



# Affine invariant points and new constructions



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## ABSTRACT

In [2] Grünbaum asked if the set of all affine invariant points of a given convex body is equal to the set of all points invariant under every affine automorphism of the body. In [3] we have proven the case of a body with no nontrivial affine automorphisms. After some partial results [6,7] the problem was solved in positive by Mordhorst [8]. In this note we provide an alternative proof of the affirmative answer, developing the ideas of [3]. Moreover, our approach allows us to construct a new large class of affine invariant points.

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## 1. Introduction

Let  $\mathbb{K}^n$  be the set of all convex bodies in  $\mathbb{R}^n$  and let  $P : \mathbb{K}^n \rightarrow \mathbb{R}^n$  be a function satisfying the following two conditions:

1. For every nonsingular affine map  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and every convex body  $K \in \mathbb{K}^n$  one has  $P(\varphi(K)) = \varphi(P(K))$ .
2.  $P(K)$  is continuous in the Hausdorff metric.

Such a function  $P$  is called an *affine-invariant point*. The centroid and the center of the John ellipsoid (the ellipsoid of maximal volume contained in a given convex body) are examples of affine-invariant points [6].

Let  $\mathcal{P}$  be the set of all affine-invariant points in  $\mathbb{R}^n$ . It was shown in [7] that  $\mathcal{P}$  is an affine subspace of the space of continuous functions on  $\mathbb{K}^n$  with values in  $\mathbb{R}^n$ . Grünbaum [2] asked a natural question: how big is the set  $\mathcal{P}$ ? In particular, how to describe the set  $\mathcal{P}(K) = \{P(K) \mid P \in \mathcal{P}\}$  for a given  $K \in \mathbb{K}^n$ ? Denote the set of points fixed under affine maps of  $K$  onto itself by  $\mathcal{F}(K)$ . Grünbaum observed that  $\mathcal{P}(K) \subset \mathcal{F}(K)$  and asked the following question:

**Question 1.1.** *Is the set  $\mathcal{P}$  big enough to ensure that  $\mathcal{P}(K) = \mathcal{F}(K)$  for every  $K \in \mathbb{K}^n$ ?*

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In [7], Meyer, Schütt and Werner proved that the set of convex bodies  $K$  for which  $\mathcal{P}(K) = \mathbb{R}^n$  is dense in  $\mathbb{K}^n$ . Then the author showed that if  $\mathcal{F}(K) = \mathbb{R}^n$  then  $\mathcal{P}(K) = \mathbb{R}^n$  [3]. Very recently, using a completely different approach, Mordhorst [8] has shown the affirmative answer to the Question 1.1. This proof used a previous development by P. Kuchment [4,5]. The purpose of this note is to show that the method of [3] can be also used to answer Question 1.1, providing a new proof. Moreover, we construct a new large class of affine invariant points.

## 2. Definitions and notation

Recall some basic notations from group theory.

The group of all invertible linear transformations of  $\mathbb{R}^n$  is denoted by  $GL(n, \mathbb{R})$ . The group of all invertible linear transformations with the determinant equal to 1, i.e. the transformations which preserve volume and orientation is denoted by  $SL(n, \mathbb{R})$ .

For the purposes of the current paper we will use the group of all linear transformations preserving volume but not necessarily preserving orientation, i.e. the transformations with the determinant equal  $\pm 1$  denoted by  $SL_n^-$ .

The group of all affine transformations of  $\mathbb{R}^n$  is denoted by  $Aff(n)$ . It may be represented as  $GL(n) \ltimes \mathbb{R}^n$  with the rule  $(r, x)(a) = r(a) + x$  where  $r \in GL(n)$ ,  $x, a \in \mathbb{R}^n$ .

The unit Euclidian ball in  $\mathbb{R}^n$  is denoted by  $B_2^n$ . The Euclidian norm of a vector is denoted by  $|x|$ . The Lebesgue measure on  $\mathbb{R}^n$  is denoted by  $\mu$ .

A right (left) Haar measure is a measure on a locally compact topological group that is preserved under multiplication by the elements of the group from the right (left). The Lebesgue measure is an example of a Haar measure on  $\mathbb{R}^n$ . Right and left Haar measures are unique up to multiplication however, not necessarily equal to each other. In this paper we always use a left Haar measure and denote the Haar measure of a set  $X$  by  $\text{meas}(X)$ .

$SAff(n)$  is the group of all affine transformations of  $\mathbb{R}^n$  preserving volume. This group may be represented as a semidirect product of the group of all matrices with determinants equal to  $\pm 1$  and  $\mathbb{R}^n$  with the rule  $(r, x)(a) = r(a) + x$  for every  $r \in GL(n)$  with  $\det(r) = \pm 1$ ,  $x \in \mathbb{R}^n$ .  $SAff(n)$  is equipped with the Haar measure, which is the product of Haar measures on the group of all matrices with the determinant equal to  $\pm 1$  and the group  $\mathbb{R}^n$ .

The Hausdorff metric is a metric on  $\mathbb{K}^n$ , defined as

$$d_H(K_1, K_2) = \min\{\lambda \geq 0 : K_1 \subset K_2 + \lambda B_2^n; K_2 \subset K_1 + \lambda B_2^n\}.$$

By  $\mathbb{K}_1^n$  we denote the set of all convex compact sets in  $\mathbb{R}^n$  with volume 1.

## 3. Affine invariant points

For a given convex body  $K \in \mathbb{K}^n$  a family of affine invariant points is constructed by taking an arbitrary point  $v$  and averaging all possible affine transformations of this point with the weight

$$F = F_K : \mathbb{K}^n \rightarrow C(SAff(n))$$

defined by

$$F_K(L)(\varphi) = \mu(\varphi^{-1}(L) \cap K), L \in \mathbb{K}^n, \varphi \in SAff(n).$$

Let  $k \geq 1$  be an integer. For  $L \in \mathbb{K}_1^n$  define the affine invariant point  $T_{k,K,v}$  by

$$T_{k,K,v}(L) = \left( \int_{SAff(n)} F^k(L)(\varphi) d\varphi \right)^{-1} \int_{SAff(n)} F^k(L)(\varphi) \varphi(v) d\varphi. \quad (1)$$

In general, for  $L \in \mathbb{K}^n$  we set

$$T_{k,K,v}(L) = |L|^{1/n} T_{k,K,v}(L/|L|^{1/n}). \quad (2)$$

**Theorem 3.1.** *For a given convex body  $K$  and a vector  $v \in \mathcal{F}(K)$ , the function  $T_{k,K,v} : \mathbb{K}^n \rightarrow \mathbb{R}^n$ , defined in (1) has the following properties:*

1. *There exists  $k_0 \in \mathbb{Z}_+$  such that for every  $k \geq k_0$ ,  $T_{k,K,v}(L)$  is defined for all  $L \in \mathbb{K}^n$ .*
2.  *$T_{k,K,v}$  is an affine invariant point if defined.*
3.  *$T_{k,K,v}(K) \rightarrow v$ ,  $k \rightarrow \infty$ .*

Theorem 3.1 implies that for every  $K \in \mathbb{K}^n$  and every  $v \in \mathcal{F}(K)$  we can find an affine invariant point  $F$  such that  $F(K)$  is arbitrarily close to  $v$ . However, this implies that every point in  $\mathcal{F}(K)$  can be obtained as an affine point of  $K$  because the set of all affine points is an affine space [7].

#### 4. Technical part

To prove Theorem 3.1 we will require some tools for integration over the group  $SAff(n)$ .

For a matrix  $A \in GL(n, \mathbb{R})$  the ordered sequence  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$  of the singular values of the matrix  $A$ , is the sequence of all eigenvalues of  $\sqrt{AA^*}$  counting multiplicities; see e.g. [1]. In the case  $A \in SL_n^-$  we have  $1 = |\det(A)| = \prod_{i=1}^n \lambda_i$ . For a matrix  $A \in GL(n, \mathbb{R})$  we denote by  $\|A\|$  its operator norm  $\ell_2 \rightarrow \ell_2$ , that is

$$\|A\| = \sup_{|x|=1} |Ax|.$$

Note that singular values of  $A$  give a convenient description of the norm  $\|A\| = \lambda_1$ .

For  $R \geq 1$  the “ball”  $S_R$  is the set of all matrices  $A \in SL_n^-$  such that  $\|A\| \leq R$ .

Note that for  $R_1, R_2 \geq 1$  the following equality holds:  $S_{R_1} S_{R_2} = S_{R_1 R_2}$ . Indeed, by the property of the operator norm,  $S_{R_1} S_{R_2} \subset S_{R_1 R_2}$ . On the other hand, according to the polar decomposition, every  $A \in S_{R_1 R_2}$  may be represented in the form  $A = UP$ , where  $U$  is a unitary matrix and  $P$  is positive Hermitian, see e.g. [9]. Then

$$A = U P^{\ln R_1 / \ln(R_1 R_2)} P^{\ln R_2 / \ln(R_1 R_2)},$$

with  $U P^{\ln R_1 / \ln(R_1 R_2)} \in S_{R_1}$ ,  $P^{\ln R_2 / \ln(R_1 R_2)} \in S_{R_2}$ .

**Lemma 4.1.** *For every  $\varepsilon > 0$  there exists a finite set  $N \subset S_{2(1+\varepsilon)}$  such that for every integer  $l \geq 0$  one has*

$$S_{2^l(1+\varepsilon)} \subset N^l S_{(1+\varepsilon)}.$$

**Proof.** Since the set  $S_{2(1+\varepsilon)}$  is compact, it can be covered by some finite collection of balls:

$$S_{2(1+\varepsilon)} \subset \cup_{N_i \in N} N_i S_{1+\varepsilon} = N S_{1+\varepsilon}.$$

We will show by induction that the set  $N$  satisfies the condition of the proposition. The base case for  $l = 0$  is trivial. Now we show the inductive step:

$$S_{2^{l+1}(1+\varepsilon)} = S_{2^l(1+\varepsilon)} S_2 \subset N^l S_{1+\varepsilon} S_2 = N^l S_{2(1+\varepsilon)} \subset N^l N S_{1+\varepsilon}. \quad \square$$

**Proposition 4.2.** *For every  $n \geq 2$ ,  $\alpha \geq 0$  there exists  $p \geq 1$  such that for any convex bodies  $K, L$  the integral*

$$\int_{SL_n^- \mathbb{R}^n} \int \mu^p(L \cap (M(K) + x)) \|M\|^\alpha dx dM$$

*converges. Here  $dM$  is a Haar measure on  $SL_n^-$ .*

**Proof.** There exists a radius  $R > 0$  such that the bodies  $K, L$  are simultaneously contained within the ball  $RB_2^n$ . Therefore,

$$\begin{aligned} & \int_{SL_n^- \mathbb{R}^n} \int \mu^p(L \cap (M(K) + x)) \|M\|^\alpha dx dM \\ & \leq \int_{SL_n^- \mathbb{R}^n} \int \mu^p(RB_2^n \cap (M(RB_2^n) + x)) \|M\|^\alpha dx dM \\ & = R^{pn} \int_{SL_n^- \mathbb{R}^n} \int \mu^p\left(B_2^n \cap \left(M(B_2^n) + \frac{x}{R}\right)\right) \|M\|^\alpha dx dM \\ & = R^{pn+n} \int_{SL_n^- \mathbb{R}^n} \int \mu^p(B_2^n \cap (M(B_2^n) + x)) \|M\|^\alpha dx dM. \end{aligned}$$

It is enough to consider the convergence of the integral

$$\int_{SL_n^- \mathbb{R}^n} \int \mu^p(B_2^n \cap (M(B_2^n) + x)) \|M\|^\alpha dx dM. \quad (3)$$

Note that the lengths of semiaxes of the ellipsoid  $M(B_2^n)$  are defined by the singular values  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  of  $M$  in particular, the diameter of  $M(B_2^n)$  equals  $2\lambda_1$  and its minimal width equals  $2\lambda_n$ . This means that for  $|x| > \lambda_1 + 1$  the volume  $\mu(B_2^n \cap (M(B_2^n) + x)) = 0$ . For all other  $x$  the ellipsoid  $MB_2^n$  is contained within the slab  $L = \{y \in \mathbb{R}^n : |\langle y, u \rangle| \leq \lambda_n\}$  for some vector  $u$ . Therefore,

$$\mu(B_2^n \cap (M(B_2^n) + x)) \leq \mu(B_2^n \cap (L + x)) \leq 2\lambda_n |B_2^{n-1}|.$$

Summing up, the integral (3) is bounded by

$$\begin{aligned} & \int_{SL_n^- \mathbb{R}^n} \int \mu^p(B_2^n \cap (M(B_2^n) + x)) \|M\|^\alpha dx dM \\ & = \int_{SL_n^-} \int_{|x| \leq \lambda_1 + 1} \mu^p(B_2^n \cap (M(B_2^n) + x)) \|M\|^\alpha dx dM \\ & \leq \int_{SL_n^-} \int_{|x| \leq \lambda_1 + 1} (2\lambda_n |B_2^{n-1}|)^p \|M\|^\alpha dx dM \end{aligned}$$

$$\begin{aligned}
&\leq \int_{SL_n^-} (2\lambda_1)^n |B_2^n| (2\lambda_n |B_2^{n-1}|)^p \|M\|^\alpha dM \\
&= 2^{n+p} |B_2^{n-1}|^p |B_2^n| \int_{SL_n^-} \lambda_1^{n+\alpha} \lambda_n^p dM.
\end{aligned}$$

Keeping in mind that  $\prod_{i=1}^n \lambda_i = 1$  one has

$$\lambda_1^{n+\alpha} \lambda_n^p \leq \lambda_1^{n+\alpha} (\lambda_2 \lambda_3 \dots \lambda_n)^{p/(n-1)} = \lambda_1^{n+\alpha} \left( \frac{1}{\lambda_1} \right)^{p/(n-1)} = \lambda_1^{n+\alpha - \frac{p}{n-1}}.$$

Finally, putting  $q = -n - \alpha + \frac{p}{n-1}$  it is enough to show that there exists sufficiently big  $q > 0$  such that the integral

$$\int_{SL_n^-} \|M\|^{-q} dM \quad (4)$$

is convergent. To prove this we split the group  $SL_n^-$  into smaller sets

$$S_{2^l} \setminus S_{2^{l-1}}, l \geq 1.$$

Then

$$\int_{SL_n^-} \|M\|^{-q} dM = \sum_{l=1}^{\infty} \int_{S_{2^l} \setminus S_{2^{l-1}}} \|M\|^{-q} dM \leq \sum_{l=1}^{\infty} 2^{-lq} \text{meas}(S_{2^l}). \quad (5)$$

According to [Lemma 4.1](#), there exists a set  $N$  such that  $\text{meas}(S_{2^l}) \leq |N|^l \text{meas}(S_{2(1+\varepsilon)})$ . Therefore, the series (5) is bounded by a geometric series with the ratio  $2^{-q} |N|$  which is convergent for  $q > \log_2 |N|$ .  $\square$

**Proposition 4.3.** *Let  $G$  be a locally compact topological group and  $dx$  be a Haar measure on  $G$ . Let continuous functions  $f, g$  satisfy the following conditions:*

1. *For every  $x \in G : 0 \leq f(x) \leq 1$ .*
2. *There exists  $x_0 \in G$  such that  $f(x_0) = 1$ . Moreover, if  $x_1 \in G$  is such that  $f(x_1) = f(x_0) = 1$  then  $g(x_1) = g(x_0)$ .*
3. *There exists a constant  $c < 1$  and a compact  $K$  such that for every  $x \in G \setminus K, f(x) < c$ .*
4. *There exists  $k_0 \geq 1$  such that for every  $k \geq k_0$  the integrals*

$$\int_G f^k(x) dx, \int_G f^k(x) |g(x)| dx$$

*are convergent.*

*Then*

$$\lim_{k \rightarrow \infty} \frac{\int_G f^k(x) g(x) dx}{\int_G f^k(x) dx} = g(x_0).$$

**Proof.** Note that the integral

$$\int_G f^k(x) |g(x) - g(x_0)| dx \leq \int_G f^k(x) |g(x)| dx + g(x_0) \int_G f^k(x) dx$$

is convergent for  $k \geq k_0$ . Passing to the new function  $g - g(x_0)$  if needed, we may assume that  $g(x_0) = 0$ .

The set  $N = f^{-1}(1) \subset K$  is closed and therefore compact. By the assumption of the proposition  $g(N) = \{0\}$ . Fix  $\varepsilon > 0$  and consider a neighborhood  $U$  of  $N$  such that  $|g| < \varepsilon$  on  $U$ . There exists a positive constant  $C < 1$  such that  $f < C$  outside of  $U$ . Indeed, outside of  $K$  the function  $f$  is bounded from above by  $c$ , on the compact set  $K \setminus U$  the function  $f$  is separated from 1 by the compactness argument. By continuity of  $f$ , there exists a constant  $D \in (C, 1)$  and a neighborhood  $V \subset U$  of  $N$  such that  $D < f \leq 1$  on  $V$ . Then

$$\begin{aligned} \frac{\int_G f^k(x) |g(x)| dx}{\int_G f^k(x) dx} &\leq \frac{\int_U f^k(x) |g(x)| dx + \int_{G \setminus U} f^k(x) |g(x)| dx}{\int_U f^k(x) dx} \\ &\leq \varepsilon + \frac{\int_{G \setminus U} f^k(x) |g(x)| dx}{\int_V f^k(x) dx} \\ &\leq \varepsilon + \frac{C^{k-k_0} \int_{G \setminus U} f^{k_0}(x) |g(x)| dx}{D^{k-k_0} \int_V f^{k_0}(x) dx} \rightarrow \varepsilon, \quad k \rightarrow \infty. \end{aligned}$$

Sending  $\varepsilon$  to 0 we obtain the required statement.  $\square$

**Proof of Theorem 3.1.** For fixed  $K$  and  $v$  we will shorten the notation by writing  $T_k$  instead of  $T_{k,K,v}$ .

1. Proposition 4.2 applied with  $\alpha = 1$  (respectively,  $\alpha = 0$ ) implies that the integral in the numerator (respectively, denominator) is convergent.

2.  $T_k(cK) = cT_k(K)$  by the definition of  $T_k$ .

For every  $\tau \in SAff(n)$  and  $L \in \mathbb{K}_1^n$  :  $T_k(\tau(L)) = \tau(T_k(L))$ .

Denote

$$c = \left( \int_{SAff(n)} F^k(L)(\varphi) d\varphi \right)^{-1}.$$

For arbitrary  $\tau \in SAff(n)$  we have

$$T(\tau L) = c \int_{SAff(n)} F^k(\tau L)(\varphi) \varphi(v) d\varphi = c \int_{SAff(n)} F^k(L)(\tau^{-1} \varphi) \varphi(v) d\varphi.$$

Replacing  $\varphi$  by  $\tau\varphi$  we get

$$T(\tau L) = c \int_{SAff(n)} F^k(L)(\varphi) \tau(\varphi(v)) d\varphi = \tau \left( c \int_{SAff(n)} F^k(L)(\varphi) \varphi(v) d\varphi \right).$$

The last equality holds because  $cF^k(L)(\varphi)d\varphi$  is a probabilistic measure. Therefore, for every affine  $\tau$  and every integrable function  $f$  one has

$$\int_{SAff(n)} \tau(f(\varphi)) cF^k(L)(\varphi) d\varphi = \tau \left( \int_{SAff(n)} f(\varphi) cF^k(L)(\varphi) d\varphi \right).$$

Note that the function

$$\frac{1}{\int_{S_R \times \mathbb{R}^n} F^k(L)(\varphi) d\varphi} \int_{S_R \times \mathbb{R}^n} F^k(L)(\varphi) \varphi(v) d\varphi$$

is continuous as a function of  $L$  by the Lebesgue's dominated convergence theorem because both integrals are uniformly bounded by a convergent integral by [Proposition 4.2](#). Then

$$T_k = \lim_{R \rightarrow \infty} \frac{1}{\int_{S_R \times \mathbb{R}^n} F^k(L)(\varphi) d\varphi} \int_{S_R \times \mathbb{R}^n} F^k(L)(\varphi) \varphi(v) d\varphi$$

is continuous.

3. Convergence is the direct application of the [Proposition 4.3](#), where  $f(\varphi) = F(K)(\varphi)$  and  $g(\varphi)$  is  $\varphi(v)$  taken coordinatewise. Similarly to the proof of the [Proposition 4.2](#) the function  $F(K)((A, x))$  is separated from 1 when either  $\|A\|$  or  $|x|$  is big. Note that  $F(K)(id) = 1$  and if  $F(K)(\varphi) = 1$  then  $\varphi(K) = K$  which means  $\varphi(v) = v$  because  $v \in \mathcal{F}(K)$ . Therefore,

$$\frac{1}{\int_{SAff(n)} F(K)^k(\varphi) d\varphi} \int_{SAff(n)} F(K)^k(\varphi) \varphi(v) d\varphi \rightarrow id(v) = v, \quad k \rightarrow \infty. \quad \square$$

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