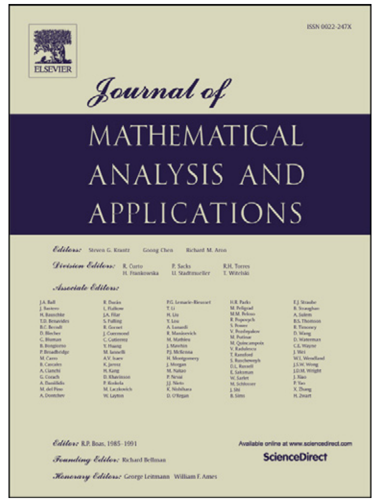

Boundedness in a fully parabolic chemotaxis system with singular sensitivity

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Boundedness in a fully parabolic chemotaxis system with singular sensitivity

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Abstract

This paper deals with a fully parabolic chemotaxis system $u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v),\ v_t = \Delta v - v + u$ with singular sensitivity $\frac{\chi}{v}$ $(\chi > 0)$ on a bounded domain $\Omega \subset \mathbb{R}^n, n \geq 2$. The main result solves the open problem of uniform-in-time boundedness of solutions for $\chi < \sqrt{\frac{2}{n}}$, which was conjectured by Winkler [16].

Key words: chemotaxis; boundedness; singular sensitivity.
AMS Classification: 92C17, 35B35, 35B45, 35K55.

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1 Introduction

We consider the Neumann initial-boundary value problem for a fully parabolic chemotaxis system with singular sensitivity

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ with smooth boundary, where $\chi > 0$ and

$$\begin{cases} u_0 \in C^0(\bar{\Omega}), \quad u_0 \geq 0 \quad \text{in } \bar{\Omega}, \quad u_0 \not\equiv 0, \\ v_0 \in W^{1,\infty}(\Omega), \quad v_0 > 0 \quad \text{in } \bar{\Omega}. \end{cases} \tag{1.2}$$

The particular choice of the sensitivity function $\frac{\chi}{v}$ in the present problem (1.1) was proposed in an original model by Keller and Segel [11] in order to account for the so-called Weber-Fechner law of stimulus perception in the process of chemotactic response.

As to the problem (1.1) with logistic source in the two dimensional setting, Aida, Osaki, Tsujikawa, Yagi and Mimura [1] asserted global existence of classical solutions, leaving open the question whether or not they are bounded. Winkler [16] proved that if $\chi < \sqrt{\frac{2}{n}}$, then (1.1) possesses a global classical solution without relying on logistic source. As pointed out in [16], the result did not rule out the possibility that the solution may become unbounded as $t \to \infty$. The question of boundedness of the solution to (1.1) has been posted as an open problem. Indeed, in this context, Kavallaris and Souplet [10] studied a precise grow-up rate and asymptotic estimates for solutions to a simplified chemotaxis system without $\frac{1}{v}$. Moreover, as to the problem (1.1) without $\frac{1}{v}$, Cieślak and Stinner [3] showed that the solutions blow up in finite time under some conditions. As to the present problem (1.1) with $\frac{1}{v}$, global existence of weak solutions was established when $\chi < \sqrt{\frac{n+2}{3n-4}}$ ([16]). In the radially symmetric setting, Stinner and Winkler [14] constructed certain weak solutions under the condition $\chi < \sqrt{\frac{n}{n-2}}$. Moreover, in virtue of additional dampening kinetic terms, Manásevich, Phan and Souplet [12] proved global existence and boundedness in a related system for all χ. As compared to the above, the parabolic-elliptic case has been studied more precisely ([2, 13, 4, 6, 5]). Many references to earlier work on chemotaxis systems can be found in Hillen and Painter [8].

In the present paper we improve the approach in [16] and establish uniform-in-time boundedness of solutions to (1.1). The main result reads as follows.

Main Theorem *Let $n \geq 2$. Assume that χ satisfies*

$$0 < \chi < \sqrt{\frac{2}{n}},$$

and suppose that u_0 and v_0 satisfy (1.2). *Then the global solution of* (1.1) *is bounded in the sense that there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C \qquad \text{for all } t > 0.$$

The above theorem states *uniform-in-time* boundedness of solutions under the same condition as in [16]. There are two difficulties in deriving boundedness. The first difficulty stems from the singularity of $\frac{1}{v}$. To overcome this difficulty we shall establish a *time-independent* pointwise lower bound for v (Lemma 2.2). Note that the strong maximum principle easily implies

$$v(\cdot,t) \geq \eta(t) := \min_{x\in\bar{\Omega}} v_0(x) \cdot e^{-t} \qquad \text{for all } t > 0.$$

However, this is useless in proving uniform-in-time boundedness of solutions, since $\eta(t) \to 0$ as $t \to \infty$. The second difficulty lies in deducing *time-independent* L^p-boundedness of solutions. Although the L^p-estimate in [16] depends on time, we shall reconstruct the method in [16] and remove the dependence. Invoking the above two *time-independent* estimates, we establish boundedness.

Remark 1.1 *In the regular case that* $\frac{1}{v}$ *is replaced with* $\frac{1}{(1+\alpha v)^k}$ $(\alpha > 0,\ k > 1)$, *global existence and boundedness were shown for all* $\chi > 0$ *by* Winkler [15]. *After the completion of the present paper, using the time-independent pointwise lower bound for* v (Lemma 2.2), *the boundedness result in* [15] *was extended to the strongly singular case* $\frac{1}{v^k}$ $(k > 1)$ [7]. *We note that the methods in* [15, 7] *cannot be applied to the critical case* $k = 1$.

This paper is organized as follows. Section 2 will be concerned with preliminaries, including the announced pointwise lower bound for v. In Section 3 we deduce time-independent L^p-boundedness of solutions and complete the proof of Main Theorem.

2 Preliminaries

We first recall the global existence result established in [16].

Lemma 2.1 *Assume that* $0 < \chi < \sqrt{\frac{2}{n}}$. *If the initial data* (u_0, v_0) *satisfies* (1.2), *then* (1.1) *has a global classical positive solution*

$$u \in C^{2,1}(\bar{\Omega} \times (0,\infty)) \cap C^0([0,\infty); C^0(\bar{\Omega})),$$
$$v \in C^{2,0}(\bar{\Omega} \times (0,\infty)) \cap C^0([0,\infty); C^0(\bar{\Omega})).$$

Moreover, the first component of the solution satisfies the mass identity

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx \qquad \textit{for all } t > 0. \tag{2.1}$$

The following lemma is a cornerstone of our work. The mass identity (2.1) plays a key role in the proof of this lemma. We shall denote by (u, v) the solution of (1.1) in the rest of the paper.

Lemma 2.2 *There exists* $\eta > 0$ *such that*

$$\inf_{x\in\Omega} v(x,t) \geq \eta > 0 \qquad \textit{for all } t \geq 0,$$

where η *does not depend on* t.

Proof. We use a known result for the Neumann heat semigroup $e^{t\Delta}$. In the same way as in the proof of [9, Lemma 3.1], we can obtain the pointwise estimate from below

$$e^{t\Delta}w(x) \geq \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{(\operatorname{diam}\Omega)^2}{4t}} \cdot \int_\Omega w > 0 \quad (x \in \Omega, t > 0) \quad \text{for all nonnegative } w \in C^0(\bar{\Omega}),$$

where $\operatorname{diam}\Omega := \max_{x,y\in\bar{\Omega}} |x-y|$. First by the positivity of $v_0 > 0$ in $\bar{\Omega}$ and the maximum principle we have

$$v(t) \geq \min_{x\in\bar{\Omega}} v_0(x) \cdot e^{-t} > 0 \qquad \text{for all } t \geq 0.$$

Now fix $\tau > 0$. Then it follows that

$$v(t) \geq \min_{x\in\bar{\Omega}} v_0(x) \cdot e^{-\tau} =: \eta_1 > 0 \qquad \text{for all } t \in [0,\tau].$$

Next, the representation formula of v, the maximal principle and (2.1) imply that

$$\begin{aligned}
v(t) &= e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}u(s)\,ds \\
&\geq \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-\left((t-s)+\frac{(\operatorname{diam}\Omega)^2}{4(t-s)}\right)} \cdot \left(\int_\Omega u(x,s)\,dx\right) ds \\
&= \|u_0\|_{L^1(\Omega)} \cdot \int_0^t \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{-\left(r+\frac{(\operatorname{diam}\Omega)^2}{4r}\right)} dr \\
&\geq \|u_0\|_{L^1(\Omega)} \cdot \int_0^\tau \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{-\left(r+\frac{(\operatorname{diam}\Omega)^2}{4r}\right)} dr =: \eta_2 > 0 \qquad \text{for all } t \in [\tau,\infty).
\end{aligned}$$

Therefore we have $v(t) \geq \min\{\eta_1,\eta_2\} =: \eta$ for all $t \geq 0$. This completes the proof. □

To achieve boundedness of the norm of $u(\cdot,t)$ in $L^p(\Omega)$ we shall use the following lemmas.

Lemma 2.3 *Let $p \in \mathbb{R}$ and $q \in \mathbb{R}$. Then the following identity holds for all $t > 0$:*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^p v^q + q\int_\Omega u^p v^q - q\int_\Omega u^{p+1}v^{q-1} \\
&= -p(p-1)\int_\Omega u^{p-2}v^q|\nabla u|^2 + \int_\Omega u^p v^{q-2} \cdot \big[-q(q-1)+pq\chi\big] \cdot |\nabla v|^2 \\
&\quad + \int_\Omega u^{p-1}v^{q-1} \cdot \big[-2pq + p(p-1)\chi\big]\nabla u \cdot \nabla v.
\end{aligned}$$

Proof. Proceeding analogously to [16, Lemma 2.3], we can prove the desired identity. □

Lemma 2.4 *Let $1 \leq \theta, \mu \leq \infty$.*

(i) *If $\frac{n}{2}(\frac{1}{\theta} - \frac{1}{\mu}) < 1$, then there exists $C > 0$ such that*

$$\|v(\cdot,t)\|_{L^\mu(\Omega)} \leq C\Big(1 + \sup_{s\in(0,\infty)} \|u(\cdot,s)\|_{L^\theta(\Omega)}\Big) \qquad \textit{for all } t > 0.$$

(ii) *If* $\frac{1}{2}+\frac{n}{2}(\frac{1}{\theta}-\frac{1}{\mu})<1$, *then there exists* $C>0$ *such that*

$$\|\nabla v(\cdot,t)\|_{L^\mu(\Omega)} \le C\Big(1+\sup_{s\in(0,\infty)}\|u(\cdot,s)\|_{L^\theta(\Omega)}\Big) \qquad \textit{for all } t>0.$$

Proof. We can argue similarly as in [16, Lemma 2.4] due to the estimate for $e^{t(\Delta-1)}$:

$$\|e^{t(\Delta-1)}\varphi\|_{L^\mu(\Omega)} \le c\,t^{-\frac{n}{2}(\frac{1}{\theta}-\frac{1}{\mu})}e^{-\delta t}\|\varphi\|_{L^\theta(\Omega)} \qquad \text{for all } t>0,\ \varphi\in L^\theta(\Omega),$$

with some constants $c,\delta>0$. □

3 Proof of Main Theorem

We follow the same way as in [16]. The difference is that our estimates are independent of time.

Lemma 3.1 *Let* $n\ge 2$ *and* $0<\chi<\sqrt{\frac{2}{n}}$. *Assume that* $p\in(1,\frac{1}{\chi^2})$ *and* $r\in(r_-(p),r_+(p))$, *where* $r_\pm(p):=\frac{p-1}{2}(1\pm\sqrt{1-p\chi^2})$. *If there exists a constant* $c>0$ *such that*

$$\|v(\cdot,t)\|_{L^{p-r}(\Omega)} \le c \qquad \textit{for all } t>0, \tag{3.1}$$

then there exists $C>0$ *such that*

$$\int_\Omega u^p(x,t)v^{-r}(x,t)\,dx \le C \qquad \textit{for all } t>0.$$

Proof. Choosing $q:=-r$ in Lemma 2.3, we obtain

$$\begin{aligned}\mathbf{I} &:= \frac{d}{dt}\int_\Omega u^pv^{-r} - r\int_\Omega u^pv^{-r} + r\int_\Omega u^{p+1}v^{-r-1}\\ &= -p(p-1)\int_\Omega u^{p-2}v^{-r}|\nabla u|^2 - \int_\Omega u^pv^{-r-2}\big[r(r+1)+pr\chi\big]\cdot|\nabla v|^2\\ &\quad + \int_\Omega u^{p-1}v^{-r-1}\big[2pr+p(p-1)\chi\big]\nabla u\cdot\nabla v \end{aligned}\tag{3.2}$$

for $t>0$. Applying Young's inequality to the last term, we have

$$\begin{aligned}&\Big|\int_\Omega u^{p-1}v^{-r-1}\big[2pr+p(p-1)\chi\big]\nabla u\cdot\nabla v\Big|\\ &\le p(p-1)\int_\Omega u^{p-2}v^{-r}|\nabla u|^2 + \frac{1}{4p(p-1)}\int_\Omega u^pv^{-r-2}\big[2pr+p(p-1)\chi\big]^2\cdot|\nabla v|^2.\end{aligned}$$

Therefore (3.2) yields

$$\mathbf{I} \le -\int_\Omega u^pv^{-r-2}h(p,r,\chi)|\nabla v|^2, \tag{3.3}$$

where

$$h(p,r,\chi) := r(r+1) + pr\chi - \frac{\big[2pr + p(p-1)\chi\big]^2}{4p(p-1)}. \tag{3.4}$$

As $p \in (1, \frac{1}{\chi^2})$ and $r \in (r_-(p), r_+(p))$, we thus obtain

$$\begin{aligned} 4(p-1)h(p,r,\chi) &= -4r^2 + 4(p-1)r - p(p-1)^2\chi^2 \\ &= 4(r_+(p) - r)(r - r_-(p)) > 0. \end{aligned}$$

In view of the positivity $h > 0$, (3.2) and (3.3) imply

$$\frac{d}{dt}\int_\Omega u^p v^{-r} + r\int_\Omega u^{p+1}v^{-r-1} \le r\int_\Omega u^p v^{-r} \qquad \text{for all } t > 0. \tag{3.5}$$

Now unlike the proof of [16, Lemma 4.2] we pay attention to the term $r\int_\Omega u^{p+1}v^{-r-1}$. Hölder's inequality implies that

$$\int_\Omega u^p v^{-r} = \int_\Omega (u^{p+1}v^{-r-1})^{\frac{p}{p+1}} \cdot v^{-r-\frac{p(-r-1)}{p+1}} \le \Big(\int_\Omega u^{p+1}v^{-r-1}\Big)^{\frac{p}{p+1}}\Big(\int_\Omega v^{p-r}\Big)^{\frac{1}{p+1}}.$$

In virtue of the assumption (3.1), we see that

$$\int_\Omega u^p v^{-r} \le c^{\frac{p-r}{p+1}}\Big(\int_\Omega u^{p+1}v^{-r-1}\Big)^{\frac{p}{p+1}}.$$

Hence we have that

$$c^{-\frac{p-r}{p}}\Big(\int_\Omega u^p v^{-r}\Big)^{\frac{p+1}{p}} \le \int_\Omega u^{p+1}v^{-r-1}. \tag{3.6}$$

Combining (3.6) with (3.5), we establish the following inequality:

$$\frac{d}{dt}\int_\Omega u^p v^{-r} \le -rc^{-\frac{p-r}{p}}\Big(\int_\Omega u^p v^{-r}\Big)^{\frac{p+1}{p}} + r\int_\Omega u^p v^{-r}.$$

Since we find $\frac{p+1}{p} > 1$, thus the standard ODE technique completes the proof. □

We are now in a position to prove our main theorem.

Proof of Main Theorem The proof is divided into two steps.

(Step 1) In this step we shall gain L^p-boundedness of solutions. We will prove that there exist some $p > \frac{n}{2}$ and $C_p > 0$ such that

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C_p \qquad \text{for all } t > 0. \tag{3.7}$$

We consider an iterative argument. First we pick a pair (p_0, r_0) such that

$$\begin{cases} p_0 \in \Big(1, \min\Big\{\frac{1}{\chi^2}, n+1, \frac{n+2}{n-2}\Big\}\Big), \\ r_0 := \frac{p_0 - 1}{2}. \end{cases} \tag{3.8}$$

Then we can confirm that

$$p_0 > r_0, \quad r_0 < \frac{n}{2}, \quad r_0 \in (r_-(p_0), r_+(p_0)) \quad \text{and} \quad p_0 - r_0 = \frac{p_0+1}{2} < \frac{n}{n-2}.$$

Since $\frac{n}{2}(1 - \frac{1}{p_0-r_0}) < 1$ due to the inequality $p_0 - r_0 < \frac{n}{n-2}$, Lemma 2.4 (i) together with the mass identity (3) allows us to find a constant $c_0 > 0$ fulfilling

$$\|v(\cdot,t)\|_{L^{p_0-r_0}(\Omega)} \le C\Big(1 + \sup_{s\in(0,\infty)} \|u(\cdot,s)\|_{L^1(\Omega)}\Big) \le c_0 \qquad \text{for all } t > 0.$$

Therefore Lemma 3.1 yields that there exists a constant $c_0' > 0$ such that

$$\int_\Omega u^{p_0} v^{-r_0} \le c_0' \qquad \text{for all } t > 0.$$

Now we claim that for all $q_0 \in (1, \min\{p_0, \frac{n(p_0-r_0)}{n-2r_0}\})$ there exists a constant $c_0'' > 0$ such that

$$\int_\Omega u^{q_0} \le c_0'' \qquad \text{for all } t > 0. \tag{3.9}$$

Indeed, applying Hölder's inequality, we obtain

$$\begin{aligned}\int_\Omega u^{q_0} &= \int_\Omega (u^{p_0} v^{-r_0})^{\frac{q_0}{p_0}} \cdot v^{\frac{r_0 q_0}{p_0}} \\ &\le \Big(\int_\Omega u^{p_0} v^{-r_0}\Big)^{\frac{q_0}{p_0}} \cdot \Big(\int_\Omega v^{\frac{q_0 r_0}{p_0-q_0}}\Big)^{\frac{p_0-q_0}{p_0}} \\ &\le c_0'^{\frac{q_0}{p_0}} \cdot \Big(\int_\Omega v^{\frac{q_0 r_0}{p_0-q_0}}\Big)^{\frac{p_0-q_0}{p_0}}. \end{aligned} \tag{3.10}$$

Since $\frac{n}{2}(\frac{1}{q_0} - \frac{p_0-q_0}{q_0 r_0}) < 1$ due to $q_0 < \frac{n(p_0-r_0)}{n-2r_0}$, it follows from Lemma 2.4 (i) that

$$\sup_{t>0} \|v(\cdot,t)\|_{L^{\frac{q_0 r_0}{p_0-q_0}}(\Omega)} \le K_0\Big(1 + \sup_{t>0}\|u(\cdot,t)\|_{L^{q_0}(\Omega)}\Big)$$

with $K_0 > 0$. Applying this estimate to (3.10), we have

$$\sup_{t>0} \|u(\cdot,t)\|_{L^{q_0}(\Omega)} \le K_0'\Big(1 + (\sup_{t>0}\|u(\cdot,t)\|_{L^{q_0}(\Omega)})^{\frac{r_0}{p_0}}\Big)$$

with $K_0' > 0$. Since $\frac{r_0}{p_0} < 1$, we can verify (3.9).
In the above argument, if $p_0 > \frac{n}{2}$, then we can pick $q_0 > \frac{n}{2}$ and we establish (3.7). On the other hand, if $p_0 \le \frac{n}{2}$, then we consequently deduce that for all $q_0 \in (1, \frac{n(p_0+1)}{2(n-p_0+1)})$ there exists $c_0'' > 0$ satisfying

$$\int_\Omega u^{q_0} \le c_0'' \qquad \text{for all } t > 0 \tag{3.11}$$

due to $p_0 \ge \frac{n(p_0-r_0)}{n-2r_0} = \frac{n(p_0+1)}{2(n-p_0+1)}$ when $p_0 \le \frac{n}{2}$.

We proceed the second iteration. We fix a pair (p_1, r_1) such that

$$\begin{cases} p_1 \in \Big(p_0, \min\Big\{\dfrac{1}{\chi^2}, n+1, \dfrac{p_0(n+2)}{n-2p_0}\Big\}\Big), \\ r_1 := \dfrac{p_1 - 1}{2}. \end{cases} \tag{3.12}$$

Then we see that

$$p_1 > r_1, \quad r_1 < \frac{n}{2} \quad \text{and} \quad r_1 \in (r_-(p_1), r_+(p_1)).$$

Moreover, we can calculate that

$$\begin{aligned} p_1 - r_1 = \frac{p_1+1}{2} &< \frac{\frac{p_0(n+2)}{n-2p_0}+1}{2} \\ &= \frac{n(p_0+1)}{2(n-2p_0)} = \frac{n(p_0+1)}{2\{(n-p_0+1)-(p_0+1)\}} = \frac{n \cdot \frac{n(p_0+1)}{2(n-p_0+1)}}{n - 2\cdot \frac{n(p_0+1)}{2(n-p_0+1)}}. \end{aligned}$$

Hence, we can find some $q_0 \in (1, \frac{n(p_0+1)}{2(n-p_0+1)})$ satisfying

$$p_1 - r_1 < \frac{nq_0}{n-2q_0}.$$

Noting that $\frac{n}{2}(\frac{1}{q_0} - \frac{1}{p_1-r_1}) < 1$, we deduce from Lemma 2.4 (i) and (3.11) that there exists a constant $c_1 > 0$ such that

$$\|v(\cdot,t)\|_{L^{p_1-r_1}(\Omega)} \le C\Big(1 + \sup_{s\in(0,\infty)} \|u(\cdot,s)\|_{L^{q_0}(\Omega)}\Big) \le c_1 \qquad \text{for all } t > 0$$

and Lemma 3.1 yields that there exists a constant $c_1' > 0$ fulfilling

$$\int_\Omega u^{p_1} v^{-r_1} \le c_1' \qquad \text{for all } t > 0.$$

Using a similar estimate as the first iteration, we have that for all $q_1 \in (1, \min\{p_1, \frac{n(p_1-r_1)}{n-2r_1}\})$ there exists a constant $c_1'' > 0$ such that

$$\int_\Omega u^{q_1} \le c_1'' \qquad \text{for all } t > 0.$$

If we can choose $p_1 > \frac{n}{2}$, then we can pick $q_1 > \frac{n}{2}$ and establish (3.7). Moreover if $p_1 \le \frac{n}{2}$, then we have that for all $q_1 \in (1, \frac{n(p_1+1)}{2(n-p_1+1)})$ there exists a constant $c_1'' > 0$ satisfying

$$\int_\Omega u^{q_1} \le c_1'' \qquad \text{for all } t > 0.$$

Consequently, we can define a pair (p_k, r_k) $(k \in \mathbb{N})$:

$$\begin{cases} p_k \in \Big(p_{k-1},\, \min\Big\{\dfrac{1}{\chi^2},\, n+1,\, \dfrac{p_{k-1}(n+2)}{n-2p_{k-1}}\Big\}\Big), \\ r_k := \dfrac{p_k - 1}{2}, \end{cases} \tag{3.13}$$

and if $p_k \le \frac{n}{2}$, then we deduce that for all $q_k \in (1, \frac{n(p_k+1)}{2(n-p_k+1)})$

$$\int_\Omega u^{q_k} \le c_k'' \qquad \text{for all } t > 0$$

with constant $c_k'' > 0$. Because $\frac{2}{n} < \min\{\frac{1}{\chi^2},\, n+1\}$ due to the condition $\chi < \sqrt{\frac{2}{n}}$ and the increasing function $f(x) := \frac{x(n+2)}{n-2x}$ satisfies $f(x) > 1\,(x > 1)$ and $f(x) \to \infty$ as $x \to \frac{n}{2}$, we can obtain some k_0 large enough such that $p_{k_0} > \frac{n}{2}$ and hence $q_{k_0} > \frac{n}{2}$. Therefore we prove (3.7).

(Step 2) In light of L^p-boundedness of solutions (Step 1), we show L^∞-boundedness in this step. Building on Lemma 2.4 (ii), we invoke the standard semigroup technique (e.g. [16, Lemma 3.4]) to imply that there exists $C > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \text{for all } t > 0.$$

Thus we can complete the proof. □

Remark 3.2 *Our method in this work can be applied to the general case:*

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v^k} \nabla v), \qquad x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, \qquad x \in \Omega,\ t > 0, \end{cases} \tag{3.14}$$

with $k > 1$. Indeed, instead of $h(p, r, \chi)$ in (3.4), set

$$\begin{aligned} h(p, r, \chi, v) :&= r(r+1) + pr\chi \cdot \frac{1}{v^{k-1}} - \frac{\big[2pr + p(p-1)\chi \cdot \frac{1}{v^{k-1}}\big]^2}{4p(p-1)} \\ &\ge r(r+1) + pr\chi \cdot \frac{1}{\eta^{k-1}} - \frac{\big[2pr + p(p-1)\chi \cdot \frac{1}{\eta^{k-1}}\big]^2}{4p(p-1)}. \end{aligned}$$

Replacing χ with $\bar{\chi} := \frac{\chi}{\eta^{k-1}}$, we can argue similarly as our proofs. Hence, if $\chi < \sqrt{\frac{2}{n}} \cdot \eta^{k-1}$ we can establish boundedness of solutions to (3.14) with $k > 1$.

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