



A note on the complexity function and entropy of pseudogroups



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ABSTRACT

We examine the connections between complexity of a pseudogroup, its equicontinuity, the mixing property and entropy. We prove that the entropy of a pseudogroup can be (under some additional assumptions) computed using a continuous and dynamically generating pseudometric.

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1. Introduction

In [9], Ghys, Langevin and Walczak introduced the notion of entropy for foliations of Riemannian manifolds and finitely generated pseudogroups of local homeomorphisms. This concept applies also to laminations as observed in [7].

Pseudogroups of local homeomorphisms are natural generalizations of group actions on topological, in particular compact metric spaces (each group action generates a pseudogroup). Another important example is the holonomy pseudogroup of a foliated space defined by a regular covering by flow boxes. Therefore it is natural to ask about connections between dynamics of a pseudogroup and its entropy. This problem was studied by many authors (see for example [4,5,9,14,19,21,22]). Note that the value of the entropy of a pseudogroup depends both on the entropy of generators and the growth type of the pseudogroup. In particular, it depends on the choice of a generating set. But if the entropy is equal to zero for one generating set, then it vanishes for all of them.

We study the structure of pseudogroups with zero entropy. It has been conjectured that every distal pseudogroup has zero entropy. To our best knowledge this problem is still open, although some special cases have been answered positively: for example the entropy of a compact minimal distal foliated bundle was shown to vanish whenever its holonomy group has linear growth [4] or for the case of the foliations

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of codimension one [14]. Dynamical properties and entropy of pseudogroups were also considered in [11–13, 17, 22].

A special class of zero entropy pseudogroups are equicontinuous pseudogroups (see for example [1, 2, 9, 15, 18, 20]). The authors of [1] generalized some properties of Riemannian foliations on closed manifolds to compact equicontinuous foliated spaces. For example, it is shown there that all holonomy covers of the leaves are quasi-isometric to each other. In [20] it is proved that equicontinuous transversely conformal foliations are Riemannian, which also follows from the main result of [2] in the case of dense leaves. In [9] it was shown that every equicontinuous pseudogroup has zero entropy. We prove that equicontinuity of a pseudogroup acting on a compact space implies its bounded complexity. We also study a connection between complexity of a pseudogroup and (a version of) the weak mixing property. This is an extension of [5], where the connection between dynamics of a system generated by a single map and its complexity is examined.

In the last part of the article we will show that entropy of a pseudogroup can be (under some additional assumptions) computed using a continuous and dynamically generating pseudometric. This result is known for the case of entropy of amenable groups and also was proved (in [16]) in the case of sofic entropy.

2. Terminology and notation

We assume that X is a compact metrizable space. Let $d: X \times X \rightarrow \mathbb{R}_+$ be a function. We say that d is a *pseudometric* if d is symmetric and fulfills the triangle inequality, that is, for all $x, y, z \in X$ one has $d(x, y) = d(y, x)$ and $d(x, y) + d(y, z) \geq d(x, z)$. We say that d is a *semimetric* if it is symmetric and definite ($d(x, y) = 0$ if and only if $x = y$).

Given a function $d: X \times X \rightarrow \mathbb{R}_+$ (not necessarily a semi- or pseudometric), a set $A \subset X$ and $x \in X$, recall the definitions

$$\text{dist}_d(A, x) := \inf_{a \in A} d(a, x) \text{ and } \text{diam}_d(A) := \sup_{a, b \in A} d(a, b).$$

Recall that a function $F: X \rightarrow \mathbb{R}$ is *lower semicontinuous* if for any $x \in X$ one has

$$\liminf_{y \rightarrow x} F(y) \geq F(x).$$

A lower semicontinuous function on a compact space attains its minimum and a pointwise supremum of lower semicontinuous functions is lower semicontinuous (see [6, Chapter 6.2]).

Let $\text{Homeo}(X)$ denote the family of all homeomorphisms between open subsets of a topological space X . For a function g let D_g denote the domain of g and let Im_g denote its image. Given $g, h \in \text{Homeo}(X)$, we denote by $g \circ h$ the restriction $g \circ h|_{h^{-1}(\text{Im}_h \cap D_g)}$. Recall that a subfamily \mathcal{G} of $\text{Homeo}(X)$ is called a *pseudogroup* if the following conditions are satisfied:

1. $g \circ h \in \mathcal{G}$ for all $g, h \in \mathcal{G}$,
2. $g^{-1} \in \mathcal{G}$ for each $g \in \mathcal{G}$,
3. $g|_U \in \mathcal{G}$ for each $g \in \mathcal{G}$ and each open set $U \subset D_g$,
4. for all $g \in \text{Homeo}(X)$ and for any open cover \mathcal{U} of D_g , if $g|_U \in \mathcal{G}$ for each $U \in \mathcal{U}$, then $g \in \mathcal{G}$,
5. $\text{id}_X \in \mathcal{G}$ (this is equivalent to the fact that $\bigcup \{D_g \mid g \in \mathcal{G}\} = X$).

We say that $G \subset \text{Homeo}(X)$ generates \mathcal{G} if \mathcal{G} is the smallest (with respect to inclusion) pseudogroup containing G . Let G be a finite and symmetric set generating a pseudogroup \mathcal{G} . We say that G is *good* if for each $g \in G$ there is a compact set $K_g \subset D_g$ such that the family $\{g|_{\text{int } K_g} \mid g \in G\}$ generates \mathcal{G} . A pseudogroup which admits a good generating set is called a *good pseudogroup*. Given a good pseudogroup

\mathcal{G} generated by a good set G , we denote by \mathcal{G}_n^c the set $\{g_1 \circ \dots \circ g_k \mid k \leq n, g_i \in G \text{ for } i \in \{1, \dots, k\}\}$. To avoid trivialities we assume that $|\bigcup_{n=1}^{\infty} \mathcal{G}_n^c| = \aleph_0$.

We say that a pseudometric d on X is *dynamically generating* for a pseudogroup \mathcal{G} if for any two distinct points x and y in X there is an element $g \in \bigcup_{n \in \mathbb{N}} \mathcal{G}_n^c$ such that $x, y \in \text{int } D_g$ and $d(gx, gy) > 0$.

For any set $A \subset X$ and for any $g \in \mathcal{G}$ we will denote by gA the set $\{x \in X \mid \exists a \in A \cap D_g \text{ such that } ga = x\}$.

Let $\mathcal{U} = \{U_1, \dots, U_m\}$ be a finite (not necessarily open) cover of X . Put $A = \{1, \dots, m\}$. A function $\omega: \bigcup_{n \in \mathbb{N}} \mathcal{G}_n^c \rightarrow A$ is called *\mathcal{U} -name of the point $x \in X$* if

$$x \in \bigcap_{g \in \bigcup_{n \in \mathbb{N}} \mathcal{G}_n^c \cap \{h \in \mathcal{G} \mid x \in D_h\}} g^{-1}U_{\omega(g)},$$

Similarly we say that a function $\omega: \mathcal{G}_k^c \rightarrow A$ is a *\mathcal{U}, k -name* (or a *\mathcal{U} -name of length k*) of a point $x \in X$ if:

$$x \in \bigcap_{g \in \mathcal{G}_k^c \cap \{h \in \mathcal{G} \mid x \in D_h\}} g^{-1}U_{\omega(g)}.$$

We say that a space is *covered by a subfamily of \mathcal{U} -names*, if it is covered by associated sets of points having those \mathcal{U} -names. Given an open cover \mathcal{U} of X , denote by $N(\mathcal{U}, n)$ the minimal cardinality of a cover of X consisting of the \mathcal{U} -names of the length n . The function $N(\mathcal{U}, n)$ (where n is a variable) will be called the *complexity of \mathcal{G} with respect to \mathcal{U}* . The cover \mathcal{U} has *bounded complexity* if the function $N(\mathcal{U}, n)$ is bounded. Since the complexity is non-decreasing, it is bounded if and only if it is eventually constant. The *entropy of a pseudogroup \mathcal{G} with respect to \mathcal{U} and G* is defined by:

$$h_c(\mathcal{G}, G, \mathcal{U}) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log N(\mathcal{U}, n).$$

Finally, we define the topological entropy of \mathcal{G} with respect to G as:

$$h_c(\mathcal{G}, G) := \sup_{\mathcal{U}} h(\mathcal{U}).$$

Given a pseudometric d on X , one can also define the entropy of a pseudogroup using (d, n, ε) -separated and (d, n, ε) -spanning sets. Namely, a set $E \subset X$ is *(d, n, ε) -separated*, if for any two distinct points x and y in E there exists $g \in \mathcal{G}_n^c$ such that $x, y \in D_g$ and $d(gx, gy) \geq \varepsilon$. Let $s(d, n, \varepsilon)$ denote the maximal cardinality of a (d, n, ε) -separated subset of X . The entropy of \mathcal{G} is defined as

$$h_s(d, \mathcal{G}, G) := \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s(d, n, \varepsilon).$$

A set $F \subset X$ is *(d, n, ε) -spanning* if for any point $x \in X$ there is a point $y \in F$ such that for every $g \in \mathcal{G}_n^c$ with $x, y \in D_g$ one has $d(gx, gy) < \varepsilon$. Let $r(d, n, \varepsilon)$ denote the minimal cardinality of (d, n, ε) -spanning set and let

$$h_r(d, \mathcal{G}, G) := \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log r(d, n, \varepsilon).$$

It is elementary to check that if d is a metric, then $h_c(\mathcal{G}, G) = h_r(d, \mathcal{G}, G) = h_s(d, \mathcal{G}, G)$ and in this paper we will also show that instead of a metric we can take a dynamically generating continuous pseudometric satisfying certain properties.

A set $G \subset \mathcal{G}$ satisfies the *uniform equicontinuity* condition if for every $\varepsilon > 0$ there exists $\eta > 0$ such that for all points $x, y \in X$ and for every $g \in G$ such that $x, y \in D_g$, if $d(x, y) < \eta$ then $d(gx, gy) < \varepsilon$.

A pseudogroup is *equicontinuous* if it has a generating subset that is closed by the operations of composition and inversion and satisfies the uniform equicontinuity condition.

A pseudogroup \mathcal{G} is *compactly generated* if the following conditions are satisfied:

1. X is locally compact and contains a relatively compact open subset U meeting all the orbits of \mathcal{G} , and
2. there exists finite set $G = \{g_1, \dots, g_k\} \subset \mathcal{G}$ generating $\mathcal{G}|_U$, so that each $g_i: V_i \rightarrow W_i$ is the restriction of an element $\tilde{g}_i \in \mathcal{G}$ whose domain \tilde{V}_i contains the closure of V_i .

A finite symmetric family G of generators of \mathcal{G} is said to be *recurrent* if there exists a relatively compact open subset $U \subset X$ and $R \in \mathbb{N}$ such that for every $x \in X$ there exists $g \in \mathcal{G}_R^c$ such that $x \in D_g$ and $gx \in U$ (see [1, Definition 4.2.] for more details). We say that \mathcal{G} is compactly generated by a recurrent system, if the corresponding pseudogroup given by restriction of \mathcal{G} to the relatively compact open set that meets every orbit is recurrent.

A cover is *trivial* if one of its elements is dense in X . A *standard cover* is a cover which consists of two non-dense open sets. A pseudogroup \mathcal{G} is *weakly mixing* if for any pair of non-empty open sets A, B there exists $g \in \mathcal{G}$ such that $(A \cap gA) \times (B \cap gA) \neq \emptyset$. Note that this is only one of the possible definitions of weak mixing for pseudogroups. For a single transformation on a compact metric space weak mixing has many characterizations ([3], [10, Theorem 3.34.2.]) and it is not clear whether the analogous statements are equivalent in the case of pseudogroups.

For two covers \mathcal{U} and \mathcal{V} we write $\mathcal{U} \prec \mathcal{V}$ if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subset U$. We denote by $\mathcal{U} \vee \mathcal{V}$ the *joining* of covers, that is the family $\{U \cup V \mid U \in \mathcal{U}, V \in \mathcal{V}\}$.

3. Equicontinuity versus bounded complexity

Lemma 1. Assume X is a metric space and \mathcal{G} is a pseudogroup acting on X and generated by a finite symmetric set G . Let $\mathcal{U} = \{U_1, \dots, U_m\}$ be a cover of X . Then the following conditions are equivalent:

1. The complexity function $N(\mathcal{U}, n)$ is bounded from above by an integer s .
2. There exists a family of cardinality s consisting of \mathcal{U} -names which covers X .

Proof. Assume that \mathcal{U} has the complexity bounded by s . Let $A = \{1, \dots, m\}$ and let $\Omega = A^{\bigcup_{n \in \mathbb{N}} \mathcal{G}_n^c}$. We equip Ω with the product topology of the discrete space A . Then Ω becomes the compact metric space homeomorphic with the Cantor set. Take $\omega \in \Omega$. By $J_n(\omega)$ we denote the set of points $x \in X$ such that $\omega|_{\mathcal{G}_n^c}$ is an n -name of x for $n \in \mathbb{N}$. Denote by $H(n)$ the family of all s -tuples $(\omega_1, \dots, \omega_s)$ of elements of Ω such that sets $\{J_n(\omega_1), \dots, J_n(\omega_s)\}$ form a cover of the space X . Since s bounds from the above the complexity of \mathcal{U} , the set $H(n)$ is non-empty. Moreover, one can easily see that for every n we have $H(n+1) \subset H(n) \subset \Omega^s$. It is also evident that the sets $H(n)$ are closed. Consequently, by the Cantor intersection theorem we get that there exists $\bar{\omega} \in \bigcap_{i=0}^{\infty} H(n)$. The s -tuple $\bar{\omega}$ consists of \mathcal{U} -names and covers the space X . The remaining implication is obvious. \square

Theorem 2. Let (X, d) be a compact metric space and \mathcal{G} be a pseudogroup acting on X . Assume that \mathcal{G} is generated by a finite symmetric set G which satisfies the uniform equicontinuity condition. Let \mathcal{U} be a finite open cover of X . Then the complexity $N(\mathcal{U}, n)$ is bounded.

Proof. Let $\varepsilon > 0$ be a Lebesgue number of \mathcal{U} . There exists $\eta > 0$ such that for every pair of points $x, y \in X$ and for every $g \in \bigcup_{n \in \mathbb{N}} \mathcal{G}_n^c$ such that $x, y \in D_g$ if $d(x, y) < \eta$, then $d(gx, gy) < \varepsilon$. Let (x_1, \dots, x_k) be a finite $\eta/2$ -net (that is, let $\bigcup_{i=1}^k B(x_i, \eta/2) = X$). Fix $g \in \bigcup_{n \in \mathbb{N}} \mathcal{G}_n^c$ and $z_i \in B(x_i, \eta/2) \cap D_g$ for every $i \in \{1, \dots, k\}$. Note that for every $y \in B(x_i, \eta/2) \cap D_g$ we have $d(y, z_i) < \eta$. Therefore $\text{diam}(g(B(x_i, \eta/2) \cap D_g)) < \varepsilon$. It

follows that there exists an element $U_{g,i} \in \mathcal{U}$ such that if $y \in B(x_i, \eta/2) \cap D_g$, then $gy \in U_{g,i}$. In other words, $g(B(x_i, \eta/2) \cap D_g) \subset U_{g,i}$. Therefore for all $i \in \{1, \dots, k\}$ and $n \in \mathbb{N}$:

$$B(x_i, \eta/2) \subset \bigcap_{g \in \mathcal{G}} g^{-1}U_{g,i} \cup (X \setminus D_g).$$

Consequently, $\omega_i^{(n)}: \mathcal{G}_n^c \rightarrow \{1, \dots, p\}$ given by $\omega_i^{(n)}(g) = p_{g,i}$, where $U_{g,i} = U_{p_{g,i}}$ is a \mathcal{U} name of length n for every $x \in B(x_i, \eta/2)$. Therefore there exists a family of cardinality at most k consisting of \mathcal{U} -names which covers X . Hence $N(\mathcal{U}, n)$ is bounded by k for every n . \square

Example 3. The above implication is not true if we do not assume that X is compact, even in case of compactly generated recurrent pseudogroups and for covers with a positive Lebesgue number.

Consider the pseudogroup \mathcal{G} acting on the real line \mathbb{R} and generated by a set $G = \{g, \text{id}, g^{-1}\}$, where $g: \mathbb{R} \rightarrow \mathbb{R}$ is defined by $g(x) = x + 1$. Notice that \mathcal{G} is compactly generated by a recurrent system. Moreover, every map in the set $\bigcup_{n \in \mathbb{N}} \mathcal{G}_n^c$ is an isometry and hence \mathcal{G} is equicontinuous. However, the complexity of $\mathcal{U} = \{(-\infty, 0), (-1, 1), (0, \infty)\}$ is unbounded, since $N(\mathcal{U}, n) \geq 2n + 1$.

Theorem 4. Let (X, d) be a metric space and \mathcal{G} be a pseudogroup acting on X and compactly generated by a finite symmetric set G . Denote by V the relatively compact open subset of X that meets every orbit. Let \mathcal{H} be a restriction of \mathcal{G} to V . Assume that for every finite open cover \mathcal{U} the function $f(n) = N(\mathcal{U}, n)$ is bounded. Then for every $x \in V$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for every $g \in \mathcal{H}$ for which there exists $\tilde{g} \in \bigcup_{n \in \mathbb{N}} \mathcal{G}_n^c$ the following holds: if $x \in D_g$, then for every $y \in B(x, \delta) \cap D_g$ one has $d(gx, gy) < \varepsilon$.

Proof. Suppose that there exist $x \in V$ and $\varepsilon > 0$ satisfying that for every $n \in \mathbb{N}$ there exists $y_n \in B(x, 1/n)$ such that one can find $g_n \in \mathcal{H}$ with $x, y_n \in D_{g_n}$ and $d(g_n x, g_n y_n) > \varepsilon$. Let $\mathcal{U} = \{B(x_1, \varepsilon/4), \dots, B(x_m, \varepsilon/4)\}$ be a finite cover of \overline{V} . Set $\mathcal{U}' = \mathcal{U} \cup \{X \setminus \overline{V}\}$. Since the complexity of \mathcal{U}' is bounded, it follows from Lemma 1 that there is a finite cover $\Xi = \{X_1, \dots, X_p\}$ of X consisting of \mathcal{U}' -names. By the definition of Ξ and \mathcal{U} it holds that if $y, z \in X_i$ (for some $i \in \{1, \dots, p\}$), then for every $g \in \mathcal{H}$ such that $y, z \in D_g$ one has $d(gy, gz) < \varepsilon/2$. Passing to a subsequence we can assume that $\{y_n\}_{n \in \mathbb{N}} \subset X_i$ for some $i \in \{1, \dots, p\}$. Hence for all $n \in \mathbb{N}$ one has $x \in \overline{X_i} \cap D_{g_n}$ and consequently one has $d(g_n x, g_n y_n) \leq \varepsilon/2$ for some n large enough, which leads to a contradiction. \square

Remark 5. In [1, Lemma 8.8] it is proved that if pseudogroups \mathcal{G} and \mathcal{G}' are equivalent in the sense of Haefliger (see [1, Definition 2.1]), then \mathcal{G} is equicontinuous if and only if \mathcal{G}' is equicontinuous. The pseudogroups \mathcal{G} and \mathcal{H} defined in the above theorem are equivalent (see [1, p. 732]). However, it is not clear how to define pointwise equicontinuity for pseudogroups and if some kind of that is equivalent to equicontinuity in the case of compactly generated pseudogroups (see [1, Problem 2]).

4. The mixing property

Theorem 6. Let \mathcal{G} be a good pseudogroup. If for every standard cover \mathcal{U} of X there exists $n > 0$ such that $N(\mathcal{U}, n) > |\mathcal{G}_n^c| + 1$, then \mathcal{G} is weakly mixing.

Proof. Assume that \mathcal{G} is not weakly mixing, that is, there are non-empty open sets A and B such that for every $g \in \mathcal{G}$ such that $D_g \cap A \neq \emptyset$ at least one of the following conditions holds: $A \cap gA = \emptyset$, or $A \cap gB = \emptyset$. Without loss of generality we can assume that $A \cap B = \emptyset$. Let $\mathcal{U} = \{U_1, U_2\}$ be a standard cover of X such that $X \setminus A \subset U_1$ and $X \setminus B \subset U_2$. Fix $n \in \mathbb{N}$ and $g \in \mathcal{G}$. If $A \cap gB = \emptyset$, then $A \subset X \setminus gB \subset gU_2 \cup (X \setminus D_{g^{-1}})$. If $A \cap gA = \emptyset$, then $A \subset gU_1 \cup (X \setminus D_{g^{-1}})$. This means that for every $g, h \in \mathcal{G}_n^c$ there exists a set $W_{g,h} \in \{U_1, U_2\}$ such that:

$$A \subset \bigcap_{g,h \in \mathcal{G}_n^c} gh^{-1}W_{g,h} \cup (X \setminus D_{hg^{-1}}).$$

Fix $x \in X$ and consider two cases:

1. If there exists $g \in \mathcal{G}_n^c$ such that $x \in D_g$ and $gx \in A$, then

$$x \in \bigcap_{h \in \mathcal{G}_n^c} (h^{-1}W_{g,h} \cup (X \setminus D_h))$$

and $\omega_g: \mathcal{G}_n^c \rightarrow \{1, 2\}$ given by $\omega_g(h) = a_h$, where $U_{a_h} = W_{g,h}$ is a \mathcal{U} -name of length n for x .

2. If $gx \in X \setminus A$ for every $g \in \mathcal{G}_n^c \cap \{g \in \mathcal{G} \mid x \in D_g\}$, then

$$x \in \bigcap_{g \in \mathcal{G}_n^c} (g^{-1}U_1 \cup (X \setminus D_g))$$

and $\omega: \mathcal{G}_n^c \rightarrow \{1, 2\}$ defined as $\omega(g) = 1$ for every $g \in \mathcal{G}_n^c$ is a \mathcal{U} -name of length n for x .

Consequently $N(\mathcal{U}, n) \leq |\mathcal{G}_n^c| + 1$. \square

5. Entropy of a pseudogroup with respect to a continuous and dynamically generating pseudometric

Lemma 7. Let τ be a topology on X . For any continuous pseudometric $d: (X, \tau) \times (X, \tau) \rightarrow \mathbb{R}_+$ and any set $A \subset X$ the function $\text{dist}(A, \cdot): X \rightarrow \mathbb{R}_+$ is τ -continuous.

Proof. For any $x, z \in X$ one has $|\text{dist}(A, x) - \text{dist}(A, z)| \leq d(x, z)$. Hence $\text{dist}(A, \cdot)$ is d -continuous and so τ -continuous. \square

Lemma 8. Let (X, τ) be a compact space and $d: X \times X \rightarrow \mathbb{R}_+$ be a semimetric such that for every closed set $A \subset X$ the function $\text{dist}_d(A, \cdot): X \rightarrow \mathbb{R}_+$ is lower semicontinuous. Then for every open cover \mathcal{U} of X there exists $\eta > 0$ such that for any $Y \subset X$ with $\text{diam}_d(Y) < \eta$ there exists $U \in \mathcal{U}$ such that $Y \subset U$.

Proof. Fix a τ -open cover \mathcal{U} and its finite subcover $\{U_1, \dots, U_n\}$. If there exists $i \in \{1, \dots, n\}$ such that $U_i = X$, then there is nothing to prove. Therefore we assume that $X \setminus U_i \neq \emptyset$ for every $i \in \{1, \dots, n\}$. For every $i \in \{1, \dots, n\}$ let $f_i: X \rightarrow \mathbb{R}_+$ be given by $f_i(x) = \text{dist}_d(X \setminus U_i, x)$ and let $f: X \rightarrow \mathbb{R}_+$ be defined by

$$f(x) = \frac{1}{n} \sum_{i=1}^n f_i(x).$$

Since \mathcal{U} is a cover, for every $x \in X$ there is $i \in \{1, \dots, n\}$ such that $x \notin X \setminus U_i$. Consider the function $\text{dist}_d(\{x\}, \cdot) = d(x, \cdot)$. By our assumption it is a lower semicontinuous function on X and hence it attains its minimum value on a compact set $X \setminus U_i$. But $\text{dist}_d(\{x\}, a) = d(x, a) = 0$ for some $a \in X \setminus U_i$ would imply that $x \in X \setminus U_i$. Hence $f(x) > 0$. Moreover f is a sum of lower semicontinuous functions, hence it is also lower semicontinuous and it attains its minimum on X , that is there is $\eta = \min_{x \in X} f(x)$ and η is positive since $f(x) > 0$ for all $x \in X$. We will show that η satisfies the requested conditions. Fix a non-empty set $Y \subset X$ such that $\text{diam}_d(Y) < \eta$ and $x_0 \in Y$. Then $Y \subset B_d(x_0, \eta)$. One also has $f(x_0) \geq \eta$ and hence there exists $i \in \{1, \dots, n\}$ such that $f_i(x_0) \geq \eta$. Consequently $B_d(x_0, \eta) \subset U_i$, hence $Y \subset U_i$. This finishes the proof. \square

Definition 9. Let $d: X \times X \rightarrow \mathbb{R}_+$ be a pseudometric. We say that $x \in X$ is (d, n, ε) -close to $y \in X$ if for all $h \in \mathcal{G}_n^c$ with $x, y \in D_h$ one has $d(hx, hy) < \varepsilon$. A pseudometric d is *compatible* with (\mathcal{G}, G) if there exists $N \in \mathbb{N}$ such that for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $n \geq N$, $g \in \mathcal{G}_n^c$, $x \in X$ and $y_1, y_2 \in D_g$ if x is (d, n, δ) -close to y_i for $i = 1, 2$, then $d(gy_1, gy_2) < \varepsilon$.

Remark 10. Notice that if $x \in D_g$ is $(d, n, \varepsilon/2)$ -close to y_i for $i = 1, 2$, then the condition $d(gy_1, gy_2) < \varepsilon$ is a consequence of the triangle inequality. In particular, if for all $g \in \mathcal{G}$ one has $D_g = X$, then every pseudometric on X is compatible with (\mathcal{G}, G) .

Theorem 11. Let \mathcal{G} be a good pseudogroup generated by a finite symmetric set G and acting on a compact metrizable topological space (X, τ) . Suppose that $d: X \times X \rightarrow \mathbb{R}_+$ is a dynamically generating pseudometric compatible with (\mathcal{G}, G) . Then $h_c(\mathcal{G}, G) = h_r(d, \mathcal{G}, G) = h_s(d, \mathcal{G}, G)$. In particular, the value of $h_r(d, \mathcal{G}, G) = h_s(d, \mathcal{G}, G)$ does not depend on the choice of such a pseudometric.

Proof. It is enough to prove that

$$h_c(\mathcal{G}, G) \leq h_r(d, \mathcal{G}, G) = h_s(d, \mathcal{G}, G) \leq h_c(\mathcal{G}, G).$$

A standard proof (see [8, Theorem 3.18]) shows that $h_r(d, \mathcal{G}, G) = h_s(d, \mathcal{G}, G) \leq h_c(\mathcal{G}, G)$. Hence it is enough to prove that $h_c(\mathcal{G}, G) \leq h_r(d, \mathcal{G}, G)$. We can assume without loss of generality that $\text{diam}_d(X) = 1$. Because for every n the set \mathcal{G}_n^c is finite, we can enumerate elements from the set $\bigcup_{n=0}^{\infty} \mathcal{G}_n^c$ as a sequence $\{g_1, g_2, \dots\}$ in such a way that, for any integer n , the elements of the set $\mathcal{G}_{n+2}^c \setminus \mathcal{G}_{n+1}^c$ will be written after the elements of the set $\mathcal{G}_{n+1}^c \setminus \mathcal{G}_n^c$. For every $i \in \mathbb{N}$ define $d_i: X \times X \rightarrow \mathbb{R}_+$ as

$$d_i(x, y) = \begin{cases} \frac{1}{2^i} d(g_i x, g_i y), & \text{if } x, y \in \text{int } D_{g_i}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $d_\infty: X \times X \rightarrow \mathbb{R}_+$ be given by the formula

$$d_\infty(x, y) = \sum_{i=1}^{\infty} d_i(x, y).$$

Note that

$$d_\infty(x, y) \leq \sum_{i=1}^{\infty} \frac{1}{2^i} = 1 < \infty \text{ for all } x, y \in X.$$

Since d is a dynamically generating pseudometric, d_∞ is a semimetric. We will show that it satisfies assumptions of Lemma 8. Fix $A \subset X$ and $x \in X$. For every $n \in \mathbb{N}$ let $s_n = d_0 + \dots + d_{n-1}$. Notice that $\text{dist}_{d_\infty}(A, \cdot) = \sup \{ \text{dist}_{s_n}(A, x) : n \in \mathbb{N} \}$. To see this fix $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $\sum_{i=N}^{\infty} 1/2^i < \varepsilon/2$. Let $a \in A$ be such that $\text{dist}_{s_N}(A, x) > s_N(a, x) - \varepsilon/2$. Then

$$\text{dist}_{s_N}(A, x) \leq \text{dist}_{d_\infty}(A, x) \leq d_\infty(a, x) \leq s_N(a, x) + \varepsilon/2 < \text{dist}_{s_N}(A, x) + \varepsilon.$$

Passing with $\varepsilon \rightarrow 0$ we get our claim.

By the above observation, to prove that $\text{dist}(A, \cdot)$ is lower semicontinuous, it is enough to show that, for every positive integer n the function $\text{dist}_{s_n}(A, \cdot)$ is lower semicontinuous. To this end, fix $n \in \mathbb{N}$ and $x \in X$. We want to prove that $\text{dist}_{s_n}(A, \cdot)$ is lower semicontinuous at x . Let $\{y_j\}_{j \in \mathbb{N}}$ tends to x with respect to τ . Let $\mathcal{F} = \{i \in \{1, \dots, n-1\} : x \in \text{int } D_{g_i}\}$. There exists $j_0 \in \mathbb{N}$ such that for every $j \geq j_0$ and every $i \in \mathcal{F}$

one has $y_j \in \text{int } D_{g_i}$. Fix $\varepsilon > 0$. It follows from Lemma 7 that there exists $j_1 > j_0$ such that for every $j > j_1$ one has

$$\text{dist}_{s_n}(A, x) - \varepsilon = \text{dist}_{\sum_{i \in \mathcal{F}} d_i}(A, x) - \varepsilon < \text{dist}_{\sum_{i \in \mathcal{F}} d_i}(A, y_j). \quad (1)$$

Then

$$\liminf_{j \rightarrow \infty} \text{dist}_{s_n}(A, y_j) \geq \liminf_{j \rightarrow \infty} \inf_{a \in A} \sum_{i \in \mathcal{F}} d_i(a, y_j) = \liminf_{j \rightarrow \infty} \text{dist}_{(\sum_{i \in \mathcal{F}} d_i)}(A, y_j) \geq \text{dist}_{s_n}(A, x) - \varepsilon,$$

where the last inequality follows from (1). Since ε is arbitrary we get the claim.

Fix an open cover \mathcal{U} of the space X and let η be provided for \mathcal{U} and d_∞ by Lemma 8. Let $N \in \mathbb{N}$ and $\delta > 0$ be provided for $\eta/4$ by the definition of the compatibility. There exist integers $M_1, M_2 > N$ such that $\sum_{i=M_1}^\infty 2^{-i} < \delta/2$ and $|\mathcal{G}_{M_2}^c| > M_1$. For any $m \in \mathbb{N}$ let $n_m = M_2 + m$. Fix a (d, n_m, δ) -spanning set F . Let $y_1, y_2 \in X$ and $x \in F$ be such that x is (d, n_m, δ) -close to both y_1 and y_2 . We will show that for every $g \in \mathcal{G}_m^c$ one has $d_\infty(gy_1, gy_2) < \eta$. We can assume that for every $i \in \{1, \dots, M_1 - 1\}$ one has $gy_1, gy_2 \in D_{g_i}$ (otherwise $d_i(gy_1, gy_2) = 0$). Then $d_i(gy_1, gy_2) = d(g_i gy_1, g_i gy_2) < \varepsilon/4$ since $g_i g \in \mathcal{G}_{n_m}^c$ and d is compatible with (\mathcal{G}, G) . Then

$$d_\infty(gy_1, gy_2) = \sum_{i=0}^{M_1-1} \frac{1}{2^i} d_i(y_1, y_2) + \sum_{j=M_1}^\infty \frac{1}{2^j} d_j(y_1, y_2) < 2 \cdot \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $r(d, n_m, \delta) \geq N(\mathcal{U}, m)$ and so

$$\frac{m + M_2}{m} \cdot \frac{1}{m + M_2} r(d, m + M_2, \delta) \geq \frac{1}{m} N(\mathcal{U}, m).$$

Consequently:

$$\limsup_{n_m \rightarrow \infty} \frac{1}{n_m} r(d, n_m, \delta) \geq \limsup_{m \rightarrow \infty} \frac{1}{m} N(\mathcal{U}, m).$$

Taking $\delta \rightarrow 0$ and the supremum over all open covers \mathcal{U} we get that $h_c(\mathcal{G}, G) \leq h_r(d, \mathcal{G}, G)$. \square

Remark 12. It is known that the above definitions of entropy are not equivalent without the assumption of the compactness even in case of \mathbb{Z} -action given by an iteration of a single map. To see this, let X be a totally bounded and non-compact space and $T: X \rightarrow X$ be the identity map. Then $h_r(T) = h_s(T) = 0$, whereas $h_c(T)$ is infinite.

Example 13. Consider the space $X = \{0, 1, \dots, m-1\}^\mathbb{Z}$ with the standard topology. For any integer q denote by $q: X \rightarrow X$ the shift by q symbols (depending on the sign of q the map q shifts right or left). Let \mathcal{G} be a pseudogroup generated by a set $G = \{-l, -k, 0, k, l\}$, where k and l are relatively prime positive numbers and $l > k$ (thus the pseudogroup \mathcal{G} is generated by \mathbb{Z}^2 -action). We will show that $h(\mathcal{G}, G) = 2l \log m$.

Let $d: X \times X \rightarrow \mathbb{R}_+$ be a pseudometric defining in the following way:

$$d(\{a_n\}_{n \in \mathbb{N}}, \{b_n\}_{n \in \mathbb{N}}) = \begin{cases} 0, & \text{if } a_0 = b_0, \\ 1, & \text{if } a_0 \neq b_0. \end{cases}$$

Because k and l are relatively prime d is dynamically generating. Other assumptions of Theorem 11 are also fulfilled. Hence, to compute the value of $h(\mathcal{G}, G)$ it is enough to examine the maximal possible cardinality of (d, n, ε) -separated set.

Let $\varepsilon \in (0, 1/2)$. For any pair $\{a_n\}_{n \in \mathbb{N}}, \{b_n\}_{n \in \mathbb{N}}$ of sequences from the fixed (d, n, ε) -separated set there exists a positive integer t such that

1. $a_t \neq b_t$,
2. $t = uk + wl$, where $|u| + |w| \leq n$.

It follows from (2) that $|t| \leq |ln|$. Consequently $s(d, n, \varepsilon) \leq m^{2ln+1}$ and

$$h(\mathcal{G}, G) \leq \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log m^{2ln+1} = 2l \log m.$$

On the other hand, because k and l are relatively prime there exist integers x and y such that $xk + yl = 1$. Hence, for any integer t one has $(tx)k + (ty)l = t$. Consequently for any p , if $|t| \leq p/(|x| + |y|)$, then t could be presented as a sum of p numbers from the set $\{-l, -k, 0, k, l\}$. We will show that, if $z \leq n - l(|x| + |y|)$, then for every s such that $s \leq zl$ it is possible to present s as a sum of n numbers from the set $\{-l, -k, 0, k, l\}$. Because the set $\{-l, -k, 0, k, l\}$ is symmetric, we can assume that $s > 0$. Let z_0 be the greatest integer such that $z_0 l \leq s$. Clearly $z_0 \leq z$. Since $s - z_0 l \leq l$ one also has $s - z_0 l \leq (n - z)/(|x| + |y|)$. Consequently $s - z_0 l$ could be presented as the sum of $n - z$ numbers from the set $\{-l, -k, 0, k, l\}$ and hence s could be presented as a sum of n such numbers.

Hence for every $\varepsilon > 0$ there exists a (n, ε) -separated set E such that:

$$|E| = m^{2l(n - l(|x| + |y|)) + 1} = m^{2nl - 2l^2(|x| + |y|) + 1}.$$

Consequently:

$$h(\mathcal{G}, G) \geq \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log m^{2nl - 2l^2(|x| + |y|) + 1} = 2l \log m.$$

This completes the proof.

Example 14. Let $X = \{0, 1\}^{\mathbb{Z}}$. Denote $[000] = \{\{x_i\}_{i \in \mathbb{Z}} : x_i = 0 \text{ for every } i \in \{-1, 0, 1\}\}$. Consider a pseudogroup \mathcal{G} generated by $\{g_2^{-1}, g_1^{-1}, \text{id}, g_1, g_2\}$, where $g_1 = \sigma|_{X \setminus [000]}$ is a restriction of the shift map and $g_2: [000] \rightarrow X$ is given by

$$(g_2(\{a_n\}_{n \in \mathbb{Z}}))_i = \begin{cases} a_i + a_{i+1} \pmod{2}, & \text{if } i > 0, \\ 0, & \text{if } i = 0, \\ a_{i-1} + a_i \pmod{2}, & \text{if } i < 0. \end{cases}$$

Define $d: X \times X \rightarrow \mathbb{R}_+$ by

$$d(\{a_n\}_{n \in \mathbb{Z}}, \{b_n\}_{n \in \mathbb{Z}}) = |a_{-1} - b_{-1}| + |a_0 - b_0| + |a_1 - b_1|.$$

Then d is a continuous dynamically generating pseudometric. Let $\varepsilon \in (0, 1/2)$ and $n \in \mathbb{N}$. Sequences $\{a_i\}_{i \in \mathbb{Z}}, \{b_i\}_{i \in \mathbb{Z}}$ are (d, n, ε) -close to each other if and only if $a_j = b_j$ for every $j \in \{-(n+1), \dots, 0, \dots, n+1\}$. This means that both numbers: $r(d, n, \varepsilon)$ and $s(d, n, \varepsilon)$ are equal to the number of symmetric cylinders of length $2n + 3$ and hence $h(\mathcal{G}, G) = 2 \log 2$.

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