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Regularity results for a class of obstacle problems with nonstandard growth

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A B S T R A C T

We consider the obstacle problem related to the following energy with nonstandard growth

$$\int_\Omega |Du|^{p(x)} \log(e + |Du|)\, dx.$$

We investigate the regularity properties of solutions to the obstacle problems along with a suitable assumptions on the variable exponent $p(\cdot)$ and the obstacle.

1. Introduction

In this paper, we study the regularity theory for solutions to a certain obstacle problem with nonstandard growth. Let Ω be a bounded open set in $\mathbb{R}^n$ $(n \geq 2)$, $p(\cdot) : \Omega \to [\gamma_1, \gamma_2]$ with $1 < \gamma_1 \leq \gamma_2 < \infty$ be a continuous function, and the functions $\Phi : \Omega \times [0, \infty) \to [0, \infty)$ and $\partial\Phi : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be denoted by

$$\Phi(x, t) := |t|^{p(x)} \log(e + t) \quad \text{and} \quad \partial\Phi(x, \xi) := D_\xi(\Phi(x, |\xi|)), \tag{1.1}$$

where D_ξ is the gradient with respect to the ξ-variable. For a function $\psi : \Omega \to [-\infty, \infty]$ called the *obstacle*, we define a functions space by

$$\mathcal{K}^\Phi_\psi(\Omega) := \left\{ f \in W^{1,\Phi}(\Omega) : f \geq \psi \text{ a.e. in } \Omega \right\}.$$

Here, $W^{1,\Phi}(\Omega)$ is a Sobolev space related to the function Φ, for which we will introduce in the next section. In this setting, we say a function $u \in \mathcal{K}^\Phi_\psi(\Omega)$ is a solution to the *obstacle problem* of $\mathcal{K}^\Phi_\psi(\Omega)$ if it satisfies

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$$\int_\Omega \partial\Phi(x,Du)\cdot D(\varphi-u)\,dx \geq 0$$

for all $\varphi \in \mathcal{K}^\Phi_\psi(\Omega)$ with $\varphi - u$ having a compact support in Ω, which is equivalent to that

$$\int_\Omega \partial\Phi(x,Du)\cdot D\varphi\,dx \geq 0 \tag{1.2}$$

for all $\varphi \in W^{1,\Phi}(\Omega)$ with a compact support and $\varphi \geq \psi - u$ a.e. in Ω. Under the above setting, we will prove the following regularity properties for solutions to the obstacle problem of Φ.

Theorem 1.1. *Suppose the variable exponent $p(\cdot)$ is log-Hölder continuous, that is, $p(\cdot)$ satisfies*

$$L := \sup_{0<r<\frac{1}{2}} \omega(r)\log\left(\frac{1}{r}\right) < \infty, \tag{1.3}$$

where $\omega(\cdot) : [0,\infty) \to [0,\infty)$ is the modulus of continuity of $p(\cdot)$, and the obstacle ψ is Hölder continuous. Let $u \in \mathcal{K}^\Phi_\psi(\Omega)$ be a solution to the obstacle problem of $\mathcal{K}^\Phi_\psi(\Omega)$. Then u is Hölder continuous.

Theorem 1.2. *Suppose the variable exponent $p(\cdot)$ and the gradient of the obstacle ψ are Hölder continuous. Let $u \in \mathcal{K}^\Phi_\psi(\Omega)$ be a solution to the obstacle problem of $\mathcal{K}^\Phi_\psi(\Omega)$. Then Du is Hölder continuous.*

The Obstacle problems are strongly related to many physical phenomena hence the study of those problems is one of main topics in the fields of the calculus of variation and the partial differential equation, see for instance the monograph [31]. Essentially, they are linked to the minimizing problems of energy functionals (for example)

$$\int_\Omega \mathcal{F}(x,Du)\,dx \tag{1.4}$$

in convex admissible sets constrained by obstacle functions. Here, the density function $\mathcal{F} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfies a suitable convexity condition. Indeed, from a basic computation, see for instance [29], we see that for $g \in W^{1,\Phi}(\Omega)$ with $\psi \leq g$ on $\partial\Omega$, the minimizer of the following energy functional

$$u \in \{w \in \mathcal{K}^\Phi_\psi(\Omega) : w = g \text{ on } \partial\Omega\} \quad \mapsto \quad \int_\Omega \Phi(x,|Du|)\,dx \tag{1.5}$$

is the solution to the obstacle problem of $\mathcal{K}^\Phi_\psi(\Omega)$ with $u = g$ on $\partial\Omega$.

The regularity theory for the elliptic obstacle problems with standard growth, i.e., $\mathcal{F}(x,\xi) \approx |\xi|^p$ with $1 < p < \infty$ in (1.4), is now well understood, for which we refer to classical works [36,7,6] and related references, and its parabolic counterpart has been recently developed, see for instance [4,39,23].

A first relevant extension of such results to the setting of non-uniformly elliptic operators has been obtained in the setting of functionals with $p(x)$-growth, i.e., $\mathcal{F}(x,\xi) \approx |\xi|^{p(x)}$. Regularity problems for minimizers of this functional have been intensively studied in last twenty years; see for instance [1,11] and related references. In particular, for the obstacle problems we refer to [17–20,29,5], where sharp regularity is obtained starting from the techniques developed in the unconstrained case.

Over the recent years, interest in so-called non-autonomous functionals with non-standard growth conditions, has been rapidly increasing. In this situation the functionals with $p(x)$-growth are a particular case. These are indeed functionals as in (1.4) having an energy density $\mathcal{F}$ with both ellipticity or growth properties strongly that are determined by the occurrence of x-variable. They can be framed in the general class of functionals with (p,q)-growth functionals, as defined by Marcellini in [34,35] (see for instance also [32,33]) in the sense that they globally satisfy

$$|\xi|^p \lesssim \mathcal{F}(x,\xi) \lesssim |\xi|^q \,.$$

Several significant examples for non-autonomous functionals have been introduced by Zhikov [40] in the context of Lavrentiev's phenomenon. These include the already reviewed class of functionals with $p(x)$-growth. Another example, whose phenomenology is also strictly related to the class of problems we are considering here, is the so-called class of double phase functionals;

$$\mathcal{F}(x,\xi) \approx |\xi|^p + a(x)|\xi|^q,$$

where $1 < p < q$ and $0 \le a(\cdot) \le \Lambda$, with a borderline situation given by $\mathcal{F}(x,\xi) \approx |\xi|^p + a(x)|\xi|^p \log(e+|\xi|)$. For recent regularity results in this case, see [2,3,8–10,12,21].

Here we focus on non-autonomous functionals having an energy (1.4) with

$$\mathcal{F}(x,\xi) = \Phi(x,|\xi|) = |\xi|^{p(x)} \log(e+|\xi|).$$

One of the main points making this functional interesting is that it combines features of both Orlicz-type settings, like for instance $\mathcal{F}(x,\xi) = |\xi|^p \log(e+|\xi|)$, and of functionals with a variable growth exponents, i.e., $\mathcal{F}(x,\xi) = |\xi|^{p(x)}$. Functionals have been first considered [24], and then studied by the author of the paper [37,38]; related classes are studied in [22]. Furthermore, we would like mention that Harjulehto, Hästö and Klén have studied generalized Orlicz spaces and related PDEs which cover the functionals referred above, see [27,28,30].

We point out that the results on the non-autonomous functionals mentioned above consider the minimizers of energy functionals or the solutions to relevant PDEs. In our knowledge, the current paper is the first one considering the obstacle problems related to the non-autonomous functionals except the functionals with $p(x)$-growth. Therefore, the ideas in this paper can be applicable to the other obstacle problems related to the non-autonomous functionals. The conditions on the variable exponent $p(\cdot)$ and the obstacle ψ in Theorem 1.1 and Theorem 1.2 are natural since those conditions have been naturally proposed for the obstacle problems of the function with $p(x)$-growth, see [17,19,20,29].

Finally, we would like to mention the methods of proofs of our results; Theorem 1.1 and Theorem 1.2. To prove Theorem 1.1, we will derive the supremum bound of the solution u by showing a certain Caccioppoli type estimate, and a weak Harnack type estimate which is in fact strongly link to supersolutions to an equation related to the function Φ, see Remark 3.5. From these two results we prove the Hölder continuity in the same argument as in [36]. Note that in this procedure we do not use any perturbation argument. As for Theorem 1.2, we will derive comparison estimates for the gradient of solutions between the obstacle problem of Φ and an equations with solutions whose gradients are Hölder continuous. To do that, we will first obtain the higher integrability of the gradient of solutions the obstacle problem of Φ and then consider a perturbation argument, that finds its roots in the methods introduced in [1,11] for the case of functionals with $p(x)$-growth, but facing here additional difficulties due to the more general growth and ellipticity conditions.

The rest of the paper is organized as follow. In the next section, Section 2, we will introduce notations and basic properties of density functions with Orlicz type growth. In Section 3, we will prove Theorem 1.1. In the final section, Section 4, we will prove Theorem 1.2.

2. Preliminaries

2.1. Notations

We shall introduce basic notations. For $y \in \mathbb{R}^n$ and $r > 0$, $B_r = B_r(y)$ is a ball in $\mathbb{R}^n$ centered y with radius r. For a real valued function f, we define $f_+ := \max\{f, 0\}$ and $f_- := \max\{-f, 0\}$. Furthermore, for $f \in L^1_{loc}(\mathbb{R}^n)$ and a bounded open set U in $\mathbb{R}^n$, $(f)_U$ is denoted by the integral average of f in U such that $(f)_U = \fint_U f\,dx = \frac{1}{|U|}\int_U f\,dx$. From now on, the variable exponent $p(\cdot) : \Omega \to [\gamma_1, \gamma_2]$ with $1 < \gamma_1 \le \gamma_2 < \infty$ is at least uniformly continuous and we define $\Omega(\cdot) : [0, \infty) \to [0, \infty)$ by the modulus of continuity of $p(\cdot)$, that is, $\omega(0) = 0$ and ω is concave and satisfies

$$|p(x) - p(y)| \le \omega(|x - y|) \quad \text{for all } x, y \in \Omega.$$

For $1 < p < \infty$, we define a function $\Phi_p : [0, \infty) \to [0, \infty)$ by $\Phi_p(t) := t^p \log(e + t)$. Then, recalling the function $\Phi(x, t)$ denoted in (1.1), we see that $\Phi(x, t) = \Phi_{p(x)}(t)$. For these functions, one can define the *Orlicz space* $L^{\Phi_p}(\Omega)$ (resp. $L^{\Phi}(\Omega)$) by the set of all measurable functions $f : \Omega \to \mathbb{R}$ satisfying

$$\int_\Omega \Phi_p(|f(x)|)\,dx < \infty \quad \left(\text{resp. } \int_\Omega \Phi(x, |f(x)|)\,dx < \infty\right),$$

and the *Orlicz–Sobolev spaces* $W^{1,\Phi_p}(\Omega)$ (resp. $W^{1,\Phi}(\Omega)$) by the set of all $f \in W^{1,1}(\Omega)$ with $f, |Df| \in L^{\Phi_p}(\Omega)$ (resp. $f, |Df| \in L^{\Phi}(\Omega)$) and $W^{1,\Phi_p}_0(\Omega)$ (resp. $W^{1,\Phi}_0(\Omega)$) by the closure of the set of all functions in $W^{1,\Phi_p}(\Omega)$ (resp. $W^{1,\Phi}(\Omega)$) with a compact support in Ω.

We further define $\partial\Phi_p : \mathbb{R}^n \to \mathbb{R}^n$ by the gradient of $\Phi_p(|\cdot|)$ such that

$$\partial\Phi_p(\xi) := D_\xi(\Phi_p(|\xi|)) = \Phi'_p(|\xi|)\frac{\xi}{|\xi|} = \left(p|\xi|^{p-1}\log(e + |\xi|) + \frac{|\xi|^p}{e + |\xi|}\right)\frac{\xi}{|\xi|}.$$

Then from (1.1) we see that

$$\partial\Phi(x, \xi) = \partial\Phi_{p(x)}(\xi) = \Phi'_{p(x)}(|\xi|)\frac{\xi}{|\xi|}.$$

2.2. Basic properties for Φ_p and $\partial\Phi_p$

Let us first introduce the conjugate of Φ_p. Define $\Phi^*_p : [0, \infty) \to [0, \infty)$ by

$$\Phi^*_p(s) := \sup_{t \ge 0}\,(st - \Phi_p(t)).$$

From this definition we see that

$$st \le \Phi^*_p(s) + \Phi_p(t) \quad \text{for all } s, t > 0. \tag{2.1}$$

We first state several properties of Φ_p.

Proposition 2.1. *Let $1 < p < \infty$, $t, s > 0$ and $\theta > 1$ and $0 < \delta < 1$.*

(1) $\Phi_p(\theta t) \le \theta^{p+1}\Phi_p(t)$ *and* $\Phi_p(\delta t) \le \delta^p\Phi_p(t)$.
(2) $\Phi^*_p(\theta s) \le \theta^{\frac{p}{p-1}}\Phi^*_p(s)$ *and* $\Phi^*_p(\delta s) \le \delta^{\frac{p+1}{p}}\Phi^*_p(s)$.

(3) $\Phi_p(t+s) \le \frac{1}{2}(\Phi_p(2t)+\Phi_p(2s)) \le 2^p(\Phi_p(t)+\Phi_p(s))$.
(4) *(Young's inequality) For any $\kappa \in (0,1]$, we have*

$$st \le \Phi_p(\kappa^{\frac{1}{p}}t) + \Phi_p^*(\kappa^{-\frac{1}{p}}s) \le \kappa\Phi_p(t) + \kappa^{-\frac{1}{p-1}}\Phi_p^*(s) \tag{2.2}$$

and

$$st \le \Phi_p^*(\kappa^{\frac{p-1}{p}}s) + \Phi_p(\kappa^{-\frac{p-1}{p}}t) \le \kappa\Phi_p^*(s) + \kappa^{-\frac{p^2-1}{p}}\Phi_p(t). \tag{2.3}$$

Proof. The inequalities in (1) can be obtained directly from the definition of Φ. In addition, the inequalities in (3) come from the convexity of Φ along with the first inequality in (1). We then prove the inequalities in (2). By using the second inequality in (1) we have

$$\Phi_p^*(\theta s) = \sup_{t\ge 0}\,(\theta st - \Phi_p(t)) \le \sup_{t\ge 0}\,\left(\theta st - \theta^{\frac{p}{p-1}}\Phi_p(\theta^{-\frac{1}{p-1}}t)\right) = \theta^{\frac{p}{p-1}}\Phi_p^*(s).$$

Similarly, by using the first inequality in (1),

$$\Phi_p^*(\delta s) = \sup_{t\ge 0}\,(\delta st - \Phi_p(t)) \le \sup_{t\ge 0}\,(\delta st - \delta^{\frac{p+1}{p}}\Phi_p(\delta^{-\frac{1}{p}}t)) = \delta^{\frac{p+1}{p}}\Phi_p^*(s).$$

Finally, the inequalities in (4) directly follow from (1) and (2). $\square$

The following proposition states some properties related to $\partial\Phi_p$.

Proposition 2.2. *Let $1 < p < \infty$ and $\xi, \xi_1, \xi_2 \in \mathbb{R}^n$.*

(1) *We have*

$$\partial\Phi_p(\xi)\cdot\xi \ge p\,\Phi_p(\xi). \tag{2.4}$$

(2) *There exists $c = c(p) > 0$ such that*

$$\Phi_p^*(|\partial\Phi_p(\xi)|) \le \Phi_p^*\left((p+1)\Phi_p(|\xi|)|\xi|^{-1}\right) \le c\Phi(x,|\xi|). \tag{2.5}$$

(3) *If $p \ge 2$, then we have*

$$\Phi_p(|\xi_1-\xi_2|) \le c(\partial\Phi(x,\xi_1) - \partial\Phi(x,\xi_2))\cdot(\xi_1-\xi_2),$$

for some $c(p) > 0$. If $1 < p < 2$, then we have for any $\kappa \in (0,1)$,

$$\Phi_p(|\xi_1-\xi_2|) \le c\kappa(\Phi_p(|\xi_1|) + \Phi_p(x,|\xi_2|)) + c\kappa^{-\frac{2-p}{p}}(\partial\Phi(\xi_1) - \partial\Phi_p(\xi_2))\cdot(\xi_1-\xi_2)$$

for some $c = c(p) > 0$. In particular, if $1 < \gamma_1 \le p \le \gamma_2$ and $\gamma_1 \le 2$, we have for any $\kappa \in (0,1)$,

$$\begin{aligned}\Phi_p(|\xi_1-\xi_2|) &\le c\kappa(\Phi_p(|\xi_1|) + \Phi_p(|\xi_2|)) \\ &\quad + c\kappa^{-\frac{2-\gamma_1}{\gamma_1}}(\partial\Phi_p(\xi_1) - \partial\Phi_p(\xi_2))\cdot(\xi_1-\xi_2)\end{aligned} \tag{2.6}$$

for some $c = c(\gamma_1,\gamma_2) > 0$.

(4) *If $p \geq 2$, we have*

$$|\partial\Phi_p(\xi_1) - \partial\Phi_p(\xi_2)| \leq c(|\xi_1| + |\xi_2|)^{p-2}|\xi_1 - \xi_2| \log(e + |\xi_1| + |\xi_2|). \tag{2.7}$$

If $1 < p < 2$, we have

$$|\partial\Phi_p(\xi_1) - \partial\Phi_p(\xi_2)| \leq c|\xi_1 - \xi_2|^{p-1} \log(e + |\xi_1| + |\xi_2|). \tag{2.8}$$

Proof. The inequality (2.4) in (1) directly comes from the definition of $\partial\Phi$. Since $|\partial\Phi_p(\xi)| = |\Phi'(|\xi|)| \leq (p+1)\Phi_p(|\xi|)/|\xi|$, applying [15, Lemma 2.6.11], we obtain (2.5) in (2). For the proof of (3) we refer to [37, Lemma 4.1]. Now we prove the inequalities in (4). From the definition of $\partial\Phi_p$, we have

$$\begin{aligned}\partial\Phi_p(\xi_1) - \partial\Phi_p(\xi_2) &= \int_0^1 \frac{d}{dt}\partial\Phi_p(t\xi_1 + (1-t)\xi_2)\, dx \\ &\leq c|\xi_1 - \xi_2| \int_0^1 |t\xi_1 + (1-t)\xi_2|^{p-2} \log(e + |t\xi_1 + (1-t)\xi_2|)\, dt \\ &\leq c|\xi_1 - \xi_2| \log(e + |\xi_1| + |\xi_2|) \int_0^1 |t\xi_1 + (1-t)\xi_2|^{p-2}\, dt.\end{aligned}$$

If $p \geq 2$, then we see $|t\xi_1 + (1-t)\xi_2|^{p-2} \leq (|\xi_1| + |\xi_2|)^{p-2}$ and hence obtain (2.7). On the other hand, if $1 < p < 2$, by using the same argument in the proof of [13, Lemma 4.4], we see that

$$\int_0^1 |t\xi_1 + (1-t)\xi_2|^{p-2}\, dt \leq c|\xi_1 - \xi_2|^{p-2},$$

which yields (2.8). $\square$

Remark 2.3. The all constants c in the previous proposition are stable with respect to p. That is, if $1 < \gamma_1 \leq p \leq \gamma_2 < \infty$, then we can find the constants $c > 0$ depending on γ_1 and γ_2 instead of p.

3. Hölder continuity

We prove Theorem 1.1. Hence we suppose that the variable exponent $p(\cdot)$ satisfies (1.3) and the obstacle ψ is in $C^\beta(\Omega)$ for some $\beta \in (0,1)$, i.e.,

$$|\psi(x) - \psi(y)| \leq [\psi]_\beta |x-y|^\beta \quad \text{for all } x, y \in \Omega,$$

for some $[\psi]_\beta > 0$. We start with recalling supremum bounds and the weak Harnack estimates related to the function Φ that have been obtained in [38].

Lemma 3.1. *(Corollary 3.3 and Remark 3.4 in [38]) Let $f \in W^{1,\Phi(B_{4r})}$. There exist sufficiently small $\delta_1, \delta_2 \in (0, 1/4)$ depending on n, γ_1, γ_2 such that if $B_{4r} \subset \Omega$,*

$$r \leq \delta_1 \left(\int_\Omega [\Phi(x, |f|) + 1]\, dx + 1 \right)^{-1} \leq \frac{1}{4} \quad \textit{and} \quad \omega(8r) \leq \delta_2$$

and

$$\int_{B_{\rho'}} \Phi(x, |D(f-k)_+|)\, dx \le c_1 \int_{B_\rho} \Phi\left(x, \frac{(f-k)_+}{\rho - \rho'}\right) dx \tag{3.1}$$

for any $k \ge 0$ *and concentric balls* $B_{\rho'} \subset B_\rho \subset B_{4r}$ *with* $0 < \rho' < \rho < 4r$ *and for some* $c_1 > 0$*, then we have for* $s > 0$*,*

$$\sup_{B_r} f_+ \le c(s) \left\{ \left(\fint_{B_{2r}} f_+^s \, dx \right)^{\frac{1}{s}} + r \right\}$$

for some $c(s) > 0$ *depending on* n*,* γ_1*,* γ_2*,* L*,* c_1*,* s*.*

Lemma 3.2. *(Theorem 5.3 and Remark 5.4 in [38]) Let* $f \in W^{1,\Phi}(\Omega)$ *be nonnegative. There exist sufficiently small* $\delta_3, \delta_4 \in (0, 1/4)$ *and* $s_0 = s_0(n, \gamma_1, \gamma_2, L, c_3) > 0$ *such that if* $B_{2r} \subset \Omega$*,*

$$r \le \delta_3 \left(\int_\Omega [\Phi(x, |f|) + 1]\, dx + 1 \right)^{-1} \le \frac{1}{4}, \quad \omega(4r) \le \delta_4,$$

$$\left(\frac{\sup_{B_r} f}{r} \right)^{\omega(2r)} \le c_2$$

for some $c_2 > 0$*, and*

$$\int_{B_{\rho'}} \Phi(x, |D(f-k)_-|)\, dx \le c_3 \int_{B_\rho} \Phi\left(x, \frac{(f-k)_-}{\rho - \rho'}\right) dx$$

for any $k \in \mathbb{R}$ *and concentric balls* $B_{\rho'} \subset B_\rho \subset B_r$ *with* $0 < \rho' < \rho < r$ *and for some* $c_3 > 0$*, then we have*

$$\left(\fint_{B_{r/2}} f^{s_0} \, dx \right)^{\frac{1}{s_0}} \le c \left(\inf_{B_r} f + r \right)$$

for some $c = c(n, \gamma_1, \gamma_2, L, c_2, c_3) > 0$*.*

Now, we derive the estimates for the upper and lower bounds of the solutions to the obstacle problem of $\mathcal{K}^\Phi_\psi(\Omega)$.

Proposition 3.3. *Let* $u \in \mathcal{K}^\Phi_\psi(\Omega)$ *be a solution to the obstacle problem of* $\mathcal{K}^\Phi_\psi(\Omega)$ *and* $\delta_1, \delta_2 \in (0, 1/4)$ *be given in Lemma 3.1. Suppose that* $B_{4r} \subset \Omega$ *and*

$$r \le \delta_1 \left(\int_\Omega \left[\Phi\left(x, |u - l|\right) + 1 \right] dx + 1 \right)^{-1} \le \frac{1}{4} \quad \text{and} \quad \omega(8r) \le \delta_2 \tag{3.2}$$

for some $l \ge \overline{\psi} := \sup_{B_{4r}} \psi$*. Then, for any* $s \in (0, \infty)$ *we have*

$$\sup_{B_r} (u-l)_+ \le c_4(s) \left\{ \left(\fint_{B_{2r}} (u-l)_+^s \, dx \right)^{\frac{1}{s}} + r \right\}$$

for some $c_4(s) \ge 1$ depending on n, γ_1, γ_2, L, s.

Proof. We claim that $f = u - l$ satisfies (3.1). Then, in view of the Lemma 3.1, we have the conclusion. Set $v = (u-l)_+$ and consider any ball B_ρ in B_{4r}. Let $k \ge 0$, $q := \gamma_2 + 1$ and $\eta \in C_0^\infty(B_\rho)$ be a cut off function such that $0 \le \eta \le 1$, $\eta \equiv 1$ in $B_{\rho'}$ and $|D\eta| \le c(n)/(\rho - \rho')$. We note from the second inequality in (2) of Proposition 2.1 that $\Phi^*(x, \eta^{q-1} t) \le \eta^q \Phi^*(x,t)$ for $x \in \Omega$ and $t \ge 0$. Then since $(v-k)_+ = (u-l-k)_+ \le u - \psi$, one can take $\varphi = -\eta^q (v-k)_+$ as a test function in (1.2) and so

$$\int_{A_{l+k,\rho}} [\partial\Phi(x,|Du|) \cdot Du] \, \eta^q \, dx + q \int_{B_\rho} [\partial\Phi(x,|Du|) \cdot D\eta] \, \eta^{q-1} (v-k)_+ \, dx \le 0,$$

where $A_{l+k,\rho} := \{x \in B_\rho : u(x) > l + k\}$. Therefore, by (2.4) and (2.5) we have

$$\begin{aligned} &\int_{A_{l+k,\rho}} \Phi(x,|Du|)\eta^q \, dx \\ &\quad \le \kappa \int_{A_{l+k,\rho}} \Phi^*(x, |\partial\Phi(x,|Du|)|\eta^{q-1}) \, dx + c_\kappa \int_{B_\rho} \Phi(x, \eta^{q-1}|D\eta|(v-k)_+) \, dx \\ &\quad \le \kappa c \int_{A_{l+k,\rho}} \Phi(x,|Du|)\eta^q \, dx + c(\kappa) \int_{B_\rho} \Phi\left(x, \frac{(v-k)_+}{\rho - \rho'}\right) dx, \end{aligned} \tag{3.3}$$

for any $\kappa \in (0,1)$. Consequently, by taking κ sufficiently small, we have (3.1) with $(f-k)_\pm$ replaced by $(v-k)_+ = (u-l-k)_+$ and c_1 depending on n, γ_1, γ_2. □

Proposition 3.4. *Let $u \in \mathcal{K}_\psi^\Phi(\Omega)$ be a solution to the obstacle problem of $\mathcal{K}_\psi^\Phi(\Omega)$. Then we have for any $k \in \mathbb{R}$ and concentric balls $B_{\rho'} \subset B_\rho \subset \Omega$ with $0 < \rho' < \rho$,*

$$\int_{B_{\rho'}} \Phi(x, |D(u-k)_-|) \, dx \le c \int_{B_\rho} \Phi\left(x, \frac{(u-k)_-}{\rho - \rho'}\right) dx \tag{3.4}$$

for some $c = c(n, \gamma_1, \gamma_2) > 0$. Moreover, if

$$r \le \delta_1 \left(\int_\Omega [\Phi(x,|u|) + 1] \, dx + 1 \right)^{-1} \le \frac{1}{4},$$

then for any $s > 0$ we have

$$\sup_{B_r} u_- \le c_5(s) \left\{ \left(\fint_{B_{2r}} u_-^s \, dx \right)^{\frac{1}{s}} + r \right\} \tag{3.5}$$

for some $c_5(s) > 0$ depending on n, γ_1, γ_2, L, s.

Proof. Fix any concentric balls $B_{\rho'} \subset B_\rho$, $0 < \rho' < \rho$ in Ω. Let $k \in \mathbb{R}$, $q := \gamma_2 + 1$ and $\eta \in C_0^\infty(B_\rho)$ be a cut off function such that $0 \le \eta \le 1$, $\eta \equiv 1$ in $B_{\rho'}$ and $|D\eta| \le c(n)/(\rho - \rho')$. Since $\eta^q(u-k)_- \ge 0 \ge \psi - u$, we take $\varphi = \eta^q(u-k)_-$ as a test function in (1.2) in order to get

$$-\int_{A^-_{k,\rho}} [\partial\Phi(x,|Du|) \cdot Du]\, \eta^q \, dx + q \int_{B_\rho} [\partial\Phi(x,|Du|) \cdot D\eta]\, \eta^{q-1}(u-k)_- \, dx \ge 0,$$

where $A^-_{k,\rho} := \{x \in B_\rho : u(x) < k\}$. In the same way as in (3.3), we have

$$\begin{aligned} &\int_{A^-_{k,\rho}} \Phi(x,|Du|)\eta^q \, dx \\ &\qquad \le \kappa c \int_{A^-_{k,\rho}} \Phi(x,|Du|)\eta^q \, dx + c(\kappa) \int_{B_\rho} \Phi\left(x, \frac{(u-k)_-}{\rho - \rho'}\right) dx \end{aligned}$$

for any $\kappa \in (0,1)$. Therefore, by taking κ sufficiently small, we have (3.4).

In addition, since $(u-k)_- = (-u+k)_+$, we see from (3.4) that the inequality (3.1) with $f = -u$ holds for any $k \ge 0$. Therefore, by Lemma 3.1, we have (3.5). □

Remark 3.5. We say $u \in W^{1,\Phi}(\Omega)$ is a (weak) supersolution to

$$-\mathrm{div}\,(\partial\Phi(x,Du)) = 0, \tag{3.6}$$

that is, it satisfies

$$\int_\Omega \partial\Phi(x,Du) \cdot D\varphi \, dx \ge 0$$

for all $\varphi \in C_0^\infty(\Omega)$ with $\varphi \ge 0$. Note that, from the definition of solution to the obstacle problem (1.2), the solution to the obstacle problem $\mathcal{K}^\Phi_\psi(\Omega)$ is also a supersolution to (3.6). Moreover, we see from the poof of the previous proposition that if u is a supersolution to (3.6), we also have the estimate (3.4).

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Since ψ is Hölder continuous, there exists $\beta \in (0,1)$ and $[\psi]_\beta > 0$ such that

$$|\psi(x) - \psi(y)| \le [\psi]_\beta |x-y|^\beta \quad \text{for all } x, y \in \Omega.$$

We first observe from the previous two lemmas that the solution $u \in \mathcal{K}^\Phi_\psi(\Omega)$ to the obstacle problem of $\mathcal{K}^\Phi_\psi(\Omega)$ is locally bounded. Let $\Omega' \subset\subset \Omega$. Then we know $\|u\|_{L^\infty(\Omega')} < \infty$. Fix any ball B_{16r_0} in Ω', where $r_0 > 0$ satisfies

$$r_0 \le \min\{\delta_1, \delta_3\} \left(\int_\Omega \left[\Phi\left(x, |u| + \|u\|_{L^\infty(\Omega')} + \|\psi\|_{L^\infty(\Omega')}\right) + 1\right] dx + 1 \right)^{-1} \tag{3.7}$$

and

$$\omega(2r_0) \le \min\{\delta_3, \delta_4\}, \tag{3.8}$$

where δ_1 and δ_3 are given in Lemma 3.1 and Lemma 3.2, and

$$\omega(2r_0) \le 1.$$

For $r \in (0, r_0]$, set

$$\overline{u}(r) := \sup_{B_r} u, \quad \underline{u}(r) := \inf_{B_r} u, \quad \overline{\psi}(r) := \sup_{B_r} \psi, \quad \underline{\psi}(r) := \inf_{B_r} \psi.$$

Then, if $\overline{\psi}(r) \le \underline{u}(r)$, applying Proposition 3.3 with r replaced by $r/4$ and $l = \underline{u}(r)$, we have for any $s > 0$,

$$\overline{u}(r/4) - \underline{u}(r) = \sup_{B_{r/4}} (u - \underline{u}(r)) \le c_4(s) \left\{ \left(\fint_{B_{r/2}} (u - \underline{u}(r))^s \, dx \right)^{\frac{1}{s}} + r \right\}.$$

Note that the assumptions in (3.2) with r replaced by $r/4$ and $l = \underline{u}(r)$ are satisfied by (3.7) and (3.8). On the other hand, if $\underline{u}(r) \le \overline{\psi}(r)$, again applying Proposition 3.3 with r replaced by $r/4$ and $l = \overline{\psi}(r)$, we have

$$\begin{aligned} \overline{u}(r/4) - \overline{\psi}(r) = \sup_{B_{r/4}} (u - \overline{\psi}(r)) &\le c_4(s) \left\{ \left(\fint_{B_{r/2}} (u - \overline{\psi}(r))_+^s \, dx \right)^{\frac{1}{s}} + r \right\} \\ &\le c_4(s) \left\{ \left(\fint_{B_{r/2}} (u - \underline{u}(r))^{s_0} \, dx \right)^{\frac{1}{s_0}} + r \right\}. \end{aligned}$$

Note that the assumptions in (3.2) with r replaced by $r/4$ and $l = \overline{\psi}(r)$ are satisfied by (3.7) and (3.8). From this estimate together with the Hölder continuity of ψ and the fact $u \ge \psi$, we have

$$\begin{aligned} \overline{u}(r/4) - \underline{u}(r) \le \overline{u}(r/4) - \underline{\psi}(r) &\le \overline{u}(r/4) - \overline{\psi}(r) + [\psi]_\beta r^\beta \\ &\le c_4(s) \left\{ \left(\fint_{B_{r/2}} (u - \underline{u}(r))^{s_0} \, dx \right)^{\frac{1}{s_0}} + r \right\} + [\psi]_\beta r^\beta. \end{aligned}$$

Consequently, we obtain

$$\overline{u}(r/4) - \underline{u}(r) \le c_4(s) \left\{ \left(\fint_{B_{r/2}} (u - \underline{u}(r))^{s_0} \, dx \right)^{\frac{1}{s_0}} + r \right\} + [\psi]_\beta r^\beta. \tag{3.9}$$

Now we consider the function $u - \underline{u}(r)$. Since $u - \underline{u}(r)$ is a nonnegative solution to the obstacle problem of $\mathcal{K}^{\Phi}_{\psi - \underline{u}(\rho)}(B_{r/2})$, in view of Proposition 3.4, we have (3.4) with u replaced by $u - \underline{u}(r)$. Moreover, applying Proposition 3.3 with $l = \overline{\psi}(4r)$ and $s = 1$ again, we have

$$\begin{aligned} \sup_{B_r} (u - \underline{u}(r)) &\le \sup_{B_r} (u - \overline{\psi}(4r))_+ + \overline{\psi}(4r) - \underline{\psi}(r) \\ &\le c \fint_{B_{2r}} (u - \overline{\psi}(4r))_+ \, dx + [\psi]_\beta r^\beta \\ &\le \frac{c}{r^n} \int_{\Omega} \left[\Phi \left(x, |u| + \|\psi\|_{L^\infty(\Omega')} \right) + 1 \right] \, dx + [\psi]_\beta r^\beta, \end{aligned}$$

which together with (1.3) implies

$$\left(\frac{\sup_{B_r}(u-\underline{u}(r))}{r}\right)^{\omega(2r)} \le c(2r)^{-(n+1)\omega(2r)} + c[\psi]_\beta^{\omega(2r)} \le c_2$$

for some $c_2 = c_2(n,\gamma_1,\gamma_2,L,[\psi]_\beta) > 0$. Therefore, by Lemma 3.2 we have

$$\left(\fint_{B_{r/2}} (u-\underline{u}(r))^{s_0}\,dx\right)^{\frac{1}{s_0}} \le c\left(\underline{u}(r/4) - \underline{u}(r) + r\right), \tag{3.10}$$

where $s_0 > 0$ depends on n, γ_1, γ_2, $[\psi]_\beta$.

Combining (3.9) with $s = s_0$ and (3.10), we have

$$\overline{u}(r/4) - \underline{u}(r) \le C_1\left(\underline{u}(r/4) - \underline{u}(r)\right) + D_1 r^\beta,$$

for some $C_1, D_1 \ge 1$ depending on n, γ_1, γ_2, L, $[\psi]_\beta$. Therefore, if $(C_1+1)(\underline{u}(r/4) - \underline{u}(r)) \le \overline{u}(r) - \underline{u}(r)$, we have

$$\overline{u}(r/4) - \underline{u}(r/4) \le \overline{u}(r/4) - \underline{u}(r) \le \frac{C_1}{C_1+1}(\overline{u}(r) - \underline{u}(r)) + D_1 r^\beta.$$

On the other hand, if $(C_1+1)(\underline{u}(r/4) - \underline{u}(r)) > \overline{u}(r) - \underline{u}(r)$, we have

$$\begin{aligned}\overline{u}(r/4) - \underline{u}(r/4) &\le \overline{u}(r/4) - \underline{u}(r) - (\underline{u}(r/4) - \underline{u}(r)) \\ &\le \overline{u}(r/4) - \underline{u}(r) - \frac{1}{C_1+1}(\overline{u}(r) - \underline{u}(r)) \\ &\le \frac{C_1}{C_1+1}(\overline{u}(r) - \underline{u}(r)).\end{aligned}$$

Finally, we have

$$\overline{u}(r/4) - \underline{u}(r/4) \le \frac{C_1}{C_1+1}(\overline{u}(r) - \underline{u}(r)) + D_1 r^\beta,$$

for all $r \in (0, r_0]$, which implies the Hölder continuity of u, see for instance [26, Lemma 7.3]. □

4. Hölder continuity for the gradient

In this section, we prove Theorem 1.2. For the variable exponent $p(\cdot)$ and the obstacle ψ we shall assume that there exists $\beta \in (0,1)$ such that

$$|p(x) - p(y)| \le [p(\cdot)]_\beta |x-y|^\beta \quad \text{and} \quad |D\psi(x) - D\psi(y)| \le [D\psi]_\beta |x-y|^\beta \tag{4.1}$$

for some $[p(\cdot)]_\beta, [D\psi]_\beta > 0$. From now on, without loss of generality, we assume that

$$1 < \gamma_1 < 2 < \gamma_2 < \infty.$$

We also define

$$M := \int_\Omega \left[\Phi(x,|Du|) + \Phi(x,|D\psi|) + 1\right] dx + 1.$$

4.1. Higher integrability

Let us first recall the result of Sobolev–Poincaré type inequality for Φ_p which can be found in [14, Theorem 7] with $\varphi = \Phi_p$ and $\omega \equiv 1$.

Lemma 4.1. *Let $1 < \gamma_1 \le p \le \gamma_2 < \infty$. Then there exists $\tau_0 = \tau_0(n, \gamma_1, \gamma_2) \in (0,1)$ such that for $f \in W^{1,\Phi_p}(B_r)$ with $r > 0$, we have*

$$\fint_{B_r} \Phi_p\left(\frac{|f-(f)_{B_r}|}{r}\right) \le c\left(\fint_{B_r} \Phi_p(|Df|)^{\tau_0}\,dx\right)^{\frac{1}{\tau_0}}$$

for some $c = c(n, \gamma_1, \gamma_2) > 0$.

Now we prove the higher integrability of Du.

Theorem 4.2. *Suppose $p(\cdot)$ and ψ satisfy (4.1). There exists $\sigma_0 = \sigma_0(n, \gamma_1, \gamma_2) \in (0,1)$ such that if $u \in \mathcal{K}^{\Phi}_{\psi}(\Omega)$ is a solution to the obstacle problem of $\mathcal{K}^{\Phi}_{\psi}(\Omega)$ and $B_{2r} \subset \Omega$ with $r > 0$ satisfying*

$$r \le \min\left\{\left(\frac{\beta}{8[p(\cdot)]_\beta}\right)^{\frac{2}{\beta}}, \frac{1}{M}\right\} \quad \text{and} \quad \omega(4r) \le \min\left\{\frac{\gamma_1(1-\tau_0)}{2}, 1\right\}, \tag{4.2}$$

then $\Phi(\cdot, |Du|) \in L^{1+\sigma_0}(B_r)$. Moreover, for any $\sigma \in (0, \sigma_0]$ we have

$$\fint_{B_r} \Phi(x, |Du|)^{1+\sigma}\,dx \le c\left(\fint_{B_{2r}} \Phi(x,|Du|)\,dx\right)^{1+\sigma} + c\fint_{B_{2r}} \Phi(x, |D\psi|)^{1+\sigma}\,dx + c \tag{4.3}$$

for some $c(n, \gamma_1, \gamma_2) > 0$.

Proof. Fix $B_{2r} \subset \Omega$ with $r > 0$ satisfying (4.2) and set $p_2 := \sup_{B_{2r}} p(\cdot)$ and $p_1 := \inf_{B_{2r}} p(\cdot)$. Let $\eta \in C_0^\infty(B_{2r})$ satisfy $0 \le \eta \le 1$, $\eta \equiv 1$ in B_r and $|D\eta| \le c(n)r^{-1}$. Since $\psi - (\psi)_{B_{2r}} - u + (u)_{B_{2r}} \ge \psi - u$ we see that $\varphi := \eta^q(\psi - (\psi)_{B_{2r}} - u + (u)_{B_{2r}}) \ge \psi - u$, where $q = \gamma_2 + 1$. With this φ we have from (1.2) that

$$\int_{B_{2r}} [\partial\Phi(x, Du)\cdot Du]\eta^q\,dx \le \int_{B_{2r}} [\partial\Phi(x,Du)\cdot D\psi]\eta^q\,dx + q\int_{B_{2r}} [\partial\Phi(x,Du)\cdot D\eta]\eta^{q-1}(\psi - (\psi)_{B_{2r}} - u + (u)_{B_{2r}})\,dx$$

Then, in the same way to estimate (3.3), we have

$$\fint_{B_r} \Phi(x,|Du|)\,dx \le c\fint_{B_{2r}} \Phi\left(x, \frac{|u-(u)_{B_2r}|}{r}\right)dx + c\int_{B_{2r}} \Phi\left(x, \frac{|\psi - (\psi)_{B_2r}|}{r}\right)dx + c\fint_{B_{2r}} \Phi(x, |D\psi|)\,dx.$$

Then, in view of Lemma 4.1 and Hölder's inequality with the fact

$$\frac{p_1}{p_2} \geq 1 - \frac{\omega(4r)}{\gamma_1} \geq \frac{\tau_0 + 1}{2} > \tau_0,$$

we have

$$\begin{aligned}
\fint_{B_{2r}} \Phi\left(x, \frac{|u-(u)_{B_{2r}}|}{r}\right) dx &\leq \fint_{B_{2r}} \Phi_{p_2}\left(\frac{|u-(u)_{B_{2r}}|}{r}\right) dx + 1 \\
&\leq c\left(\fint_{B_{2r}} \Phi_{p_2}(|Du|)^{\tau_0}\, dx\right)^{\frac{1}{\tau_0}} + 1 \\
&\leq c\left(\fint_{B_{2r}} \Phi(x,|Du|)^{\frac{\tau_0+1}{2}}\, dx\right)^{\frac{p_2}{p_1}\frac{2}{\tau_0+1}} + c
\end{aligned}$$

and

$$\begin{aligned}
\fint_{B_{2r}} \Phi\left(x, \frac{|\psi-(\psi)_{B_{2r}}|}{r}\right) dx &\leq \fint_{B_{2r}} \Phi_{p_2}\left(\frac{|\psi-(\psi)_{B_{2r}}|}{r}\right) dx + 1 \\
&\leq c\left(\fint_{B_{2r}} \Phi_{p_2}(|D\psi|)^{\tau_0}\, dx\right)^{\frac{1}{\tau_0}} + 1 \\
&\leq c\left(\fint_{B_{2r}} \Phi(x,|D\psi|)\, dx\right)^{\frac{p_2}{p_1}} + c.
\end{aligned}$$

We note from (4.1) and (4.2) that

$$(p_2-p_1)\log\left(\frac{1}{r}\right) \leq [p(\cdot)]_\beta (4r)^\beta \frac{2}{\beta}\left(\frac{1}{r}\right)^{\frac{\beta}{2}} = \frac{8[p(\cdot)]_\beta}{\beta} r^{\frac{\beta}{2}} \leq 1,$$

from which and again (4.2) imply

$$\left(\fint_{B_{2r}} \Phi(x,|Du|)\, dx\right)^{p_2-p_1} \leq c\left(\frac{M}{r^n}\right)^{p_2-p_1} \leq c\left(\frac{1}{r^{n+1}}\right)^{p_2-p_1} \leq c. \tag{4.4}$$

Similarly, we have

$$\left(\fint_{B_{2r}} \Phi(x,|D\psi|)\, dx\right)^{p_2-p_1} \leq c. \tag{4.5}$$

Combining the above results we have

$$\fint_{B_r} \Phi(x,|Du|)\, dx \leq c\left(\fint_{B_{2r}} \Phi(x,|Du|)^{\frac{\tau_0+1}{2}}\, dx\right)^{\frac{2}{\tau_0+1}} + c\fint_{B_{2r}} \Phi(x,|D\psi|)\, dx + c.$$

Finally, since $D\psi \in L^\infty(\Omega)$, we obtain (4.3) from Gehring's lemma, see for instance [26, Theorem 6.6]. □

Remark 4.3. From the proof of the previous theorem, one can deduce that the result of Theorem 4.2 still holds true if $p(\cdot)$ satisfies so-called vanishing log-Hölder continuity:

$$\lim_{t\to 0} \omega(r)\log\left(\frac{1}{r}\right) = 0$$

and $D\psi \in L^{1+\tilde{\sigma}}_{loc}(\Omega)$ for some $\tilde{\sigma} > 0$. The proof is almost same, and the restrictions of r in (4.2) and the constants σ_0 and c are modified in the reasonable way. In particular, σ_0 and c depend on n, γ_1, γ_2, $\tilde{\sigma}$.

4.2. Comparison estimates

Let $u \in \mathcal{K}^{\Phi}_{\psi}(\Omega)$ be a solution to the obstacle problem of $\mathcal{K}^{\Phi}_{\psi}(\Omega)$ and $B_{2r} \subset \Omega$, where $r > 0$ is a sufficiently small number satisfying (4.2) and (4.11) below. Set

$$\overline{D\psi} := \sup_{\Omega} |D\psi|, \quad p_2 := \sup_{B_r} p(\cdot) \quad \text{and} \quad p_1 := \inf_{B_r} p(\cdot).$$

We start with the comparison principle for $\partial\Phi$.

Lemma 4.4. *Suppose that $w \in W^{1,\Phi}(U)$ satisfies*

$$\begin{cases} -\mathrm{div}\,(\partial\Phi_p(D\psi)) \le -\mathrm{div}\,(\partial\Phi_p(Dw)) & \text{in} \quad B_r, \\ \psi \le w & \text{on} \quad \partial B_r, \end{cases}$$

in the weak sense, that is, $(\psi - w)_+ \in W^{1,\Phi_p}_0(B_r)$ and

$$\int_U (\partial\Phi_p(D\psi) - \partial\Phi_p(Dw)) \cdot D\varphi \, dx \le 0$$

for all $\varphi \in W^{1,\Phi_p}_0(B_r)$ with $\varphi \ge 0$. Then we have $\psi \le w$ a.e. in B_r.

Proof. By taking $\varphi = (\psi - w)_+$ in the above weak inequality and (2.6), we have

$$\int_{\{x\in B_r:\psi(x)>w(x)\}} \Phi_p(|D\psi - Dw|)\,dx \le \kappa \int_{\{x\in B_r:\psi(x)>w(x)\}} [\Phi_p(|D\psi|) + \Phi_p(|Dw|)]\,dx,$$

for any $\kappa \in (0, 1)$. Since κ is arbitrary, we see that $\psi \le w$ a.e. in B_r. □

Next, we consider the following two comparison maps with so-called Orlicz growth. Here, we suppose that

$$p_2 - p_1 \le \omega(2r) \le \frac{\sigma_0}{2} \tag{4.6}$$

Then we have $p_2 \le p(x)(1+p_2-p_1) \le p(x)(1+\sigma_0/2)$ and so, by (4.3), (4.4) and (4.5), we have $\Phi_{p_2}(|Du|) \in L^1(B_r)$ with the estimate

$$\begin{aligned}\fint_{B_r} \Phi_{p_2}(|Du|)\,dx &\le \fint_{B_r} \Phi(x,|Du|)^{1+p_2-p_1}\,dx + 1\\ &\le c \fint_{B_{2r}} \Phi(x,|Du|)\,dx + c\Phi_{\gamma_2}(\overline{D\psi}) + c.\end{aligned}$$

Let $w \in W^{1,\Phi_{p_2}}(B_r)$ be the unique weak solution to

$$\begin{cases} -\mathrm{div}\,(\partial\Phi_{p_2}(Dw)) = -\mathrm{div}\,(\partial\Phi_{p_2}(D\psi)) & \text{in} \quad B_r,\\ w = u & \text{on} \quad \partial B_r, \end{cases} \tag{4.7}$$

and $v \in W^{1,\Phi_{p_2}}(B_r)$ be the unique weak solution to

$$\begin{cases} -\mathrm{div}\,(\partial\Phi_{p_2}(x,Dv)) = 0 & \text{in} \quad B_r,\\ v = w & \text{on} \quad \partial B_r. \end{cases} \tag{4.8}$$

Then by a standard energy estimate we have

$$\begin{aligned}\int_{B_r} \Phi_{p_2}(|Dw|)\,dx &\le c\int_{B_r} \Phi_{p_2}(|Du|)\,dx + c\int_{B_r} \Phi_{p_2}(|D\psi|)\,dx\\ &\le c\left(\int_{B_{2r}} \Phi_{p_2}(|Du|)\,dx + r^n\right)\end{aligned} \tag{4.9}$$

and

$$\int_{B_r} \Phi_{p_2}(|Dv|)\,dx \le c\int_{B_r} \Phi_{p_2}(|Dw|)\,dx \le c\left(\int_{B_{2r}} \Phi(x,|Du|)\,dx + r^n\right), \tag{4.10}$$

where $c > 0$ depends on n, γ_1, γ_2, $\overline{D\psi}$.

Lemma 4.5. *Suppose that $r > 0$ satisfies that*

$$\omega(2r) \le \frac{\sigma_1}{2}, \quad \textit{where} \ \ \sigma_1 := \min\left\{\frac{(\gamma_1-1)\beta}{4n}, \sigma_0\right\}. \tag{4.11}$$

Then we have

$$\int_{B_r} \Phi_{p_2}(Du - Dw)\,dx \le cr^{\frac{(\gamma_1-1)\beta}{2}}\left\{M^{\sigma_1}\int_{B_{2r}} \Phi_{p_2}(|Du|)\,dx + r^n\right\} \tag{4.12}$$

for some $c(n,\gamma_1,\gamma_2,[p(\cdot)]_\beta,[D\psi]_\beta,\overline{D\psi}) \ge 1$.

Proof. Form (4.7), we see that

$$\int_{B_r} \partial\Phi_{p_2}(Dw)\cdot(Du - Dw)\,dx = \int_{B_r} \partial\Phi_{p_2}(D\psi)\cdot(Du - Dw)\,dx. \tag{4.13}$$

Since $w = u \geq \psi$ on ∂B_r, in view of Lemma 4.4, we have $w \geq \psi$ a.e. in B_r. By putting $w = u$ in $\Omega \setminus B_r$, we have $w \in W^{1,\Phi}(\Omega)$ and $w \geq \psi$ a.e. in Ω, that is, $w \in \mathcal{K}^{\Phi}_{\psi}(\Omega)$. Therefore, by testing $\varphi = w - u$ in (1.2), we have

$$\int_{B_r} \partial\Phi(x, Du) \cdot (Dw - Du)\, dx \geq 0. \tag{4.14}$$

Combining (4.13) and (4.14) we have

$$\begin{aligned} I_1 &:= \int_{B_r} (\partial\Phi_{p_2}(Du) - \partial\Phi_{p_2}(Dw)) \cdot (Du - Dw)\, dx \\ &\leq \int_{B_r} (\partial\Phi_{p_2}(Du) - \partial\Phi(x, Du)) \cdot (Du - Dw)\, dx + \int_{B_r} \partial\Phi_{p_2}(D\psi) \cdot (Dw - Du)\, dx \\ &=: I_2 + I_3. \end{aligned}$$

Then, applying (2.6) and (4.9), we have

$$\int_{B_r} \Phi_{p_2}(|Du - Dw|)\, dx \leq c\kappa \left(\int_{B_r} \Phi_{p_2}(|Du|)\, dx + r^n \right) + c\kappa^{-\frac{2-\gamma_1}{2}} I_1 \tag{4.15}$$

for any $\kappa \in (0, 1)$.

We next estimate I_2. Applying the mean value theorem to the map $t \in [0, 1] \mapsto |Du|^{t(p_2 - p(x))}$ we obtain

$$\begin{aligned} |\partial\Phi_{p_2}(Du) - \partial\Phi(x, Du)| &= \left| (|Du|^{p_2 - p(x)} - 1)\partial\Phi(x, Du) + (p_2 - p(x))|Du|^{p_2 - 2} \log(e + |Du|)Du \right| \\ &\leq c(p_2 - p_1) \left(|Du|^{t_x(p_2 - p(x))} |\log|Du|| + |Du|^{p_2 - p(x)} \right) |Du|^{p(x)-1} \log(e + |Du|) \end{aligned}$$

for some $t_x \in (0, 1)$, where $x \in B_r$. Then using the elementary inequalities $t^{(\gamma_1 - 1)}|\log t| \leq c(\gamma_1)$ for $0 < t \leq 1$ and $\log t \leq c(\sigma)t^{\sigma}$ for $t \geq 1$ with $\sigma > 0$, we have

$$|\partial\Phi_{p_2}(Du) - \partial\Phi(x, Du)| \leq c(p_2 - p_1) \left(|Du|^{\sigma_2} \Phi_{p_2}(|Du|)|Du|^{-1} + 1 \right),$$

where $\sigma_2 := \frac{(\gamma_1 - 1)\sigma_1}{2\gamma_1}$ and σ_1 is denoted in (4.11). From this estimate, (4.1), (2.1), (2) of Proposition 2.1, (2.5) and (4.9), we have

$$\begin{aligned} |I_2| &\leq c(p_2 - p_1) \int_{B_r} \left[|Du - Dw| + |Du|^{\sigma_2} \Phi_{p_2}(|Du|)|Du|^{-1}|Du - Dw| \right] dx \\ &\leq cr^{\beta} \int_{B_r} \left[\Phi_{p_2}(|Du - Dw|) + \Phi^*_{p_2}(|Du|^{\sigma_2} \Phi_{p_2}(|Du|)|Du|^{-1}) \right] dx \\ &\leq cr^{\beta} \int_{B_r} \left[\Phi_{p_2}(|Du - Dw|) + |Du|^{\frac{\sigma_2 p_2}{p_2 - 1}} \Phi_{p_2}(|Du|) + 1 \right] dx \\ &\leq cr^{\beta} \int_{B_r} \left[\Phi_{p(x) + \omega(2r) + \frac{\sigma_2 \gamma_1}{\gamma_1 - 1}}(|Du|) + 1 \right] dx \\ &\leq cr^{\beta} \int_{B_r} \left[\Phi(x, |Du|)^{1 + \sigma_1} + 1 \right] dx. \end{aligned}$$

Note that in the last inequality, we have used the fact $\omega(2r) \le \frac{\sigma_1}{2}$ and $\frac{\sigma_2\gamma_1}{\gamma_1-1} \le \frac{\sigma_1}{2}$. Moreover, applying Theorem 4.2,

$$\begin{aligned} |I_2| &\le cr^{\beta}\left\{r^n\left(\fint_{B_{2r}} \Phi(x,|Du|)\,dx\right)^{1+\sigma_1} + \int_{B_{2r}} \left[\Phi(x,|D\psi|)^{1+\sigma_1}+1\right]\,dx\right\} \\ &\le cr^{\beta}\left(r^{-n\sigma_1}M^{\sigma_1}\int_{B_{2r}} \Phi_{p_2}(|Du|)\,dx + r^n\right), \end{aligned} \tag{4.16}$$

where $c>0$ depends on n, γ_1, γ_2, $[p(\cdot)]_\beta$, $\overline{D\psi}$.

Finally we estimate I_3. We first observe that

$$I_3 = \left|\int_{B_r} (\partial\Phi_{p_2}(D\psi) - \partial\Phi_{p_2}(D\psi(x_0))\cdot(Dw-Du)\,dx\right|,$$

where x_0 is the center of B_r. If $p_2>2$, then by (2.7), (4.1) and (4.9), we have

$$\begin{aligned} |I_3| &\le \int_{B_r} |\partial\Phi_{p_2}(D\psi) - \partial\Phi_{p_2}(D\psi(x_0))||Dw-Du|\,dx \\ &\le \int_{B_r} (|D\psi|+|D\psi(x_0)|)^{p_2-2}|D\psi - D\psi(x_0)| \\ &\qquad\qquad \times \log(e+|D\psi|+|D\psi(x_0)|)|Dw-Du|\,dx \\ &\le cr^{\beta}\left(\int_{B_r}[|Du|+|Dw|+1]\,dx\right) \\ &\le cr^{\beta}\left(\int_{B_r}\Phi_{p_2}(|Du|)\,dx + r^n\right), \end{aligned}$$

where $c>0$ depends on γ_1, γ_2, $[D\psi]_\beta$, $\overline{D\psi}$. Similarly, if $p_2<2$, then applying (2.8), (4.1) and (4.9), we have

$$|I_3| \le cr^{(p_2-1)\beta}\left(\int_{B_r}\Phi_{p_2}(|Du|)\,dx + r^n\right).$$

From those two cases, we obtain

$$|I_3| \le cr^{(\gamma_1-1)\beta}\left(\int_{B_r}\Phi_{p_2}(|Du|)\,dx + r^n\right). \tag{4.17}$$

Consequently, by (4.15), (4.16), (4.17) and the fact $I_1 = I_2 + I_3$ we obtain

$$\int_{B_r}\Phi_{p_2}(Du-Dw)\,dx \le c\kappa\left(\int_{B_r}\Phi_{p_2}(Du)\,dx + r^n\right)$$

$$+ c\kappa^{-\frac{2-\gamma_1}{2}} r^{(\gamma_1-1)\beta} \left\{ r^{-n\sigma_1} M^{\sigma_1} \int_{B_{2r}} \Phi_{p_2}(|Du|)\, dx + r^n \right\}.$$

Hence, by taking $\kappa = r^{\frac{(\gamma_1-1)\beta}{2(2-\gamma_1)}}$ we have (4.12). $\square$

Lemma 4.6. *Then we have*

$$\int_{B_r} \Phi_{p_2}(Dw - Dv)\, dx \le c r^{\frac{(\gamma_1-1)\beta}{2}} \left(\int_{B_{2r}} \Phi_{p_2}(|Du|)\, dx + r^n \right) \tag{4.18}$$

for some $c(n, \gamma_1, \gamma_2, [D\psi]_\beta, \overline{D\psi}) \ge 1$.

Proof. Since $w - v \in W_0^{1,\Phi_2}(B_r)$, we deduce from (4.7) and (4.8) that

$$\int_{B_r} (\partial\Phi_{p_2}(Dw) - \partial\Phi_{p_2}(Dv)) \cdot (Dw - Dv)\, dx$$
$$= \int_{B_r} (\partial\Phi_{p_2}(D\psi) - \partial\Phi_{p_2}(D\psi(x_0))) \cdot (Dw - Dv)\, dx,$$

where x_0 is the center of the ball B_r. Applying (2.6), (4.9) and (4.10), we have

$$\int_{B_r} \Phi_{p_2}(|Dw - Dv|)\, dx \le c\kappa \left(\int_{B_r} \Phi_{p_2}(|Du|)\, dx + r^n \right)$$
$$+ c\kappa^{-\frac{2-\gamma_1}{2}} \int_{B_r} (\partial\Phi_{p_2}(Dw) - \partial\Phi_{p_2}(Dv)) \cdot (Dw - Dv)\, dx.$$

Moreover, in the same way to estimate I_3 in the proof of the previous lemma, we have

$$\int_{B_r} (\partial\Phi_{p_2}(D\psi) - \partial\Phi_{p_2}(D\psi(x_0))) \cdot (Dw - Dv)\, dx$$
$$\le c r^{(\gamma_1-1)\beta} \left(\int_{B_r} \Phi_{p_2}(|Du|)\, dx + r^n \right).$$

Hence, we have

$$\int_{B_r} \Phi_{p_2}(Dw - Dv) \le c\kappa \left(\int_{B_r} \Phi_{p_2}(Du)\, dx + r^n \right)$$
$$+ c\kappa^{-\frac{2-\gamma_1}{2}} r^{(\gamma_1-1)\beta} \left(\int_{B_r} \Phi_{p_2}(|Du|)\, dx + r^n \right)$$

Consequently, by taking $\kappa = r^{\frac{(\gamma_1-1)\beta}{2-\gamma_1}}$, we obtain (4.18). $\square$

4.3. Hölder continuity of Du

We first observe the following two lemmas. The first one is a technical iteration lemma.

Lemma 4.7. *[25, Lemma 2.1 of Chapter 3] Let ϕ be a nonnegative and nondecreasing function. Suppose that*

$$\phi(\rho) \le A\left\{\left(\frac{\rho}{r}\right)^{\alpha_1} + \epsilon\right\}\phi(r) + Br^{\alpha_2}$$

for all $0 < \rho < r < r_0$, with nonnegative constants A, B, α_1, α_2 ($\alpha_1 > \alpha_2$). Then there exists $\epsilon_0 = \epsilon(A, \alpha_1, \alpha_2) > 0$ such that if $\epsilon < \epsilon_0$, for all $0 < \rho < r \le r_0$ we have

$$\phi(\rho) \le c\left\{\left(\frac{\rho}{r}\right)^{\alpha_2}\phi(r) + Br^{\alpha_2}\right\}$$

for some $c = c(A, \alpha_1, \alpha_2) > 0$.

The second lemma is the results of Hölder continuity for function Φ_{p_2}.

Lemma 4.8. *([16], see also [24, Lemma 2.6 and Corollary 2.7]) Let $1 < \gamma_1 \le p \le \gamma_2 < \infty$ and $v \in W^{1,\Phi_p}(B_r)$ be a weak solution to*

$$\operatorname{div}(\partial\Phi_p(Dv)) = 0 \quad \textit{in} \ \ B_r.$$

There exists $\beta_1 \in (0,1)$ and $c > 0$ depending on n, γ_1, γ_2 such that for any $0 < \rho < r$,

$$\fint_{B_\rho} \Phi_p(|Dv - (Dv)_{B_\rho}|)\,dx \le c\left(\frac{\rho}{r}\right)^{\beta_1} \fint_{B_r} \Phi_p(|Dv|)\,dx \tag{4.19}$$

and

$$\fint_{B_\rho} \Phi_p(|Dv|)\,dx \le c \fint_{B_r} \Phi_p(|Dv|)\,dx. \tag{4.20}$$

From the comparison estimates and the previous lemmas, we obtain the following decay estimates.

Proposition 4.9. *Let $u \in \mathcal{K}^{\Phi}_{\psi}(\Omega)$ is a solution to the obstacle problem of $\mathcal{K}^{\Phi}_{\psi}(\Omega)$. There exists $\delta = \delta(n, \gamma_1, \gamma_2, \omega(\cdot), L, \psi) \in (0,1)$ such that if $r > 0$ satisfies (4.2), (4.11) and $r \le \delta M^{-\frac{2\sigma_1}{(\gamma_1-1)\beta}}$, where σ_1 is denoted in (4.11), and $B_{2r} \subset \Omega$, then we have for any $\tau \in (0,n)$ and $\rho \in (0,r)$,*

$$\fint_{B_\rho} \Phi_{p_2}(|Du|)\,dx \le c\rho^{-\tau}\left(\fint_{B_r} \Phi_{p_2}(|Du|)\,dx + 1\right), \tag{4.21}$$

for some $c = c(n, \gamma_1, \gamma_2, [p(\cdot)]_\beta, [D\psi]_\beta, \overline{D\psi}, \tau) > 0$.

Proof. By (4.12), (4.18), (4.20) and (4.10),

$$\int_{B_\rho} \Phi_{p_2}(|Du|)\,dx \le c\left(\int_{B_\rho} \Phi_{p_2}(|Du - Dw|)\,dx + \int_{B_\rho} \Phi_{p_2}(|Dw - Dv|)\,dx + \int_{B_\rho} \Phi_{p_2}(|Dv|)\,dx\right)$$

$$\leq cr^{\frac{(\gamma_1-1)\beta}{2}}\left\{M^{\sigma_1}\int\limits_{B_{2r}}\Phi_{p_2}(|Du|)\,dx+r^n\right\}+\left(\frac{\rho}{r}\right)^n\int\limits_{B_r}\Phi_{p_2}(|Dv|)\,dx$$

$$\leq c_6\left\{\left(\frac{\rho}{r}\right)^n+r^{\frac{(\gamma_1-1)\beta}{2}}M^{\sigma_1}\right\}\int\limits_{B_{2r}}\Phi_{p_2}(|Du|)\,dx+c_7r^{n-\tau},$$

for some $c_6, c_7 > 0$ depending only on n, γ_1, γ_2, $[p(\cdot)]_\beta$, $[D\psi]_\beta$, $\overline{D\psi}$ and any $\tau \in (0,n)$. At this point, we take $\delta > 0$ sufficiently small so that

$$r^{\frac{(\gamma_1-1)\beta}{2}}M^{\sigma_1}\leq\delta^{\frac{(\gamma_1-1)\beta}{2}}\leq\epsilon_0,$$

where $\epsilon_0 > 0$ is given in Lemma 4.7 with $(A, B, \tau_1, \tau_2) = (c_6, c_7, n, n-\tau)$, from which we obtain

$$\int\limits_{B_\rho}\Phi_{p_2}(|Du|)\,dx\leq c\left(\frac{\rho}{r}\right)^{n-\tau}\int\limits_{B_r}\Phi_{p_2}(|Du|)\,dx+c\rho^{n-\tau}$$

$$\leq c\rho^{n-\tau}\left(\fint\limits_{B_r}\Phi_{p_2}(|Du|)\,dx+1\right).$$

This implies (4.21). □

Now, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $r_0 > 0$ satisfy (4.2), (4.6), (4.11) when $r = r_0$ and

$$r_0^{\frac{(\gamma_1-1)\beta}{4}}M^{\sigma_1}\leq 1.$$

Suppose $0 < \rho < r \leq r_0/2$, and set

$$p_+ := \sup_{B_{r_0}} p(\cdot),\quad p_- := \inf_{B_{r_0}} p(\cdot)\ \text{ and }\ p_2 := \sup_{B_r} p(\cdot).$$

From the results in the previous subsection, we observe

$$\int\limits_{B_\rho}|Du-(Du)_{B_\rho}|^{p_2}\,dx\leq c\int\limits_{B_\rho}|Du-(Dv)_{B_\rho}|^{p_2}\,dx$$

$$\leq c\int\limits_{B_\rho}|Dv-(Dv)_{B_\rho}|^{p_2}\,dx+c\int\limits_{B_\rho}|Du-Dv|^{p_2}\,dx.$$

Applying (4.19), (4.10) and (4.21) with (ρ, r, p_2) replaced by (r, r_0, p_+), we have

$$\int\limits_{B_\rho}|Dv-(Dv)_{B_\rho}|^{p_2}\,dx\leq\int\limits_{B_\rho}\Phi_{p_2}(|Dv-(Dv)_{B_\rho}|)\,dx$$

$$\leq c\rho^n\left(\frac{\rho}{r}\right)^{\beta_1}\left(\fint\limits_{B_r}\Phi_{p_2}(|Du|)\,dx+1\right)$$

$$\leq c\rho^n \left(\frac{\rho}{r}\right)^{\beta_1} \left(\fint_{B_r} \Phi_{p_+}(|Du|)\,dx + 1 \right)$$

$$\leq c\rho^n \left(\frac{\rho}{r}\right)^{\beta_1} r^{-\tau} \left(\fint_{B_{r_0}} \Phi_{p_+}(|Du|)\,dx + 1 \right).$$

On the other hand, by (4.12), (4.18) and (4.21) with (ρ, r, p_2) replaced by $(2r, r_0, p_+)$, we have

$$\int_{B_\rho} |Du - Dv|^{p_2}\,dx \leq \int_{B_r} \Phi_{p_2}(|Du - Dv|)\,dx$$

$$\leq cr^{\frac{(\gamma_1-1)\beta}{2}+n} \left(M^{\sigma_1} \fint_{B_{2r}} \Phi_{p_2}(|Du|)\,dx + 1 \right)$$

$$\leq cr^{\frac{(\gamma_1-1)\beta}{4}+n} \left(\fint_{B_{2r}} \Phi_{p_+}(|Du|)\,dx + 1 \right)$$

$$\leq cr^{\frac{(\gamma_1-1)\beta}{4}+n-\tau} \left(\fint_{B_{r_0}} \Phi_{p_+}(|Du|)\,dx + 1 \right),$$

for any $\tau \in (0, n)$. Therefore, combining the previous estimates, we obtain

$$\int_{B_\rho} |Du - (Du)_{B_\rho}|^{p_2}\,dx \leq c(\rho^{n+\beta_1} r^{-\beta_1-\tau} + r^{\beta_2+n-\tau}) \left(\fint_{B_{r_0}} \Phi_{p_+}(|Du|)\,dx + 1 \right),$$

where $\beta_2 := \frac{(\gamma_1-1)\beta}{4}$. Now we choose $\rho = r^{1+\mu}$ with $\mu = \frac{\beta_2}{n+\beta_1}$, so that

$$\int_{B_\rho} |Du - (Du)_{B_\rho}|^{p_2}\,dx \leq c\rho^{\frac{n+\beta_2-\tau}{1+\mu}} \left(\fint_{B_{r_0}} \Phi_{p_+}(|Du|)\,dx + 1 \right).$$

Finally, choosing $\tau = \frac{\beta_2 - n\mu}{2} = \frac{\beta_1\beta_2}{2(n+\beta_1)}$, we have

$$\int_{B_\rho} |Du - (Du)_{B_\rho}|^{p_2}\,dx \leq c\rho^{n+\frac{\beta_1\beta_2}{2(n+\beta_1+\beta_2)}} \left(\fint_{B_{r_0}} \Phi_{p_+}(|Du|)\,dx + 1 \right),$$

which implies

$$\left(\fint_{B_\rho} |Du - (Du)_{B_\rho}|^{\gamma_1}\,dx \right)^{\frac{1}{\gamma_1}} \leq c\rho^{\frac{\beta_1\beta_2}{2\gamma_2(n+\beta_1+\beta_2)}} \left(\fint_{B_{r_0}} \Phi_{p_+}(|Du|)\,dx + 1 \right)^{\frac{1}{p_-}}.$$

Therefore, we conclude that for $B_{2r_0}(x_0) \subset \Omega$ we have

$$\left(\fint_{B_\rho(y)} |Du - (Du)_{B_\rho}|^{\gamma_1}\, dx \right)^{\frac{1}{\gamma_1}} \le c\rho^{\frac{\beta_1\beta_2}{2\gamma_2(n+\beta_1+\beta_2)}} \left(\fint_{B_{2r_0(x_0)}} \Phi_{p_+}(|Du|)\, dx + 1 \right)^{\frac{1}{p_-}}$$

for any $B_\rho(y) \subset B_{(r_0/2)^{1+\mu}}(x_0)$. By Campanto's theorem, see for example [25, Theorem 1.2 in Chapter 3], this estimate implies $Du \in C^\alpha(B_{(r_0/2)^{1+\mu}}(x_0))$ with $\alpha = \frac{\beta_1\beta_2}{2\gamma_2(n+\beta_1+\beta_2)}$. $\square$

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