega\big\|^2+\big\|\Lambda^{1-s}\omega\big\|\|\nabla S_1\|_{L^{\frac{1}{5/6+s/3}}} \\
&\lesssim \big\|\Lambda^{1-s}\omega\big\|^2+\big\|\Lambda^{1-s}\omega\big\|\Big(\big\|\nabla^2\sigma\big\|\|\omega\|_{L^{\frac{3}{s+1}}}+\|\nabla\sigma\|\|\nabla\omega\|_{L^{\frac{3}{s+1}}}+\|\sigma\|_{L^{\frac{3}{s+1}}}\big\|\nabla^2\omega\big\| \\
&\quad+\big\|(1+|x|)\nabla^2\bar{\rho}\big\|_{L^{\frac{3}{s+1}}}\left\|\frac{\omega}{1+|x|}\right\|+\|\nabla\bar{\rho}\|_{L^{\frac{3}{s+1}}}\|\nabla\omega\|+\|\bar{\rho}\|_{L^{\frac{3}{s+1}}}\big\|\nabla^2\omega\big\|\Big) \\
&\lesssim \big\|\Lambda^{1-s}\omega\big\|^2+\big\|\Lambda^{1-s}\omega\big\|\big(\big\|\nabla^2\sigma\big\|\|\omega\|_1+\|\nabla\sigma\|\|\nabla\omega\|_1+\|\sigma\|_1\big\|\nabla^2\omega\big\| \\
&\quad+\big\|(1+|x|)\nabla^2\bar{\rho}\big\|_1\|\nabla\omega\|+\|\bar{\rho}\|_2\|\nabla\omega\|_1\big) \\
&\lesssim \big\|\Lambda^{-s}\nabla\omega\big\|^2+\delta\|\nabla\sigma\|_1^2+(\delta+\epsilon)\|\nabla\omega\|_1^2. \qquad (4.12)
\end{aligned}$$

Similarly, we have

$$V_4 \le \frac{1}{2}\|\Lambda^{-s}\nabla\sigma\|^2 + C\|\nabla\sigma\|_1 + C\|\nabla\omega\|_2^2. \tag{4.13}$$

By using the Young inequality and Lemma A.5, it is obvious that

$$\begin{aligned} V_2 + V_3 &\le \frac{1}{2}\|\Lambda^{-s}\nabla\sigma\|^2 + C\|\Lambda^{2-s}\omega\|^2 \\ &\le \frac{1}{2}\|\Lambda^{-s}\nabla\sigma\|^2 + C\|\nabla^2\omega\|^{1-s}\|\nabla\omega\|^s \\ &\lesssim \frac{1}{2}\|\Lambda^{-s}\nabla\sigma\|^2 + C\|\nabla\omega\|^2 + C\|\nabla^2\omega\|^2. \end{aligned} \tag{4.14}$$

Then, substituting (4.12) and (4.14) into (4.11) and using the smallness of η_0, we complete the proof of Lemma 4.2. □

Multiplying (4.10) by $2C_3\eta_0$ and adding it with (4.1), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\Lambda^{-s}(\sigma,\omega)(t)\|^2 + \langle\Lambda^{-s}\omega(t), \Lambda^{-s}\nabla\sigma(t)\rangle\big) + C_3\eta_0\|\Lambda^{-s}\nabla\sigma(t)\|^2 + \|\Lambda^{-s}\nabla\omega(t)\|^2 \\ &\le C\eta_0\|\Lambda^{-s}\nabla\omega(t)\|^2 + C\|\nabla\sigma(t)\|_1^2 + C\|\nabla\omega(t)\|_2^2. \end{aligned} \tag{4.15}$$

Integrating (4.15) with respect to t, by the smallness of η_0 and the Young inequality, Lemma A.4 and (2.10), we have

$$\begin{aligned} &\|\Lambda^{-s}(\sigma,\omega)(t)\|^2 + \int_0^t \|\Lambda^{-s}\nabla(\sigma,\omega)(\tau)\|^2 \mathrm{d}\tau \\ &\le \|\Lambda^{-s}(\sigma_0,\omega_0)\|^2 + 2C_3\eta_0|\langle\Lambda^{-s}\omega_0, \Lambda^{1-s}\sigma_0\rangle| + 2C_3\eta_0|\langle\Lambda^{-s}\omega(t), \Lambda^{1-s}\sigma(t)\rangle| \\ &\quad + C\|\Lambda^{1-s}\omega(t)\|^2 + C\int_0^t \big(\|\nabla\sigma(\tau)\|_1^2 + \|\nabla\omega(\tau)\|_2^2\big)\mathrm{d}\tau \\ &\le C\|\Lambda^{-s}(\sigma_0,\omega_0)\|^2 + C\|\Lambda^{1-s}\sigma_0\|^2 + C\|\Lambda^{1-s}(\sigma,\omega)(t)\|^2 + C\|(\sigma_0,\omega_0)\|_3^2 \\ &\le C\|\Lambda^{-s}(\sigma_0,\omega_0)\|^2 + C\|\nabla\sigma_0\|^{2(1-s)}\|\sigma_0\|^{2s} + C\|\nabla\sigma(t)\|^{2(1-s)}\|\sigma(t)\|^{2s} \\ &\quad + C\|\nabla\omega(t)\|^{2(1-s)}\|\omega(t)\|^{2s} + C\|(\sigma_0,\omega_0)\|_3^2 \\ &\le C\|\Lambda^{-s}(\sigma_0,\omega_0)\|^2 + C\|(\sigma_0,\omega_0)\|_3^2 + C\|\sigma(t)\|_1^2 + C\|\omega(t)\|_1^2 \\ &\le C\|\Lambda^{-s}(\sigma_0,\omega_0)\|^2 + C\|(\sigma_0,\omega_0)\|_3^2. \end{aligned} \tag{4.16}$$

Then, we obtain the negative Sobolev estimates (2.7).

4.2. Time-decay rates in H^2-framework

We consider decay-in-time estimates on (σ,ω) in H^2-framework. Precisely, we have the following lemma.

Lemma 4.3. *Under the assumptions of Theorem 2.1, the solution (σ,ω) to the problem (2.2) satisfies*

$$\|(\sigma,\omega)(t)\|_2 \le C(1+t)^{-\frac{s}{2}}. \tag{4.17}$$

Proof. Define the temporal energy functional

$$\mathcal{H}(t) = \left\|(\sigma,\omega)(t)\right\|_2^2 + \frac{2C_1(\delta+\epsilon)}{C_2}\left\{\left\langle\nabla\sigma(t),\omega(t)\right\rangle + \left\langle\nabla\nabla\sigma(t),\nabla\cdot\omega(t)\right\rangle\right\},$$

where it is noticed that $\mathcal{H}(t)$ is equivalent to $\|(\sigma,\omega)(t)\|_2^2$ since the positive constants δ and ϵ can be sufficiently small. Then, from (3.43), we have

$$\frac{\mathrm{d}\mathcal{H}(t)}{\mathrm{d}t} + \left\|\nabla\sigma(t)\right\|_1^2 + \left\|\nabla\omega(t)\right\|_2^2 \le 0. \tag{4.18}$$

In view of Lemma A.4, for $s \in (0,1/2)$, we have

$$\left\|\nabla(\sigma,\omega)(t)\right\| \ge C\left\|\Lambda^{-s}(\sigma,\omega)(t)\right\|^{-\frac{1}{s}}\left\|(\sigma,\omega)(t)\right\|^{1+\frac{1}{s}}.$$

By using (4.16), there exists a constant $C_4 > 0$ such that

$$\left\|\nabla(\sigma,\omega)(t)\right\|^2 \ge C_4\left\{\left\|(\sigma,\omega)(t)\right\|^2\right\}^{1+\frac{1}{s}}.$$

Then, we have

$$\frac{\mathrm{d}\mathcal{H}(t)}{\mathrm{d}t} + C_4\mathcal{H}(t)^{1+\frac{1}{s}} \le 0.$$

Solving this inequality directly gives

$$\mathcal{H}(t) \le \left(\mathcal{H}(0)^{-\frac{2}{s}} + \frac{C_4 t}{s}\right)^{-s} \le C_0(1+t)^{-s}.$$

Thus, we complete the proof of Lemma 4.3. □

4.3. Time decay rates in H^3-framework

In order to obtain time decay estimates of solutions stated in Theorem 2.2, we first show the L^p–L^q estimate on linearized system for later use. The linearized equations corresponding to system (2.2) takes the form

$$\begin{cases} \sigma_t + \gamma\nabla\cdot u = 0, \\ \omega_t - \mu_1\Delta\omega - \mu_2\nabla\operatorname{div}\omega + \gamma\nabla\sigma = 0, \\ (\sigma,\omega)|_{t=0} = (\sigma_0,\omega_0). \end{cases} \tag{4.19}$$

Then, the solution (σ,ω) to the problem (4.19) can be defined by $(\sigma,\omega)(x,t) = \mathrm{e}^{-tA}(\sigma_0,\omega_0)(x)$ $(t \ge 0)$ with $A = A(D_x)$ being a matrix-valued differential operator given by

$$A(D_x) = \begin{pmatrix} 0 & \gamma\operatorname{div} \\ \gamma\nabla & -\mu_1\Delta - \mu_2\nabla\operatorname{div} \end{pmatrix}.$$

The semigroup e^{-tA} has the following properties on the decay in time, cf. [11,12].

Lemma 4.4. *Let $k \ge 0$ be an integer with $1 \le p \le 2 \le q < \infty$. Then for any $t \ge 0$, it holds that*

$$\left\|\nabla^k\mathrm{e}^{-tA}(\sigma_0,\omega_0)\right\|_{L^q} \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{k}{2}}\left\|(\sigma_0,\omega_0)\right\|_{L^p\cap H^k}.$$

By using the L^p–L^q estimates on linearized system, we obtain the time decay rates of the solution to the problem (2.2) as follows.

Lemma 4.5. *Under the assumptions of Theorem 2.3, the solution (σ,ω) to the initial value problem (2.2) satisfies*

$$\left\|\nabla^k(\sigma,\omega)(t)\right\|_2 \le C(1+t)^{-\frac{1+s}{2}}, \quad k=1,2,3. \tag{4.20}$$

Proof. Firstly, we denote that

$$M(t) := \sup_{0\le\tau\le t}\left\{(1+\tau)^{1+s}\mathcal{L}(\tau)\right\}.$$

Then, we have

$$\left\|\nabla(\sigma,\omega)(\tau)\right\|_2 \lesssim \sqrt{\mathcal{L}(\tau)} \lesssim (1+\tau)^{-\frac{1+s}{2}}\sqrt{M(t)}, \quad 0\le\tau\le t.$$

By applying the Plancherel theorem, Lemma 4.4 and Lemma A.4, we have, for any $s\in(0,1/2)$ and $k=0,1$,

$$\begin{aligned}
\left\|\nabla^k(\sigma,\omega)(t)\right\| &\lesssim \left\|\nabla^{k+s}\mathrm{e}^{-tA}\Lambda^{-s}(\sigma_0,\omega_0)\right\| + \int_0^t \left\|\nabla^k\mathrm{e}^{-tA}(S_1,S_2)(\tau)\right\|\mathrm{d}\tau \\
&\lesssim \left\|\nabla^k\mathrm{e}^{-tA}\Lambda^{-s}(\sigma_0,\omega_0)\right\|^{1-s}\left\|\nabla^{k+1}\mathrm{e}^{-tA}\Lambda^{-s}(\sigma_0,\omega_0)\right\|^{s} \\
&\quad + \int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}\left\|(S_1,S_2)(\tau)\right\|_{L^1\cap H^k}\mathrm{d}\tau \\
&\lesssim (1+t)^{-\frac{k+s}{2}}\left\|\Lambda^{-s}(\sigma_0,\omega_0)\right\|_{L^2\cap H^{k+1}} \\
&\quad + (\delta+\epsilon)\int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}(1+\tau)^{-\frac{1+s}{2}}\sqrt{M(t)}\mathrm{d}\tau \\
&\lesssim (1+t)^{-\frac{k+s}{2}}\left[L_0+\epsilon_1\sqrt{M(t)}\right],
\end{aligned} \tag{4.21}$$

where $L_0 := \|(\sigma_0,\omega_0)\|_{\dot{H}^{-s}\cap H^{k+1}}$ and we have made use of the Hölder inequality, the Hardy inequality and (2.10) to estimate the right-hand side term as

$$\begin{aligned}
\left\|(S_1,S_2)(t)\right\|_{L^1} &\lesssim \left(\left\|(\sigma,\omega)(t)\right\|_1 + \|\bar\rho\| + \left\|\left(1+|x|\right)\nabla\bar\rho\right\|\right)\left\|\nabla(\sigma,\omega)(t)\right\|_1 \\
&\lesssim \epsilon_1\left\|\nabla(\sigma,\omega)(t)\right\|_1,
\end{aligned}$$

and

$$\begin{aligned}
\left\|(S_1,S_2)(t)\right\|_{H^1} &\lesssim \left(\left\|(\sigma,\omega)(t)\right\|_{W^{1,\infty}} + \|\bar\rho\|_{L^\infty} + \sum_{1\le k\le 2}\left\|\left(1+|x|\right)\nabla^k\bar\rho\right\|_{L^\infty}\right)\left\|\nabla(\sigma,\omega)(t)\right\|_2 \\
&\lesssim \epsilon_1\left\|\nabla(\sigma,\omega)(t)\right\|_2.
\end{aligned}$$

Then, by using Gronwall inequality and (4.21), from (2.11), we have

$$\begin{aligned}\mathcal{L}(t) &\lesssim \mathcal{L}(0)\mathrm{e}^{-t} + \int_0^t \mathrm{e}^{-(t-\tau)} \big\|\nabla(\sigma,\omega)(\tau)\big\|^2 \mathrm{d}\tau \\ &\lesssim \mathcal{L}(0)\mathrm{e}^{-t} + \int_0^t \mathrm{e}^{-(t-\tau)}(1+\tau)^{-(1+s)}\big(L_0^2 + \epsilon_1^2 M(t)\big)\mathrm{d}\tau \\ &\lesssim (1+t)^{-(1+s)}\big\{\mathcal{L}(0) + L_0^2 + \epsilon_1^2 M(t)\big\}.\end{aligned}$$

Noticing the definition of $M(t)$ and the smallness of ϵ_1, we have

$$M(t) \lesssim \mathcal{L}(0) + L_0^2. \tag{4.22}$$

Thus, we complete the proof of Lemma 4.5. □

Moreover, from (4.21) and (4.22), we have

$$\big\|(\sigma,\omega)(t)\big\| \le C(1+t)^{-\frac{s}{2}}.$$

Thus, we complete the proof of Theorem 2.3. □

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Appendix A

In this appendix, we state some useful inequalities in the Sobolev space. The proof of the following lemma can be found in [1].

Lemma A.1. *Let $f \in H^2(\mathbb{R}^3)$. Then*

(i) $\|f\|_{L^\infty} \le C\|\nabla f\|^{1/2}\|\nabla f\|_{H^1}^{1/2} \le C\|\nabla f\|_{H^1}$;
(ii) $\|f\|_{L^6} \le C\|\nabla f\|$;
(iii) $\|f\|_{L^q} \le C\|f\|_{H^1}$, $2 \le q \le 6$.

The following is the usual Sobolev interpolation of the Gagliardo–Nirenberg–Sobolev inequality.

Lemma A.2. *Let $0 \le m, \alpha \le l$, then we have*

$$\big\|\nabla^\alpha f\big\|_{L^p} \lesssim \big\|\nabla^m f\big\|_{L^q}^{1-\theta}\big\|\nabla^l f\big\|_{L^r}^{\theta} \tag{A.1}$$

where $0 \le \theta \le 1$ and α satisfies

$$\frac{\alpha}{3} - \frac{1}{p} = \left(\frac{m}{3} - \frac{1}{q}\right)(1-\theta) + \left(\frac{l}{3} - \frac{1}{r}\right)\theta. \tag{A.2}$$

Here when $p = \infty$ we require that $0 < \theta < 1$.

We recall the following two lemmas. One can find them in [13,23].

Lemma A.3. *Let $m \geq 1$ be an integer, then we have*

$$\left\|\nabla^m (fg)\right\|_{L^p} \lesssim \|f\|_{L^{p_1}} \left\|\nabla^m g\right\|_{L^{p_2}} + \left\|\nabla^m f\right\|_{L^{p_2}} \|g\|_{L^{p_1}}, \tag{A.3}$$

and

$$\left\|\nabla^m (fg) - f\nabla^m g\right\|_{L^p} \lesssim \|\nabla f\|_{L^{p_3}} \left\|\nabla^{m-1} g\right\|_{L^{p_4}} + \left\|\nabla^m f\right\|_{L^{p_5}} \|g\|_{L^{p_6}}, \tag{A.4}$$

where $1 \leq p_i \leq +\infty$ $(i = 1, \ldots, 6)$ and

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{p_5} + \frac{1}{p_6}. \tag{A.5}$$

Lemma A.4. *Let $s \in (0, 1/2)$, then we have*

$$\left\|\nabla^{1-s} f\right\| \leq \|\nabla f\|^{1-s} \|f\|^s.$$

The Hardy–Littlewood–Sobolev theorem implies the following L^p type inequality for the Riesz potential, cf. [21].

Lemma A.5. *Let $0 < s < 3, 1 < p < q < \infty, 1/q + s/3 = 1/p$, then*

$$\left\|\Lambda^{-s} f\right\|_{L^q} \lesssim \|f\|_{L^p}.$$

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