



# Super WCG Banach spaces



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## ABSTRACT

A Banach space  $X$  admits an equivalent strongly uniformly Gâteaux smooth norm if and only if it contains the dense range of a super weakly compact operator, which is equivalent to say that  $X$  is generated by a convex super weakly compact set. Moreover, if  $X$  is strongly generated by a convex super weakly compact set, then there is an equivalent norm on  $X$  such that its restriction to any reflexive subspace of  $X$  is both uniformly convex and uniformly Fréchet smooth.

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## 1. Introduction

The notion of weakly compactly generated Banach space (WCG) is the first and most remarkable attempt to generalize separable Banach spaces keeping quite a few good structural, geometrical and topological properties. Recall that a Banach space  $X$  is WCG if there exists a weakly compact  $K \subset X$  such that  $\overline{\text{span}}(K) = X$ . The deep impact of WCG spaces in Banach space theory began with the seminal paper [1], and nowadays the amount of material is overwhelming, see [37] for an account of properties of WCG Banach spaces in the frame of nonseparable Banach space theory. Let us recall that we are dealing only with real Banach spaces and any operator here is always linear and bounded. As usual, if  $X$  is a Banach space, then  $B_X$  and  $S_X$  denote its unit ball and its unit sphere respectively. The well known Davis–Figiel–Johnson–Pelczynski interpolation theorem [10], see also [18, Theorem 13.22], gives easily the following characterization of WCG Banach spaces.

**Theorem 1.1.** *For a Banach space  $X$  the following are equivalent:*

- (a)  $X$  is weakly compactly generated;
- (b) there exists a weakly compact operator  $T : Z \rightarrow X$  with dense range;
- (c) there exists a reflexive space  $Z$  and operator  $T : Z \rightarrow X$  with dense range.

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In other words, WCG is the same as “weakly compact operator generated” or “reflexive generated”. Recent results depend on the possibility of changing reflexivity in (c) by a stronger condition, as *superreflexivity* or *Hilbert*, leading to the classes of superreflexive-generated Banach spaces and Hilbert-generated Banach spaces. In particular, we have in mind the paper [16] where several particular classes of “space-generated” properties are involved with smoothness conditions on equivalent renormings. We shall need the following notions.

**Definition 1.2.** The norm of the Banach space  $(X, \|\cdot\|)$  is said to be uniformly Gâteaux (UG) smooth if for every  $h \in X$

$$\sup\{\|x + th\| + \|x - th\| - 2 : x \in S_X\} = o(t) \text{ when } t \rightarrow 0.$$

Given a bounded set  $H \subset X$ , the norm is said to be  $H$ -UG smooth if

$$\sup\{\|x + th\| + \|x - th\| - 2 : x \in S_X, h \in H\} = o(t) \text{ when } t \rightarrow 0.$$

Finally, the norm is said to be strongly UG smooth if it is  $H$ -UG smooth for some bounded and linearly dense subset  $H \subset X$ .

To our knowledge, the most representative result showing different classes of space generation and its relationships is the following result of M. Fabian, G. Godefroy, P. Hájek and V. Zizler.

**Theorem 1.3.** (See Theorem 1 of [16].) For a Banach space  $X$  consider the assertions:

- (i)  $X$  is Hilbert-generated.
- (ii)  $X$  is superreflexive-generated.
- (iii)  $X$  is generated by the  $\ell_2$ -sum of superreflexive spaces.
- (iv)  $X$  admits an equivalent strongly UG smooth norm.
- (v)  $X$  is WCG and admits an equivalent UG smooth norm.
- (vi)  $X$  is a subspace of a Hilbert-generated space.

Then (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (v)  $\Rightarrow$  (vi). Moreover, no one of these implications can be reversed in general.

Our aim is to study the suitable “ideal-generated” or “subset-generated” version of superreflexive-generated Banach spaces, that is, in the spirit of (b) or (a) from Theorem 1.1. For this purpose we shall need the operator version of superreflexivity. Among several equivalent definitions, we shall give one based on ultrapowers. An operator  $T : X \rightarrow Y$  induces an operator between the ultrapowers of the spaces  $T^{\mathcal{U}} : X^{\mathcal{U}} \rightarrow Y^{\mathcal{U}}$  for a free ultrafilter  $\mathcal{U}$  on an index set.

**Definition 1.4.** An operator  $T : X \rightarrow Y$  is said to be super weakly compact if  $T^{\mathcal{U}}$  is weakly compact for any ultrafilter  $\mathcal{U}$  (equivalently, a free ultrafilter on  $\mathbb{N}$ ). The class of all super weakly compact operators will be denoted  $\mathfrak{W}^{super}$ .

We are following the notation of [13]. The class  $\mathfrak{W}^{super}$  is an operator ideal that was first studied by B. Beauzamy in [2–4] under the name of *uniformly convexifying* operators, being the link between the alternative definitions a sort of Enflo’s renorming theorem for operators. See also [24, Theorem 5.1] for some more characterizations. Note that  $\mathfrak{W}^{super}$  lies strictly between the compact and the weakly compact operators, and like them, it is a symmetric ideal as well, that is,  $T \in \mathfrak{W}^{super}$  if and only if  $T^* \in \mathfrak{W}^{super}$ .

Clearly, the identity map of a superreflexive Banach space is a natural example of super weakly compact operator. We can state now the definition that gives the title to this paper.

**Definition 1.5.** A Banach space  $X$  is said to be super weakly compactly generated (super WCG for short) if there exist a Banach space  $Z$  and a super weakly compact operator  $T : Z \rightarrow X$  such that  $T(Z)$  is dense in  $X$ .

We proved in [34, Proposition 4.6] that an operator  $T : Z \rightarrow X$  is super weakly compact if and only if  $\overline{T(B_Z)}$  is *finitely dentable* (see Definition 2.1). Moreover, if  $K \subset X$  is a finitely dentable bounded closed convex subset, then there exists a reflexive Banach space  $Z$  and an operator  $T : Z \rightarrow X$  such that  $K \subset T(B_Z)$  [34, Theorem 4.5]. In [8] the authors introduced the suggestive notion of *super weakly compact set* (see Definition 2.3) which is, for bounded closed convex sets, equivalent to being finitely dentable (see Proposition 2.4). From now on, we will adopt the terminology of [8] for consistency. Therefore, our results yield that  $X$  is super WCG if and only if there is  $K \subset X$  convex super weakly compact such that  $\overline{\text{span}}(K) = X$ , which agrees with [9, Definition 4.3]. In other words, in this setting “ideal-generated” and “subset-generated” essentially coincide.

The ideal  $\mathfrak{M}^{\text{super}}$  does not have the factorization property [2, p. 122] or [4, p. 80], see also Example 3.11. In particular, that means that there are super WCG Banach spaces which are not superreflexive-generated. It is natural to wonder how the class super WCG is related to the six classes in Theorem 1.3. The answer is the following.

**Theorem 1.6.** *A Banach space  $X$  is super WCG if and only if it admits an equivalent strongly UG smooth norm.*

Bearing in mind that  $X$  admits an equivalent UG smooth norm if and only if  $B_{X^*}$  is *uniform Eberlein* for the weak\* topology [17], Theorem 1.6 improves our previous result [34, Corollary 4.8] where we first dealt with Banach spaces generated by bounded closed convex finitely dentable subsets. A key ingredient for the proof is the symmetry of the ideal  $\mathfrak{M}^{\text{super}}$ .

A stronger notion of generation for Banach spaces is necessary in order to transfer properties from a super weakly compact generator to all the weakly compact subsets of the space.

**Definition 1.7.** A Banach space  $X$  is said to be strongly generated by a subset  $K \subset X$  if for any weakly compact  $H \subset X$  and  $\varepsilon > 0$  there is  $n \in \mathbb{N}$  such that  $H \subset nK + \varepsilon B_X$ .

This definition admits “space-generated” and “ideal-generated” variations in a quite obvious way. Banach spaces strongly generated by a weakly compact subset are called strongly WCG Banach spaces, and denoted SWCG (or  $\beta$ WCG). Their interesting properties have been studied by G. Schlüchtermann and R.F. Wheeler [35], see also [23]. For instance, if  $X$  is SWCG then it is weakly sequentially complete, and so the subspaces of  $X$  either contain  $\ell_1$  or are reflexive, by Rosenthal’s theorem. Here we shall consider Banach spaces strongly generated by a convex super weakly compact subset. That has been considered in [9, Definition 4.3] too.

**Definition 1.8.** A Banach space  $X$  is said to be strongly super weakly compactly generated ( $S^2$ WCG for short) if there is a convex super weakly compact set  $K \subset X$  that strongly generates  $X$ .

In spite of the length of the name, the notion of  $S^2$ WCG has very natural examples. It is well known that  $L_1(\mu)$  for a finite measure  $\mu$  is strongly Hilbert-generated and so it is  $S^2$ WCG. Moreover, if  $X$  is superreflexive then the Lebesgue–Bochner space  $L_1(\mu, X)$  is strongly superreflexive generated. Indeed, we

may assume that  $\mu$  is a probability. If  $H \subset L_1(\mu, X)$  is weakly compact, then it is uniformly integrable, that is, the sequence defined by

$$a_n = \sup_{\|f\| \geq n} \left\{ \int \|f\| d\mu : f \in H \right\}$$

converges to 0. The decomposition  $\mathbf{1}_{\|f\| \geq n} f + \mathbf{1}_{\|f\| < n} f$  for  $f \in H$  shows that  $H \subset a_n B_{L_1(\mu, X)} + n B_{L_2(\mu, X)}$  where  $L_2(\mu, X)$  is identified with a subset of  $X$  by its continuous injection into  $L_1(\mu, X)$ . That finishes the proof since  $L_2(\mu, X)$  is superreflexive [33].

**Theorem 1.9.** *Let  $X$  be a  $S^2$ WCG Banach space. Then there is an equivalent norm on  $X$  such that its restriction to any reflexive subspace of  $X$  is both uniformly convex and uniformly Fréchet smooth.*

This result extends qualitatively renorming results done for the spaces  $L_1(\mu)$  [14,27] and  $L_1(\mu, X)$  with  $X$  superreflexive [19]. Note that in [20] it is established that for a strongly superreflexive generated Banach space there is a renorming which is uniformly Fréchet smooth on its reflexive subspaces. Example 3.10 shows that the class of  $S^2$ WCG Banach spaces is strictly larger than the class of strongly superreflexive generated Banach spaces.

The structure of the paper is the following. The second section is a survey on super weak compactness, which includes the main equivalences from [8,34] in the convex case (Proposition 2.4) and an account of the properties which are relevant for the results of the paper (Proposition 2.7). We also describe the relationships with the uniformly convexifying operators of Beauzamy which are extremely useful for the proofs. The third section is devoted to super WCG Banach spaces and their renormings, including the proof of the main results and two examples. We believe that our notation is totally standard and can be found in reference books of Banach space theory as [7,11,18,26] for instance.

## 2. Super weak compactness

Maybe the most interesting example of a *super property* is the *superreflexivity*, introduced by James in [25]. A Banach space  $X$  is superreflexive if any ultrapower  $X^{\mathcal{U}}$  is reflexive for  $\mathcal{U}$  any free ultrafilter. Enflo's famous result [15] states that  $X$  is superreflexive if and only if it has an equivalent uniformly convex renorming. Beauzamy [2] extended the notion of superreflexivity to operators, under the name of “opérateurs uniformément convexifiants”. An operator  $T : Z \rightarrow X$  is *uniformly convex* if given  $\varepsilon > 0$  there is  $\delta > 0$  such that  $\|T(x) - T(y)\| < \varepsilon$  whenever  $\|x\| = \|y\| = 1$  and  $\|x + y\| > 2 - \delta$ . An operator  $T : Z \rightarrow X$  is said *uniformly convexifying* if it becomes uniformly convex after a suitable renorming of  $Z$ . Of course, uniformly convexifying operators coincide with the super weakly compact operators [24].

A localized version of superreflexivity for subsets was introduced in [34] using as a reference the previous work of G. Lancien in his Ph.D. Thesis [29,30]. Let  $C$  be a bounded closed convex set of a Banach space  $X$  and let  $\rho$  be a metric defined on  $X$  (the norm metric, for instance). We say that  $C$  is  $\rho$ -dentable if for any nonempty closed convex subset  $D \subset C$  and  $\varepsilon > 0$  it is possible to find an open halfspace  $H$  intersecting  $D$  such that  $\rho\text{-diam}(D \cap H) < \varepsilon$ . If  $C$  is  $\rho$ -dentable we may consider the following “derivation”

$$[D]_{\varepsilon}' = \{x \in D : \rho\text{-diam}(D \cap H) > \varepsilon, \forall H \in \mathbb{H}, x \in H\}.$$

Here  $\mathbb{H}$  denotes the set of all the open halfspaces of  $X$ . Clearly,  $[D]_{\varepsilon}'$  is what remains of  $D$  after removing all the slices of  $\rho$ -diameter less or equal than  $\varepsilon$ . Consider the sequence of sets defined by  $[C]_{\varepsilon}^0 = C$  and, for every  $n \in \mathbb{N}$ , inductively by

$$[C]_{\varepsilon}^n = [[C]_{\varepsilon}^{n-1}]_{\varepsilon}'.$$

Such a process can be extended to transfinite ordinal numbers in a quite natural way, and for any dentable set the process finishes at the empty set. However, we are only interested in sets for which the iteration process is finite.

**Definition 2.1.** The subset  $C \subset X$  is said to be  $\rho$ -finitely dentable if for every  $\varepsilon > 0$  there is  $n \in \mathbb{N}$  such that  $[C]_\varepsilon^n = \emptyset$ , where the set derivation is made with respect to  $\rho$ . If  $\rho$  is the norm metric, then we simply say that  $C$  is finitely dentable. The first  $n \in \mathbb{N}$  such that  $[C]_\varepsilon^n = \emptyset$  is called the index of  $\varepsilon$ -dentability and it is denoted  $Dz(C, \varepsilon)$ .

The motivation is the following result of Lancien [30] (see also [31]):  $X$  is superreflexive if and only if  $B_X$  is finitely dentable. Moreover, if  $X$  is uniformly convex then  $Dz(B_X, \varepsilon) \leq 1 + \delta_X(\varepsilon)^{-1}$  where  $\delta_X$  is the modulus of convexity of  $X$ . Note that Pisier's celebrated renorming result [33] implies that for a superreflexive space  $X$  there exists  $c > 0$  and  $p \geq 2$  such that  $Dz(B_X, \varepsilon) \leq c\varepsilon^{-p}$  for every  $\varepsilon \in (0, 1]$ . Part of our paper [34] is devoted to the study of the properties of finitely dentable sets in Banach spaces. The most relevant properties are that convex finitely dentable sets are weakly compact and uniform Eberlein with respect to the weak topology. Another interesting fact is that they characterize the super weak compactness of operators in the following sense, see [34, Proposition 4.6].

**Proposition 2.2.** A linear operator  $T : X \rightarrow Y$  is super weakly compact (equivalently, uniformly convexifying) if and only if  $\overline{T(B_X)}$  is finitely dentable.

This result, as stated, suggests that “finitely dentable” is not the best name. In [8] the authors introduced the notion of super weakly compact set and proved several properties similar to those already established for finitely dentable sets in [34]. See also [9] and some references therein showing that super weakly compactness has become a subject of active research. They say that a bounded closed convex set  $K$  is super weakly compact if any subset of a Banach space which is finitely representable in  $K$  is weakly compact. Finite representability for sets is done in a similar fashion as the one for Banach spaces, see [6] or [18] for instance. However, it is easy to give a shorter equivalent formulation using ultrapowers.

**Definition 2.3.** A subset  $K \subset X$  is said to be super weakly compact if  $K^\mathcal{U}$  is a weakly compact subset of  $X^\mathcal{U}$  for any free ultrafilter  $\mathcal{U}$ .

The relation of equivalence here is the same as in the case of Banach spaces, that is,  $(x_i)_{i \in I} \sim (y_i)_{i \in I}$  if and only if  $\lim_{i \in \mathcal{U}} \|x_i - y_i\| = 0$  where  $\mathcal{U}$  is a free ultrafilter on a set  $I$ . Note that it is enough to consider free ultrafilters on  $\mathbb{N}$  since the weak compactness is separably determined by the Eberlein–Šmul'yan theorem [18, Theorem 3.109]. We shall need some assorted definitions. A convex set  $C \subset X$  is said to have the *finite tree property* if there exists  $\varepsilon > 0$  such that  $C$  contains  $\varepsilon$ -separated dyadic trees of arbitrary height. Recall that a dyadic tree of height  $n \in \mathbb{N}$  is a set of the form  $\{x_s : |s| \leq n\}$ , indexed by finite sequences  $s \in \bigcup_{k=0}^n \{0, 1\}^k$  of length  $|s| \leq n$ , such that  $x_s = 2^{-1}(x_{s \smallfrown 0} + x_{s \smallfrown 1})$  for every  $|s| < n$ , where  $\{0, 1\}^0 := \{\emptyset\}$  indexes the root  $x_\emptyset$  and “ $\smallfrown$ ” denotes concatenation. We say that a dyadic tree  $\{x_s : |s| \leq n\}$  is  $\varepsilon$ -separated if  $\|x_{s \smallfrown 0} - x_{s \smallfrown 1}\| \geq \varepsilon$  for every  $|s| < n$ . A function  $f : C \rightarrow \mathbb{R}$  defined on a convex subset  $C \subset X$  is said *uniformly convex* if for every  $\varepsilon > 0$  there is  $\delta > 0$  such that  $\|x - y\| < \varepsilon$  whenever  $x, y \in C$  are such that

$$\frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) < \delta.$$

The most typical convex function on a Banach space, its norm  $\|\cdot\|$ , cannot be an uniformly convex function (neither a *strictly convex* function), so we shall modify the definition for norms. We say that a norm  $\|\cdot\|$  is uniformly convex on some bounded convex set  $K \subset X$  if  $f(x) = \|x\|^2$  is a uniformly convex function on  $K$ .

Note that a space  $X$  is uniformly convex if and only if its norm is uniformly convex (in the previous sense) on  $B_X$ , equivalently on any bounded convex subset  $K \subset X$ .

As we announced in the introduction, super weak compactness coincide with finite dentability for bounded closed convex subsets of a Banach space. The following result establishes the equivalence between both properties and several others studied in [8] and [34].

**Proposition 2.4.** *Let  $X$  be a Banach space and  $K \subset X$  a bounded closed convex subset. The following conditions are equivalent:*

- (i)  $K$  is super weakly compact;
- (ii)  $K$  is finitely dentable;
- (iii)  $K$  does not have the finite tree property;
- (iv) there is a reflexive Banach space  $Z$  and a super weakly compact operator  $T : Z \rightarrow X$  such that  $K \subset T(B_Z)$ ;
- (v) there is a bounded uniformly convex function  $f : K \rightarrow \mathbb{R}$ ;
- (vi) there is an equivalent norm  $\|\cdot\|$  on  $X$  which is uniformly convex on  $K$ .

**Proof.** The equivalences (ii) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) follow from [34, Theorem 2.2] applied to the identity map on  $K$ . On the other hand, (iv) $\Rightarrow$ (ii) is consequence of Proposition 2.2, and (ii) $\Rightarrow$ (iv) is contained in [34, Theorem 1.3]. Note that if  $K$  contains a  $\varepsilon$ -separated dyadic tree of height  $n$ , then  $Dz(K, \varepsilon/2) > n$ , following that (ii) $\Rightarrow$ (iii). The equivalence (i) $\Leftrightarrow$ (iii) is [8, Theorem 2.14]. In order to close the loop, assume (ii). Then  $H = K - K$  is finitely dentable by [34, Proposition 4.4] (see also Proposition 2.7). Clearly, if the norm  $\|\cdot\|$  of  $X$  is uniformly convex on  $H$ , then  $f_{x_0}(x) := \|x - x_0\|^2$  is uniformly convex on  $K$  for every  $x_0 \in K$ , and thus  $K$  is uniformly convexifiable in terms of [8]. Applying [8, Main Theorem], we get that  $K$  is super weakly compact and so (ii) $\Rightarrow$ (i), which completes the proof.  $\square$

Note that if a bounded closed convex subset  $K \subset X$  has the finite tree property, then there is  $\varepsilon > 0$  such that  $K^{\mathcal{U}}$  contains an infinite  $\varepsilon$ -separated dyadic tree for any free ultrafilter  $\mathcal{U}$  on  $\mathbb{N}$  (just follow the ideas in [6, p. 228]), and therefore  $K^{\mathcal{U}}$  is not weakly compact. That means that super weak compactness for closed convex sets can be checked just by one free ultrafilter on  $\mathbb{N}$ .

Property (vi) suggests to reproduce arguments involving uniformly convex norms for super weakly compact sets. For the next two examples we shall need a couple of definitions. It is said that a subset  $K$  has the *Banach–Saks property* if every sequence  $(x_n) \subset K$  has a subsequence  $(x_{n_k})$  such that its Cesàro averages  $k^{-1} \sum_{j=1}^k x_{n_j}$  are norm converging to some point of  $X$ . A bounded convex set  $K \subset X$  is said to have *normal structure* if every nonsingleton convex subset  $H \subset K$  has a nondiametral point  $x \in H$ , that is,  $\sup\{\|y - x\| : y \in H\} < \text{diam}(H)$ .

**Proposition 2.5.** *Convex super weakly compact sets of Banach spaces have the Banach–Saks property.*

**Proof.** It is possible to adapt the proof of Kakutani’s theorem as presented in [12, p. 78], but it is easier to use that a super weakly compact operator has the Banach–Saks property [5].  $\square$

**Proposition 2.6.** *If  $K \subset X$  is a convex super weakly compact set, then there is a renorming of  $X$  such that  $K$  has normal structure.*

**Proof.** We follow the ideas from the proof of [7, Proposition A.9]. By Proposition 2.7 (3) the set  $H = K - K$  is a convex super weakly compact set. By Proposition 2.4 (vi) there is a norm  $\|\cdot\|$  of  $X$  which is uniformly convex on  $H$ . Let  $S \subset K$  be a convex subset containing at least two different points  $u, v \in S$ . Take

$d = \text{diam}(S)$  and for  $\varepsilon = \|u - v\|$  find  $\delta > 0$  witnessing the uniform convexity of  $\|\cdot\|^2$  on  $H$ . Put  $x = (u + v)/2$  and observe that for any  $z \in S$  we have

$$\delta \leq \frac{\|u - z\|^2 + \|v - z\|^2}{2} - \|x - z\|^2 \leq d^2 - \|x - z\|^2$$

since  $u - z, v - z, x - z \in H$  and the uniform convexity of  $\|\cdot\|^2$ . Therefore  $\|x - z\| \leq \sqrt{d^2 - \delta} < d$  for every  $z \in S$  and thus  $x$  is a nondiametral point of  $S$ .  $\square$

Note that the normal structure implies the *fixed point property* for nonexpansive mappings. In [9], the authors produce a renorming having the super fixed point property.

Observe that the Proposition 2.4 requires the hypothesis of convexity, and implicitly our main results Theorem 1.6 and Theorem 1.9 too. Actually, we do not know if the closed convex hull of a super weakly compact is again super weakly compact, that is, a sort of Krein's theorem. We know that the answer is negative for finite dentability [34, Example 4.9] and it is also negative for the somehow related property of Banach-Saks [32].

In order to make this paper more self-contained, we include the following proposition which gathers [34, Proposition 4.4] and [34, Lemma 4.2] formulated here in terms of super weak compactness. The estimations of the dentability indices are implicit in the proofs. These results also can be found in [8] (Proposition 3.10, Corollary 3.11 and Lemma 4.5).

**Proposition 2.7.** “*Stability properties of convex super weakly compact sets*”.

1. A closed convex subset  $H \subset K$  of a convex super weakly compact is again super weakly compact. Moreover,  $Dz(H, \varepsilon) \leq Dz(K, \varepsilon)$ .
2. The image of a convex super weakly compact  $K$  set through an operator  $T$  is again super weakly compact. Moreover,  $Dz(T(K), \varepsilon) \leq Dz(K, \varepsilon/2\|T\|)$ .
3. The product of convex super weakly compact sets in a finite direct sum of Banach spaces is super weakly compact.
4. The sum and the convex hull of two convex super weakly compact sets are again super weakly compact. In particular, the absolute convex hull of a convex super weakly compact is super weakly compact.
5. Let  $K \subset X$  be such that for every  $\varepsilon > 0$  there is a convex super weakly compact set  $H_\varepsilon$  such that  $K \subset H_\varepsilon + \varepsilon B_X$ . Then  $K$  is super weakly compact.

### 3. Renormings in super WCG spaces

The most remarkable result in renorming of WCG spaces is Troyanski's theorem [36] (see also [11] for generalizations) that ensures the existence of equivalent locally uniformly convex norms. As super weakly compact sets are exactly the sets supporting uniformly convex functions, we may expect that renormings for super WCG should be “more uniform”. Actually, the uniform convexity of the norm given by Proposition 2.4 (vi) only extends to certain family of weakly compact sets which satisfy a local version of Definition 1.7.

**Definition 3.1.** Let  $K, H \subset X$  be subsets and suppose moreover that  $K$  is absolutely convex. The set  $H$  is said to be strongly generated by  $K$  if for every  $\varepsilon > 0$  there is  $n \in \mathbb{N}$  such that  $H \subset nK + \varepsilon B_X$ .

This definition is necessary since there are scenarios where the strongly generated subsets are known. For instance, consider a SWCG Banach  $X$  space and a probability measure space  $(\Omega, \Sigma, \mu)$ . Then [28, Theorem 1] says that there exists a symmetric weakly compact  $K \subset L_1(\mu, X)$  that strongly generates any weakly



compact decomposable set of  $L_1(\mu, X)$ . A set  $H \subset L_1(\mu, X)$  is called decomposable if  $\mathbf{1}_A f + \mathbf{1}_{\Omega \setminus A} g \in H$  whenever  $f, g \in H$  and  $A \in \Sigma$ .

We shall begin with an improvement of statement (vi) of [Proposition 2.4](#).

**Theorem 3.2.** *Let  $K \subset X$  be an absolutely convex super weakly compact. There is an equivalent norm  $\|\cdot\|$  on  $X$  such that its restriction to convex sets strongly generated by  $K$  is uniformly convex.*

**Proof.** Without loss of generality we may assume that  $K = T(B_Z)$  where  $T : Z \rightarrow X$  is a uniformly convex operator and  $Z$  is reflexive by [Proposition 2.4](#) (iv). Consider the sequence of equivalent norms on  $X$

$$\|x\|_k^2 = \inf\{\|x - T(z)\|^2 + k^{-1}\|z\|^2 : z \in Z\}.$$

Note that the infimum is actually attained since  $Z$  is reflexive. Fix  $H$  a subset strongly generated by  $K$ . Note that  $\lim_k \|x\|_k = 0$  uniformly on  $H$ . We claim that the norm  $\|\cdot\|$  on  $X$  defined by  $\|\cdot\|^2 = \sum_{k=1}^{\infty} 2^{-k} \|\cdot\|_k^2$  has the desired property. Fix  $\varepsilon > 0$  and suppose that  $(x_n), (y_n) \subset H$  are such that

$$\lim_n (2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2) = 0.$$

An standard convexity argument (see for instance [\[11, Fact II.2.3\]](#)) yields that

$$\lim_n (2\|x_n\|_k^2 + 2\|y_n\|_k^2 - \|x_n + y_n\|_k^2) = 0$$

for any  $k \in \mathbb{N}$ . Fix  $k \in \mathbb{N}$  such that  $\|x\|_k < \varepsilon$  for every  $x \in H$  and find  $(z_n), (w_n) \subset Z$  such that

$$\begin{aligned} \|x_n\|_k^2 &= \|x_n - T(z_n)\|^2 + k^{-1}\|z_n\|^2, \\ \|y_n\|_k^2 &= \|y_n - T(w_n)\|^2 + k^{-1}\|w_n\|^2. \end{aligned}$$

Note that the sequences  $(z_n), (w_n)$  are bounded. For the sum of the points we have

$$\|x_n + y_n\|_k^2 \leq \|x_n + y_n - T(z_n + w_n)\|^2 + k^{-1}\|z_n + w_n\|^2$$

and so, using the convexity of the squared norm, we obtain that

$$k^{-1}(2\|z_n\|^2 + 2\|w_n\|^2 - \|z_n + w_n\|^2) \leq 2\|x_n\|_k^2 + 2\|y_n\|_k^2 - \|x_n + y_n\|_k^2.$$

Therefore

$$\lim_n (2\|z_n\|^2 + 2\|w_n\|^2 - \|z_n + w_n\|^2) = 0$$

which implies that  $\lim_n \|T(z_n) - T(w_n)\| = 0$  by the uniform convexity of  $T$ . Take an  $N \in \mathbb{N}$  such that  $\|T(z_n) - T(w_n)\| < \varepsilon$  if  $n \geq N$ . Then we have

$$\|x_n - y_n\| \leq \|x_n - T(z_n)\| + \|T(z_n) - T(w_n)\| + \|y_n - T(w_n)\| < 3\varepsilon$$

for  $n \geq N$ . That shows  $\lim_n \|x_n - y_n\| = 0$  as we wanted.  $\square$

**Remark 3.3.** The formula used for the norm is a transfer trick well known for locally uniformly rotund renorming (LUR) [\[22\]](#), see also [\[11, Theorem II.2.1\]](#). In particular, if  $X$  is super WCG the norm provided by [Theorem 3.2](#) is LUR.



Dual WCG Banach spaces admit equivalent dual LUR norms [22], see also [11]. We have the following.

**Proposition 3.4.** *Let  $X$  be a dual Banach space generated by a super weakly compact convex set  $K$ . There is an equivalent dual norm  $\|\cdot\|$  on  $X$  such that its restriction to convex sets strongly generated by  $K$  is uniformly convex.*

**Proof.** Let  $\|\cdot\|$  be a dual norm on  $X$ . We construct  $\|\cdot\|$  as in Theorem 3.2. We only need to check that it is  $w^*$ -lower semicontinuous which is easy using the fact that the infimum in the definition of  $\|\cdot\|_k$  is attained.  $\square$

This is another observation about dual renormings and, actually, a missing detail for the proof of [34, Corollary 4.8].

**Lemma 3.5.** *Suppose that  $X$  is a dual Banach space and  $T : X \rightarrow Y$  is super weakly compact and  $w^*$ - $w$ -continuous. Then there is an equivalent dual norm on  $X$  such that  $T$  becomes uniformly convex.*

**Proof.** Suppose that  $X$  is endowed with a (nondual) norm such that  $T$  is uniformly convex. We claim that the norm  $\|\cdot\|$  on  $X$  having  $\overline{B_X}^{w^*}$  as the unit ball makes  $T$  also uniformly convex. Given  $\varepsilon > 0$  there is  $\delta > 0$  such that  $x, y \in B_X$  and  $\|\frac{x+y}{2}\| > 1 - \delta$  implies  $\|T(x) - T(y)\| < \varepsilon$ . As a consequence, if  $H$  is a halfspace such that  $H \cap (1 - \delta)B_X = \emptyset$  then  $\text{diam}(T(H \cap B_X)) \leq \varepsilon$ . Take  $x, y \in X$  with  $\|x\| = \|y\| = 1$  and  $\|x + y\| > 2 - 2\delta$ , that is,  $x, y \in \overline{B_X}^{w^*}$  and  $\frac{x+y}{2} \notin (1 - \delta)\overline{B_X}^{w^*}$ . Take  $H$  a  $w^*$ -open halfspace with  $\frac{x+y}{2} \in H$  and  $H \cap (1 - \delta)\overline{B_X}^{w^*} = \emptyset$ . Observe that  $\|x - y\| \leq 2 \text{diam}(H \cap \overline{B_X}^{w^*})$ . Now, by the  $w^*$ - $w$ -continuity of  $T$  we have

$$T(H \cap \overline{B_X}^{w^*}) \subset \overline{T(H \cap B_X)}^w = \overline{T(H \cap B_X)}.$$

As  $\text{diam}(\overline{T(H \cap B_X)}) = \text{diam}(T(H \cap B_X)) \leq \varepsilon$ , we get that  $\|T(x) - T(y)\| \leq 2\varepsilon$  and so the uniform convexity of  $T$  with respect to  $\|\cdot\|$ .  $\square$

Given  $H \subset X$ , the seminorm on  $X^*$  of uniform convergence on  $H$  is denoted  $\mathfrak{p}_H$ , that is,  $\mathfrak{p}_H(x^*) = \sup\{|x^*(x)| : x \in H\}$ . The following Šmulyan's criterion for  $H$ -UG smooth norms is contained in the proof of [16, Theorem 4] and it is analogous to [11, Theorem II.6.7].

**Lemma 3.6.** *Let  $X$  be a Banach space and  $H \subset X$  a bounded subset. The norm on  $X$  is  $H$ -UG smooth if and only if  $\mathfrak{p}_H(x_n^* - y_n^*) = 0$  whenever  $(x_n^*), (y_n^*) \subset S_{X^*}$  are such that  $\lim_n \|x_n^* + y_n^*\| = 2$ .*

**Lemma 3.7.** *Suppose that  $X$  is  $K$ -UG smooth and  $H$  is strongly generated by  $K$ , then  $X$  is  $H$ -UG smooth as well.*

**Proof.** Let  $(x_n^*), (y_n^*) \subset S_{X^*}$  such that  $\lim_n \|x_n^* + y_n^*\| = 2$ . Fix  $\varepsilon > 0$  and find  $m \in \mathbb{N}$  such that  $H \subset mK + \varepsilon B_X$ . By Lemma 3.6, take  $N \in \mathbb{N}$  such that  $\mathfrak{p}_K(x_n^* - y_n^*) < \varepsilon/m$  for  $n \geq N$ . It is easy to see that  $\mathfrak{p}_H(x_n^* - y_n^*) < 3\varepsilon$  for  $n \geq N$ , and thus the norm of  $X$  is  $H$ -UG smooth, again by Lemma 3.6.  $\square$

This result is [9, Theorem 4.5] with small changes. Let us recall that it is a natural consequence of the symmetry of  $\mathfrak{W}^{super}$ , as we already sketched in the proof of [34, Corollary 4.8].

**Theorem 3.8.** *Let  $K \subset X$  be an absolutely convex super weakly compact set. There is an equivalent norm  $\|\cdot\|$  on  $X$  such that it is  $H$ -UG smooth for any  $H \subset X$  bounded and strongly generated by  $K$ .*

**Proof.** Take  $T : Z \rightarrow X$  a super weakly compact operator such that  $K \subset T(B_Z)$  where  $Z$  is reflexive (see Proposition 2.4 (iv)). Then the adjoint  $T^* : X^* \rightarrow Z^*$  is super weakly compact as well. By Lemma 3.5 we may renorm  $X^*$  with a dual norm  $\|\cdot\|$  such that  $T^*$  is uniformly convex. Moreover, we may assume that  $X$  is endowed with the induced predual norm. We claim that this norm is  $K$ -UG smooth. Indeed, applying Lemma 3.6, take  $(x_n^*), (y_n^*) \subset S_{X^*}$  such that  $\lim_n \|x_n^* + y_n^*\| = 2$ . Since  $T^*$  is uniformly convex, we have

$$\begin{aligned} 0 &= \lim_n \|T^*(x_n^*) - T^*(y_n^*)\| = \limsup_n \{|T^*(x_n^* - y_n^*)(z)| : z \in B_Z\} \\ &= \limsup_n \{|(x_n^* - y_n^*)(T(z))| : z \in B_Z\} \geq \lim_n \mathfrak{p}_K(x_n^* - y_n^*). \end{aligned}$$

Therefore,  $\lim_n \mathfrak{p}_K(x_n^* - y_n^*) = 0$  and the norm on  $X$  is  $K$ -UG smooth as desired. Now, by Lemma 3.7 the norm  $\|\cdot\|$  built on  $X$  is  $H$ -UG smooth for every  $H \subset X$  strongly generated by  $K$ .  $\square$

**Remark 3.9.** If  $T : Z \rightarrow X$  is super weakly compact and  $K = \overline{T(B_Z)}$ , then the fact that  $T^*$  is super weakly compact implies that  $(B_{X^*}, w^*)$  is finitely dentable with respect to the seminorm  $\mathfrak{p}_K$ , see [34, Corollary 4.8]. In [21] the authors develop the duality between smoothness of renormings of  $X$  and indices of dentability of  $(B_{X^*}, w^*)$ . In fact, [21, Theorem 5] can be used to provide another proof of Theorem 3.8 as it is done in [9].

**Proof of Theorem 1.6.** If  $X$  is generated by an absolutely convex super weakly compact  $K$ , Theorem 3.8 implies that  $X$  has an equivalent  $K$ -UG smooth renorming. Suppose now that  $X$  is strongly UG smooth, so there is  $K \subset X$  total such that the norm is  $K$ -UG smooth. Lemma 3.6 implies that we may suppose  $K$  to be absolutely convex and closed. By [16, Lemma 1],  $K$  is weakly compact (see also [18, Proposition 14.18]). Use the Davis–Figiel–Johnson–Pelczynski interpolation theorem to find a reflexive Banach space  $Z$  and a operator  $T : Z \rightarrow X$  such that  $T(B_Z) \subset 2^n K + 2^{-n} B_X$  for every  $n \in \mathbb{N}$ , see the proof of [18, Theorem 13.22]. Note that  $T(B_Z)$  is strongly generated by  $K$ , and so  $X$  is  $T(B_Z)$ -UG smooth by Lemma 3.7. Now Lemma 3.6 yields that

$$\begin{aligned} 0 &= \lim_n \mathfrak{p}_{T(B_Z)}(x_n^* - y_n^*) = \limsup_n \{|(x_n^* - y_n^*)(T(z))| : z \in B_Z\} \\ &= \limsup_n \{|T^*(x_n^* - y_n^*)(z)| : z \in B_Z\} = \lim_n \|T^*(x_n^*) - T^*(y_n^*)\| \end{aligned}$$

whenever  $(x_n^*), (y_n^*) \subset S_{X^*}$  are such that  $\lim_n \|x_n^* + y_n^*\| = 2$ , that is,  $T^*$  is uniformly convex. Therefore  $T$  is a super weakly compact operator and so  $X$  is super WCG.  $\square$

**Proof of Theorem 1.9.** By Theorem 3.2 we know that there is an equivalent norm  $\|\cdot\|_1$  on  $X$  such that its restriction to any reflexive subspace of  $X$  is uniformly convex. On the other hand, by Theorem 3.8 there is an equivalent norm  $\|\cdot\|_2$  on  $X$  such that given a reflexive subspace  $Y \subset X$ , then  $\|\cdot\|_2$  is  $B_Y$ -UG smooth. In particular, the restriction of  $\|\cdot\|_2$  to  $Y$  is uniformly Fréchet smooth. Our aim is to show that the norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  can be “averaged” in Asplund’s sense, see [11, II.4]. Let  $P$  denote the set of equivalent norms on  $X$  endowed with the distance  $\rho(p, q) = \sup\{|p(x) - q(x)| : \|x\| = 1\}$ . The metric space  $(P, \rho)$  is a Baire space [11, p. 52].

We claim that the set of norms sharing the property of  $\|\cdot\|_1$  is a residual subset of  $(P, \rho)$ , that is, it contains a dense  $\mathcal{G}_\delta$ -set. For any  $p \in P$  consider the set

$$G(p, j) = \{q \in P : \sup\{|p(x)^2 + j^{-1}\|x\|_1^2 - q(x)^2| : \|x\| = 1\} < j^{-2}\}.$$

By construction  $G(p, j)$  is open in  $(P, \rho)$  and  $G_k = \bigcup_{p \in P} \bigcup_{j \geq k} G(p, j)$  is dense. We will show that any  $q \in \bigcap_{k=1}^\infty G_k$  is uniformly convex restricted to any  $Y \subset X$  reflexive. Suppose that  $(x_n), (y_n) \subset B_Y$  are such that

$$\lim_n (2q(x_n)^2 + 2q(y_n)^2 - q(x_n + y_n)^2) = 0.$$

Given  $k \in \mathbb{N}$  then  $q \in G(p, j)$  for some  $p \in P$  and some  $j \geq k$ . Using convexity we deduce that

$$j^{-1}(2\|x_n\|_1^2 + 2\|y_n\|_1^2 - \|x_n + y_n\|_1^2) < 8j^{-2} + 2q(x_n)^2 + 2q(y_n)^2 - q(x_n + y_n)^2.$$

Taking limits as  $n \rightarrow \infty$  we have

$$j^{-1} \limsup_n (2\|x_n\|_1^2 + 2\|y_n\|_1^2 - \|x_n + y_n\|_1^2) \leq 8j^{-2}.$$

That is,  $\limsup_n (2\|x_n\|_1^2 + 2\|y_n\|_1^2 - \|x_n + y_n\|_1^2) \leq 8j^{-1} \leq 8k^{-1}$ . Since  $k \in \mathbb{N}$  was arbitrary, we have  $\lim_n (2\|x_n\|_1^2 + 2\|y_n\|_1^2 - \|x_n + y_n\|_1^2) = 0$ , and the uniform convexity of  $\|\cdot\|_1$  implies that  $\lim_n \|x_n - y_n\| = 0$ . Therefore  $q$  is uniformly convex on  $Y$  as desired.

In order to show that the set of norms sharing the property of  $\|\cdot\|_2$  is a residual subset of  $(P, \rho)$  too it is enough to work on the set of equivalent dual norms on  $X^*$  because the duality map is a homeomorphism.

By Lemma 3.6 it is enough to show that there is a residual set of equivalent dual norms  $\|\cdot\|$  on  $X^*$  such that  $\lim_n \mathfrak{p}_{B_Y}(x_n^* - y_n^*) = 0$  whenever  $Y \subset X$  is reflexive and  $(x_n^*), (y_n^*) \subset B_{X^*}$  are such that

$$\lim_n (2\|x_n^*\|^2 + 2\|y_n^*\|^2 - \|x_n^* + y_n^*\|^2) = 0.$$

It is obvious that the same argument as the one used before for  $\|\cdot\|_1$  will give the desired result. Now, the intersection of two residual sets in the Baire space  $(P, \rho)$  is nonempty, thus there exist norms sharing the properties of  $\|\cdot\|_1$  and  $\|\cdot\|_2$ .  $\square$

The following example shows that there exist  $S^2$ WCG Banach spaces which are not superreflexive generated. Note that such spaces cannot be reflexive because a reflexive  $S^2$ WCG Banach space is superreflexive, and they must be nonseparable since separable Banach spaces are Hilbert generated.

**Example 3.10.** Let  $(p_k)$  be an enumeration of  $(1, 2] \cap \mathbb{Q}$ . Then the space

$$X = \left( \sum_{k=1}^{\infty} \ell_{p_k}(\omega_1) \right)_1$$

is  $S^2$ WCG, but  $X$  is not superreflexive generated.

**Proof.** We claim that  $K = \prod_{k=1}^{\infty} 2^{-k} B_{\ell_{p_k}(\omega_1)}$  is super weakly compact. Indeed, observe that  $K \subset \prod_{k=1}^n 2^{-k} B_{\ell_{p_k}(\omega_1)} + 2^{-n} B_X$  for every  $n \in \mathbb{N}$ . Since  $\prod_{k=1}^n 2^{-k} B_{\ell_{p_k}(\omega_1)}$  is a finite sum of convex super weakly compact subsets it is again super weakly compact by Proposition 2.7 (2). Now Proposition 2.7 (4) implies that  $K$  is super weakly compact.

Let  $H \subset X$  be weakly compact, and let  $\pi_k$  be the projection on the  $k$ -th summand of  $X$ . We claim that for every  $\varepsilon > 0$  there is  $n \in \mathbb{N}$  such that

$$\sup \left\{ \sum_{k=n+1}^{\infty} \|\pi_k(x)\| : x \in H \right\} \leq \varepsilon. \quad (1)$$

Indeed, assume that for some  $\varepsilon > 0$  the property does not hold. Then we can find  $x_1 \in H$ ,  $n_1 \in \mathbb{N}$  and  $w_k \in \ell_{p_k}(\omega_1)^*$  with  $\|w_k\| = 1$  for  $k \leq n_1$  such that

$$\sum_{k=1}^{n_1} w_k(\pi_k(x_1)) > \varepsilon \quad \text{and} \quad \sum_{k=n_1+1}^{\infty} \|\pi_k(x_1)\| < \varepsilon/2.$$

Find now  $x_2 \in H$ ,  $n_2 > n_1$  and  $w_k \in \ell_{p_k}(\omega_1)^*$  with  $\|w_k\| = 1$  for  $n_1 < k \leq n_2$  such that

$$\sum_{k=n_1+1}^{n_2} w_k(\pi_k(x_2)) > \varepsilon \quad \text{and} \quad \sum_{k=n_2+1}^{\infty} \|\pi_k(x_2)\| < \varepsilon/2.$$

Repeating inductively this argument we get sequences  $(x_k) \subset H$ ,  $(n_k) \subset \mathbb{N}$  and norm one functionals  $w_k \in \ell_{p_k}(\omega_1)^*$  satisfying analogous estimations. Consider the operator  $T : X \rightarrow \ell_1$  defined by  $T(x) = (w_k(\pi_k(x)))_{k=1}^{\infty}$ . Since  $\ell_1$  has the Schur property, we have that  $T(H)$  is a norm compact subset of  $\ell_1$ . On the other hand, by the previous construction we have  $\|T(x_k) - T(x_j)\| > \varepsilon/2$  for  $k \neq j$ , and thus  $T(H)$  cannot be norm compact. This contradiction proves the claim.

Now we are ready to show that  $K$  strongly generates  $X$ . Let  $H \subset X$  be a weakly compact subset and  $\varepsilon > 0$ . Take  $n \in \mathbb{N}$  such that inequality (1) is satisfied. If  $m > 0$  is such that  $\pi_k(H) \subset m2^{-k}B_{\ell_{p_k}(\omega_1)}$  for every  $k \leq n$ , then  $H \subset mK + \varepsilon B_X$ .

In order to prove the second statement, consider the space

$$Y = \left( \sum_{k=1}^{\infty} \ell_{p_k}(\omega_1) \right)_2.$$

The identity map  $J : X \rightarrow Y$  is an operator with dense range. If  $X$  were superreflexive generated, then  $Y$  would be superreflexive generated too. But  $Y$  is not superreflexive generated since this space is the example given in [16] showing the nonreversibility of (ii) $\Rightarrow$ (iii) in Theorem 1.3.  $\square$

The construction given in Example 3.10 easily implies that  $\mathfrak{W}^{super}$  does not have the factorization property, but it has the disadvantage of being nonseparable. The following is an example of separable convex super weakly compact set that cannot be interpolated by a superreflexive space.

**Example 3.11.** Consider  $X = (\sum_{k=2}^{\infty} \ell_k)_2$  and  $K = \prod_{k=2}^{\infty} 2^{-k}B_{\ell_k}$ . Then  $X$  is reflexive and separable,  $K$  is a super weakly compact set that generates  $X$  and

$$\sup\{\varepsilon^p Dz(K, \varepsilon) : \varepsilon \in (0, 1)\} = +\infty$$

for every  $p > 1$ . In particular, we have  $K \not\subset T(B_Z)$  for any superreflexive space  $Z$  and any operator  $T : Z \rightarrow X$ .

**Proof.** Some of the statements can be easily checked and the super weak compactness of  $K$  follows by the same proof as in the previous example. Only the estimation of the growth of  $Dz(K, \varepsilon)$  needs a proof. Fix  $k \in \mathbb{N}$  and take  $\varepsilon \in (0, 2^{-k})$ . A simple homogeneity argument gives that

$$Dz(K, \varepsilon) \geq Dz(B_{\ell_k}, 2^k \varepsilon) \geq (2^k \varepsilon)^{-k} = 2^{-k^2} \varepsilon^{-k}$$

where we are using that  $Dz(B_{\ell_p}, \varepsilon) \geq \varepsilon^{-p}$ . Such an estimation is obtained as follows. Start a  $2\varepsilon$ -separated dyadic tree in  $\ell_p$  with root 0. Set the first level as  $(\varepsilon, 0, 0, \dots)$  and  $(-\varepsilon, 0, 0, \dots)$ , the second level as  $(\varepsilon, \varepsilon, 0, \dots)$ ,  $(\varepsilon, -\varepsilon, 0, \dots)$ ,  $(-\varepsilon, \varepsilon, 0, \dots)$ ,  $(-\varepsilon, -\varepsilon, 0, \dots)$ , and so on until the  $n$ -th level. If  $n\varepsilon^p \leq 1$ , then that tree is contained in the unit ball and, in such a case,  $Dz(B_{\ell_p}, \varepsilon) > n$ . Taking  $n$  as the integer part of  $\varepsilon^{-p}$  we get the desired bound.  $\square$

We want to finish this paper with a reflection on an alternative meaning for the sentence “super WCG”: What are the Banach spaces  $X$  such that their ultraproducts  $X^{\mathcal{U}}$  are WCG? We believe that such class of Banach spaces might be very restrictive as the next partial result hints.

**Proposition 3.12.** *Let  $X$  be a Banach space, let  $K$  be a convex weakly compact set and let  $\mathcal{U}$  be a free ultrafilter on  $\mathbb{N}$ . Assume that  $K^{\mathcal{U}}$  is weakly compact and generates  $X^{\mathcal{U}}$ , then  $X$  is superreflexive.*

**Proof.** First note that  $K$  is a super weakly compact and by Proposition 2.7 (3) we may assume that  $K$  is absolutely convex taking  $\text{conv}(K \cup (-K))$ . We claim that  $B_X$  is strongly generated by  $K$ . Assume the contrary, so there is  $\varepsilon > 0$  such that for every  $n \in \mathbb{N}$  we can find  $x_n \in B_X \setminus (nK + \varepsilon B_X)$ . By construction, the element  $(x_n) \in X^{\mathcal{U}}$  satisfies  $\|(x_n) - (y_n)\| \geq \varepsilon$  for every  $(y_n) \in \bigcup_{m=1}^{\infty} mK^{\mathcal{U}}$  which is a contradiction. Now  $B_X$  is weakly compact since it is strongly generated by a weak compact. Moreover  $B_X$  is super weakly compact by Proposition 2.7 (4), and thus  $X$  is superreflexive.  $\square$

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