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On the first exterior p-harmonic Steklov eigenvalue

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Dedicated to Professor Giles Auchmuty on the occasion of his seventy's birthday

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ABSTRACT

In this short paper we study the Sobolev function property of the Rayleigh's quotient $\delta(q) = \inf_{u\in E^{1,p}(U)} \frac{\|u\|^p_\nabla}{\|u\|^p_{q,\partial U}}$ as a function of $q \in [1, p_*]$ for $p_* = \frac{p(N-1)}{N-p}$ when $p \in (1, N)$, as well as the asymptotic behavior of positive solutions with minimal energy of the following problem

$$-\Delta_p u = 0 \text{ in } U, \quad \text{subject to } |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda\,|u|^{q-2}\,u \text{ on } \partial U$$

to the first p-harmonic Steklov eigenpair on an exterior region $U \subsetneq \mathbb{R}^N$ when $N \geq 3$.
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1. Introduction

The study of harmonic Steklov eigenvalue problems on bounded regions has a long history, yet even nowadays it is still an active research field for both theoretical and applied reasons (see, for example, Kuznetsov et al. [17,18]). Only quite recently, this problem on exterior regions, say, exterior to the unit ball in dimension 3, was treated by Auchmuty and Han [5,6,14]. Notice that, as shown in Section 8 of [5], this problem is closely related to the classical exterior Laplace's spherical harmonics. See also Chapter 5 in Axler et al. [7]. A more physical description of this subject may be found in Section 12.6 of Arfken and Weber [4].

In this paper, we consider the asymptotic behavior of positive solutions of

$$\begin{cases} -\Delta_p u := -\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right) = 0 \quad \text{in} \quad U, \\ \text{subject to } |\nabla u|^{p-2}\,\frac{\partial u}{\partial \nu} = \lambda\,|u|^{q-2}\,u \quad \text{on} \quad \partial U \end{cases} \tag{1.1}$$

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of minimal energy, when $q \to p$, to the first p-harmonic Steklov eigenpair $(\delta_1, \mathfrak{s}_1)$

$$-\Delta_p \mathfrak{s}_1 = 0 \text{ in } U \quad \text{and} \quad |\nabla \mathfrak{s}_1|^{p-2} \frac{\partial \mathfrak{s}_1}{\partial \nu} = \delta_1 \mathfrak{s}_1^{p-1} \text{ on } \partial U. \tag{1.2}$$

Besides, we also study the absolute continuity of the best Sobolev trace constant.

Here, U is a nonempty, open, connected subset of $\mathbb{R}^N$ with $N \geq 3$ whose complement $\mathbb{R}^N \setminus U$ is nonempty and compact. Without loss of generality, assume $0 \notin U$ and denote the boundary of U by ∂U. Also, we take $p \in (1, N)$, $q \in [1, p_*]$ with $p_* := \frac{p(N-1)}{N-p}$ and $\lambda \in (0, \infty)$ a positive constant, and consider problems (1.1) and (1.2) in the weak sense.

Our general assumption on U is the following condition.

Condition B. *$U \subsetneq \mathbb{R}^N$ is an exterior region with $0 \notin U$ when $N \geq 3$, and ∂U is the union of finitely many disjoint, closed, Lipschitz surfaces with the total surface area 1.*

From now on, we use σ and $d\sigma$ to represent Hausdorff $(N-1)$-dimensional measure and integration with respect to this measure. Given $1 \leq p, q \leq \infty$, $L^p(U)$ and $L^q(\partial U, d\sigma)$ denote the usual spaces of extended, real-valued, Lebesgue measurable functions on U and ∂U, with their standard norms written as $\|u\|_{p,U}$ and $\|u\|_{q,\partial U}$, respectively.

One notices here, under condition **(B)**, $\delta_1 > 0$ is simple and isolated, and $\mathfrak{s}_1 > 0$ is bounded on $\overline{U}$ such that $\mathfrak{s}_1 \to 0$ as $|x| \to \infty$ (see [14, Section 3 and Appendix B]). In particular, when $U = \mathbb{R}^N \setminus \overline{\mathbb{B}_1}$ with $\mathbb{B}_1$ the unit, open ball in $\mathbb{R}^N$, centered at the origin, then $\delta_1 = \left(\frac{N-p}{p-1}\right)^{p-1}$ and $\mathfrak{s}_1 = \frac{1}{|x|^{\frac{N-p}{p-1}}}$ so that one has $\delta_1 = 1$ and $\mathfrak{s}_1 = \frac{1}{|x|}$ for $N = 3$ and $p = 2$.

We consider weak solutions to problems (1.1)–(1.2) in the space $E^{1,p}(U)$ of functions where $u \in L^{p^*}(U)$ and $|\nabla u| \in L^p(U)$ for $p^* := \frac{pN}{N-p}$ and the weak gradient $\nabla u = (D_1 u, D_2 u, \ldots, D_N u)$ of u. Recall (see [6, Section 3] and [14, Sections 2–3]) when $p \in (1, N)$ and under condition **(B)**, $E^{1,p}(U)$ is a real Banach space with respect to the gradient L^p-norm

$$\|u\|_{\nabla} := \left(\int_U |\nabla u|^p \, dx \right)^{\frac{1}{p}}. \tag{1.3}$$

Moreover, the family $C^1(\overline{U}) = \left\{\psi : \psi = \varphi|_{\overline{U}} \text{ for some } \varphi \in C_c^1(\mathbb{R}^N)\right\}$ of functions is dense in $E^{1,p}(U)$ with respect to the norm (1.3), and $E^{1,p}(U)$ is continuously embedded into the space $L^q(\partial U, d\sigma)$. That is, for a constant $C_1 > 0$ depending on p, q, N, U, one has

$$\|u\|_{q,\partial U} \leq C_1 \|u\|_{\nabla} \qquad \text{for all } u \in E^{1,p}(U). \tag{1.4}$$

Here, $1 \leq q \leq p_*$ and when $q < p_*$, then this embedding is compact as well.

Now, in view of (1.4), we can define, for every $q \in [1, p_*]$, $\delta(q) := \inf_{u \in E^{1,p}(U)} \frac{\|u\|_{\nabla}^p}{\|u\|_{q,\partial U}^p} \in (0, \infty)$ via standard Rayleigh's quotient (write $\frac{\|u\|_{\nabla}^p}{\|u\|_{q,\partial U}^p} := +\infty$ if $u \equiv 0$ on ∂U). When $q = p$, then $\delta_1 = \delta(p)$. On the other hand, define the energy functional $\mathcal{J} : E^{1,p}(U) \to \mathbb{R}$ by

$$\mathcal{J}(u) := \frac{1}{p} \int_U |\nabla u|^p \, dx - \frac{\lambda}{q} \int_{\partial U} (u^+)^q \, d\sigma. \tag{1.5}$$

One easily observes that positive weak solutions to (1.1) are critical points of $\mathcal{J}$. Denote by $u_{\lambda,q} \in E^{1,p}(U)$ positive solutions to (1.1) of minimal energy. Then, our main results are as follows.[1]

Theorem 1.1. *Assume that $p \in (1, N)$ and condition* **(B)** *holds. Then, $\delta(q) : [1, p_*] \to (0, \infty)$ is an absolutely continuous function and thus one has $\delta(q) \in W^{1,1}([1, p_*])$.*

Theorem 1.2. *When $p \in (1, N)$, condition* **(B)** *holds and $u_{\lambda,q} (> 0) \in E^{1,p}(U)$ are weak solutions to (1.1) of minimal energy, then, for some constants $\mathfrak{c}_1 \geq \mathfrak{c}_2 > 0$, we have*

$$\begin{cases} \lim\limits_{q\to p^-} \left(\frac{\lambda}{\delta_1}\right)^{\frac{p}{q-p}} \|u_{\lambda,q}\|_{\nabla}^{p} = \mathfrak{c}_1, \\ \lim\limits_{q\to p^+} \left(\frac{\lambda}{\delta_1}\right)^{\frac{p}{q-p}} \|u_{\lambda,q}\|_{\nabla}^{p} = \mathfrak{c}_2. \end{cases} \tag{1.6}$$

One may be reminded here we only assume Lipschitz regularity on ∂U.

We remark problems like this were initially considered on bounded regions by Huang [16], and his results were substantially extended recently by Anello et al. [1–3] and Ercole [10–12]. Theorems 1.1 and 1.2 here correspond to Theorems 2.1 and 2.2 of [16]. Finally, it is worth to mention that our assumption $\sigma(\partial U) = 1$ is not essential, but it simplifies the proofs later and can be attained through the probability measure $\frac{d\sigma}{\sigma(\partial U)}$.

2. Proof of the main results

In this section, we will carry out the detailed proofs of our main results which in turn depend on several lemmas. We start with the following elementary observation.

Lemma 2.1. *When $u \in E^{1,p}(U) \cap L^\infty(\partial U, d\sigma)$ satisfies $u \not\equiv 0$ on ∂U, then*

$$\frac{\|u\|_{s_2,\partial U}}{\|u\|_{s_1,\partial U}} = \exp\left(\int_{s_1}^{s_2} \frac{K(t,u)}{t^2}\, dt\right) \geq 1. \tag{2.1}$$

Here, $s_1, s_2 \in [1, p_*]$ with $s_1 \leq s_2$ and $K(t,u) := \|u\|_{t,\partial U}^{-t} \int_{\partial U} |u|^t \ln |u|^t \, d\sigma - \ln \|u\|_{t,\partial U}^t \geq 0$.

The boundedness hypothesis is used to guarantee that $\int_{\partial U} |u|^t \ln |u|^t \, d\sigma \leq \|u\|_{t,\partial U}^t \ln \|u\|_{\infty,\partial U}^t$ and thus the finiteness of $K(t,u)$, which will be made clearer from the other results below.

Proof. First, note $x \ln x$ is convex when $x \geq 0$ (if we define $0 \ln 0 := 0$). Therefore, one has, using Jensen's inequality, $\|u\|_{t,\partial U}^t \ln \|u\|_{t,\partial U}^t \leq \int_{\partial U} |u|^t \ln |u|^t \, d\sigma$ so that $K(t,u) \geq 0$.

Next, for $\ln \|u\|_{t,\partial U}$ (viewed as a function of t), take derivative to see

$$\frac{d}{dt}\left(\ln \|u\|_{t,\partial U}\right) = \frac{d}{dt}\left[\frac{1}{t} \ln\left(\int_{\partial U} |u|^t \, d\sigma\right)\right]$$

$$= -\frac{1}{t^2} \ln \|u\|_{t,\partial U}^t + \frac{1}{t} \frac{\int_{\partial U} \left(|u|^t \ln |u|\right) d\sigma}{\|u\|_{t,\partial U}^t} = \frac{K(t,u)}{t^2}.$$

Taking integral of the above equality from s_1 to s_2 readily yields (2.1). □

[1] See also Remarks 2.5 and 2.6 at the end of this paper.

The next results[2] are of independent interest and extend [15, Theorem 0.1], which are proved by similar strategies of Cuesta and Leadi [9, Appendix A], and Ercole [10, Lemma 5].

Proposition 2.2. *Suppose that $p \in (1, N)$ and condition* **(B)** *holds. Then, every weak solution $u \in E^{1,p}(U)$ of (1.1) associated with $q \in [1, p_*)$ and $\lambda > 0$ satisfies $\|u\|_{\infty,U} + \|u\|_{\infty,\partial U} \le C_2$. Here, $C_2 > 0$ is a constant that depends on p, q, N, U, $\lambda > 0$ and $\|u\|_{p_*,\partial U}$.*

Proof. As common practice, write $u^+ = \max\{u, 0\} \ge 0$ and $u^- = \max\{-u, 0\} \ge 0$. Below, we only consider u^+ and define as usual the truncation $u_M := u^+ \chi_{\{u^+ \le M\}}$ of u^+.

Define $\phi := u_M^{kp+1}$ for some (temporarily) fixed $k, M \ge 0$. Notice $0 \le \phi \le M^{kp}(u^+)$ so that $\phi \in E^{1,p}(U)$. Using the week form of (1.1), described as below

$$\int_U |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx \;-\; \lambda \int_{\partial U} |u|^{q-2} u \, v \, d\sigma \;=\; 0 \qquad \text{for all} \;\; v \in E^{1,p}(U), \tag{2.2}$$

and substituting $v = \phi$ into (2.2), together with (1.4), it follows that

$$\begin{aligned} \left\{ \int_{\partial U} \left(u_M^{k+1}\right)^{p_*} d\sigma \right\}^{\frac{p}{p_*}} &\le \frac{1}{\delta(p_*)} \int_U \left|\nabla \left(u_M^{k+1}\right)\right|^p dx \\ = \frac{1}{\delta(p_*)} \frac{(k+1)^p}{kp+1} \int_U |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx &= \frac{\lambda}{\delta(p_*)} \frac{(k+1)^p}{kp+1} \int_{\partial U} u_M^{kp+q} \, d\sigma. \end{aligned}$$

Letting $M \to \infty$ leads to

$$\left\{ \int_{\partial U} \left(u^+\right)^{p_*(k+1)} d\sigma \right\}^{\frac{1}{p_*(k+1)}} \le \left\{ \frac{\lambda}{\delta(p_*)} \frac{(k+1)^p}{kp+1} \int_{\partial U} \left(u^+\right)^{kp+q} d\sigma \right\}^{\frac{1}{p(k+1)}},$$

that is, $\|u^+\|_{p_*(k+1),\partial U}$ can actually be controlled by $\left(\|u^+\|_{kp+q,\partial U}\right)^{\frac{kp+q}{kp+p}}$, provided one has the *a priori* fact $u^+ \in L^{kp+q}(\partial U, d\sigma)$ which is true when $k = k_0 = 0$. Thus, we write $k_l := \frac{p_*(k_{l-1}+1)-q}{p} = \left[\left(\frac{p_*}{p}\right)^l - 1\right] \frac{p_*-q}{p_*-p}$ inductively and conclude that $u^+ \in L^{p_*(k_l+1)}(\partial U, d\sigma)$ for all integers $l \ge 1$. Since $k_l \to \infty$ when $l \to \infty$, we certainly have $u^+ \in L^r(\partial U, d\sigma)$ for each $r \in [1, \infty)$.

Next, we introduce a sequence $\left\{q_0 := p_*,\, q_{n+1} := p_* \left(\frac{q_n}{sp} + \frac{p-1}{p}\right)\right\}_{n=0}^{\infty}$ of real numbers, with a fixed constant s satisfying $1 < s < \frac{p_*}{p}$. One notices that, for $p' := \frac{p}{p-1}$,

$$\begin{aligned} q_n &= \frac{p_*^{n+1}}{(sp)^n} + \left(\frac{p_*}{sp}\right)^{n-1} \frac{1}{p'} + \frac{p_*^{n-1}}{(sp)^{n-2}} \frac{1}{p'} + \cdots + \frac{p_*^2}{sp} \frac{1}{p'} + \frac{p_*}{p'} \\ &= \frac{p_*^{n+1}}{(sp)^n} + \left(\frac{p_*}{sp}\right)^{n-1} \frac{p-1}{p} + \left[\left(\frac{p_*}{sp}\right)^{n-1} - 1\right] \frac{sp_*(p-1)}{p_* - sp}. \end{aligned} \tag{2.3}$$

[2] When $q = 1$, we always consider positive solutions to (1.1) that is sufficient for our purpose.

Define $\tilde{\phi} := u_M^{\frac{q_n}{s}} \in E^{1,p}(U)$ and substitute $v = \tilde{\phi}$ into (2.2) to deduce

$$\left\{ \int_{\partial U} \left(u_M^{\frac{q_n}{sp} + \frac{p-1}{p}} \right)^{p_*} d\sigma \right\}^{\frac{p}{p_*}} \le \frac{1}{\delta(p_*)} \int_U \left| \nabla \left(u_M^{\frac{q_n}{sp} + \frac{p-1}{p}} \right) \right|^p dx$$

$$= \frac{1}{\delta(p_*)} \frac{(q_{n+1}/p_*)^p}{q_n/s} \int_U |\nabla u|^{p-2} \nabla u \cdot \nabla \tilde{\phi}\, dx = \frac{\lambda}{\delta(p_*)} \frac{(q_{n+1}/p_*)^p}{q_n/s} \int_{\partial U} u^{q-1} \tilde{\phi}\, d\sigma.$$

Hölder's inequality says $\int_{\partial U} u^{q-1} \tilde{\phi}\, d\sigma \le \|u_M\|_{s'(q-1),\partial U}^{q-1} \|u_M\|_{q_n,\partial U}^{\frac{q_n}{s}}$, so that one has

$$\|u^+\|_{q_{n+1},\partial U}^{q_{n+1}} \le \left\{ \frac{\lambda}{\delta(p_*)} \frac{s}{p_*^p} \|u^+\|_{s'(q-1),\partial U}^{q-1} \right\}^{\frac{p_*}{p}} \left\{ \frac{q_{n+1}^p}{q_n} \right\}^{\frac{p_*}{p}} \|u^+\|_{q_n,\partial U}^{\frac{p_* q_n}{sp}}$$

after letting $M \to \infty$ with $s' := \frac{s}{s-1}$. One recalls here $\|u^+\|_{s'(q-1),\partial U}$ can be controlled by $\|u\|_{p_*,\partial U}$ (as well as p, q, N, U and $\lambda > 0$) as the discussion of the first part indicated.

Now, for $\rho_n := \ln \|u^+\|_{q_n,\partial U}^{q_n}$ and $\varrho_n := \ln \left(\frac{\lambda}{\delta(p_*)} \frac{s}{p_*^p} \|u^+\|_{s'(q-1),\partial U}^{q-1} \frac{q_{n+1}^p}{q_n} \right)$,

$$\rho_{n+1} \le \frac{p_*}{p} \varrho_n + \frac{p_*}{sp} \rho_n \le \cdots \le \frac{p_*}{p} \sum_{j=0}^{n} \left(\frac{p_*}{sp} \right)^{n-j} \varrho_j + \left(\frac{p_*}{sp} \right)^{n+1} \rho_0$$

follows easily. By virtue of (2.3), we have $\frac{p_*}{p} \sum_{j=0}^{n} \left(\frac{p_*}{sp} \right)^{n-j} \varrho_j \le C_2' \left(\frac{p_*}{sp} \right)^{n+1}$ for a constant $C_2' > 0$ depending on p, q, N, U, $\lambda > 0$ and $\|u\|_{p_*,\partial U}$, since s is fixed. As a matter of fact,

$$\frac{p_*}{p} \ln \left(\frac{\lambda}{\delta(p_*)} \frac{s}{p_*^p} \|u^+\|_{s'(q-1),\partial U}^{q-1} \right) \sum_{j=0}^{n} \left(\frac{p_*}{sp} \right)^{j} \le C_3 \left(\frac{p_*}{sp} \right)^{n+1}$$

holds trivially, and by the identity $\sum_{j=0}^{n} j a^{n-j} = \frac{a^{n+1} - a(n+1) + n}{(a-1)^2}$ for $a \ne 1$, we get

$$\frac{p_*}{p} \sum_{j=0}^{n} \left\{ \left(\frac{p_*}{sp} \right)^{n-j} \ln \left(\frac{q_{j+1}^p}{q_j} \right) \right\} \le p_* \sum_{j=0}^{n} \left\{ \left(\frac{p_*}{sp} \right)^{n-j} \ln q_{j+1} \right\}$$

$$\le C_3' \ln \left(\frac{p_*}{sp} \right) \sum_{j=0}^{n} (j+1) \left(\frac{p_*}{sp} \right)^{n-j} \le C_3'' \left(\frac{p_*}{sp} \right)^{n+1},$$

where $C_3, C_3', C_3'' > 0$ are constants depending on the same parameters as C_2'.

Thus, one has $\rho_n \le (\rho_0 + C_2') \left(\frac{p_*}{sp} \right)^n$ and noticing $q_n \ge \frac{p_*^{n+1}}{(sp)^n}$, it leads to

$$\|u^+\|_{\infty,\partial U} = \lim_{n\to\infty} \|u^+\|_{q_n,\partial U} \le \sup_n \left\{ \exp \left(\frac{\rho_n}{q_n} \right) \right\} \le \exp \left(\frac{\rho_0 + C_2'}{p_*} \right) < \infty.$$

So, $u^+ \in L^\infty(\partial U, d\sigma)$, and similarly $u^- \in L^\infty(\partial U, d\sigma)$ and $u = u^+ - u^- \in L^\infty(\partial U, d\sigma)$ will follow. Via the maximum principle, we have $u \in L^\infty(U)$ and the desired estimate. $\square$

Proposition 2.3. *When $u \in E^{1,p}(U)$ is a weak solution of (1.1) associated with $q \in [1, p_*)$ and $\lambda > 0$, then, for every $\tau \geq 1$, one has the following estimate*

$$\frac{\tau}{2}\left(\frac{\delta(p_*)}{\lambda}\right)^{\frac{N-1}{p-1}}\left(\frac{1}{N}\right)^{\frac{pN-1}{p-1}} \boldsymbol{\beta}(N,\tau)\left\{\|u\|_{\infty,U}^{\frac{(p-q)(N-1)}{p-1}+\tau} + \|u\|_{\infty,\partial U}^{\frac{(p-q)(N-1)}{p-1}+\tau}\right\} \leq \|u\|_{\tau,\partial U}^{\tau}. \tag{2.4}$$

Here, $\boldsymbol{\beta}(N,\tau) = \int_0^1 (1-\theta)^{N-1}\theta^{\tau-1}\,d\theta > 0$ denotes the Euler beta function *at (N,τ).*

Proof. Given $u \in E^{1,p}(U)$ and $k \geq 0$, one recalls $(u-k)^+ := \max\{u-k, 0\} \in E^{1,p}(U)$ as observed in [14, Appendix B]. Now, set $\vartheta_k := (u-k)^+ \in E^{1,p}(U)$ for a weak solution u of (1.1), and use this ϑ_k as a test function in (2.2). Accordingly, we see

$$\int_{A_k} |\nabla u|^p\,dx = \lambda \int_{B_k} u^{q-1}\vartheta_k\,d\sigma. \tag{2.5}$$

Here, $A_k := \{x \in U : u(x) > k\}$ and $B_k := \overline{A_k} \cap \partial U = \overline{\operatorname{supp}(\vartheta_k)} \cap \partial U$.

Next, take some $0 < k < \|u\|_{\infty,\partial U}$, and use (1.4) and (2.5) to derive

$$\left\{\int_{B_k}(u-k)\,d\sigma\right\}^{\frac{p(N-1)}{pN-1}} \leq \left\{\frac{\lambda}{\delta(p_*)}\|u\|_{\infty,\partial U}^{q-1}\left(\|u\|_{\infty,\partial U} - k\right)\right\}^{\frac{N-1}{pN-1}} \sigma(B_k), \tag{2.6}$$

where we used $\int_{A_k}|\nabla u|^p\,dx = \int_U |\nabla\vartheta_k|^p\,dx$ and $\int_{B_k}(u-k)\,d\sigma = \int_{\partial U}\vartheta_k\,d\sigma$. Notice the derivation of (2.6) is standard and follows from the same argument as [10, (20)–(22)].

Set $f(k) := \int_{B_k}(u-k)\,d\sigma$. Using the standard result about the level sets of measurable function and its integral (see Lieb and Loss [19, Section 1.5]), we have

$$f(k) = \int_0^\infty \sigma(\{z \in B_k : u(z) - k > t\})\,dt = \int_k^\infty \sigma(\{z \in B_k : u(z) > t\})\,dt$$

so that $f'(k) = -\sigma(B_k)$, from which together with (2.6) we can deduce

$$\left\{\frac{\lambda}{\delta(p_*)}\|u\|_{\infty,\partial U}^{q-1}\left(\|u\|_{\infty,\partial U} - k\right)\right\}^{-\frac{N-1}{pN-1}} \leq -\{f(k)\}^{-\frac{p(N-1)}{pN-1}} f'(k).$$

Take integral of the above inequality with respect to k, from k to $\|u\|_{\infty,\partial U}$, to see

$$\left\{\frac{\lambda}{\delta(p_*)}\|u\|_{\infty,\partial U}^{q-1}\right\}^{-\frac{N-1}{p-1}}\left(\frac{1}{N}\right)^{\frac{pN-1}{p-1}}\left(\|u\|_{\infty,\partial U} - k\right)^N \leq f(k).$$

Now, define $\widetilde{B}_k := \{z \in \partial U : u(z) > k\}$. Then, the density of $C^1(\overline{U})$ in $E^{1,p}(U)$ and Lemma 2 of Faraci, Iannizzotto and Varga [13] imply $B_k \subseteq \widetilde{B}_k$,[3] so that we have

$$\left\{\frac{\lambda}{\delta(p_*)}\|u\|_{\infty,\partial U}^{q-1}\right\}^{-\frac{N-1}{p-1}}\left(\frac{1}{N}\right)^{\frac{pN-1}{p-1}}\left(\|u\|_{\infty,\partial U} - k\right)^{N-1} \leq \sigma(\tilde{B}_k)$$

[3] Should $B_k = \widetilde{B}_k$ for all $k \geq 0$, the proof of [14, Appendix B] could be simplified substantially.

for $0 \le f(k) \le \left(\|u\|_{\infty,\partial U} - k\right)\sigma(B_k)$. Multiply both sides of the above estimate by $\tau k^{\tau-1}$, with any $\tau \ge 1$, and take integral with respect to k from 0 to $\alpha := \|u\|_{\infty,\partial U}$ to derive

$$\left\{\frac{\lambda}{\delta(p_*)}\|u\|^{q-1}_{\infty,\partial U}\right\}^{-\frac{N-1}{p-1}}\left(\frac{1}{N}\right)^{\frac{pN-1}{p-1}}\tau\int_0^\alpha\left(\|u\|_{\infty,\partial U}-k\right)^{N-1}k^{\tau-1}\,dk$$

$$\le \tau\int_0^\alpha \sigma\left(\{z\in\partial U : u(z)>k\}\right)k^{\tau-1}\,dk = \int_{\partial U}\left(u^+\right)^\tau\,d\sigma \le \|u\|^\tau_{\tau,\partial U}$$

by virtue of the layer cake representation theorem (see [19, Section 1.13]). Finally, substitute $k := \theta\|u\|_{\infty,\partial U}$ into the term $\int_0^\alpha\left(\|u\|_{\infty,\partial U}-k\right)^{N-1}k^{\tau-1}\,dk$ to deduce

$$\|u\|^{N+\tau-1}_{\infty,\partial U}\int_0^1(1-\theta)^{N-1}\theta^{\tau-1}\,d\theta = \|u\|^{N+\tau-1}_{\infty,\partial U}\,\beta(N,\tau),$$

which yields (2.4) immediately in combination with the maximum principle. □

Notice that (1.1) always has a solution for each $\lambda > 0$ when $q \in [1,p)$ via Ekeland's variational principle or when $q \in (p, p_*)$ via the mountain pass theorem, in view of the compact embedding $E^{1,p}(U) \hookrightarrow L^q(\partial U, d\sigma)$ as described in the introduction. On the other hand, because $\sigma(\partial U) = 1$, a simple application of Hölder's inequality says

$$\|u\|_{1,\partial U} \le \|u\|_{s_1,\partial U} \le \|u\|_{s_2,\partial U} \le \|u\|_{p_*,\partial U}\left(\le \|u\|_{\infty,\partial U}\right) \tag{2.7}$$

for all $u \in E^{1,p}(U)$ whenever $1 \le s_1 \le s_2 \le p_*$ or $1 \le s_1 \le s_2 \le \infty$.

Below, we prove Theorem 1.1. Before to proceed to that, we remark the homogeneity of Rayleigh's quotient implies $\delta(q)$ is scale invariant. Standard variational method provides the existence of a minimizer $\omega_q > 0$ for $q \in [1, p_*)$, satisfying $\|\omega_q\|_{q,\partial U} = 1$. Clearly when $q = p$, we recover $\delta(p) = \delta_1$ and $\omega_p = \frac{\mathfrak{s}_1}{\|\mathfrak{s}_1\|_{p,\partial U}}$. Next, from (2.1) and the definition of $\delta(q)$, one sees $\delta(q) : [1, p_*] \to (0,\infty)$ is decreasing. In fact, when $1 \le s_1 < s_2 \le p_*$, we have

$$\delta(s_1) = \frac{\|\omega_{s_1}\|^p_\nabla}{\|\omega_{s_1}\|^p_{s_2,\partial U}}\exp\left(p\int_{s_1}^{s_2}\frac{K(t,\omega_{s_1})}{t^2}\,dt\right) \ge \frac{\|\omega_{s_1}\|^p_\nabla}{\|\omega_{s_1}\|^p_{s_2,\partial U}} \ge \delta(s_2).$$

Proof of Theorem 1.1. We first observe the continuity of $\delta(q) : (1, p_*] \to (0,\infty)$ by proving $\lim\limits_{s\to q^-}\delta(s) = \delta(q)$ for any $q \in (1, p_*)$, as $\delta(q)$ is decreasing. As a matter of fact, (2.1) yields

$$\delta(q) \le \delta(s) \le \frac{\|\omega_q\|^p_\nabla}{\|\omega_q\|^p_{s,\partial U}} = \delta(q)\exp\left(p\int_s^q\frac{K(t,\omega_q)}{t^2}\,dt\right). \tag{2.8}$$

As for $K(t,\omega_q) \ge 0$, we have, in view of (2.7), the following upper bound

$$\|\omega_q\|^{-t}_{t,\partial U}\int_{\partial U}\omega_q^t\ln\omega_q^t\,d\sigma - \ln\|\omega_q\|^t_{t,\partial U} \le t\left[\ln\left(\frac{\|\omega_q\|_{\infty,\partial U}}{\|\omega_q\|_{1,\partial U}}\right)\right], \tag{2.9}$$

which along with (2.8) yields $\delta(q) \le \delta(s) \le \delta(q)\left(\frac{q}{s}\right)^{\ln\left(\frac{\|\omega_q\|_{\infty,\partial U}}{\|\omega_q\|_{1,\partial U}}\right)^p} \to \delta(q)$ as $s \to q^-$.

Next, we consider the situation where $q = p_*$. Like what has been remarked in [10, p. 426], the solvability of (1.1) is subtle and $\delta(p_*)$ may actually have no minimizer ω_{p_*}. [I believe the answer is no in general without any mean curvature hypothesis on ∂U.] Take $\psi \in C^1\left(\overline{U}\right)$ to deduce $K(t,\psi) \le t\left[\ln\left(\frac{\|\psi\|_{\infty,\partial U}}{\|\psi\|_{1,\partial U}}\right)\right]$ in the same way as of (2.9). Therefore, one can observe

$$\delta(p_*) \le \liminf_{s\to p_*^-} \delta(s) \le \limsup_{s\to p_*^-} \delta(s) \le \limsup_{s\to p_*^-} \inf_{\psi\in C^1(\overline{U})} \frac{\|\psi\|_\nabla^p}{\|\psi\|_{p_*,\partial U}^p}\left(\frac{p_*}{s}\right)^{\ln\left(\frac{\|\psi\|_{\infty,\partial U}}{\|\psi\|_{1,\partial U}}\right)^p} = \delta(p_*)$$

and $\delta(q)$ is continuous at p_*. Here, we used the density of $C^1\left(\overline{U}\right)$ in $E^{1,p}(U)$ and the upper semi-continuity of the function $\inf\limits_{\psi\in C^1(\overline{U})} \frac{\|\psi\|_\nabla^p}{\|\psi\|_{p_*,\partial U}^p}\left(\frac{p_*}{s}\right)^{\ln\left(\frac{\|\psi\|_{\infty,\partial U}}{\|\psi\|_{1,\partial U}}\right)^p}$ about $s \in [1,p_*]$.

In addition, we show $\delta(q)$ is Lipschitz continuous when $q \in [1, p_* - \epsilon]$ for each $\epsilon > 0$. Take again, without loss of generality, $s < q$ in $[1, p_* - \epsilon]$. From (2.8), we get

$$0 \le \delta(s) - \delta(q) \le \delta(q)\left[\exp\left(p\int_s^q \frac{K(t,\omega_q)}{t^2}\,dt\right) - 1\right]. \tag{2.10}$$

When $1 < q \le p$, we let $\tau = 1$ in (2.4) and see, as $\boldsymbol{\beta}(N,1) = \frac{1}{N}$ and $\delta(q) \le \delta(1)$,

$$\frac{\|\omega_q\|_{\infty,\partial U}}{\|\omega_q\|_{1,\partial U}} \le N^{\frac{pN+p-2}{p-1}}\left(\frac{\delta(1)}{\delta(p_*)}\right)^{\frac{N-1}{p-1}} \frac{1}{\|\omega_q\|_{\infty,\partial U}^{\frac{(p-q)(N-1)}{p-1}}},$$

which combined with (2.7) and the condition $\|\omega_q\|_{q,\partial U} = 1$ gives an upper bound of $\frac{\|\omega_q\|_{\infty,\partial U}}{\|\omega_q\|_{1,\partial U}}$, independent of q. When $p < q \le p_* - \epsilon$, then we let $\tau = q$ in (2.4) and have

$$\|\omega_q\|_{\infty,\partial U}^{\frac{(q-p)(N-1)}{p-1}} \le \left\{\frac{N^{\frac{pN-1}{p-1}}}{q\boldsymbol{\beta}(N,q)}\left(\frac{\delta(1)}{\delta(p_*)}\right)^{\frac{N-1}{p-1}}\right\}^{\frac{(q-p)(N-1)}{(p_*-q)(N-p)}},$$

since $pN - p - qN + pq = (p_* - q)(N - p)$ and $\|\omega_q\|_{q,\partial U} = 1$. Take supremum of the right hand side with respect to q to give an upper bound of $\frac{\|\omega_q\|_{\infty,\partial U}}{\|\omega_q\|_{1,\partial U}}$ about $\epsilon > 0$.

Finally, we come back to (2.10) and observe $\exp\left(p\int_s^q \frac{K(t,\omega_q)}{t^2}\,dt\right) \le \left(\frac{q}{s}\right)^{K(\epsilon)}$, with $K(\epsilon) > 0$ a constant depending only on p, N, U and $\epsilon > 0$. Hence, we have

$$0 \le \frac{\delta(s) - \delta(q)}{q - s} \le \delta(1)\left[\frac{\left(\frac{q}{s}\right)^{K(\epsilon)} - 1}{\frac{q}{s} - 1}\right] \le L(\epsilon)$$

in view of $\lim\limits_{x\to 1} \frac{x^{K(\epsilon)}-1}{x-1} = K(\epsilon)$. Here, $L(\epsilon) > 0$ depends on ϵ, p, N, U. Since $\delta(q)$ is decreasing (therefore of bounded variation), continuous on $[1, p_*]$ and Lipschitz continuous on $[1, p_* - \epsilon]$, it is absolutely continuous on $[1, p_*]$ because $\delta'(q)$ can only have a *singularity* concentrated at p_* that is smoothed out by the continuity of $\delta(q)$ there. This finishes our proof. □

Below, we shall give the proof of Theorem 1.2. Notice, given $q \in [1, p_*) \setminus \{p\}$ and $\lambda > 0$, each positive solution of (1.1) can be found as a critical point of $\mathcal{J}$, defined by (1.5), in $E^{1,p}(U)$; in fact, $\left(\frac{\lambda}{\delta(q)}\right)^{\frac{1}{p-q}}\omega_q > 0$ (with $\|\omega_q\|_{q,\partial U} = 1$) is such a solution, since we have

$$\int_U |\nabla\omega_q|^{p-2}\,\nabla\omega_q\cdot\nabla v\;dx \;=\; \delta(q)\int_{\partial U}\omega_q^{q-1}\,v\;d\sigma \qquad \text{for all}\quad v\in E^{1,p}(U)\,.$$

This implies that the existence of a weak solution to (1.1) associated with one (given) $\lambda > 0$ will ensure the existence of a family of weak solutions associated with every $\lambda > 0$. One may be reminded that Theorem 2.1 in [16] is a simple consequence of Theorem 8 in [10].

We are interested in the positive solutions to problem (1.1) of minimal energy (in some sense). Notice when $1 \le q < p$, $\mathcal{J}$ is convex. So, $\mathcal{J}$ admits of a global minimizer $u_{\lambda,q}(>0) \in E^{1,p}(U)$, as $\mathcal{J}(u) \ge \mathcal{J}(u^+)$, with $\mathcal{J}(u_{\lambda,q}) < 0$ and $\mathcal{J}'(u_{\lambda,q}) = 0$, corresponding to (λ, q). That is, $u_{\lambda,q}$ belongs to the associated Nehari manifold $\mathscr{N}$ of $\mathcal{J}$ in $E^{1,p}(U)$ defined as

$$\mathscr{N} \;:=\; \left\{u\in E^{1,p}(U) : u\not\equiv 0 \quad\text{and}\quad \|u\|_\nabla^p \;=\; \lambda\left\|u^+\right\|_{q,\partial U}^q\right\}.$$

Necessarily, $u_{\lambda,q}$ is also a minimizer of $\mathcal{J}$ in $\mathscr{N}$ and we have $\mathcal{J}(u) = \mathcal{L}(u) := \left(\frac{1}{p} - \frac{1}{q}\right)\|u\|_\nabla^p$ on $\mathscr{N}$, so that $\mathcal{J}(u_{\lambda,q}) = \mathcal{L}(u_{\lambda,q}) = \inf\limits_{u\in\mathscr{N}} \mathcal{L}(u)$ and $\|u_{\lambda,q}\|_\nabla \le \lambda^{\frac{1}{p-q}}\delta(q)^{\frac{q}{p(q-p)}}$.

When $p < q < p_*$, the situation is delicate and like in [2], we define

$$\mathscr{A} \;:=\; \left\{u\in E^{1,p}(U) : u\not\equiv 0 \quad\text{and}\quad \|u\|_\nabla^p \;\le\; \lambda\left\|u^+\right\|_{q,\partial U}^q\right\}.$$

Then, $\|u\|_\nabla \ge \lambda^{\frac{1}{p-q}}\delta(q)^{\frac{q}{p(q-p)}} > 0$ in $\mathscr{A}$ and $\sup\limits_{u\in\mathscr{A}} \mathcal{L}(u) = \infty$. To guarantee the admissibility of a minimizer $u_{\lambda,q}$ of $\mathcal{L}$ in $\mathscr{A}$, we only need to see $\mathscr{A}$ is sequentially weakly closed in $E^{1,p}(U)$. In fact, let $\{u_n : n \ge 1\}$ be a sequence in $\mathscr{A}$ convergent weakly to $u_0 \in E^{1,p}(U)$. Then, via a subsequence, $u_n \to u_0$ in $L^q(\partial U, d\sigma)$, and $u_0^+ > 0$ as $\|u_n^+\|_{q,\partial U} \ge \left(\frac{\delta(q)}{\lambda}\right)^{\frac{1}{q-p}} > 0$. The lower semi-continuity of norms says $\|u_0\|_\nabla^p \le \liminf\limits_{n\to\infty}\|u_n\|_\nabla^p \le \lambda\left\|u_0^+\right\|_{q,\partial U}^q$. So, $u_0 \in \mathscr{A}$.

We now follow [2, Lemma 2] to make the following observations.

Lemma 2.4. *Assume $q \in [1, p_*)\setminus\{p\}$, $\lambda > 0$ and $u_0 \in \mathscr{A}$ is such that $\mathcal{L}(u_0) = \inf\limits_{u\in\mathscr{A}}\{\mathcal{L}(u)\}$. Then, u_0 satisfies these conditions:* (a) $u_0 \in \mathscr{N}$, (b) $\mathcal{J}(u_0) = \inf\limits_{u\in\mathscr{N}}\{\mathcal{J}(u)\}$, (c) $\mathcal{J}'(u_0) = 0$ *and* (d) $\|u_0\|_\nabla^p = \lambda^{\frac{p}{p-q}}\delta(q)^{\frac{q}{q-p}}$. *Here, $\lambda > 0$ is assumed to be any number.*

Proof. When $1 \le q < p$, then u_0 is the global minimizer $u_{\lambda,q}$ of $\mathcal{J}$ in view of

$$\mathcal{L}(u_{\lambda,q}) \;=\; \inf_{\mathscr{N}}\mathcal{L} \;=\; \inf_{E^{1,p}(U)}\mathcal{J} \;\le\; \inf_{\mathscr{A}}\mathcal{J} \;\le\; \inf_{\mathscr{A}}\mathcal{L} \;\le\; \inf_{\mathscr{N}}\mathcal{L},$$

since $\mathcal{J} \le \mathcal{L}$ on $\mathscr{A}$ and $\mathscr{N} \subsetneq \mathscr{A}$. So, conditions (a)–(c) are automatically satisfied.

When $p < q < p_*$, then condition (a) follows easily. Actually, if $u_0 \notin \mathscr{N}$, u_0 must be an interior point of $\mathscr{A}$, as $(1+t)\,u_0 \in \mathscr{A}$ for all $t \ge -\varepsilon$ with $0 \le \varepsilon \le 1 - \left(\frac{\|u\|_\nabla^p + \lambda\|u\|_{q,\partial U}^q}{2\lambda\|u\|_{q,\partial U}^q}\right)^{\frac{1}{q-p}}$. Then, $\mathcal{L}'(u_0) = 0$ and hence $u_0 \equiv 0$, a contradiction. Condition (b) is due to $\mathcal{J} = \mathcal{L}$ on $\mathscr{N}$. Condition (c) is a simple consequence of Lagrange multiplier theorem (on η). In fact,

$$\mathcal{J}'(u_0)(v) \;=\; \eta\left\{p\int_U |\nabla u_0|^{p-2}\,\nabla u_0\cdot\nabla v\,dx - \lambda\,q\int_{\partial U}\left(u_0^+\right)^{q-1} v\,d\sigma\right\}$$

holds for all $v \in E^{1,p}(U)$. As $u_0 \in \mathscr{N}$ is a minimizer of $\mathcal{J}$ on $\mathscr{N}$, $\mathcal{J}'(u_0)(u_0) = 0$. Thus, after substituting $v = u_0$, the above equation becomes $\eta\,(p-q)\,\|u_0\|_\nabla^p = 0$, so that $\eta = 0$.

Finally, we prove the last condition (d). Notice that

$$\mathscr{N} = \left\{ \left(\frac{\|u\|_{\nabla}^{p}}{\lambda \|u^{+}\|_{q,\partial U}^{q}} \right)^{\frac{1}{q-p}} u \quad \text{for each} \quad u(\not\equiv 0) \in E^{1,p}(U) \right\}.$$

When $1 \le q < p$, seeing $\mathcal{L}(u_0) = \inf\limits_{u\in\mathscr{N}} \left(\frac{1}{p} - \frac{1}{q}\right) \|u\|_{\nabla}^{p}$, one has

$$\begin{aligned}\|u_0\|_{\nabla}^{p} = \sup_{u\in\mathscr{N}} \|u\|_{\nabla}^{p} &= \sup_{u\in E^{1,p}(U)} \left\| \left(\frac{\|u\|_{\nabla}^{p}}{\lambda \|u^{+}\|_{q,\partial U}^{q}} \right)^{\frac{1}{q-p}} u \right\|_{\nabla}^{p} \\ &= \lambda^{\frac{p}{p-q}} \left\{ \inf_{u\in E^{1,p}(U)} \frac{\|u\|_{\nabla}^{p}}{\|u^{+}\|_{q,\partial U}^{p}} \right\}^{\frac{q}{q-p}} = \lambda^{\frac{p}{p-q}}\, \delta(q)^{\frac{q}{q-p}}. \end{aligned} \tag{2.11}$$

When $p < q < p_*$, then $\|u_0\|_{\nabla}^{p} = \inf\limits_{u\in\mathscr{N}} \|u\|_{\nabla}^{p}$ and thus (2.11) follows similarly. □

From now on, any such a function $u_0 > 0$ is said to be of minimal energy and denoted by $u_{\lambda,q}$. Note that, using $\|\omega_q\|_{\nabla}^{p} = \delta(q)$, one as a matter of fact has $\left\| \left(\frac{\lambda}{\delta(q)}\right)^{\frac{1}{p-q}} \omega_q \right\|_{\nabla}^{p} = \lambda^{\frac{p}{p-q}} \delta(q)^{\frac{q}{q-p}}$.

Proof of Theorem 1.2. A simple calculation leads to

$$\left(\frac{\lambda}{\delta_1}\right)^{\frac{p}{q-p}} \|u_{\lambda,q}\|_{\nabla}^{p} = \delta_1 \left(\frac{\delta(q)}{\delta_1}\right)^{\frac{q}{q-p}},$$

from which it follows that, recalling $\delta(p) = \delta_1$,

$$\begin{aligned} &\lim_{q\to p^-} \left(\frac{\lambda}{\delta_1}\right)^{\frac{p}{q-p}} \|u_{\lambda,q}\|_{\nabla}^{p} = \delta_1 \lim_{q\to p^-} \left(\frac{\delta(q)}{\delta_1}\right)^{\frac{q}{q-p}} \\ &= \delta_1 \exp \left\{ \left(\lim_{q\to p^-} \frac{q}{\delta_1} \right) \left[\lim_{q\to p^-} \frac{\ln\left(\frac{\delta(q)}{\delta_1}\right)}{\frac{\delta(q)}{\delta_1} - 1} \right] \left(\lim_{q\to p^-} \frac{\delta(q) - \delta_1}{q-p} \right) \right\}. \end{aligned} \tag{2.12}$$

An exactly the same analysis derives this limit estimate (2.12) when $q \to p^+$.

Notice that $\lim\limits_{x\to 1} \frac{\ln x}{x-1} = 1$. Define $I(s,u) := \int_{\partial U} |u|^s \ln|u|\, d\sigma$ and, for $\|\omega_q\|_{s,\partial U}^{p}$ (viewed as a function of s), take derivative and evaluate at $s = q \in [1, p_*)$ to observe

$$\begin{aligned} \frac{d}{ds}\left(\|\omega_q\|_{s,\partial U}^{p}\right)\Big|_{s=q} &= \frac{d}{ds}\left\{ \exp\left[\frac{p}{s} \ln\left(\int_{\partial U} \omega_q^s \, d\sigma \right) \right] \right\}\Big|_{s=q} \\ = \|\omega_q\|_{q,\partial U}^{p} &\left\{ -\frac{p}{q} \ln \|\omega_q\|_{q,\partial U} + \frac{p}{q} \frac{\int_{\partial U} \left(\omega_q^q \ln \omega_q\right) d\sigma}{\|\omega_q\|_{q,\partial U}^{q}} \right\} = \frac{p}{q} I(q, \omega_q) \end{aligned}$$

since $\|\omega_q\|_{q,\partial U} = 1$.

On the other hand, by definition, we know $\delta(s) \le \frac{\|\omega_q\|_{\nabla}^{p}}{\|\omega_q\|_{s,\partial U}^{p}} = \frac{\delta(q)}{\|\omega_q\|_{s,\partial U}^{p}}$ for all $s \in [1, p_*]$. As a consequence, one has

$$\delta'_-(q) = \lim_{s\to q^-} \frac{\delta(s)-\delta(q)}{s-q} \geq \liminf_{s\to q^-} \delta(s)\,\frac{1-\|\omega_q\|^p_{s,\partial U}}{s-q}$$
$$= -\,\delta(q)\left\{\frac{d}{ds}\left(\|\omega_q\|^p_{s,\partial U}\right)\Big|_{s=q}\right\} = -\,\delta(q)\,\frac{p}{q}\,I(q,\omega_q).$$

In an analogous manner, one can deduce

$$\delta'_+(q) = \lim_{s\to q^+} \frac{\delta(s)-\delta(q)}{s-q} \leq \limsup_{s\to q^+} \delta(s)\,\frac{1-\|\omega_q\|^p_{s,\partial U}}{s-q} = -\,\delta(q)\,\frac{p}{q}\,I(q,\omega_q).$$

Here, the monotonicity and Lipschitz continuity of $\delta(q)$ were applied to ensure the existence (and finiteness) of $\delta'_\pm(q)$.

The preceding discussions combined with (2.12) lead to

$$\lim_{q\to p^-}\left(\frac{\lambda}{\delta_1}\right)^{\frac{p}{q-p}} \|u_{\lambda,q}\|^p_\nabla = \delta_1 \exp\left\{\frac{p}{\delta_1}\,\delta'_-(p)\right\}$$
$$\geq \delta_1 \exp\left\{\frac{p}{\delta_1}\,\delta'_+(p)\right\} = \lim_{q\to p^+}\left(\frac{\lambda}{\delta_1}\right)^{\frac{p}{q-p}} \|u_{\lambda,q}\|^p_\nabla\,.$$

Define $\mathfrak{c}_1 := \delta_1 e^{\frac{p}{\delta_1}\delta'_-(p)}$ and $\mathfrak{c}_2 := \delta_1 e^{\frac{p}{\delta_1}\delta'_+(p)}$ to finish our proof of Theorem 1.2. $\square$

When $1 \leq q < p$, then one has $\left(\frac{\lambda}{\delta(q)}\right)^{\frac{1}{p-q}} \omega_q > 0$ is the only solution to (1.1) of minimal energy, since the uniqueness property holds in this situation that can be proved very similarly, if not identical, to the analysis of Lindqvist [20] and the argument in [16, Lemma 2.2][4] (some other proofs are available and the interested reader may find Belloni and Kawohl [8] helpful where more references are provided). When $p < q < p_*$, this property is however far from clear and actually may depend on the geometry of the domain involved.

Remark 2.5. Concerning Theorem 1.1, it is easy to observe that after a parallel application of the analyses in [1, pp. 2062–2063] and [2, pp. 134–135], the results (in particular, Proposition 5) of [12, Section 2] can be adapted to our setting, and hence $\delta(q)$ is continuously differentiable on $[1, p]$ as $\lim_{q\to p^-} \delta'(q)$ exists. The key is to guarantee $\omega_{q_n} \to \omega_q$ in $E^{1,p}(U)$ when $q_n \to q$, which does not depend on any regularity result about ω_q but depends only on the uniqueness of ω_q in the range $1 \leq q \leq p$. Also, more can be said about the regularity of $\delta(q)$ near p_*, like Theorem 12 of [12, Section 3], under an extra hypothesis $\limsup_{q\to p_*^-} (p_* - q)\,\|\omega_q\|^\gamma_{\infty,\partial U} < \infty$ for some $\gamma > 0$.

Remark 2.6. Concerning Theorem 1.2, one has $\mathfrak{c}_1 = \mathfrak{c}_2$ provided $\delta(q)$ is differentiable around p. To describe the differentiability of $\delta(q)$ around p, the proofs of [11] cannot be adapted here, because they depend heavily on the $C^{1,\alpha}$-regularity of ω_q. This is an open question in our setting.

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