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Local approach to Kadec–Klee properties in symmetric function spaces ☆

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A B S T R A C T

We prove several results concerning local approach to Kadec–Klee properties with respect to global (local) convergence in measure in symmetric Banach function spaces which may be of independent interest. Moreover, we prove characterizations of these properties in the Lorentz spaces. Finally, we show applications of H_g and H_l points to the local best dominated approximation problems in Banach lattices.

1. Introduction

Geometry of Banach spaces has been intensively developed during the last decades, since it has found a lot of applications in many branches of mathematics. The metric geometry deals with properties invariant under isometries (for example rotundity, uniform rotundity and many intermediate properties). The monotonicity properties (strict and uniform monotonicity) play an analogous role in the geometry of Banach lattices. However, the studies of global properties are not always sufficient. When the Banach space (Banach lattice) has not the global property then it is natural to ask about the local structure. This leads among others to the notion of an extreme point. The respective role in the theory of Banach lattices play the points of lower and upper monotonicity. The local geometry has been deeply investigated recently (see [7,14,16,26–28]) and one of the important reasons is an application to local best dominated approximation problems in Banach lattices (see [7]).

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In Section 2 we recall the necessary terminology.

Section 3 is devoted to symmetric Banach function spaces. The essential question in the global geometry is whether a geometric property can be equivalently considered only on the positive cone E_+ of E (see [21,22] for further references). We prove the local version of such result, namely a point x is an H_g point if and only if $|x|$ is an H_g point. The more delicate question is whether a point x has some local property P if and only if its nonincreasing rearrangement x^* has the same property P and the positive answer is very useful in verifying local properties in particular classes of symmetric function spaces (see [7]). The goal of this paper is to study the structure of H_g and H_l points from that point of view. Moreover, we will show the relationships between H_g, H_l points and points of upper monotonicity, generalizing the global characterization from [5]. Furthermore, we prove that, for an H_g point, the norm is lower semicontinuous with respect to the global convergence in measure, similarly as, for the point of order continuity, the norm is lower semicontinuous with respect to the convergence a.e.

Section 4 concerns the Lorentz spaces $\Gamma_{p,w}$ and $\Lambda_{p,w}$. We give the full characterization of H_g and H_l points. Several corollaries concerning respective global properties are also deduced.

In the last section, we show applications of H_g and H_l points to local best dominated approximation problems in Banach lattices. It is known that global monotonicity properties (strict and uniform monotonicity) play an analogous role in the best dominated approximation problems in Banach lattices as the respective rotundity properties (strict and uniform rotundity) do in the best approximation problems in Banach spaces (see [30]). The points of lower (upper) monotonicity of a Banach lattice E play an analogous role like the extreme points in a Banach space X. Similarly, the role of points of upper (lower) local uniform monotonicity in Banach lattices is analogous to that of points of local uniform rotundity in Banach spaces. The role of lower (upper) monotonicity points and points of order continuity in local best dominated approximation problems in Banach lattices has been investigated in [7]. We will show that although the order continuity and property H_g are not comparable each to other, H_g point has a similar impact in local best dominated approximation problems in Banach lattices as a point of order continuity. Recall that global properties of H_g and H_l points have been investigated among others in [10,22,23]. The uniform versions of these properties have been studied in [36].

2. Preliminaries

Let $\mathbb{R}$ and $\mathbb{N}$ be the sets of real and positive integers, respectively. As usual $S(X)$ (resp. $B(X)$) stands for the unit sphere (resp. the closed unit ball) of a Banach space $(X, \|\cdot\|_X)$.

Denote by L^0 the set of all (equivalence classes of) extended real valued Lebesgue measurable functions on $[0, \alpha)$, where $\alpha = 1$ or $\alpha = \infty$. Let m be the Lebesgue measure on $[0, \alpha)$.

A Banach lattice $(E, \|\cdot\|_E)$ is called a *Banach function space* (or a *Köthe space*) if it is a sublattice of L^0 satisfying the following conditions:

(1) if $x \in L^0$, $y \in E$ and $|x| \leq |y|$ a.e., then $x \in E$ and $\|x\|_E \leq \|y\|_E$;
(2) there exists a strictly positive element $x \in E$.

By E_+ we denote the positive cone of E, that is, $E_+ = \{x \in E : x \geq 0\}$. We use the notation $A^c = [0, \alpha)\backslash A$ for any measurable set A.

A point $x \in E$ is said to have an *order continuous norm* if for any sequence (x_n) in E such that $0 \leq x_n \leq |x|$ and $x_n \to 0$ m-a.e. we have $\|x_n\|_E \to 0$. A Köthe space E is called *order continuous* ($E \in (OC)$) if every element of E has an order continuous norm (see [20,31]). As usual E_a stands for the subspace of order continuous elements of E.

We will assume in the whole paper (unless it is stated otherwise) that E has the *Fatou property* ($E \in (FP)$), that is, if $0 \leq x_n \uparrow x \in L^0$ with $(x_n)_{n=1}^{\infty}$ in E and $\sup_{n \in \mathbb{N}} \|x_n\|_E < \infty$, then $x \in E$ and

$\lim_n \|x_n\|_E = \|x\|_E$. A space E has the semi-*Fatou property* ($E \in (s-FP)$) if conditions $0 \leqslant x_n \uparrow x \in E$ with $x_n \in E$ imply $\|x_n\|_E \uparrow \|x\|_E$.

A point $x \in E_+ \setminus \{0\}$ is said to be a *point of upper monotonicity* if for any $y \in E_+$ such that $x \leq y$ and $y \neq x$, we have $\|x\|_E < \|y\|_E$. A point $x \in E_+$ is called a *point of upper local uniform monotonicity* if $\|x_n - x\|_E \to 0$ for any sequence $x_n \in E$ such that $x \leq x_n$ and $\|x_n\|_E \to \|x\|_E$. We will write shortly that x is a *UM-point* and *ULUM-point*, respectively. Recall that if each point of $E_+ \setminus \{0\}$ is a *UM* point, then we say that E is *strictly monotone* ($E \in (SM)$) (see [2,13]). Similarly, if each point of $E_+ \setminus \{0\}$ is a *ULUM* point, then we say that E is *upper locally uniformly monotone* ($E \in (ULUM)$).

A point $x \in E$ is said to be an H_g point (resp. H_l point) in E if for any $(x_n) \subset E$ such that $x_n \to x$ globally (resp. locally) in measure and $\|x_n\|_E \to \|x\|_E$, we have $\|x_n - x\|_E \to 0$. We say that the space E has Kadec–Klee property globally (resp. locally) in measure if each $x \in E$ is an H_g point (resp. H_l point) in E.

For $x \in L^0$ we denote its *distribution function* by

$$d_x(\lambda) = m\{s \in [0,\alpha) : |x(s)| > \lambda\}, \quad \lambda \geq 0,$$

and its *decreasing rearrangement* by

$$x^*(t) = \inf\{\lambda > 0 : d_x(\lambda) \leq t\}, \quad t \geq 0.$$

A function $x \in L^0$ is said to be **regular* if

$$m(\{t \in \operatorname{supp} x : |x(t)| < x^*(\alpha)\}) = 0.$$

The above equality works under the convention $x^*(\infty) = \lim_{t\to\infty} x^*(t)$. It is easy to see that every $x \in L^0$ is *regular whenever $\alpha = 1$. Moreover, if $\alpha = \infty$, then every $x \in L^0$ with $x^*(\infty) = 0$ is *regular.

Given $x \in L^0$ we define the *maximal function* of x^* by

$$x^{**}(t) = \frac{1}{t}\int_0^t x^*(s)ds.$$

It is well known that $x^* \leq x^{**}$, x^{**} is nonincreasing and subadditive, i.e.

$$(x+y)^{**} \leq x^{**} + y^{**} \tag{1}$$

for any $x, y \in L^0$. For the properties of d_x, x^* and x^{**}, the reader is referred to [1,29].

Two functions $x, y \in L^0$ are called *equimeasurable* ($x \sim y$ for short) if $d_x = d_y$. We say that a Banach function space $(E, \|\cdot\|_E)$ is *rearrangement invariant* (r.i. for short) or symmetric if whenever $x \in L^0$ and $y \in E$ with $x \sim y$, then $x \in E$ and $\|x\|_E = \|y\|_E$. Given an r.i. Banach function space E, by ϕ_E we denote its *fundamental function*, that is $\phi_E(t) = \|\chi_{(0,t)}\|_E$ for any $t \in [0,\alpha)$ (see [1]).

The relation $\prec$ is defined for any x, y in $L^1 + L^\infty$ by

$$x \prec y \quad \Leftrightarrow \quad x^{**}(t) \leq y^{**}(t) \quad \text{for all } t > 0.$$

Recall that a symmetric space E is *K-monotone* (*KM* for short) or has the *majorant property* if for any $x \in L^1 + L^\infty$ and $y \in E$ such that $x \prec y$, we have $x \in E$ and $\|x\|_E \leq \|y\|_E$.

It is well known that a symmetric space is K-monotone iff it is exact interpolation space between L^1 and L^∞. Moreover, symmetric spaces with Fatou property as well as separable symmetric spaces are K-monotone (see [29]).

3. Symmetric Banach function spaces

Lemma 3.1. *Let E be a symmetric Banach function space on $[0,\alpha)$, where $\alpha = 1$ or $\alpha = \infty$. The following conditions are equivalent:*

(i) $x \in E$ *is an* H_g *(resp.* H_l*) point in* E*;*
(ii) $|x|$ *is an* H_g *(resp.* H_l*) point in* E*;*
(iii) $|x|$ *is an* $(H_g)_+$ *(resp.* $(H_l)_+$*) point in* E*, that is for any sequence* (x_n) *in* E_+ *with* $x_n \to |x|$ *globally (resp. locally) in measure and* $\|x_n\|_E \to \|x\|_E$ *we have* $\|x_n - |x|\|_E \to 0$.

Proof. We prove only the lemma for H_g points, because the proof for H_l points is similar. Moreover, the global version of the lemma for H_l points was shown in [15].

The implication $(ii) \Rightarrow (i)$ follows the same way as in the proof of Lemma 3.5 from [22]. We prove $(i) \Rightarrow (ii)$. Let $x_n \to |x|$ in measure and $\|x_n\|_E \to \|x\|_E$. Set

$$A_+ = \{t \in [0,\alpha) : x(t) \geq 0\} \quad \text{and} \quad A_- = \{t \in [0,\alpha) : x(t) < 0\}.$$

Define

$$y_n(t) = \begin{cases} x_n(t) & \text{if } x \in A_+, \\ -x_n(t) & \text{if } x \in A_-. \end{cases}$$

Then $y_n \to x$ in measure. Clearly, $\|y_n\|_E \to \|x\|_E$. By the assumption we have $\|y_n - x\|_E \to 0$. Moreover,

$$\begin{aligned} \big\|x_n - |x|\big\|_E &\leq \big\|\big(x_n - |x|\big)\chi_{A_+}\big\|_E + \big\|\big(x_n - |x|\big)\chi_{A_-}\big\|_E \\ &= \big\|(y_n - x)\chi_{A_+}\big\|_E + \big\|(-y_n + x)\chi_{A_-}\big\|_E \to 0. \end{aligned}$$

The implication $(ii) \Rightarrow (iii)$ is obvious. Finally, $(iii) \Rightarrow (ii)$ follows again as in the proof of Lemma 3.5 from [22]. □

The following lemma is a local version of the implication $(ii) \Rightarrow (iii)$ from Lemma 3.2 in [22].

Lemma 3.2. *Let E be a symmetric Banach function space on $[0,\alpha)$ with $\alpha = 1$ or $\alpha = \infty$. If x is an H_g point, then $\|x\chi_{A_n}\|_E \to 0$ for any sequence (A_n) of measurable sets satisfying $m(A_n) \to 0$.*

Proof. Take a sequence (A_n) of measurable sets with $m(A_n) \to 0$. Since the convergence of the sequence $\|x\chi_{A_n}\|_E$ we will prove by using the double extract subsequence theorem, without loss of generality we can assume that $\sum_{i=1}^{\infty} m(A_i) < \infty$. Define

$$B_n := \bigcup_{i=n}^{\infty} A_i.$$

Obviously, $B_{n+1} \subset B_n$ for every $n \in \mathbb{N}$ and $m(B_n) \to 0$. For any $n \in \mathbb{N}$ define

$$x_n = x\chi_{[0,\alpha)\setminus B_n}.$$

Clearly, $0 \leq x_n \uparrow x$ in measure, whence, by $E \in (FP)$, $\lim_{n\to\infty}\|x_n\|_E = \|x\|_E$. Consequently, $\|x_n - x\|_E \to 0$, because x is an H_g-point. Therefore

$$\|x\chi_{A_n}\|_E \leq \|x\chi_{B_n}\|_E = \|x_n - x\|_E \to 0. \qquad \square$$

Theorem 3.3. *Let E be a symmetric Banach function space on $[0,\alpha)$, where $\alpha = 1$ or $\alpha = \infty$. If a *regular element $x \in E$ is an H_g-point in E, then x^* is an H_g-point in E.*

Proof. Suppose $y_n \to x^*$ in measure and $\|y_n\|_E \to \|x^*\|_E$. By Lemma 3.1, we may take (y_n) in E_+. We divide the proof in two parts.

I. Let $\alpha = 1$. There exists a measure preserving transformation $\sigma : [0,1) \to [0,1)$ such that $x^* \circ \sigma = |x|$ a.e. (see [1]). Then $y_n \circ \sigma \to |x|$ in measure and $\|y_n \circ \sigma\|_E \to \||x|\|_E$. Moreover,

$$\left\|y_n - x^*\right\|_E = \left\|\left(y_n - x^*\right) \circ \sigma\right\|_E = \left\|y_n \circ \sigma - |x|\right\|_E \to 0$$

because, by Lemma 3.1, $|x|$ is an H_g-point.

II. Suppose $\alpha = \infty$. Let $x^*(\infty) = 0$. If $m(\operatorname{supp} x) < \infty$ ($m(\operatorname{supp} x) = \infty$), by Lemma 2 in [17], there is a measure preserving transformation $\sigma : I \to I$ ($\sigma : \operatorname{supp} x \to [0,\infty)$) with $x^* \circ \sigma = |x|$ a.e. ($x^* \circ \sigma = |x|$ a.e. on $\operatorname{supp} x$). In the first case the proof can be easily finished with the sequence $z_n = y_n \circ \sigma$. If $m(\operatorname{supp} x) = \infty$ then we follow with the sequence

$$z_n(t) = \begin{cases} y_n(\sigma(t)) & \text{if } t \in \operatorname{supp} x, \\ 0 & \text{if } t \notin \operatorname{supp} x. \end{cases}$$

Now suppose that $x^*(\infty) > 0$. First we claim that without loss of generality we may assume that

$$y_n \geq x^*(\infty)\chi_{[0,\infty)}.$$

Otherwise, we set

$$A_n = \left\{t \in [0,\infty) : y_n(t) \leq x^*(\infty)\right\}, \qquad B_n = \left\{t \in [0,\infty) : y_n(t) > x^*(\infty)\right\}$$

and

$$\tilde{y}_n = y_n\chi_{B_n} + x^*(\infty)\chi_{A_n}.$$

Then $|\tilde{y}_n - x^*| \leq |y_n - x^*|$, whence $\tilde{y}_n \to x^*$ in measure. We prove that

$$\left\|\left(y_n - x^*\right)\chi_{A_n}\right\|_E \to 0. \tag{2}$$

Since $x^*(\infty) > 0$, $\chi_{[0,\infty)} \in E$. Set $\varepsilon > 0$ and

$$A_n^\varepsilon = \left\{t \in A_n : x^*(t) - y_n(t) > \frac{\varepsilon}{2\|\chi_{[0,\infty)}\|_E}\right\}.$$

Define

$$C = \left\{t \in [0,\infty) : \left|x(t)\right| > x^*(\infty)\right\} \quad \text{and} \quad D = \left\{t \in [0,\infty) : \left|x(t)\right| = x^*(\infty)\right\}.$$

By Lemma 2.2 in [7], there is a measure preserving transformation $\sigma : C \to [0, m(C))$ such that $x^* \circ \sigma = |x|$ a.e. on C. In the case of $m(C) < \infty$ and $m(D) = \infty$ we apply the construction from the proof of Proposition 2.3 in [24], i.e. we employ a measure preserving transformation $\beta : D \to [m(C), \infty)$ and define

$$\gamma = \sigma\chi_C + \beta\chi_D.$$

Observe that $\gamma : C \cup D \to [0,\infty)$ is also a measure preserving transformation and

$$x^* \circ \gamma = x^* \circ \sigma \chi_C + x^* \circ \beta \chi_D = |x|\chi_C + x^*(\infty)\chi_D = |x|\chi_{C\cup D}.$$

For simplicity, in both cases when $m(C) = \infty$ and $m(C) < \infty$, we use one notation γ as a measure preserving transformation that recovers $|x|$ from x^* a.e. on C and $C \cup D$, respectively. Then

$$\begin{aligned}\left\|\left(y_n - x^*\right)\chi_{A_n}\right\|_E &\le \left\|\left(x^* - y_n\right)\chi_{A_n^\varepsilon}\right\|_E + \left\|\left(x^* - y_n\right)\chi_{A_n\setminus A_n^\varepsilon}\right\|_E \le \left\|x^*\chi_{A_n^\varepsilon}\right\|_E + \varepsilon/2 \\ &= \left\|\left(x^*\circ\gamma\right)\left(\chi_{A_n^\varepsilon}\circ\gamma\right)\right\|_E + \varepsilon/2 = \left\||x|\chi_{\gamma^{-1}[A_n^\varepsilon]}\right\|_E + \varepsilon/2.\end{aligned}$$

By Lemma 3.2, we get $\||x|\chi_{\gamma^{-1}[A_n^\varepsilon]}\|_E \to 0$ because $m(\gamma^{-1}[A_n^\varepsilon]) = m(A_n^\varepsilon) \to 0$ and x is an H_g-point in E. Consequently, (2) is proved. Since

$$\|y_n\|_E \le \|\tilde{y}_n\|_E = \left\|y_n - y_n\chi_{A_n} + x^*(\infty)\chi_{A_n}\right\|_E \le \|y_n\|_E + \left\|\left(x^* - y_n\right)\chi_{A_n}\right\|_E$$

and

$$\lim_{n\to\infty}\|y_n\|_E = \lim_{n\to\infty}\left(\|y_n\|_E + \left\|\left(x^* - y_n\right)\chi_{A_n}\right\|_E\right) = \left\|x^*\right\|_E,$$

by the squeeze theorem, $\lim_{n\to\infty}\|\tilde{y}_n\|_E = \|x^*\|_E$.

Moreover,

$$\begin{aligned}\left\|\left(\tilde{y}_n - x^*\right)\right\|_E &\le \left\|y_n - x^*\right\|_E \\ &= \left\|\left(y_n - x^*\right)\chi_{A_n} + \left(y_n - x^*\right)\chi_{B_n}\right\|_E \\ &= \left\|\left(x^*(\infty) - x^*\right)\chi_{A_n} + \left(y_n - x^*\right)\chi_{B_n} + \left(y_n - x^*(\infty)\right)\chi_{A_n}\right\|_E \\ &\le \left\|\left(\tilde{y}_n - x^*\right)\right\|_E + \left\|\left(x^* - y_n\right)\chi_{A_n}\right\|_E,\end{aligned}$$

whence

$$\lim_{n\to\infty}\left\|\left(\tilde{y}_n - x^*\right)\right\|_E = \lim_{n\to\infty}\left\|y_n - x^*\right\|_E,$$

which finishes the proof of our claim.

Define

$$x_n = y_n \circ \sigma\chi_C + x^*(\infty)\chi_{\mathrm{supp}(x)\setminus C}.$$

We have

$$\begin{aligned}m\left(\left\{t \in C : \left|x_n(t) - |x(t)|\right| > \delta\right\}\right) &= m\left(\left\{t \in C : \left|y_n\left(\sigma(t)\right) - x^*\left(\sigma(t)\right)\right| > \delta\right\}\right) \\ &= m\left(\left\{t \in \left[0, m(C)\right) : \left|y_n(t) - x^*(t)\right| > \delta\right\}\right)\end{aligned}$$

for $\delta > 0$. Hence, by the assumption,

$$x_n\chi_C \to |x|\chi_C \tag{3}$$

in measure. Now we proceed the proof in two cases.

Case 1. Let $m(C) = \infty$. Clearly, $x_n \to |x|$ in measure. Then,

$$m\left(\left\{t \in [0,\infty) : |x|\chi_C(t) > \delta\right\}\right) = m\left(\left\{t \in [0,\infty) : |x|(t) > \delta\right\}\right)$$

for every $\delta > x^*(\infty)$. Furthermore,

$$m\big(\{t \in [0,\infty) : |x|(t) > \delta\}\big) \geq m\big(\{t \in [0,\infty) : |x|\chi_C(t) > \delta\}\big) = \infty$$

for any $\delta \leq x^*(\infty)$. Hence, $|x|\chi_C \sim |x|$. Moreover, by the equality

$$\begin{aligned} m\big(\{t : \big|(y_n \circ \sigma\chi_C)(t)\big| > \delta\}\big) &= m\big(\{t \in C : \big|y_n\big(\sigma(t)\big)\big| > \delta\}\big) \\ &= m\big(\sigma^{-1}\{s \in [0,\infty) : \big|y_n(s)\big| > \delta\}\big) \\ &= m\big(\{s \in [0,\infty) : \big|y_n(s)\big| > \delta\}\big), \end{aligned}$$

it follows that $y_n \circ \sigma\chi_C \sim y_n$. Consequently,

$$\|x_n\chi_C\|_E = \|y_n \circ \sigma\chi_C\|_E = \|y_n\|_E \to \|x\|_E = \big\||x|\chi_C\big\|_E. \tag{4}$$

Now we show that

$$\|x_n\|_E \to \|x\|_E. \tag{5}$$

If $\delta \geq x^*(\infty)$, then

$$m\big(t \in [0,\infty) : |x_n|(t) > \delta\big) = m\big(t \in [0,\infty) : |y_n \circ \sigma\chi_C|(t) > \delta\big).$$

For each $0 < \delta < x^*(\infty)$ we have $d_{y_n\circ\sigma\chi_C}(\delta) = \infty$ for all n, because $y_n \geq x^*(\infty)\chi_{[0,\infty)}$. Then $d_{x_n}(\delta) \geq d_{y_n\circ\sigma\chi_C}(\delta) = \infty$ for all n, whence $\|x_n\|_E = \|y_n \circ \sigma\chi_C\|_E = \|y_n\|_E$ and condition (5) is proved. Since, by Lemma 3.1, $|x|$ is an H_g-point, it follows that

$$\big\|x_n\chi_C - |x|\chi_C\big\|_E = \big\|x_n - |x|\big\|_E \to 0.$$

Consequently,

$$\big\|y_n - x^*\big\|_E = \big\|y_n \circ \sigma - x^* \circ \sigma\big\|_E = \big\|x_n\chi_C - |x|\chi_C\big\|_E \to 0.$$

Case 2. Let $m(C) < \infty$. Then $m(D) = \infty$. Now, according to the construction of the measure preserving transformation γ, we get

$$z = x^* \circ \gamma = |x|\chi_{C\cup D}.$$

Define

$$z_n = y_n \circ \gamma = y_n \circ \sigma\chi_C + y_n \circ \beta\chi_D.$$

Then

$$\begin{aligned} &m\big(\{t \in C \cup D : \big|z_n(t) - z(t)\big| > \epsilon\}\big) \\ &\quad = m\big(\{t \in C : \big|y_n\big(\sigma(t)\big) - x^*\big(\sigma(t)\big)\big| > \epsilon\}\big) + m\big(\{t \in D : \big|y_n\big(\beta(t)\big) - x^*(\infty)\big| > \epsilon\}\big) \\ &\quad = m\big(\{t \in \big[0, m(C)\big) : \big|y_n(t) - x^*(t)\big| > \epsilon\}\big) + m\big(\{t \in \big[m(C),\infty\big) : \big|y_n(t) - x^*(\infty)\big| > \epsilon\}\big) \\ &\quad = m\big(\{t \in [0,\infty) : \big|y_n(t) - x^*(t)\big| > \epsilon\}\big) \end{aligned}$$

for any $\epsilon > 0$ and $n \in \mathbb{N}$. Hence, z_n converges to z in measure. It is easy to observe that

$$\|z_n\|_E = \|y_n \circ \gamma\chi_{C\cup D}\|_E = \|y_n\|_E \to \|x\|_E = \|z\|_E.$$

Since $z = |x|\chi_{C\cup D} = |x|$ is an H_g-point in E, then

$$\|y_n - x^*\|_E = \|y_n \circ \gamma - x^* \circ \gamma\|_E = \|z_n - z\|_E \to 0. \qquad \square$$

Lemma 3.4. *Let E be a symmetric Banach function space on $[0, \alpha)$, where $\alpha = 1$ or $\alpha = \infty$. If an element $x \in E$ is an H_l point then x^* is an H_l point in E.*

Proof. We follow similarly as in the proof of Theorem 3.3, case II, because each H_l point is a point of order continuity, whence $x^*(\infty) = 0$. $\square$

Theorem 3.5. *Let E be a symmetric Banach function space on $[0, \alpha)$, where $\alpha = 1$ or $\alpha = \infty$. If $x \in E_a$ and x^* is an H_g point in E, then x is an H_g point in E.*

Proof. Suppose that $x_n \to x$ in measure and $\|x_n\|_E \to \|x\|_E$. Then

$$\|x_n^*\|_E = \|x_n\|_E \to \|x\|_E = \|x^*\|_E. \tag{6}$$

By property 11^o in [29], x_n^* converges to x^* a.e. We will show that $x_n^* \to x^*$ in measure. Only the case $[0, \infty)$ should be considered. Since $x \in E_a$, we have $x^*(\infty) = 0$. Hence, for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$x^*(t) < \varepsilon \quad \text{and} \quad x_n^*(t_\varepsilon) \to x^*(t_\varepsilon)$$

for all $t \geq t_\varepsilon$. Furthermore, since x_n^*, for $n \in \mathbb{N}$, and x^* are decreasing functions, there is $N_\varepsilon \in \mathbb{N}$ such that

$$|x_n^*(t) - x^*(t)| < \varepsilon$$

for all $n \geq N_\varepsilon$ and $t \geq t_\varepsilon$. Consequently,

$$m\big(\{t \in [t_\varepsilon, \infty) : |x_n^*(t) - x^*(t)| > \varepsilon\}\big) \to 0$$

for every $\varepsilon > 0$. Since $x_n^* \to x^*$ a.e., x_n^* converges to x^* locally in measure. Thus

$$m\big(\{t \in [0, t_\varepsilon] : |x_n^*(t) - x^*(t)| > \varepsilon\}\big) \to 0.$$

Hence x_n^* converges to x^* in measure. Now, in view of condition (6) and the assumption that x^* is H_g point in E, we have

$$\|x_n^* - x^*\|_E \to 0.$$

Since $x \in E_a$, $x_n^* \to x^*$ in measure and also in norm of E, by Proposition 2.4 in [9], it follows that

$$\|x_n - x\|_E \to 0. \qquad \square$$

The above two theorems imply immediately

Corollary 3.6. *Suppose $x \in E_a$. Then x is an H_g-point in E if and only if x^* is an H_g-point in E.*

The following lemma shows a nice analogy to a characterization of point of order continuity. Namely, replacing the convergence a.e. by the convergence globally in measure, we get

Lemma 3.7. *Let E be a symmetric Banach function space on $[0, \alpha)$ with the semi-Fatou property, where $\alpha = 1$ or $\alpha = \infty$. If $x \geq 0$ is an H_g-point, then for each sequence (d_n) in L^0 with $0 \leq d_n \leq x$ and $d_n \to 0$ globally in measure we have $\|d_n\|_E \to 0$.*

Proof. Assume for the contrary that $0 \leq d_n \leq x$, $d_n \to 0$ globally in measure and $\|d_n\|_E \nrightarrow 0$. Passing to a subsequence if necessary, we have $\|d_n\|_E \geq \delta > 0$ for some $\delta > 0$. Clearly, $m(A_n(\varepsilon)) \to 0$ for each $\varepsilon > 0$, where $A_n(\varepsilon) = \{t \in [0, \alpha) : d_n(t) > \varepsilon\}$. Since x is an H_g point, by Lemma 3.2,

$$\|d_n \chi_{A_n(\varepsilon)}\|_E \leq \|x \chi_{A_n(\varepsilon)}\|_E \to 0$$

for each $\varepsilon > 0$. For $\varepsilon_1 = 1$ we find an index n_1 with $\|d_{n_1} \chi_{A_{n_1}(\varepsilon_1)}\|_E < 1$ and $m(A_{n_1}(\varepsilon_1)) < 1$. Next, for $\varepsilon_2 = 1/2$ we find an index $n_2 > n_1$ with $\|d_{n_2} \chi_{A_{n_2}(\varepsilon_2)}\|_E < 1/2$ and $m(A_{n_2}(\varepsilon_2)) < 1/2$. Consequently, passing to a subsequence if necessary, we may assume that

$$\|d_n \chi_{A_n(\varepsilon_n)}\|_E < 1/n \quad \text{and} \quad m\big(A_n(\varepsilon_n)\big) < 1/n$$

for $\varepsilon_n = 1/n$. Suppose $x^*(\infty) > 0$. Then $\|\chi_{[0,\alpha)}\|_E < \infty$. Notice that $\|d_n \chi_{[0,\alpha)\setminus A_n(\varepsilon_n)}\|_E \geq \delta/2$ for sufficiently large $n \in \mathbb{N}$. Therefore

$$\delta/2 \leq \|d_n \chi_{[0,\alpha)\setminus A_n(\varepsilon_n)}\|_E \leq \frac{1}{n} \|\chi_{[0,\alpha)}\|_E$$

for $n \in \mathbb{N}$ large enough, a contradiction.

Assume $x^*(\infty) = 0$. By Theorem 3.3, x^* is an H_g-point. We have $0 \leq d_n^* \leq x^*$. Moreover, it is not difficult to show that $d_n^* \to 0$ globally in measure (see for example the proof of Theorem 3.5). Similarly as above, take sequences $\varepsilon_n \downarrow 0$ and $B_n(\varepsilon_n) = \{t \in [0, \alpha) : d_n^*(t) > \varepsilon_n\}$ such that

$$m\big(B_n(\varepsilon_n)\big) \to 0 \quad \text{and} \quad \big\|d_n^* \chi_{B_n(\varepsilon_n)}\big\|_E \to 0.$$

For each ε_n there is t_{ε_n} satisfying $x^*(t_{\varepsilon_n}) \leq \varepsilon_n$. Set $t_{\varepsilon_n} = \inf\{t : x^*(t) \leq \varepsilon_n\}$. First we claim that $\|\varepsilon_n \chi_{[0,t_{\varepsilon_n})}\|_E \to 0$. Otherwise, set $z_n = (x^* - \varepsilon_n)\chi_{[0,t_{\varepsilon_n})}$. Then z_n is nondecreasing. Moreover, $|z_n - x^*| = \varepsilon_n \chi_{[0,t_{\varepsilon_n})} + x^* \chi_{[t_{\varepsilon_n},\infty)}$. Note that $t_{\varepsilon_n} \to \infty$ when $m(\operatorname{supp} x^*) = \infty$ and $t_{\varepsilon_n} \to m(\operatorname{supp} x^*)$ if $m(\operatorname{supp} x^*) < \infty$. In both cases we conclude that $z_n \uparrow x^*$ globally in measure. Consequently, by $E \in (s-FP)$, $\|z_n\|_E \to \|x^*\|_E$. On the other hand,

$$\big\|z_n - x^*\big\|_E \geq \|\varepsilon_n \chi_{[0,t_{\varepsilon_n})}\|_E \nrightarrow 0,$$

a contradiction with x^* is an H_g point. This proves the claim. Note that

$$\big\|d_n^* \chi_{[0,t_{\varepsilon_n})\cap B_n(\varepsilon_n)}\big\|_E \leq \big\|d_n^* \chi_{B_n(\varepsilon_n)}\big\|_E \to 0.$$

Therefore,

$$\big\|d_n^* \chi_{[0,t_{\varepsilon_n})}\big\|_E \leq \big\|d_n^* \chi_{[0,t_{\varepsilon_n})\cap B_n(\varepsilon_n)}\big\|_E + \big\|d_n^* \chi_{[0,t_{\varepsilon_n})\setminus B_n(\varepsilon_n)}\big\|_E \to 0.$$

Consequently,

$$\delta/2 \leq \big\|d_n^* \chi_{(t_{\varepsilon_n},\infty)}\big\|_E \leq \big\|x^* \chi_{(t_{\varepsilon_n},\infty)}\big\|_E$$

for sufficiently large $n \in \mathbb{N}$. Taking $y_n = x^* \chi_{[0,t_{\varepsilon_n})}$, we conclude $y_n \to x^*$ globally in measure and $\|y_n\|_E \to \|x^*\|_E$. On the other hand, $\|y_n - x^*\|_E \geq \delta/2$ for sufficiently large $n \in \mathbb{N}$. This means x^* is not an H_g-point, a contradiction with Theorem 3.3. This finishes the proof. $\square$

It is well known that the norm is lower semicontinuous with respect to weak convergence. Moreover, if $E \in (OC)$, the norm is lower semicontinuous with respect to convergence almost everywhere. Namely, if $x_n \to x \in E_a$ a.e. then $\|x\|_E \leq \liminf \|x_n\|_E$. We will prove the analogous result for global convergence in measure and for an H_g point in symmetric Banach function spaces (recall that properties H_g and OC are not comparable in general, see Section 5 below).

Lemma 3.8. *Let E be a symmetric Banach function space on $[0, \alpha)$ with the semi-Fatou property, where $\alpha = 1$ or $\alpha = \infty$. Suppose x is an H_g point. If $x_n \to x$ globally in measure, then $\|x\|_E \leq \liminf \|x_n\|_E$.*

Proof. Assume for a moment that $x, x_n \geq 0$. Set

$$w_n = \max\{x, x_n\} \quad \text{and} \quad d_n = (x - x_n)\chi_{A_n}, \quad \text{where } A_n = \{t : x_n(t) \leq x(t)\}.$$

Then $0 \leq d_n \leq x$ and $d_n \to 0$ globally in measure. By Lemma 3.7, $\|d_n\|_E \to 0$. Moreover, $x_n = w_n - d_n$ and $w_n \geq x \geq d_n$. Consequently,

$$\|w_n\|_E - \|d_n\|_E \leq \|x_n\|_E \leq \|w_n\|_E + \|d_n\|_E,$$

whence

$$\|x\|_E \leq \liminf \|w_n\|_E = \liminf \|x_n\|_E.$$

Take arbitrary x, x_n such that $x_n \to x$ globally in measure. By the inequality

$$m\big(\big\{t : \big||x(t)| - |x_n(t)|\big| > \varepsilon\big\}\big) \leq m\big(\big\{t : \big|x(t) - x_n(t)\big| > \varepsilon\big\}\big)$$

we conclude that $|x_n| \to |x|$ globally in measure. By Lemma 3.1, $|x|$ is an H_g point. Since the lemma holds for elements of positive cone of E, we get $\||x|\|_E \leq \liminf \||x_n|\|_E$, which finishes the proof. □

Lemma 3.9. *Let E be a symmetric Banach function space on $[0, \alpha)$, where $\alpha = 1$ or $\alpha = \infty$. If $x \in E_a$ and a sequence (x_n) of elements in E converges to x locally in measure, then there exists a sequence (A_n) of measurable subsets of finite measure such that $\chi_{A_n} \to \chi_{\operatorname{supp} x}$ locally in measure and a subsequence (x_{n_k}) of the sequence (x_n) such that*

$$\|x_{n_k}\chi_{A_k} - x\|_E \to 0 \quad \textit{and} \quad \|x\chi_{[0,\alpha)\setminus A_k}\|_E \to 0.$$

Proof. Let us consider two cases.

Case 1. Suppose that $m(\operatorname{supp} x) < \infty$. Then $x_n\chi_{\operatorname{supp} x} \to x$ in measure. Hence there is an increasing sequence of positive integers (n_k) such that

$$m\left(\left\{t \in \operatorname{supp} x : \big|x_{n_k}(t) - x(t)\big| > \frac{1}{k}\right\}\right) < \frac{1}{k}$$

for any $k \in \mathbb{N}$. Denote

$$A_k = \left\{t \in \operatorname{supp} x : \big|x_{n_k}(t) - x(t)\big| \leq \frac{1}{k}\right\}$$

for any $k \in \mathbb{N}$. Obviously, $m(\operatorname{supp} x \setminus A_k) < \frac{1}{k}$ for $k \in \mathbb{N}$. Therefore, by the order continuity of x, we have

$$\|x\chi_{[0,\alpha)\setminus A_n}\|_E = \|x\chi_{\operatorname{supp} x\setminus A_n}\|_E \to 0.$$

Moreover,

$$\|x_{n_k}\chi_{A_k} - x\|_E \le \big\|(x_{n_k} - x)\chi_{A_k}\big\|_E + \|x\chi_{[0,\alpha)\setminus A_k}\|_E$$
$$\le \frac{1}{k}\|\chi_{\operatorname{supp} x}\|_E + \|x\chi_{[0,\alpha)\setminus A_k}\|_E \to 0.$$

Case 2. Suppose that $m(\operatorname{supp} x) = \infty$. Then $\alpha = \infty$. For all $n \in \mathbb{N}$ define

$$C_n = \left\{t \in \operatorname{supp} x : |x(t)| \ge \frac{1}{n}\right\}.$$

By the order continuity of x, we have $m(C_n) < \infty$ for any $n \in \mathbb{N}$. Taking into account that $(\operatorname{supp} x\setminus C_n) \downarrow \varnothing$, we get $\|x\chi_{\operatorname{supp} x\setminus C_n}\|_E \to 0$. For any $k \in \mathbb{N}$ we obtain $x_n\chi_{C_k} \to x\chi_{C_k}$ in measure. Hence, there exists a positive integer n_k such that

$$m\left(\left\{t \in C_k : |x_{n_k}(t) - x(t)| > \frac{1}{2^k}\right\}\right) < \frac{1}{2^k}.$$

Define for any $k \in \mathbb{N}$,

$$A_k = \left\{t \in C_k : |x_{n_k}(t) - x(t)| \le \frac{1}{2^k}\right\}.$$

Since $m(C_k\setminus A_k) < \frac{1}{2^k}$, by the order continuity of x, we have

$$\|x\chi_{[0,\infty)\setminus A_k}\|_E = \|x\chi_{\operatorname{supp} x\setminus A_k}\|_E \le \|x\chi_{\operatorname{supp} x\setminus C_k}\| + \|x\chi_{C_k\setminus A_k}\|_E \to 0.$$

Notice that

$$\big\|(x_{n_k} - x)\chi_{A_k}\big\|_E \le \frac{1}{2^k}\|\chi_{A_k}\|_E \le \frac{1}{2^k}\|\chi_{C_k}\|_E = \frac{k}{2^k}\left\|\frac{1}{k}\chi_{C_k}\right\|_E$$
$$\le \frac{k}{2^k}\|x\chi_{C_k}\|_E \le \frac{k}{2^k}\|x\|_E \to 0,$$

whence

$$\|x_{n_k}\chi_{A_k} - x\|_E \le \big\|(x_{n_k} - x)\chi_{A_k}\big\|_E + \|x\chi_{[0,\infty)\setminus A_k}\|_E \to 0,$$

which completes the proof. □

The global version of the next result has been proved in [5, Theorem 3.2]. Although we use partially some methods from [5] and [7], the local approach has required also new techniques.

Theorem 3.10. *Let E be a symmetric Banach function space on $[0,\infty)$ and $x \in E_+ \setminus \{0\}$. Then the following conditions are equivalent:*

a) x is an H_l-point in the space E;
b) $x \in E_a$, x is a UM point and x is an H_g-point;
c) $x \in E_a$ and x is a ULUM point.

Proof. *a)* $\Rightarrow$ *b)*. Let x be an H_l point in the space E. Then $x \in E_a$ (see Lemma 6 in [16] for LLUM point but the same proof works for H_l point, see also Theorem 2.1 in [10]) and x is an H_g point. Suppose for the contrary that x is not a UM point. Then there is $y \in E$ such that $x \leq y$, $x \neq y$ and $\|x\|_E = \|y\|_E$. Since $x \in E_a$ gives $x^*(\infty) = 0$, the conditions $x \leq y$ with $x \neq y$ imply that $x^* \leq y^*$ with $x^* \neq y^*$ (see Lemma 3.2 in [18]). Take $t_0 > 0$ with $x^*(t_0) < y^*(t_0)$. Define

$$z = y^* \chi_{[\frac{1}{2}t_0, 2t_0)} + x^* \chi_{[0,\frac{1}{2}t_0) \cup [2t_0,\infty)}.$$

Note that $x^* \in E_a$ (see Lemma 2.6 in [7]). Obviously, $z \in E_a$, $x^* \leq z \leq y^*$, $x^* \neq z$ and $z^* \chi_{[2t_0,\infty)} = x^* \chi_{[2t_0,\infty)}$. Moreover,

$$\|x^*\|_E \leq \|z\|_E \leq \|y^*\|_E = \|x^*\|_E,$$

whence $\|x^*\|_E = \|z\|_E$. Define $v = z^* - x^*$. Then $\operatorname{supp} v \subset [0, 2t_0)$. Set

$$v_n(s) = \begin{cases} 0 & \text{if } s < n; \\ v(s-n) & \text{if } s \geq n, \end{cases}$$

for every $n \in \mathbb{N}$. By Lemma 3.1 in [5],

$$x_n = x^* + v_n \prec x^* + v = z^*$$

for any $n \in \mathbb{N}$. It is clear that $x_n \to x^*$ locally in measure. Moreover,

$$\|x^*\|_E \leq \|x^* + v_n\|_E = \|x_n\|_E \leq \|z^*\|_E = \|z\|_E = \|x^*\|_E,$$

whence $\|x^*\|_E = \|x_n\|_E$ for any $n \in \mathbb{N}$. Since x is an H_l-point, we have $x^*(\infty) = 0$. By Lemma 3.4, we conclude that x^* is an H_l-point. Thus

$$\|v_n\|_E = \|x_n - x^*\|_E \to 0.$$

Since $\|v\|_E = \|v_n\|_E$ for any $n \in \mathbb{N}$, we conclude that $\|v\|_E = 0$. Consequently, $z^* = x^*$ and in particular $y^*(t_0) = z^*(t_0) = x^*(t_0)$. A contradiction, because $y^*(t_0) > x^*(t_0)$.

b) $\Rightarrow$ *c*). This implication has been proved in [7, Theorem 2.2 (*i*)]. However, the assumption that $x \in E_a$ should be added there. Moreover, some steps of that proof require modifications. Hence we present the whole proof for the convenience of the reader.

Let $x \leq x_n$ and $\|x_n\|_E \to \|x\|_E$. By Helly's selection principle, passing to subsequence if necessary, we may assume that $x_n^* \to z^*$ a.e. Moreover, $x^* \leq x_n^*$ and consequently $x^* \leq z^*$. We claim that

$$x_n^*(\infty) \to 0.$$

If this is not true, then there is $\delta > 0$ such that $x_{n_k}^*(\infty) \geq \delta$ for some $(n_k) \subset \mathbb{N}$. Passing to subsequence and relabeling we get $x_n^*(\infty) \geq \delta$ for all $n \in \mathbb{N}$. Since x is a point of order continuity, it follows that x^* is also a point of order continuity and $x^*(\infty) = 0$ (see [7]). Hence $x_n^*(\infty) \geq \delta > x^*(\infty) = 0$, so there exists $t_0 > 0$ such that $x^*(t) < \delta$ for any $t \geq t_0$. We have

$$x^* \leq x^* \chi_{[0,t_0]} + \delta \chi_{(t_0,\infty)} \leq x_n^*$$

for all $n \in \mathbb{N}$ and

$$x^* \neq x^* \chi_{[0,t_0]} + \delta \chi_{(t_0,\infty)}.$$

Since x is a UM point, we obtain that x^* is a UM point (see Proposition 2.2 in [7]). Therefore,

$$\|x^*\|_E < \|x^*\chi_{[0,t_0]} + \delta\chi_{(t_0,\infty)}\|_E \le \|x_n^*\|_E.$$

By the assumption that $\|x_n^*\|_E \to \|x^*\|_E$, we obtain a contradiction, which proves the claim. Now, by convergence of x_n^* to z^* a.e., there exist $(n_k) \subset \mathbb{N}$ and $(t_k) \subset [0,\infty)$ such that $n_k \to \infty$, $t_k \to \infty$ and $x_{n_k}^*(t_k) \to 0$ and also for any $k \in \mathbb{N}$,

$$|x_{n_k}^*(t_k) - z^*(t_k)| < \frac{1}{k}.$$

Therefore, $z^*(\infty) = 0$ and since $x_n^* \to z^*$ a.e., we may easily show that $x_n^* \to z^*$ in measure on $[0,\infty)$. We prove that

$$x^* = z^*.$$

Suppose for the contrary that $x^* \ne z^*$. Let

$$A_n = \left\{t \in [0,\infty) : z^*(t) - x^*(t) > \frac{1}{n}\right\}$$

for any $n \in \mathbb{N}$. In consequence, $m(A_{n_0}) > 0$ for some $n_0 \in \mathbb{N}$. Moreover, since $z^*(\infty) = x^*(\infty) = 0$, it follows that $m(A_{n_0}) < \infty$. For every $n \in \mathbb{N}$ we define

$$y_n = x_n^*\chi_{A_{n_0}} + x^*\chi_{A_{n_0}^c}.$$

Notice that for any $n \in \mathbb{N}$,

$$\|x^*\|_E \le \|y_n\|_E \le \|x_n^*\|_E.$$

Thus, by $\|x_n^*\|_E \to \|x^*\|_E$, we have $\|y_n\|_E \to \|x^*\|_E$. Set

$$B_n = \left\{t \in A_{n_0} : |z^*(t) - x_n^*(t)| \le \frac{1}{2^n}\right\}$$

for any $n \in \mathbb{N}$. Since x_n^* converges to z^* in measure, there is a subsequence $(x_{n_k}^*)$ of (x_n^*) such that, for all $k \in \mathbb{N}$,

$$m\left(t \in A_{n_0} : |z^*(t) - x_{n_k}^*(t)| > \frac{1}{2^k}\right) \le \frac{1}{2^k}.$$

Passing to subsequence if necessary, we obtain

$$m\left(t \in A_{n_0} : |z^*(t) - x_n^*(t)| > \frac{1}{2^n}\right) \le \frac{1}{2^n}$$

for any $n \in \mathbb{N}$. Clearly, $m(B_n) \to m(A_{n_0})$. We have for each $t \in B_n$,

$$x_n^*(t) \ge z^*(t) - \frac{1}{2^n}.$$

Then, for sufficiently large $n \in \mathbb{N}$, we get $m(B_n) > 0$, $\frac{2n_0}{2^n} < 1$ and

$$\begin{aligned} y_n = x_n^*\chi_{A_{n_0}} + x^*\chi_{A_{n_0}^c} &\geq \left(z^* - \frac{1}{2^n}\right)\chi_{B_n} + x_n^*\chi_{B_n^c} + x^*\chi_{A_{n_0}^c} \\ &\geq \left(z^* - \frac{1}{2n_0}\frac{2n_0}{2^n}\right)\chi_{B_n} + x^*\chi_{A_{n_0}^c\cup B_n^c} \\ &\geq \left(z^* - \frac{1}{2n_0}\right)\chi_{B_n} + x^*\chi_{A_{n_0}^c\cup B_n^c} \\ &> \left(x^* + \frac{1}{2n_0}\right)\chi_{B_n} + x^*\chi_{A_{n_0}^c\cup B_n^c} \\ &= x^* + \frac{1}{2n_0}\chi_{B_n}. \end{aligned}$$

Thus,

$$\|x_n\|_E \geq \|y_n\|_E \geq \left\|\left(z^* - \frac{1}{2n_0}\right)\chi_{B_n} + x^*\chi_{A_{n_0}^c\cup B_n^c}\right\|_E \geq \left\|x^* + \frac{1}{2n_0}\chi_{B_n}\right\|_E$$

for sufficiently large $n \in \mathbb{N}$. Since $z^* \geq x^*$ and $z^*(\infty) = 0$, by definition of the set A_{n_0}, there is $t_0 > 0$ such that $z^*(t) \leq \frac{1}{n_0}$ and $x^*(t) \leq \frac{1}{n_0}$ for all $t \geq t_0$ and $A_{n_0} \subset [0, t_0]$. Consequently,

$$\left(x^* + \frac{1}{2n_0}\chi_{B_n}\right)^{**} \geq \left(x^* + \frac{1}{2n_0}\chi_{[t_0,t_0+m(B_n)]}\right)^{**}$$

for all $n \in \mathbb{N}$. Since $m(B_n) \to m(A_{n_0})$ and $B_n \subset A_{n_0}$, we may assume that $m(B_n) > m(A_{n_0})/2$ for sufficiently large $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} \left(x^* + \frac{1}{2n_0}\chi_{B_n}\right)^{**} &\geq \left(x^* + \frac{1}{2n_0}\chi_{[t_0,t_0+m(B_n)]}\right)^{**} \\ &\geq \left(x^* + \frac{1}{2n_0}\chi_{[t_0,t_0+m(A_{n_0})/2]}\right)^{**}. \end{aligned} \tag{7}$$

Set

$$w = x^* + \frac{1}{2n_0}\chi_{[t_0,t_0+m(A_{n_0})/2]} \neq x^*.$$

Clearly, $w \geq x^*$ and $w \neq x^*$. Since x^* is a *UM* point (see Proposition 2.2 in [7]), it follows that $\|w\|_E > \|x^*\|_E$. Hence, by (7) and Corollary 4.7 in [1], for sufficiently large $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \|x_n\|_E \geq \|y_n\|_E &\geq \left\|x^* + \frac{1}{2n_0}\chi_{B_n}\right\|_E \\ &\geq \left\|x^* + \frac{1}{2n_0}\chi_{[t_0,t_0+m(B_n)]}\right\|_E \geq \|w\|_E > \left\|x^*\right\|_E, \end{aligned}$$

which contradicts the assumption $\|x_n\|_E \to \|x\|_E$. Therefore $x^* = z^*$. We claim that $x_n^* \to x^*$ in measure. Since $x^*(\infty) = 0$, for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$x^*(t) < \varepsilon/2 \quad \text{for all } t \geqslant t_\varepsilon \quad \text{and} \quad x_n^*(t_\varepsilon) \to x^*(t_\varepsilon).$$

Furthermore, since x_n^* and x^* are decreasing functions, there is $N_\varepsilon \in \mathbb{N}$ such that

$$\left|x_n^*(t) - x^*(t)\right| < \varepsilon$$

for all $n \geqslant N_\varepsilon$ and $t \geqslant t_\varepsilon$. Consequently, for every $\varepsilon > 0$,

$$m\big(t \in [t_\varepsilon, \infty) : \big|x_n^*(t) - x^*(t)\big| > \varepsilon\big) \to 0.$$

Since $x_n^* \to x^*$ pointwisely, x_n^* converges to x^* locally in measure and

$$m\big(t \in [0, t_\varepsilon] : \big|x_n^*(t) - x^*(t)\big| > \varepsilon\big) \to 0.$$

Therefore x_n^* converges to x^* in measure.

By the assumption and Theorem 3.3, we conclude that x^* is an H_g point. Therefore, $\|x_n^* - x^*\| \to 0$. Now, following the proof of Theorem 3.2 in [5], the implications $(iii) \Rightarrow (ii)$, we conclude that $x_n \to x$ in measure. Finally, since x is an H_g point, we obtain $\|x_n - x\| \to 0$.

$c) \Rightarrow b)$. It follows immediately from Theorem 2.2 (ii) in [7].

$c) \Rightarrow a)$. Assume $x \in E_a$, x is a *ULUM* point. By the implication $c) \Rightarrow b)$, x is also an H_g-point. Without loss of generality, we may assume that $\|x\|_E = 1$. Take $\{x_n\} \subset E_+$ such that $\|x_n\|_E \to 1$ and $x_n \to x$ locally in measure (see Lemma 3.1). By Lemma 3.9, passing to a subsequence if necessary, there exists a sequence (A_n) of measurable subsets of finite measure such that $\chi_{A_n} \to \chi_{\operatorname{supp} x}$ locally in measure,

$$\|x_n\chi_{A_n} - x\|_E \to 0 \quad \text{and} \quad \|x\chi_{[0,\infty)\setminus A_n}\|_E \to 0. \tag{8}$$

Hence, by the symmetry of E, $x_n\chi_{A_n} \to x$ in measure. Set $y_n = x_n\chi_{[0,\infty)\setminus A_n}$.

We claim that $y_n \to 0$ in measure. If this is not so, there exists $\varepsilon > 0$ and measurable subsets $C_n \subset [0,\infty)\setminus A_n$ $(n = 1, 2, \dots)$ such that $m(C_n) = \varepsilon$ and $y_n\chi_{C_n} \geq \varepsilon\chi_{C_n}$ for any $n \in \mathbb{N}$. Define a sequence (z_n) by the following formula

$$z_n = |x_n|\chi_{A_n} + |x - x_n|\chi_{A_n} + |x|\chi_{[0,\infty)\setminus A_n} + \varepsilon\chi_{C_n}$$

for any $n \in \mathbb{N}$. Since $|x|\chi_{A_n} \leq |x_n|\chi_{A_n} + |x - x_n|\chi_{A_n}$ for every $n \in \mathbb{N}$, we have

$$0 \leq |x| \leq |x_n|\chi_{A_n} + |x - x_n|\chi_{A_n} + |x|\chi_{[0,\infty)\setminus A_n} + \varepsilon\chi_{C_n} = z_n$$

for each $n \in \mathbb{N}$. Moreover,

$$\begin{aligned} \|x_n\chi_{A_n}\|_E \leq \|z_n\|_E &\leq \big\||x_n| + |x - x_n|\chi_{A_n} + |x|\chi_{[0,\infty)\setminus A_n}\big\|_E \\ &\leq \|x_n\|_E + \|x - x_n\chi_{A_n}\|_E + 2\|x\chi_{[0,\infty)\setminus A_n}\|_E \to 1. \end{aligned}$$

Thus $\|z_n\|_E \to 1$. Consequently, $\|z_n - |x|\|_E \to 0$, because $x = |x|$ is a *ULUM* point. Therefore, by (8),

$$\begin{aligned} 0 < \varepsilon\|\chi_{[0,\varepsilon)}\|_E = \varepsilon\|\chi_{C_n}\|_E &= \big\|z_n - |x_n|\chi_{A_n} - |x - x_n|\chi_{A_n} - |x|\chi_{[0,\infty)\setminus A_n}\big\|_E \\ &= \big\|\big(z_n - |x|\big) + |x| - |x_n|\chi_{A_n} - |x - x_n|\chi_{A_n} - |x|\chi_{[0,\infty)\setminus A_n}\big\|_E \\ &\leq \big\|z_n - |x|\big\|_E + \big\|\big(|x| - |x_n|\big)\chi_{A_n}\big\|_E + \big\|(x - x_n)\chi_{A_n}\big\|_E \\ &\leq \big\|z_n - |x|\big\|_E + 2\big\|(x - x_n)\chi_{A_n}\big\|_E \to 0, \end{aligned}$$

whence $\varepsilon\|\chi_{[0,\varepsilon)}\|_E = 0$, a contradiction. Thus $y_n \to 0$ in measure as we claimed.

Therefore $x_n = x_n\chi_{A_n} + y_n \to x$ in measure. Since also $\|x_n\|_E \to \|x\|_E$, by the fact that x is an H_g point, we conclude that $\|x_n - x\|_E \to 0$. Consequently, x is an H_l-point in the space E. □

Corollary 3.11. *Let E be a symmetric Banach function space on $[0, \alpha)$, where $\alpha = 1$ or $\alpha = \infty$. An element $x \in E$ is an H_l point if and only if x^* is an H_l point in E.*

Proof. The case $\alpha = 1$ follows from Corollary 3.6, because H_l point is a point of order continuity. Let $\alpha = \infty$. The necessity follows from Lemma 3.4.

The sufficiency. By the assumption and Theorem 3.10 we conclude that $x^* \in E_a$, x^* is a UM point and x^* is an H_g point. Now, Proposition 2.2, Lemma 2.6 from [7] and Theorem 3.5 imply that $x \in E_a$, x is a UM-point and x is an H_g point. Again, Theorem 3.10 yields that x is an H_l point. $\square$

Now, we take into account the special case of symmetric space.

Proposition 3.12. *An element x is an H_l-point in the space $(L^1 + L^\infty)([0,\infty))$ if and only if $x^*(1^-) = 0$.*

Proof. The necessity. Assume that $x^*(1^-) = a > 0$. By the assumption $x^*(\infty) = 0$, we find $t_1 > 1$ with $x^*(t) < a/2$ for $t \geq t_1$. Set

$$x_n = x^* \chi_{[0,\infty)} + a/2 \chi_{[t_1+n-1, t_1+n]}.$$

Then

$$\|x_n\|_{L^1+L^\infty} = \int_0^1 x_n^*(t)dt = \int_0^1 x^*(t)dt = \|x\|_{L^1+L^\infty}$$

for all $n \geq 1$. Furthermore, $x_n \to x^*$ locally in measure. On the other hand, $\|x_n - x^*\|_{L^1+L^\infty} \geq a/4$ for n large enough. By Lemma 3.4, both x^* and x cannot be H_l points.

The sufficiency. Suppose $x^*(1^-) = 0$. Then it is easy to conclude that $x \in E_a$. Moreover, by Theorem 3.1 in [10], x is an H_g point. In view of Theorem 3.10, it is enough to show that x is a UM point. Let $x \leq y$, $x \neq y$. Since $x^*(\infty) = 0$, the conditions $x \leq y$ with $x \neq y$ imply that $x^* \leq y^*$ with $x^* \neq y^*$ (see Lemma 3.2 in [18]). There is $t_0 < 1$ such that $x^*(t_0) < y^*(t_0)$, because $x^*(1^-) = 0$. Thus $\|x\|_E < \|y\|_E$. $\square$

Remark 3.13. Recall that each point $x \in L^1 + L^\infty$ is an H_g point (see [10]). Consider now the space $L^1 \cap L^\infty$ on $[0,\alpha)$ with $\alpha = 1$ or $\alpha = \infty$ and $\|x\|_{L^1 \cap L^\infty} := \max(\|x\|_{L^1}, \|x\|_{L^\infty})$. Clearly, $(L^1 \cap L^\infty)_a = \{0\}$, whence this space has no H_l points. It is easy to notice that also it has no H_g points. Indeed, let $x \in L^1 \cap L^\infty$ with $x \neq 0$. We find $\delta > 0$ such that $m(A) > 0$, where $A = \{t : |x(t)| \geq \delta\}$. Take a sequence $(A_n) \subset A$ with $0 < m(A_n) \downarrow 0$. Set $x_n = x\chi_{[0,\alpha)\setminus A_n}$. Then $x_n \to x$ globally in measure. Since $x_n \uparrow x$, by the Fatou property, $\|x_n\|_{L^1 \cap L^\infty} \to \|x\|_{L^1 \cap L^\infty}$. On the other hand, $\|x_n - x\|_{L^1 \cap L^\infty} \geq \delta$. Thus x is not an H_g point.

Remark 3.14. Recall that each H_l-point of E is a point of order continuity of E (see the proof of Lemma 5 in [16]). Observe that the reverse implication is not satisfied. Indeed, it is enough to consider the space $L^1 + L^\infty$ on $[0,\infty)$ and an element x with $x^*(1^-) > 0$ and $x^*(\infty) = 0$. By Proposition 3.12, x is not an H_l point. Moreover, x is a point of order continuity because $x^*(\infty) = 0$.

By Theorem 3.10, we also get

Corollary 3.15. *Let E be a symmetric Banach function space on $[0,\infty)$. Then the following conditions are equivalent:*

- (i) *E has the Kadec–Klee property with respect to local convergence in measure;*
- (ii) *E has the Kadec–Klee property with respect to global convergence in measure, E is order continuous and strictly monotone;*
- (iii) *E is order continuous and upper locally uniformly monotone.*

It is worth to mention that the above result has been also obtained in [5, Theorem 3.2]. Note also that the uniform Kadec–Klee with respect to local convergence in measure is equivalent to uniform monotonicity in the symmetric space over $[0,\infty)$ (see Theorem 3 in [36]).

For any global property P the symbol $E \in (P^*)$ denotes that E has property P only for elements in the cone of nonnegative and nonincreasing functions in E. Clearly, if $E \in (P)$, then $E \in (P^*)$. The natural question of the converse implication has been considered in [4] for rotundity properties. Applying Lemma 3.4 and Corollary 3.6 for Kadec–Klee properties, we obtain

Corollary 3.16. *Let E be a symmetric Banach function space on $[0,\alpha]$, where $\alpha = 1$ or $\alpha = \infty$. Then:*

(i) *$E \in (H_l)$ if and only if $E \in (H_l)^*$.*
(ii) *Suppose $E \in (OC)$. Then $E \in (H_g)$ if and only if $E \in (H_g)^*$.*

4. Lorentz spaces $\Gamma_{p,w}$ and $\Lambda_{p,w}$

Given $0 < p < \infty$ and a nonnegative weight function $w \in L^0$, the Lorentz space $\Gamma_{p,w}$ is a subspace of L^0 such that

$$\|x\|_{\Gamma_{p,w}} := \left(\int_0^\alpha \left(x^{**}\right)^p(t)w(t)dt\right)^{1/p} < \infty.$$

In order to get $\Gamma_{p,w} \neq \{0\}$, we need to assume that w is from class D_p that is

$$W(s) := \int_0^s w(t)dt < \infty \quad \text{and} \quad W_p(s) := s^p \int_s^\alpha t^{-p}w(t)dt < \infty$$

for all $0 < s \leq 1$ if $\alpha = 1$ and for all $0 < s < \infty$ otherwise. It is well known that $(\Gamma_{p,w}, \|\cdot\|_{\Gamma_{p,w}})$ is an r.i. quasi-Banach function space with the Fatou property. Notice that

$$\phi_{\Gamma_{p,w}}(s) = \left(W(s) + W_p(s)\right)^{1/p}$$

for any $0 < s \leq 1$ if $\alpha = 1$ and for all $0 < s < \infty$ if $\alpha = \infty$. It was proved [19] that in the case $\alpha = \infty$ the space $\Gamma_{p,w}$ has order continuous norm if and only if $\int_0^\infty w(t)dt = \infty$.

The spaces $\Gamma_{p,w}$ were introduced by A.P. Calderón in [3] in an analogous way as the classical Lorentz spaces

$$\Lambda_{p,w} = \left\{x \in L^0 : \|x\|_{\Lambda_{p,w}} = \left(\int_0^\alpha \left(x^*(t)\right)^p w(t)dt\right)^{1/p} < \infty\right\},$$

where $p \geq 1$ and the weight function w is nonnegative and nonincreasing (see [32]). The spaces $\Lambda_{p,w}$ are p-convexification of the Lorentz space $\Lambda_{1,w}$. The space $\Gamma_{p,w}$ is an interpolation space between L^1 and L^∞ yielded by the Lions–Peetre K-method [1,29]. Obviously, $\Gamma_{p,w} \subset \Lambda_{p,w}$. The reverse inclusion $\Lambda_{p,w} \subset \Gamma_{p,w}$ holds iff $w \in B_p$ (cf. [19]). Moreover, the spaces $\Gamma_{p,w}$ and $\Lambda_{p,w}$ are also related by Sawyer's result (Theorem 1 in [34]; see also [35]), which states that the Köthe dual of $\Lambda_{p,w}$, for $1 < p < \infty$ and $\int_0^\infty w(t)dt = \infty$, coincides with the space $\Gamma_{p',\widetilde{w}}$, where $1/p + 1/p' = 1$ and $\widetilde{w}(t) = (t/\int_0^t w(s)ds)^{p'}w(t)$.

It is easy to observe that if $\alpha = 1$, then by the Lebesgue dominated convergence theorem, $\Gamma_{p,w}$ is order continuous. For more details about the properties of $\Gamma_{p,w}$ the reader is referred to [7,19].

Theorem 4.1. *The Lorentz space $\Gamma_{p,w}(0,\alpha)$ has the Kadec–Klee property with respect to global convergence in measure, i.e. each point x in Lorentz space $\Gamma_{p,w}(0,\alpha)$ is an H_g point.*

Proof. Note that, for $\alpha = 1$ and $\alpha = \infty$ with $W(\infty) = \infty$, we have $\Gamma_{p,w} \in (OC)$ (see [6,19]). Since OC does not imply property H_g in general (see Example 2.8 from [5]), this case also should be proved directly.

Let $x, x_n \in \Gamma_{p,w}$ for any $n \in \mathbb{N}$, $\|x_n\|_{\Gamma_{p,w}} \to \|x\|_{\Gamma_{p,w}}$ and $x_n \to x$ globally in measure.

Case I. Let $W(\alpha) < \infty$. Define $y_n = x_n - x$,

$$A_n = \left\{ s : |y_n(s)| > \frac{1}{n} \right\} \quad \text{and} \quad \epsilon_n = d_{y_n}(1/n) = m(A_n)$$

for any $n \in \mathbb{N}$. Since $y_n \to 0$ in measure, passing to a subsequence if necessary, we may assume that $\epsilon_n \to 0$. Denote $u_n = y_n \chi_{A_n}$ and $v_n = y_n \chi_{A_n^c} = y_n - u_n$ for all $n \in \mathbb{N}$. Notice that

$$\|v_n\|_{\Gamma_{p,w}} = \left(\int_0^\infty (y_n \chi_{A_n^c})^{**p} w \right)^{1/p} \le \frac{1}{n} W(\infty)^{1/p} \to 0. \tag{9}$$

Since $u_n^* = u_n^* \chi_{[0,\epsilon_n]}$ for any $n \in \mathbb{N}$, by Theorem II.3.1 in [29, p. 82], we get

$$\begin{aligned}
(x + u_n)^{**}(t) &\ge \frac{1}{t} \int_0^t (x^* - u_n^*)^* \ge \frac{1}{t} \int_0^t |x^* - u_n^*| \\
&\ge \frac{1}{t} \int_0^t (u_n^* - x^*) \chi_{[0,\epsilon_n]} + \frac{1}{t} \int_0^t (x^* - u_n^*) \chi_{(\epsilon_n,\infty)} \\
&= u_n^{**}(t) + x^{**}(t) - 2\frac{1}{t} \int_0^t x^* \chi_{[0,\epsilon_n]}.
\end{aligned}$$

Consequently, by the triangle inequality $(x + u_n)^{**} \le x^{**} + u_n^{**}$, it follows that

$$0 \le x^{**}(t) + u_n^{**}(t) - (x + u_n)^{**}(t) \le \frac{2}{t} \int_0^t x^* \chi_{[0,\epsilon_n]} \tag{10}$$

for any $n \in \mathbb{N}$. Moreover,

$$\|y_n\|_{\Gamma_{p,w}} \le \|u_n\|_{\Gamma_{p,w}} + \|v_n\|_{\Gamma_{p,w}} \le \|y_n\|_{\Gamma_{p,w}} + \|v_n\|_{\Gamma_{p,w}} \tag{11}$$

for any $p \ge 1$ and any $n \in \mathbb{N}$. In case when $0 < p < 1$, the subadditivity of the power function u^p yields

$$\begin{aligned}
\|u_n\|_{\Gamma_{p,w}}^p \le \|y_n\|_{\Gamma_{p,w}}^p &\le \int_0^\infty (u_n^{**} + v_n^{**})^p w \\
&\le \int_0^\infty (u_n^{**p} + v_n^{**p}) w = \|u_n\|_{\Gamma_{p,w}}^p + \|v_n\|_{\Gamma_{p,w}}^p
\end{aligned} \tag{12}$$

for any $n \in \mathbb{N}$. Therefore, by (9), (10) and (11), we obtain

$$\begin{aligned} \|y_n\|_{\Gamma_{p,w}} &= \|u_n\|_{\Gamma_{p,w}} + o(1) \quad \text{for } p \geq 1, \\ \|y_n\|_{\Gamma_{p,w}}^p &= \|u_n\|_{\Gamma_{p,w}}^p + o(1) \quad \text{for } 0 < p < 1. \end{aligned} \tag{13}$$

Since $\epsilon_n \to 0$, it is obvious that

$$\frac{1}{t}\int_0^t x^* \chi_{[0,\epsilon_n]} \to 0$$

for any $t > 0$. Consequently, the Lebesgue dominated convergence theorem implies

$$\left\| x^* \chi_{[0,\epsilon_n]} \right\|_{\Gamma_{p,w}} \to 0. \tag{14}$$

We divide the proof in two subcases.

(Subcase 1). Let $p \geq 1$. Then, by condition (9), we have

$$\begin{aligned} \|x + y_n\|_{\Gamma_{p,w}} &\leq \|x + u_n\|_{\Gamma_{p,w}} + \|v_n\|_{\Gamma_{p,w}} \leq \|x + u_n\|_{\Gamma_{p,w}} + \frac{1}{n} W(\infty)^{1/p} \\ &\leq \|x + y_n\|_{\Gamma_{p,w}} + \frac{2}{n} W(\infty)^{1/p} \end{aligned} \tag{15}$$

for any $n \in \mathbb{N}$. Applying condition (10), by superadditivity of the power function u^p for $p \geq 1$, we obtain

$$\begin{aligned} \int_0^\infty \left((x + u_n)^{**}(t) + \frac{2}{t}\int_0^t x^* \chi_{[0,\epsilon_n]} \right)^p w(t)dt &\geq \int_0^\infty \left(x^{**}(t) + u_n^{**}(t) \right)^p w(t)dt \\ &\geq \int_0^\infty \left(x^{**} \right)^p (t) w(t)dt + \int_0^\infty \left(u_n^{**} \right)^p (t) w(t)dt \\ &= \|x\|_{\Gamma_{p,w}}^p + \|u_n\|_{\Gamma_{p,w}}^p \end{aligned}$$

for any $n \in \mathbb{N}$. Now, by Minkowski's inequality, we get

$$\begin{aligned} \|x\|_{\Gamma_{p,w}}^p + \|u_n\|_{\Gamma_{p,w}}^p &\leq \int_0^\infty \left((x + u_n)^{**}(t) + \frac{2}{t}\int_0^t x^* \chi_{[0,\epsilon_n]} \right)^p w(t)dt \\ &\leq \left(\|x + u_n\|_{\Gamma_{p,w}} + 2\left\| x^* \chi_{[0,\epsilon_n]} \right\|_{\Gamma_{p,w}} \right)^p \end{aligned}$$

for all $n \in \mathbb{N}$, whence, by (15),

$$\begin{aligned} \|u_n\|_{\Gamma_{p,w}}^p &\leq \left(\|x + u_n\|_{\Gamma_{p,w}} + 2\left\| x^* \chi_{[0,\epsilon_n]} \right\|_{\Gamma_{p,w}} \right)^p - \|x\|_{\Gamma_{p,w}}^p \\ &\leq \left(\|x + y_n\|_{\Gamma_{p,w}} + 2\left\| x^* \chi_{[0,\epsilon_n]} \right\|_{\Gamma_{p,w}} + \frac{1}{n} W(\infty)^{1/p} \right)^p - \|x\|_{\Gamma_{p,w}}^p \\ &= \left(\|x_n\|_{\Gamma_{p,w}} + 2\left\| x^* \chi_{[0,\epsilon_n]} \right\|_{\Gamma_{p,w}} + \frac{1}{n} W(\infty)^{1/p} \right)^p - \|x\|_{\Gamma_{p,w}}^p \end{aligned}$$

for each $n \in \mathbb{N}$. Consequently, by conditions (13), (14) and assumptions that $\|x_n\|_{\Gamma_{p,w}} \to \|x\|_{\Gamma_{p,w}}$ and $W(\infty) < \infty$, it follows that

$$\|y_n\|_{\Gamma_{p,w}} = \|x_n - x\|_{\Gamma_{p,w}} \to 0,$$

which finishes the proof of the case when $p \geq 1$.

(Subcase 2). Let $0 < p < 1$. Since $v_n^* \leq \frac{1}{n}$, by Theorem II.3.1 in [29, p. 82], we get

$$\begin{aligned}(x+y_n)^{**}(t) &\geq \frac{1}{t}\int_0^t \left((x+u_n)^* - v_n^*\right)^* \geq \frac{1}{t}\int_0^t (x+u_n)^* - \frac{1}{t}\int_0^t v_n^* \\ &\geq (x+u_n)^{**}(t) - \frac{1}{n}\end{aligned}$$

for all $n \in \mathbb{N}$ and $t > 0$. According to (10), we have

$$\begin{aligned}\int_0^\infty \left(x^{**}(t) + u_n^{**}(t)\right)^p w(t)dt &\leq \int_0^\infty \left((x+u_n)^{**}(t) + \frac{2}{t}\int_0^t x^*\chi_{[0,\epsilon_n]}\right)^p w(t)dt \\ &\leq \|x+u_n\|_{\Gamma_{p,w}}^p + \left\|2x^*\chi_{[0,\epsilon_n]}\right\|_{\Gamma_{p,w}}^p\end{aligned}$$

for all $n \in \mathbb{N}$. Hence, by subadditivity of maximal and power functions, we get

$$\begin{aligned}\|x_n\|_{\Gamma_{p,w}}^p + \frac{W(\infty)}{n^p} + \left\|2x^*\chi_{[0,\epsilon_n]}\right\|_{\Gamma_{p,w}}^p &= \|x+y_n\|_{\Gamma_{p,w}}^p + \frac{W(\infty)}{n^p} + \left\|2x^*\chi_{[0,\epsilon_n]}\right\|_{\Gamma_{p,w}}^p \\ &\geq \int_0^\infty \left((x+y_n)^{**} + \frac{1}{n}\right)^p w + \left\|2x^*\chi_{[0,\epsilon_n]}\right\|_{\Gamma_{p,w}}^p \\ &\geq \|x+u_n\|_{\Gamma_{p,w}}^p + \left\|2x^*\chi_{[0,\epsilon_n]}\right\|_{\Gamma_{p,w}}^p \\ &\geq \int_0^\infty \left(x^{**} + u_n^{**}\right)^p w \geq \|x\|_{\Gamma_{p,w}}^p\end{aligned}$$

for any $n \in \mathbb{N}$. Therefore, by condition (14) and assumptions $\|x_n\|_{\Gamma_{p,w}} \to \|x\|_{\Gamma_{p,w}}$ and $W(\infty) < \infty$, we conclude

$$\int_0^\infty \left(x^{**} + u_n^{**}\right)^p w \to \int_0^\infty \left(x^{**}\right)^p w. \tag{16}$$

Clearly, $x^{**} + u_n^{**} \geq x^{**}$ for each $n \in \mathbb{N}$. Let $\lambda > 0$. Define

$$B_n = \left\{s : \left(x^{**}(s) + u_n^{**}(s)\right)^p w(s) - \left(x^{**}\right)^p(s)w(s) > \lambda\right\}$$

for any $n \in \mathbb{N}$. Observe that for every $n \in \mathbb{N}$,

$$\int_0^\infty \left(\left(x^{**} + u_n^{**}\right)^p w - \left(x^{**}\right)^p w\right) \geq \int_{B_n} \left(\left(x^{**} + u_n^{**}\right)^p w - \left(x^{**}\right)^p w\right) \geq \lambda m(B_n).$$

Consequently, by condition (16), we get $m(B_n) \to 0$. Hence $(x^{**} + u_n^{**})^p w - (x^{**})^p w$ converges to zero globally in measure. By Lemma 3.9 in [22], it follows that

$$u_n^{**} w^{1/p} = \left(x^{**} + u_n^{**}\right) w^{1/p} - x^{**} w^{1/p} \to 0$$

globally in measure. Passing to a subsequence if necessary, we conclude that $u_n^{**}w^{1/p} \to 0$. Since $(u_n^{**})^p w \leq (u_n^{**}+x^{**})^p w$, applying condition (16) and generalized Lebesgue dominated convergence theorem (see [33]), we get

$$\int_0^\infty (u_n^{**})^p w \to 0.$$

Finally, according to condition (13), we obtain $\|y_n\|_{\Gamma_{p,w}} = \|x - x_n\|_{\Gamma_{p,w}} \to 0$, which completes the proof of Case I.

Case II. Let $\alpha = \infty$ and $W(\infty) = \infty$.

Denote $y_n = x_n - x$ for every $n \in \mathbb{N}$. Then (y_n) converges to zero globally in measure. Let $0 < \delta < \beta < \infty$. Applying Theorem II.3.1 in [29], we get

$$\begin{aligned}
(x + y_n)^{**}(t) &\geq \frac{1}{t}\int_0^t (x^* - y_n^*)^* \geq \frac{1}{t}\int_0^t |x^* - y_n^*| \\
&\geq \frac{1}{t}\int_0^\delta (y_n^* - x^*)\chi_{(0,t)} + \frac{1}{t}\int_\delta^\beta (x^* - y_n^*)\chi_{(0,t)} + \frac{1}{t}\int_\beta^\alpha (y_n^* - x^*)\chi_{(0,t)} \\
&= x^{**}(t) + y_n^{**}(t) - \frac{2}{t}\int_0^t \left(x^*\chi_{[0,\delta)\cup(\beta,\alpha)} + y_n^*\chi_{[\delta,\beta]}\right)
\end{aligned}$$

for any $t \in (0,\infty)$. Therefore, by the triangle inequality for the maximal function, we have

$$0 \leq x^{**}(t) + y_n^{**}(t) - (x + y_n)^{**}(t) \leq \frac{2}{t}\int_0^t \left(x^*\chi_{[0,\delta)\cup(\beta,\alpha)} + y_n^*\chi_{[\delta,\beta]}\right) \tag{17}$$

for any $t \in (0,\infty)$. Clearly, for any $0 < p < \infty$ there exists $M > 0$ such that

$$\int_0^\alpha \left(\frac{1}{t}\int_0^t x^*\chi_{[0,\delta)\cup(\beta,\alpha)} + y_n^*\chi_{[\delta,\beta]}\right)^p w(t)dt \leq M\left(\left\|x^*\chi_{[0,\delta)\cup(\beta,\alpha)}\right\|_{\Gamma_{p,w}}^p + \left\|y_n^*\chi_{[\delta,\beta]}\right\|_{\Gamma_{p,w}}^p\right). \tag{18}$$

By the assumption $W(\infty) = \infty$, $\Gamma_{p,w}$ is order continuous. Thus, for any $\epsilon > 0$ there exist $\delta_\epsilon, \beta_\epsilon \in (0,\infty)$ such that $\delta_\epsilon < \beta_\epsilon$ and

$$\left\|x^*\chi_{[0,\delta_\epsilon)\cup(\beta_\epsilon,\alpha)}\right\|_{\Gamma_{p,w}}^p \leq \frac{\epsilon}{2M}.$$

Since $y_n \to 0$ globally in measure, it follows that (y_n^*) converges to zero at each $t \in (0,\infty)$ (see [29]). In consequence, there exists $\eta > 0$ such that

$$y_n^*\chi_{[\delta_\epsilon,\beta_\epsilon]} \leq y_n^*(\delta_\epsilon)\chi_{[\delta_\epsilon,\beta_\epsilon]} \leq \eta\chi_{[\delta_\epsilon,\beta_\epsilon]}$$

for all $n \in \mathbb{N}$. By order continuity of $\Gamma_{p,w}$, there is $N_\epsilon \in \mathbb{N}$ such that

$$\left\|y_n^*\chi_{[\delta_\epsilon,\beta_\epsilon]}\right\|^p < \frac{\epsilon}{2M}$$

for any $n \geq N_\epsilon$. Hence, by condition (18), for any $\epsilon > 0$ there exist $0 < \delta_\epsilon < \beta_\epsilon < \infty$ and $N_\epsilon \in \mathbb{N}$ such that

$$\int_0^\alpha \left(\frac{1}{t} \int_0^t x^* \chi_{[0,\delta_\epsilon)\cup(\beta_\epsilon,\alpha)} + y_n^* \chi_{[\delta_\epsilon,\beta_\epsilon]} \right)^p w(t)dt < \epsilon \tag{19}$$

for any $n \geq N_\epsilon$. Consider two subcases.

(Subcase 1). Let $p \geq 1$. By condition (17) and the superadditivity of power function u^p, we have

$$\begin{aligned} &\int_0^\alpha \left((x+y_n)^{**}(t) + \frac{2}{t} \int_0^t x^* \chi_{[0,\delta_\epsilon)\cup(\beta_\epsilon,\alpha)} + \frac{2}{t} \int_0^t y_n^* \chi_{[\delta_\epsilon,\beta_\epsilon]} \right)^p w(t)dt \\ &\geq \int_0^\alpha \left(x^{**}(t) + y_n^{**}(t) \right)^p w(t)dt \geq \|x\|_{\Gamma_{p,w}}^p + \|y_n\|_{\Gamma_{p,w}}^p \end{aligned} \tag{20}$$

for all $n \in \mathbb{N}$. It is easy to observe that

$$\begin{aligned} \|x+y_n\|_{\Gamma_{p,w}} &\leq \left(\int_0^\alpha \left((x+y_n)^{**}(t) + \frac{1}{t} \int_0^t x^* \chi_{[0,\delta_\epsilon)\cup(\beta_\epsilon,\alpha)} + y_n^* \chi_{[\delta_\epsilon,\beta_\epsilon]} \right)^p w(t)dt \right)^{1/p} \\ &\leq \|x+y_n\|_{\Gamma_{p,w}} + \left(\int_0^\alpha \left(\frac{1}{t} \int_0^t x^* \chi_{[0,\delta_\epsilon)\cup(\beta_\epsilon,\alpha)} + y_n^* \chi_{[\delta_\epsilon,\beta_\epsilon]} \right)^p w(t)dt \right)^{1/p} \end{aligned}$$

for any $n \in \mathbb{N}$. Since $x_n = x + y_n$, by condition (19), we conclude

$$0 \leq \int_0^\alpha \left((x+y_n)^{**}(t) + \frac{1}{t} \int_0^t x^* \chi_{[0,\delta_\epsilon)\cup(\beta_\epsilon,\alpha)} + y_n^* \chi_{[\delta_\epsilon,\beta_\epsilon]} \right)^p w(t)dt - \|x_n\|_{\Gamma_{p,w}}^p < \epsilon$$

for any $n \geq N_\epsilon$. Finally, in view of $\|x_n\|_{\Gamma_{p,w}} \to \|x\|_{\Gamma_{p,w}}$ and condition (20), it follows that

$$\|x_n - x\|_{\Gamma_{p,w}} = \|y_n\|_{\Gamma_{p,w}} < \epsilon$$

for any $n \geq N_\epsilon$, which finishes the proof of Subcase 1.

(Subcase 2). Let $0 < p < 1$. By the subadditivity of map u^p, we get

$$0 \leq \left(x^{**} + y_n^{**} \right)^p - (x+y_n)^{**p} \leq \left(x^{**} + y_n^{**} - (x+y_n)^{**} \right)^p$$

for any $n \in \mathbb{N}$. Consequently, by (17), we obtain

$$\begin{aligned} 0 &\leq \int_0^\alpha \left(\left(x^{**}(t) + y_n^{**}(t) \right)^p - (x+y_n)^{**p}(t) \right) w(t)dt \\ &\leq \int_0^\alpha \left(\frac{2}{t} \int_0^t x^* \chi_{[0,\delta_\epsilon)\cup(\beta_\epsilon,\alpha)} + y_n^* \chi_{[\delta_\epsilon,\beta_\epsilon]} \right)^p w(t)dt \end{aligned}$$

for any $n \in \mathbb{N}$. Now, according to the assumptions $\|x_n\|_{\Gamma_{p,w}}^p \to \|x\|_{\Gamma_{p,w}}^p$, $x_n = x + y_n$ and condition (19), we have

$$0 \leq \int_0^\alpha \left(x^{**}(t) + y_n^{**}(t) \right)^p w(t)dt - \|x\|_{\Gamma_{p,w}}^p < \epsilon \tag{21}$$

for all $n \geq N_\epsilon$. Finally, by reverse Minkowski inequality for $0 < p < 1$, we get

$$\left(\int_0^\alpha \left(x^{**}(t) + y_n^{**}(t)\right)^p w(t)dt\right)^{1/p} \geq \|x\|_{\Gamma_{p,w}} + \|y_n\|_{\Gamma_{p,w}}.$$

Hence, by condition (21), the proof is completed. □

Recall that each point $x \in \Lambda_{p,w}[0,\infty)$ with $\alpha = 1$ or $\alpha = \infty$ is an H_g point (Corollary 3.21 in [22]).

By Theorem 3.2, Proposition 3.1 from [7], Corollary 3.21 from [22], Theorem 3.10 and Theorem 4.1, we get immediately

Corollary 4.2. *Let $p \geq 1$, $E = \Gamma_{p,w}[0,\infty)$ or $E = \Lambda_{p,w}[0,\infty)$ and $x \in E$. Then x is an H_l point if and only if*

(i) *$x^*(\infty) = 0$ whenever $\int_0^\infty w < \infty$;*
(ii) *$m\{s \in [0,\infty) : x(s) < x^*(\gamma)\} = 0$;*
(iii) *$x^*(\gamma) = x^*(\gamma^-)$, where $\gamma = \inf\{s \in [0,\infty) : m(\mathrm{supp}(w) \cap (s,\infty)) = 0\}$ under the convention $\inf \varnothing = \infty$ and $x^*(\gamma^-) = \lim_{t\to\gamma^-} x^*(t)$.*

Applying Corollary 3.15, Theorem 4.1 from present paper, Corollary 4.6 from [7], Corollary 4.4 from [12] and Lemma 3.2 from [21], we conclude the following global characterization.

Corollary 4.3. *Let $p \geq 1$, $E = \Gamma_{p,w}[0,\infty)$ or $E = \Lambda_{p,w}[0,\infty)$. Then the following conditions are equivalent:*

(i) *E has Kadec–Klee property for local convergence in measure;*
(ii) *E is upper locally uniformly monotone;*
(iii) *E is strictly monotone;*
(iv) *$\int_0^\infty w(t)dt = \infty$.*

The above result for the space $\Lambda_{1,w}[0,\infty)$ has been also showed in [11, Corollary 1].

Remark 4.4. Now we show that Corollary 4.3 does not hold in the case when $\alpha = 1$ (for the space $\Lambda_{1,w}[0,\infty)$ see [11]). More precisely, we claim that the Kadec–Klee property for local convergence in measure does not imply the strict monotonicity of Lorentz spaces $\Gamma_{p,w}[0,1)$ and $\Lambda_{p,w}[0,1)$. Assume $\gamma < 1$ and $m(\mathrm{supp}(w) \cap (\gamma,1)) = 0$. Then, by Theorem 2.2 in [6] and Corollary 4.5, we conclude that $\Gamma_{p,w}[0,1)$ is not strictly monotone, but it has the Kadec–Klee property for local convergence in measure. Similarly, $\Lambda_{1,w}[0,1)$ with nonincreasing weight function w vanishing on $(\gamma,1)$ has the Kadec–Klee property for local convergence in measure (see Corollary 1.3 in [5]), although it is not strictly monotone (see Theorem 3.1 in [21]).

The coincidence of local and global convergences in measure on $[0,1)$ leads immediately to the following result.

Corollary 4.5. *Let $p \geq 1$. Then the Lorentz space $\Gamma_{p,w}[0,1)$ has Kadec–Klee property for local convergence in measure.*

Recall that a symmetric space E is said to be *strictly K-monotone* (*SKM* for short) if for any $x, y \in E$ such that $x \prec y$ and $x^* \neq y^*$ we have $\|x\|_E < \|y\|_E$. A symmetric space E is called *locally uniformly strictly K-monotone* if for any $x, x_n \in E$ such that $x \prec x_n$ and $\|x_n\|_E \to \|x\|_E$ we have $\|x_n^* - x^*\|_E \to 0$ (see [5]).

Notice that $W(u) = \int_0^u w$ is strictly increasing if and only if for any $(a,b) \in [0,\alpha)$ we have $m((a,b) \cap \operatorname{supp}(w)) > 0$ (see [8]). For the definition of Kadec–Klee property (with respect to $L^1 \cap L^\infty$) we refer to [5].

Theorem 4.6. *Let $1 \le p < \infty$ and $W(\infty) = \infty$, whenever $\alpha = \infty$. Then the following statements are equivalent:*

- (i) *$W(u)$ is strictly increasing;*
- (ii) *The norm $\|\cdot\|_{\Gamma_{p,w}}$ is strictly K-monotone;*
- (iii) *The norm $\|\cdot\|_{\Gamma_{p,w}}$ is locally uniformly strictly K-monotone;*
- (iv) *$\Gamma_{p,w}$ has the Kadec–Klee property with respect to $L^1 \cap L^\infty$;*
- (v) *$\Gamma_{p,w}$ has the Kadec–Klee property.*

Proof. Since $\Gamma_{p,w}$ is order continuous symmetric space, $L^1 \cap L^\infty \subset \Gamma'_{p,w} = \Gamma^*_{p,w}$ [1,29]. Consequently, the implication $(v) \Rightarrow (iv)$ is true. Further, by Theorem 2.7 in [5], it follows that $(iv) \Rightarrow (iii)$. Clearly, $(iii) \Rightarrow (ii)$. Immediately, by Theorem 2.10 in [8], we get $(i) \Leftrightarrow (ii)$.

Now, we show the implication $(ii) \Rightarrow (v)$. Assume that the norm on $\Gamma_{p,w}$ is strictly K-monotone. Let $x, x_n \in \Gamma_{p,w}$ for $n \in \mathbb{N}$, (x_n) be weakly convergent to x and $\|x_n\|_{\Gamma_{p,w}} \to \|x\|_{\Gamma_{p,w}}$. Since $\Gamma_{p,w}$ is order continuous, we have $\Gamma'_{p,w} = \Gamma^*_{p,w}$ [31]. Moreover, $L^1 \cap L^\infty \subset (\Gamma_{p,w})'$ (see [1,29]), by Lemma 2.6 in [5], we conclude that x_n^* converges to x^* globally in measure. Applying Theorem 4.1, we have

$$\left\|x_n^* - x^*\right\|_{\Gamma_{p,w}} \to 0. \tag{22}$$

By order continuity of $\Gamma_{p,w}$ on $[0,\alpha)$ and Theorem 1.5 from [5], there exists an equivalent symmetric norm $\|\cdot\|_0$ on $\Gamma_{p,w}$ such that $(\Gamma_{p,w}, \|\cdot\|_0)$ has the Kadec–Klee property for weak convergence with respect to $L^1 \cap L^\infty$. Hence, by condition (22), we get $\|x_n\|_0 \to \|x\|_0$ as $n \to \infty$. Furthermore, by the weak convergence of x_n to x with respect to $L^1 \cap L^\infty$ and by the Kadec–Klee property with respect to $L^1 \cap L^\infty$ of $(\Gamma_{p,w}, \|\cdot\|_0)$, we conclude $\|x_n - x\|_0 \to 0$. This implies that $\|x_n - x\|_{\Gamma_{p,w}} \to 0$, which completes the proof. $\square$

5. Application to local best dominated approximation problems

Suppose E is a Banach lattice (see [31]) and $K \subset E$ is a sublattice, that is K is closed with respect to finite suprema and infima (K does not need to be a linear subspace). The order interval $[u, v]$ is a typical example of a sublattice. The notation $f \leqslant K$ for $f \in E$ means that $f \leqslant g$ for any $g \in K$. Given the system $f \leqslant K$ set

$$P_K(f) = \left\{u \in K : \|u - f\|_E = \inf_{w \in K} \|w - f\|_E\right\}.$$

We say that the best dominated approximation problem is *solvable* (*unique*) whenever $P_K(f) \neq \emptyset$ ($P_K(f)$ is a singleton).

Analogously we may consider such problems for $f \geqslant K$. It is known that:

- (i) For all closed sublattices K and all $f \leqslant K$ ($f \geqslant K$) the best dominated approximation problem is solvable if and only if $E \in (OC)$ (see Proposition 3.3 in [30]).
- (ii) For all sublattices K and all $f \leqslant K$ ($f \geqslant K$) the set $P_K(f)$ is at most a singleton if and only if $E \in (SM)$ (see Proposition 3.1 in [30]).

For more facts concerning these problems we refer also to [7].

Recall that the reflexivity and rotundity play an analogous important role in the best approximation problems in Banach spaces. The local version of (i) [(ii)] from above has been proved in [7] showing the role of points of order continuity [points of upper monotonicity], respectively.

On the other hand, the order continuity and property H_g are not comparable in general. First, to show that the property OC does not imply property H_g, it is enough to consider Example 2.8 from [5] with the additional conditions $\phi_1(0+) = \phi_2(0+) = 0$ and $\phi_1(\infty) = \phi_2(\infty) = \infty$. Second, taking $E = \Gamma_{p,w}[0,\infty)$ with $\int_0^\infty w < \infty$ we get $E \in (H_g)$ and $E \notin (OC)$ (see Theorem 4.1 and [19]). However, it appears that property H_g is in some sense "a weaker version" of OC (see Lemma 3.2 in [22]). The following lemma also shows this phenomenon.

Lemma 5.1. *Let E be a Banach function space with the semi-Fatou property and let $K \geq 0$ be a closed sublattice of E with $\inf_{z\in K}\{z\} \in K$ and $x \in E$, $x \leq K$. If $v - x$ is H_g point of E for some $v \in K$, then $P_K(x) \neq \emptyset$.*

Proof. *Case 1.* First suppose that there is $y \in K$ with $y^*(\infty) = 0$. Let $(h_n) \subset K$ be a minimizing sequence, i.e.

$$d = \inf_{h\in K} \|x - h\|_E = \lim_{n\to\infty} \|x - h_n\|_E. \tag{23}$$

Without loss of generality we may assume that $h_n \leq v$, because otherwise it is enough to replace h_n by $h_n \wedge v$. Since K is a sublattice, we have $u_n = \bigwedge_{k=1}^n h_k \in K$. Moreover, for any $n \in \mathbb{N}$, $0 \leq u_n - x \leq h_n - x$, whence $d \leq \|u_n - x\|_E \leq \|h_n - x\|_E$ for each $n \in \mathbb{N}$. Therefore, in view of condition (23), (u_n) is a minimizing sequence. Setting $u = \bigwedge_{k=1}^\infty u_k$, we have $x \leq u \leq u_1$, whence $u \in E$ and $0 \leq u_n - u \downarrow 0$ pointwisely. In view of the fact that $y^*(\infty) = 0$, we are able to consider only the case $u_n^*(\infty) = 0$ for each $n \in \mathbb{N}$. Indeed, by the assumption that K is a sublattice of E it follows that $y \wedge u_n \in K$ and

$$d \leq \|y \wedge u_n - x\|_E \leq \|u_n - x\|_E$$

for any $n \in \mathbb{N}$. Thus $(y \wedge u_n)_{n\in\mathbb{N}}$ is a minimizing sequence and $(y \wedge u_n)^*(\infty) = 0$ for every $n \in \mathbb{N}$. We claim that $u_n \to u$ globally in measure. Let $\epsilon > 0$. Since $u^*(\infty) = 0$, there is $t_\epsilon > 0$ such that

$$(u_1 - u)^*(t_\epsilon) \leq \epsilon.$$

We have $0 \leq u_n - u \leq u_1 - u$, whence

$$m\big(\big\{t : \big|u_n(t) - u(t)\big| > \epsilon\big\}\big) = m\big(\big\{t : (u_n - u)^*(t) > \epsilon\big\}\big) = m\big(\big\{t \in [0, t_\epsilon) : (u_n - u)^*(t) > \epsilon\big\}\big).$$

Since $0 \leq u_n - u \downarrow 0$ pointwisely, by property 12^o in [29], we conclude that $(u_n - u)^* \to 0$ pointwisely. So, passing to a subsequence if necessary, we get

$$m\big(\big\{t \in [0, t_\epsilon) : (u_n - u)^*(t) > \epsilon\big\}\big) \to 0,$$

which proves the claim. Clearly,

$$v + (u_n - u) - x \to v - x \in E \tag{24}$$

globally in measure. Moreover, $0 \leq v - u_n + u - x \leq v - x$, because $v \geq u_n$. Consequently, by the semi-Fatou property of E, we get

$$\big\|v - (u_n - u) - x\big\|_E \uparrow \|v - x\|_E.$$

Hence, since $v - x$ is an H_g point, by condition (24), it follows that

$$\|u_n - u\|_E \to 0.$$

Finally, by the assumption that K is closed, we get $u \in K$ and

$$d \le \|u - x\|_E \le \|u_n - x\|_E \to d,$$

which implies $u \in P_K(x)$.

Case 2. Suppose $y^*(\infty) > 0$ for all $y \in K$. Set $w = \inf_{z \in K}\{z\} \in K$. Letting $K' = K - w^*(\infty)$ and $x' = x - w^*(\infty)$ and applying Case 1 we find $z' = z - w^*(\infty) \in K'$ for some $z \in K$ such that

$$\|x - z\|_E = \|x' - z'\|_E = dist(x', K') = dist(x, K). \qquad \square$$

Remark 5.2. The opposite implication in Lemma 5.1 is not true in general. Consider the Marcinkiewicz function spaces M_ϕ and $M_\phi^{(*)}$ given by

$$M_\phi^{(*)} = \Big\{x : \|x\|_{M_\phi} = \sup_{t>0}\{\phi(t)x^*(t)\} < \infty\Big\},$$
$$M_\phi = \Big\{x : \|x\|_{M_\phi^*} = \sup_{t>0}\{\phi(t)x^{**}(t)\} < \infty\Big\},$$

where $\phi(t) = \sqrt{t}$ (see [25] for more information). Note that $M_\phi = M_\phi^{(*)}$ in this case (see [20,25]). Define $x = \sum_{i=1}^\infty (\sqrt{i} - \sqrt{i-1})\chi_{[i-1,i)}$. Then $x = x^*$ and $x^*(\infty) = 0$. Notice that

$$\begin{aligned}\|x\|_{M_\phi^{(*)}} &= \sup_{t>0}\{x^*(t)\phi(t)\} = \sup_{t>0}\left\{\sqrt{t}\sum_{i=1}^\infty (\sqrt{i} - \sqrt{i-1})\chi_{[i-1,i)}(t)\right\} \\ &= \sup_{i\in\mathbb{N}}\{\sqrt{i}(\sqrt{i} - \sqrt{i-1})\} = \sup_{i\in\mathbb{N}}\left\{\frac{\sqrt{i}}{\sqrt{i} + \sqrt{i-1}}\right\} \\ &= \sup_{i\in\mathbb{N}}\left\{\frac{1}{1 + \sqrt{1 - \frac{1}{i}}}\right\} = 1,\end{aligned}$$

whence $x \in M_\phi$. Define a sequence (x_n) by $x_n = x\chi_{[0,n)} = \sum_{i=1}^n (\sqrt{i} - \sqrt{i-1})\chi_{[i-1,1)}$ for any $n \in \mathbb{N}$. Clearly, $x_n^* = x_n$ and x_n converges to x globally in measure. Indeed, since $\sqrt{n} - \sqrt{n-1} \to 0$, we get

$$0 \le x - x_n = \sum_{i=n+1}^\infty (\sqrt{i} - \sqrt{i-1})\chi_{[i-1,i)} \to 0 \quad \text{globally in measure.}$$

Moreover, by the Fatou property, $\|x_n\|_{M_\phi} \uparrow \|x\|_{M_\phi}$. On the other hand,

$$\begin{aligned}\|x - x_n\|_{M_\phi} \ge \|x - x_n\|_{M_\phi^{(*)}} &= \sup_{t>0}\left\{\sqrt{t}\sum_{i=n+1}^\infty (\sqrt{i} - \sqrt{i-1})\chi_{[i-1,i)}(t)\right\} \\ &= \sup_{i\in\mathbb{N},\, i>n}\{\sqrt{i}(\sqrt{i} - \sqrt{i-1})\} = \sup_{i\in\mathbb{N},\, i>n}\left\{\frac{1}{1 + \sqrt{1 - \frac{1}{i}}}\right\} \\ &= \frac{1}{1 + \sqrt{1 - \frac{1}{n+1}}} \ge \frac{1}{2},\end{aligned}$$

for any $n \in \mathbb{N}$. Thus x is not an H_g point. Let $K = \{nx\}_{n=2}^\infty$. Certainly, $P_K(x) \neq \emptyset$ and $v - x$ is not an H_g point of E for any $v \in K$.

Lemma 5.3. *Let E be a Banach function space and let K be a closed sublattice of E and $x \in E$, $x \le K$. If $v - x$ is H_l point of E for some $v \in K$, then $P_K(x) \neq \emptyset$.*

Proof. Take elements h_n, u_n and u as in the proof of Lemma 5.1. Since we need only to show that u_n converges to u locally in measure, we may apply some parts of the proof of the previous lemma. □

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