J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A characterization of the Schur property through the disk algebra ☆

Domingo García [a,*], Enrique Jordá [b], Manuel Maestre [a]

[a] *Departamento de Análisis Matemático, Universidad de Valencia, Doctor Moliner 50, 46100 Burjasot (Valencia), Spain*
[b] *Departamento de Matemática Aplicada, E. Politécnica Superior de Alcoy, Universidad Politécnica de Valencia, Plaza Ferrándiz y Carbonell 2, 03801 Alcoy (Alicante), Spain*

A R T I C L E I N F O

Article history:
Received 12 October 2015
Available online xxxx
Submitted by D. Ryabogin

This paper is dedicated to our dear friend Richard Aron

Keywords:
Banach space
Schur property
Disk algebra

A B S T R A C T

In this paper we give a new characterization of when a Banach space E has the Schur property in terms of the disk algebra. We prove that E has the Schur property if and only if $A(\mathbb{D}, E) = A(\mathbb{D}, E_w)$.

1. Introduction

The disk algebra, whether for a single, finitely many, or infinite variables is an area of intensive research (see e.g. [1–5,9,11–15]). In this paper we consider the natural vector-valued extension of the disk algebra $A(\mathbb{D})$.

Let X and E be complex Banach spaces. As usual, B_X and $\overline{B}_X$ will stand for the open (respectively closed) unit ball of X. By $H(B_X, E)$ we denote the space of all mappings $f : B_X \to E$ holomorphic (i.e. complex-Fréchet differentiable) on B_X. As in the scalar valued case, the vector-valued extension of the disk algebra has two natural and equivalent definitions. One, denoted by $A_u(B_X, E)$, is the Banach space of all uniformly continuous functions $f : B_X \to E$ that, moreover, are holomorphic on B_X, endowed with the supremum norm. The other natural definition is the following.

$$A_u(\overline{B}_X, E) := \{f : \overline{B}_X \to E : f \in H(B_X, E) \text{ and } f \text{ uniformly continuous on } \overline{B}_X\}.$$

☆ The first and third authors were supported by MINECO MTM2014-57838-C2-2-P and Prometeo II/2013/013. The second author was supported by MINECO MTM2013-43540-P.

* Corresponding author.
E-mail addresses: domingo.garcia@uv.es (D. García), ejorda@mat.upv.es (E. Jordá), manuel.maestre@uv.es (M. Maestre).

http://dx.doi.org/10.1016/j.jmaa.2016.01.028

Clearly the mapping $R : A_u(\overline{B}_X, E) \to A_u(B_X, E)$ that associates to each element in $A_u(\overline{B}_X, E)$ its restriction to the open unit ball B_X is an isometric isomorphism, since uniformly continuous functions defined on the open unit ball B_X of a Banach space X and with values in another Banach space are bounded and admit a unique extension to the closed unit ball $\overline{B}_X$ which is also uniformly continuous. Thus, from now on, we write $A_u(B_X, E) = A_u(\overline{B}_X, E)$. For $\mathbb{C}$-valued functions we simply denote $A_u(\overline{B}_X, \mathbb{C}) = A_u(B_X)$.

With E_τ we denote E endowed with the topology τ which is either the weak topology $w(E, E^*)$ or, whenever E is a dual space, i.e. there exists a complex Banach space Y such that $E = Y^*$, the weak-star topology $w^*(Y^*, Y)$.

A very classical result by Dunford of 1938 [6, Theorem 76, p. 354] or [10, Theorem 3.10.1, p. 93 combined with Theorem 3.17.1, p. 112], states that $H(B_X, E_w) = H(B_X, E)$. This means that a mapping $f : B_X \to E$ is holomorphic if and only if $u \circ f : B_X \to \mathbb{C}$ is holomorphic for every $u : E \to \mathbb{C}$ continuous linear form (in short for every $u \in E^*$).

Moreover, if $E = Y^*$, then $H(B_X, E_{w*}) = H(B_X, E)$. Again a mapping $f : B_X \to Y^*$ is holomorphic if and only if $u \circ f : B_X \to \mathbb{C}$ is holomorphic for every $u \in Y$ where we consider Y as a subspace of $E^* = Y^{**}$.

The main goal of this paper is to discuss if analogues of Dunford's results are true in the context of vector-valued algebras of the disk (or more properly called, algebras of the ball).

For that reason, we are going to consider the following spaces.

$$A_u(\overline{B}_X, E_\tau) := \{f : \overline{B}_X \to E : f \in H(B_X, E) \text{ and } f \text{ is } \tau - \text{uniformly continuous on } \overline{B}_X\},$$

and

$$A_u(B_X, E_\tau) := \{f : B_X \to E : f \in H(B_X, E) \text{ and } f \text{ is } \tau - \text{uniformly continuous on } B_X\},$$

where τ denotes either the topology w or w^*. Observe that when considering the norm topology in the range space, we simply write E. All of these spaces are Banach spaces when endowed with the supremum norm topology.

We explore the connections between these algebras of the disk,

$$A_u(\overline{B}_X, E) = A_u(B_X, E), \qquad A_u(B_X, E_w)$$

and the space of mapping defined in the closed unit ball $A_u(\overline{B}_X, E_w)$. Since the mapping $R : A_u(\overline{B}_X, E_w) \to A_u(B_X, E_w)$ defined as $R(f)(x) = f(x)$ for every x in B_X is well defined, injective, and actually an isometry into, one can consider $A_u(\overline{B}_X, E_w)$ as a subset of $A_u(B_X, E_w)$, and we have the following chain of inclusions.

$$A_u(B_X, E) = A_u(\overline{B}_X, E) \subseteq A_u(\overline{B}_X, E_w) \subseteq A_u(B_X, E_w). \tag{1.1}$$

Contrary to the Dunford's first stated result for holomorphic mapping both inclusions can be strict. This claim is shown in Section 2, where in Theorem 2.3 a necessary and sufficient condition for the equality $A_u(\overline{B}_X, E_w) = A_u(B_X, E_w)$ is given. Moreover, our main result, Theorem 2.7, proves that given a complex Banach space X, the equality $A_u(B_X, E) = A_u(B_X, E_w)$ holds if and only if E has the Schur property. Therefore, we give a new characterization of that property. We recall that a Banach space E has the *Schur property* if every weakly convergent sequence is norm convergent (see [7, p. 253]). The classical Banach sequence space ℓ_1 has this property [7, Theorem 5.36].

In Section 3 we give two different sufficient conditions for the Banach space $A_u(B_X, E_w)$ to be a Banach algebra whenever the space E is a Banach algebra.

We refer to [7] for notation and background information on Banach spaces. We will use the following classical Banach sequence spaces. The space c_0 of all null sequences endowed with the supremum norm, the space ℓ_∞ of all bounded sequences also endowed with the supremum norm and the space ℓ_1 of all absolutely summable sequences $(x_n)_n$ endowed with the usual norm given by $\|(x_n)_n\| := \sum_{n=1}^{\infty} |x_n|$, $(x_n)_n \in \ell_1$.

2. Spaces of holomorphic and uniformly continuous vector valued functions

The objective in this section is to clarify in which cases these inclusions are strict. If X is finite dimensional, then $f : \overline{B}_X \to E_\tau$ is continuous if and only if f is uniformly continuous, since E_τ is always a space with a uniformity. Thus we shall omit the subindex u, putting for example $A(\mathbb{D}, E)$ to denote the (uniformly) continuous functions on the closed disc $\overline{\mathbb{D}}$ with values in a Banach space E which are holomorphic in the interior.

First we consider the question of when the injective mapping $R : A_u(\overline{B}_X, E_w) \to A_u(B_X, E_w)$ defined above as $R(f)(x) = f(x)$ for every x in B_X is onto.

In the space $A_u(B_X, E_w)$, a priori we only have that for each $u \in E^*$ there is a unique uniformly continuous extension $\widehat{u \circ f} : \overline{B}_X \to \mathbb{C}$. This allows us to produce a unique uniformly continuous extension but taking values in $E^{**}_{w^*}$ as next Lemma shows.

Lemma 2.1. *Let X be a Banach space. Given $f \in A_u(B_X, E_w)$ there exists a unique $\hat{f} \in A_u(\overline{B}_X, E^{**}_{w^*})$ such that $\hat{f}|_{B_X} = f$.*

Proof. Let $f \in A_u(B_X, E_w)$. For each $u \in E^*$ there exists a function $\widehat{u \circ f} : \overline{B}_X \to \mathbb{C}$ which extends $u \circ f$ continuously. If $(x_n)_n$ is a sequence in B_X convergent to $x \in \overline{B_X} \setminus B_X$ then we define $\langle \hat{f}(x), u\rangle = \lim_n u \circ f(x_n)$. The continuity of $\widehat{u \circ f}$ yields that $\hat{f}(x)$ is a well defined bounded linear mapping since $f(B_X)$ is (weakly) bounded. The uniform continuity of $\widehat{u \circ f}$ and its holomorphy on B_X implies $\hat{f} \in A_u(\overline{B}_X, E^{**}_{w^*})$. □

In the next Lemma we give a sufficient condition for the strict inclusion $A_u(\overline{B}_X, E_w) \subsetneq A_u(B_X, E_w)$ to hold.

Lemma 2.2. *If X and E are Banach spaces and we assume that there exists $f \in A_u(B_X, E_w)$ satisfying that $\hat{f}(\partial B_X) \cap E^{**} \setminus E \neq \emptyset$, then*

(a) *$f \in A_u(B_X, E_w) \setminus A_u(\overline{B}_X, E_w)$.*
(b) *$\hat{f} \in A_u(\overline{B}_X, (E^{**})_{w^*}) \setminus A(\overline{B}_X, (E^{**})_w)$.*
(c) *Moreover, if we consider $g : B_X \to E^{**}$, defined by $z \mapsto f(z)$, then $g \in A_u(B_X, (E^{**})_w)$. Let $\hat{g} \in A_u(\overline{B}_X, (E^{(4)})_{w^*})$ be the extension given by Lemma 2.1. Then, $\hat{g}(\partial B_X) \cap E^{(4)} \setminus E^{**} \neq \emptyset$ (i.e. $g = f$ but $\hat{g} \neq \hat{f}$).*

Proof. Part (a) is obvious since if we assume that there exists $g \in A_u(\overline{B}_X, E_w)$ such that $f(x) = g(x)$ for every $x \in B_X$. We can consider $g : \overline{B}_X \to (E^{**})_{w^*}$ and it is a (uniformly) continuous mapping. Since $\hat{f} : \overline{B}_X \to (E^{**})_{w^*}$ is continuous too and both coincide with f in the dense subset B_X, we have $g = \hat{f}$ and then $\hat{f}(\partial B_X) \subset E$. A contradiction.

The assertion (b) follows from the fact that $f(B_X) \subseteq E$, $f(\partial B_X) \cap E^{**} \setminus E \neq \emptyset$ and $\overline{E}^{(E^{**})_w} = \overline{E}^{\|\cdot\|} = E$.

To see (c) we fix $z_0 \in \partial B_X$ with $\hat{f}(z_0) \in E^{**} \setminus E$. Since $\hat{g} : \overline{B}_X \to ((E^{**})^{**}, w^*)$ is continuous, if $\hat{g}(z_0) \in E^{**}$ then for each sequence $(z_k)_k \subset B_X$ convergent to z_0 and for each $u \in (E^{**})^*$ we have $u(\hat{g}(z_0)) = \lim_k u \circ g(z_k) = \lim_k u \circ f(z_k)$, i.e. $f(z_k)$ converges to $\hat{g}(z_0)$ in (E^{**}, w) and $f(z_k)$ converges to $\hat{f}(z_0)$ in the weaker topology (E^{**}, w^*) and we obtain that $\hat{g}(z_0) = \hat{f}(z_0)$. But, as $\overline{E}^{(E^{**},w)} = \overline{E}^{(E^{**},\|\cdot\|)} = E$, we get $\hat{g}(z_0) \in E$, but, by hypothesis, $\hat{f}(z_0) \in E^{**} \setminus E$. A contradiction. □

Observe that in part (c) above if we assume that, $\hat{f}(z) \in E^{**} \setminus E$ for each $z \in \partial B_X$, then, $\hat{g}(z) \in E^{(4)} \setminus E^{**}$ for each $z \in \partial B_X$.

These two lemmas give the following characterization.

Please cite this article in press as: D. García et al., A characterization of the Schur property through the disk algebra, J. Math. Anal. Appl. (2016), http://dx.doi.org/10.1016/j.jmaa.2016.01.028

Theorem 2.3. *Let X and E be complex Banach spaces, the equality*

$$A_u(\overline{B}_X, E_w) = A_u(B_X, E_w),$$

holds if and only if every $f \in A_u(B_X, E_w)$ satisfies that $\hat{f}(\partial B_X) \subset E$.

An immediate consequence is the following Corollary.

Corollary 2.4. *If E is a reflexive Banach space, then*

$$A_u(\overline{B}_X, E_w) = A_u(B_X, E_w),$$

for every Banach space X.

A basic example fulfilling the hypothesis of Lemma 2.2 is the following.

Example 2.5. For $f : \mathbb{D} \to c_0$, $z \mapsto (z^n)_n$, we have $\hat{f} : \overline{\mathbb{D}} \to l_\infty$, $z \mapsto (z^n)_n$, and hence,

(a) $f \in A(\mathbb{D}, (c_0)_w) \setminus A(\overline{\mathbb{D}}, (c_0)_w)$.
(b) $\hat{f} \in A(\overline{\mathbb{D}}, (l_\infty)_{w^*}) \setminus A(\overline{\mathbb{D}}, (l_\infty)_w)$.
(c) If we consider $g : \mathbb{D} \to l_\infty$, $z \mapsto f(z)$ then $g \in A(\mathbb{D}, (l_\infty)_w)$. Let $\hat{g} \in A_\infty(\overline{\mathbb{D}}, (l_\infty^{**})_w^*)$ be the extension given by Lemma 2.1. Then, $\hat{g}(z) \in l_\infty^{**} \setminus l_\infty$ for each $z \in \partial\mathbb{D}$.

Proof. Take $h : \overline{\mathbb{D}} \to l_\infty$, defined by $h(z) = (z^n)_n$. If $u = (a_n) \in \ell_1$, we have $u(h(z)) = \sum_{n=1}^\infty a_n z^n$, that is an element of the algebra of the disk $A(\mathbb{D})$. Thus $h \in A(\overline{\mathbb{D}}, (l_\infty)_{w^*})$ and it is an extension of f. By Lemma 2.1, the extension is unique. Hence $\hat{f} = g$. $\square$

Of course, in the above example $u \circ \hat{g}(z) = u \circ \hat{f}(z)$ for each $u \in l_1 \subseteq l_\infty^*$ and each $z \in \overline{\mathbb{D}}$, but l_1 is not $\sigma(l_\infty^*, l_\infty^{**})$ dense (i.e. separating in l_∞^{**}). Thus, if we want continuity in the extension composing with the functionals of l_∞^*, it is possible but we need the extension to take values in $l_\infty^{**} \setminus l_\infty$. The argument can be reiterated to get different extensions in further even duals.

The precise difference between $A(\overline{\mathbb{D}}, (c_0)_w)$ and $A(\mathbb{D}, c_0)$ is illustrated below.

Proposition 2.6. *Let $f : \overline{\mathbb{D}} \to c_0$, $f(z) = (f_n(z))_n$. Then*

(a) *$f \in A(\overline{\mathbb{D}}, (c_0)_w)$ if and only if $(f_n)_n$ converges weakly to 0 in $A(\mathbb{D})$.*
(b) *$f \in A(\mathbb{D}, c_0)$ if and only if $(f_n)_n$ converges in norm to 0 in $A(\mathbb{D})$.*

Proof. We see (a). Since $A(\mathbb{D})$ is a subspace of $C(\overline{\mathbb{D}})$, the Banach space of continuous functions on $\overline{\mathbb{D}}$, we conclude from Riesz Representation theorem and Hahn–Banach theorem that each functional $u \in A(\mathbb{D})^*$ can be represented with a regular complex measure μ, i.e. $u(f) = \int_{\overline{\mathbb{D}}} f d\mu$. If $f \in A(\overline{\mathbb{D}}, (c_0)_w)$ then $f(\overline{\mathbb{D}})$ is (weakly) bounded in c_0, i.e. there exists $M > 0$ such that $|f_n(z)| \leq M$ for each $z \in \overline{\mathbb{D}}$ and $n \in \mathbb{N}$. Moreover $\lim_n f_n(z) = 0$ for each $z \in \overline{\mathbb{D}}$. Hence we can apply Dominated Convergence Lebesgue's theorem to get that $\lim_n \int_{\overline{\mathbb{D}}} f_n d\mu = 0$ for each regular complex measure μ on $\overline{\mathbb{D}}$, and hence $\lim_n u(f_n) = 0$ for each $u \in A(\mathbb{D})^*$. Conversely, if $(f_n)_n$ tends weakly to 0 in $A(\mathbb{D})$ then $(f_n)_n$ is weakly bounded, and then norm bounded. Hence for each $(\alpha_n)_n \subset l_1$, the series $\sum_n \alpha_n f_n$ is uniformly convergent in $A(\mathbb{D})$.

To see (b) we observe that $(f_n)_n$ converges in norm to 0 in $A(\mathbb{D})$ if and only if (f_n) is equicontinuous and pointwise convergent to 0, if and only if $(f_n)_n$ converges weakly to 0 and it is equicontinuous. This is a consequence of Arzelà–Ascoli theorem, since $A(\mathbb{D})$ is a subspace of $C(\overline{\mathbb{D}})$, and the fact that in a compact space there is not any strictly weaker topology which is Hausdorff. Now for the sequence $(f_n) \subset A(\mathbb{D})$ to be equicontinuous and weakly convergent to 0 is equivalent to be $f : \overline{\mathbb{D}} \to c_0$ continuous and weakly holomorphic in $\mathbb{D}$ by (a). That in turn, as $\overline{\mathbb{D}}$ is a compact set, is equivalent to f be uniformly continuous and holomorphic on $\mathbb{D}$. □

Now we address the question of characterizing the complex Banach spaces E satisfying that $A_u(B_X, E) = A_u(B_X, E_w)$ for every Banach space X. The answer leads us to give a new characterization of complex Banach spaces having the Schur property.

Theorem 2.7. *Let X and E be complex Banach spaces. The following are equivalent.*

(i) *E has the Schur property.*
(ii) *$A_u(B_X, E) = A_u(B_X, E_w)$.*
(iii) *$A_u(B_X, E) = A_u(\overline{B}_X, E_w)$.*

Proof. *(i)* ⇒ *(ii)*. Assume that E has the Schur property and there are a Banach space X and $f \in A_u(B_X, E_w)$ such that there exist $\varepsilon > 0$ and sequences $(x_n)_n$ and $(y_n)_n$ in B_X such that $\|x_n - y_n\|$ tends to 0 and $\|f(x_n) - f(y_n)\| \geq \varepsilon$. The hypothesis $f \in A_u(B_X, E_w)$ yields that $u \circ f$ is uniformly continuous, and then $|u \circ f(x_n) - u \circ f(y_n)|$ tends to 0 for each $u \in E^*$, i.e. $f(x_n) - f(y_n)$ tends to 0 weakly in E, and hence also in norm since E has the Schur property, a contradiction.

(ii) ⇒ *(iii)*. Since $A_u(B_X, E) \subset A_u(\overline{B}_X, E_w) \subset A_u(B_X, E_w)$, we get

$$A_u(B_X, E) = A_u(\overline{B}_X, E_w)$$

for every X.

(iii) ⇒ *(i)*. Let E be a Banach space without the Schur property, and let $(e_n)_n$ be a sequence in the unit sphere S_E of E which is weakly convergent to 0. Let $x_0 \in S_X$. We consider a linear mapping $\varphi : X \to \mathbb{C}$, such that $\varphi(x_0) = 1$. Now we take a sequence $(z_n)_n$ in the unit circle $\partial\mathbb{D}$ and a sequence $(r_n)_n$ of positive numbers (convergent to zero) such that $\overline{D(z_j, r_j)} \cap \overline{D(z_k, r_k)} = \emptyset$ if $j \neq k$. Take $g_n(z) := (z + z_n)/2$ and $f_n(z) = g_n(z)^{k(n)}$ for $k(n)$ being a natural number such that $|g_n(z)|^{k(n)} < 1/4^n$ in $\overline{\mathbb{D}} \setminus D(z_n, r_n)$. We define $f(x) = \sum_n f_n(\varphi(x))e_n$, $x \in \overline{B}_X$.

Let us show that $f \in A_u(\overline{B}_X, E_w) \setminus A_u(B_X, E)$. The series is well defined since $\varphi(x)$ belongs at most to one ball $D(z_k, r_k)$ for each $x \in \overline{B}_X$. For each $n \in \mathbb{N}$ and y_n in the boundary of $D(z_n, r_n) \cap \mathbb{D}$ we have that $\|z_n x_0 - y_n x_0\| = |z_n - y_n| = r_n$ tends to 0 and

$$\|f(z_n x_0) - f(y_n x_0)\| \geq |f_n(z_n)| - \sum_{j \neq n} |f_j(z_n)| - \sum_{j \in \mathbb{N}} |f_j(y_n)| \geq 1 - 1/3 - 1/3 = 1/3.$$

Hence f is not uniformly continuous on $\overline{B}_X$.

If we take $u \in E^*$ then $u(f)(x) = \sum f_n(\varphi(x))u(e_n)$ is a convergent series in $A(\overline{B}_X)$ since $(u(e_n))_n$ tends to 0 and $\sum_n |f_n(z)| \leq 4/3$ for all $z \in \overline{\mathbb{D}}$. Hence $f \in A_u(\overline{B}_X, E_w)$. □

In particular we have the following Corollary.

Corollary 2.8. *Let E be complex Banach space. The following are equivalent.*

(i) *E has the Schur property.*
(ii) *$A(\mathbb{D}, E) = A(\mathbb{D}, E_w)$.*
(iii) *$A(\mathbb{D}, E) = A(\overline{\mathbb{D}}, E_w)$.*

Remark 2.9. If E has the Schur property then

$$A_u(B_X, E_w) = A_u(\overline{B}_X, E_w)(= A_u(B_X, E)),$$

but this does not give a characterization. Corollary 2.4 shows that for reflexive spaces

$$A_u(B_X, E_w) = A_u(\overline{B}_X, E_w),$$

and from the Jossefson–Nizenweig theorem it follows that no infinite dimensional reflexive space has the Schur property.

Remark 2.10. Example 2.5 together Proposition 2.6 and Theorem 2.7 give a proof of the well known fact that the space $A(\mathbb{D})$ does not have the Schur property.

Let us observe that in the above proof one can take the sequence $(z_n)_n$ convergent to 1. Consequently, the constructed f is in fact not continuous in x_0. Actually, the characterization gives that E has the Schur property if and only if $A(\mathbb{D}, E_w) = A(\mathbb{D}, E)$. Now, it is a natural question to ask if $A_u(\overline{B}_X, E_w) \cap C(\overline{B}_X, E) = A_u(B_X, E)$ when E does not have the Schur property. We see below that in general the answer is negative.

Theorem 2.11. *If the complex Banach space E does not have the Schur property then there exists a complex Banach space X such that*

$$A_u(\overline{B}_X, E_w) \cap C(\overline{B}_X, E) \nsubseteq A_u(B_X, E).$$

Proof. Let (x_n) be a sequence on the unit sphere of E weakly convergent to 0. We consider $f : \overline{B}_{l_2} \to E$, $(z_n)_n \to \sum z_n^n x_n$. For each $(z_n)_n$ in $\overline{B}_{l_2}$ we have $\sum |z_n|^n \leq 1 + \sum_{n\geq 2} |z_n|^2 \leq 2$, hence f is well defined and bounded. We check now that f is continuous. We fix $z = (z_n)_n \in \overline{B}_{l_2}$. Let $0 < \varepsilon < 1$. There exists n_0 such that

$$\|(z_n)_{n\geq n_0}\|_2 = \left(\sum_{n\geq n_0} |z_n|^2\right)^{1/2} < \varepsilon/4.$$

We have

$$\sum_{n\geq n_0} |z_n|^n \leq \sum_{n\geq n_0} |z_n|^2 = \|(z_n)_{n\geq n_0}\|_2^2 < \|(z_n)_{n\geq n_0}\|_2 < \varepsilon/4.$$

If $t = (t_n)_n \in \overline{B}_{l_2}$ and $\|z - t\|_2 < \varepsilon/4$ then also

$$\|(z_n)_{n\geq n_0} - (t_n)_{n\geq n_0}\|_2 < \varepsilon/4,$$

and hence

$$\|(t_n)_{n\geq n_0}\|_2 \leq \|(t_n - z_n)_{n\geq n_0}\|_2 + \|(z_n)_{n\geq n_0}\|_2 < \varepsilon/2.$$

This yields

$$\sum_{n\geq n_0} |t_n|^n \leq \sum_{n\geq n_0} |t_n|^2 = \|(t_n)_{n\geq n_0}\|_2^2 \leq \|(t_n)_{n\geq n_0}\|_2 < \varepsilon/2.$$

Please cite this article in press as: D. García et al., A characterization of the Schur property through the disk algebra, J. Math. Anal. Appl. (2016), http://dx.doi.org/10.1016/j.jmaa.2016.01.028

We get now $0 < \delta < \varepsilon/4$ such that $\|z - t\|_2 < \delta$ implies

$$\sum_{n=1}^{n_0-1} |z_n^n - t_n^n| < \frac{\varepsilon}{4}.$$

For this δ we get that $\|z - t\|_2 < \delta$ implies

$$\|f(z) - f(t)\| = \|\sum_n (z_n^n - t_n^n)x_n\| \leq \sum_{n=1}^{n_0-1} |z_n^n - t_n^n| + \sum_{n\geq n_0} |z_n|^n + \sum_{n\geq n_0} |t_n|^n < \varepsilon.$$

Hence f is continuous.

Let $u \in E^*$ and $z = (z_n)_n \in \overline{B}_{l_2}$. The series

$$u \circ f(z) = \sum z_n^n u(x_n), \quad z = (z_n)_n \in \overline{B}_{l_2}$$

is uniformly convergent on $\overline{B}_{l_2}$, i.e. the series converges in $A_u(\overline{B}_{l_2})$. This follows from the convergence to 0 of $(u(x_n))_n$ and the estimate

$$\sum_n |z_n|^n \leq 1 + \|z\|_2^2 \leq 2$$

for each $z = (z_n)_n \in \overline{B}_{l_2}$.

To finish we observe that

$$\left\| f(e_n) - f\left(\frac{n-1}{n} e_n\right) \right\| = 1 - \left(\frac{n-1}{n}\right)^n \to 1 - e^{-1},$$

and hence f is not uniformly continuous. □

There exists another natural extension to the infinite dimensional setting of the algebra of the disk, it is the Banach algebra of holomorphic, bounded and continuous functions on the closed ball of a Banach space defined as

$$A_\infty(\overline{B}_X, E) = \{f : \overline{B}_X \to E : f \in H^\infty(B_X, E) \cap C(\overline{B}_X, E)\},$$

endowed with the supremum norm, where $H^\infty(B_X, E)$ denotes the space of all holomorphic and bounded mappings from B_X into E.

The corresponding analogue with the weak topology is

$$A_\infty(\overline{B}_X, E_w) = \{f : \overline{B}_X \to E_w) : f \in H^\infty(B_X, E_w)) \cap C(\overline{B}_X, E_w)\}.$$

One has here the following inclusions

$$A_u(B_X, E) \subseteq A_\infty(\overline{B}_X, E) \subseteq A_\infty(\overline{B}_X, E_w).$$

Analogues to Theorems 2.7 and 2.11 hold, and give in part (a) the following characterization.

Theorem 2.12.

(a) *Let X and E be complex Banach spaces. The space E has the Schur property if and only if*

$$A_\infty(\overline{B}_X, E) = A_\infty(\overline{B}_X, E_w).$$

(b) *If the complex Banach space E does not have the Schur property there exists a complex Banach space X for which*

$$A_u(B_X, E) \nsubseteq A_\infty(\overline{B}_X, E) \cap A_u(\overline{B}_X, E_w)$$

3. Banach algebras

Since holomorphic functions, continuous and bounded uniformly continuous functions remain stable under products, it is immediate that $A_u(B_X, E)$ and $A_\infty(\overline{B}_X, E)$ are Banach algebras whenever E is. Additionally, $A_u(B_X, E_w)$ and $A_\infty(\overline{B}_X, E_w)$ are contained in $H_\infty(B_X, E)$, the Banach space of all bounded holomorphic functions from B_X into E. Hence, for $f, g \in A_u(B_X, E_w)$, respectively $f, g \in A_\infty(\overline{B}_X, E_w)$, we have

$$\|fg\| = \sup_{x \in B_X} \|f(x)g(x)\| \leq \sup_{x \in B_X} \|f(x)\|\|g(x)\| = \|f\|\|g\|.$$

Thus, $fg \in H_\infty(B_X, E)$, respectively $fg \in H_\infty(B_X, E) \cap C_\infty(\overline{B}_X, F)$, where $C_\infty(\overline{B}_X, F)$ is the Banach space of all bounded continuous mappings on the closed unit ball $\overline{B_X}$ with values in F.

Our aim in this section is to study, if we assume that E is a Banach algebra, when the Banach spaces $A_u(B_X, E_w)$ and $A_\infty(\overline{B}_X, E_w)$ are Banach algebras.

Let us observe that as a consequence of Theorem 2.7, if E is a Banach algebra with the Schur property (e.g. ℓ_1) then $A_u(B_X, E_w) = A_u(B_X, E)$, and hence, $A_u(B_X, E_w)$ is a Banach algebra too.

Proposition 3.1. *Let E be a Banach subalgebra of $C(K)$, the Banach algebra of complex valued continuous functions on a Hausdorff compact space K endowed with the supremum norm. The following hold.*

(a) $A_\infty(\overline{B}_X, E_w)$ *is a Banach algebra.*
(b) $A_u(B_X, E_w)$ *is a Banach algebra.*

Proof. To see (a), let $f, g \in A_\infty(\overline{B}_X, E_w)$ and $(x_n)_n$ be a sequence in $\overline{B}_X$ convergent in norm to $x \in \overline{B}_X$. Then $(f(x_n)g(x_n))_n$ is a sequence of continuous functions on K which is uniformly bounded and pointwise convergent to $f(x)g(x)$, i.e. for each $z \in K$ $f(x_n)(z)g(x_n)(z)$ converges to $f(x)(z)g(x)(z)$. By Dominated Convergence Lebesgue's theorem, for each regular measure μ in K we get

$$\lim_n \int_K f(x_n)(z)g(x_n)(z)d\mu(z) = \int_K f(x)(z)g(x)(z)d\mu(z).$$

We conclude from Hahn–Banach theorem and Riesz representation theorem that $f(x_n)g(x_n)$ tends weakly to $f(x)g(x)$, and we have obtained that $fg \in A_\infty(\overline{B}_X, E_w)$.

Now we prove (b). Let $M(E)$ be the maximal ideal space of E and $f, g \in A_u(B_X, E_w)$. Since $A_u(B_X, E_w)$ is a subspace of $A_\infty(B_X, E_w)$, part (a) implies that $u \circ fg$ is continuous for each $u \in M(E)$. Suppose that there exists $u \in E^*$ such that $u \circ fg$ is not uniformly continuous. Then there exist $\varepsilon > 0$ and two sequences $(x_n)_n$ and $(y_n)_n$ in B_X such that $\|x_n - y_n\| \to 0$ but $|u \circ fg(x_n) - u \circ fg(y_n)| > \varepsilon$ for every n. Let $z \in K$.

$$\begin{aligned}
|\delta_z(f(x_n)g(x_n) - f(y_n)g(y_n))| &= |f(x_n)(z)g(x_n)(z) - f(y_n)(z)g(y_n)(z)| \\
&\leq |f(x_n)(z)(g(x_n)(z) - g(y_n)(z))| + |g(y_n)(z)(f(x_n)(z) - f(y_n)(z))| \\
&\leq \|f\||g(x_n)(z) - g(y_n)(z)| + \|g\||f(x_n)(z) - f(y_n)(z)|,
\end{aligned}$$

and the right hand side tends to 0 since $\delta_z \in E^*$ and $\delta_z \circ f$ and $\delta_z \circ g$ are uniformly continuous. Now, the Riesz representation theorem and the Hahn–Banach theorem yield that there exists a regular measure μ on K such that

$$u(x) = \int_K x(z) d\mu(z)$$

for each $x \in E$. Since $(f(x_n)g(x_n) - f(y_n)g(y_n))_n \subset E$ is a bounded sequence which tends pointwise to 0, we have that, Dominated Convergence Lebesgue's theorem yields

$$u \circ fg(x_n) - u \circ fg(y_n) = \lim_n \int_K (f(x_n)(z)g(x_n)(z) - f(y_n)(z)g(y_n)(z)) d\mu(z) = 0,$$

a contradiction. $\square$

Proposition 3.2. *Let E be a Banach algebra such that span$(M(E))$ is dense in E^*. Then both $A_\infty(\overline{B}_X, E_w)$ and $A_u(B_X, E_w)$ are Banach algebras.*

Proof. We prove the statement for $A_u(B_X, E_w)$, the proof for $A_\infty(\overline{B}_X, E_w)$ is analogous. We observe that if $f, g \in A_u(B_X, E_w)$, then $u \circ fg$ is uniformly continuous for each $u \in \operatorname{span}(M(E))$, since such a u can be written $u = \sum_{i=1}^k a_i m_i$, with $a_i \in \mathbb{C}$ and $m_i \in M(E)$, and $m \circ (fg) = (m \circ f)(m \circ g)$ is a uniformly continuous function for each $m \in M(E)$. Assume that both f and g are in the unit ball of $A_u(B_X, E)$, and then

$$\sup_{x \in B_X} \|f(x)g(x)\| \le \sup_{x \in B_X} \|f(x)\| \|g(x)\| \le 1.$$

Let $v \in E^*$, let $\varepsilon > 0$ and let $u \in \operatorname{span}(M(E))$ such that $\|v - u\| < \varepsilon/3$. There exists $0 < \delta < 1$ such that $x, y \in \overline{B}_X$ and $\|x - y\| < \delta$ imply $|u \circ (fg)(x) - u \circ (fg)(y)| < \varepsilon/3$. Now, for $x, y \in \overline{B}_X$ and $\|x - y\| < \delta$, we have

$$|v \circ (fg)(x) - v \circ (fg)(y)| \le |(v - u) \circ (fg)(x)| + |(u - v) \circ (fg)(y)| + |u \circ (fg)(x) - u \circ (fg)(y)| < \varepsilon. \qquad \square$$

Let G be a compact topological group, let $1 \le p < \infty$ and let $L_p(G)$ be the space of all functions $f : G \to \mathbb{C}$ with f measurable and $|f|^p$ integrable with respect to the Haar measure. These spaces are Banach algebras with respect to the convolution [8, 5.21, p. 135].

Corollary 3.3. *Let G be an abelian compact topological group and let $1 < p < \infty$, we have that $A_u(\overline{B}_X, L_p(G)_w)$ is a Banach algebra.*

Proof. Let denote $\widehat{G}$ its dual group formed by all the characters. For $\xi \in \widehat{G}$ the abstract Fourier transform $\hat{f} : \widehat{G} \to \mathbb{C}$ is defined by

$$\hat{f}(\xi) = \int_G \langle x, \xi \rangle f(x) dx.$$

By [8, Theorem 4.2], the maximal ideal space $M(L_1(G))$ of $L_1(G)$ is completely determined by the Fourier transform, i.e. for $\xi \in \widehat{G}$, if we define

$$\Lambda_\xi(f) := \hat{f}(\xi), \quad f \in L_1(G),$$

then

$$M(L_1(G)) = \{\Lambda_\xi : \ \xi \in \widehat{G}\}.$$

Moreover, $L_p(G) \subset L_1(G)$ for $1 < p < \infty$. Thus

$$M(L_p(G)) = \{\Lambda_\xi : \ \xi \in \widehat{G}\}, \ 1 < p < \infty.$$

Now, for each $1 < p < \infty$, $L_p(G)$ is a reflexive Banach algebra and $M(L_p(G))$ is separating in $L_p(G)$ by the Fourier Uniqueness theorem [8, 4.33]. Hence, the reflexivity of $L_p(G)$ yields that the span of $M(L_p(G))$ is dense in $L_q(G)$, with $1/p + 1/q = 1$. The result now follows from Proposition 3.2. □

Remark 3.4.

(a) As a particular case of the above corollary we can take as G the torus $\mathbb{T}$, the finite product of the torus $\mathbb{T}^N$ or the countable product of the copies of the torus $\mathbb{T}^{\mathbb{N}}$. In these cases the characters are $\mathbb{Z}$, the finite product $\mathbb{Z}^N$ and the countable direct sum $\mathbb{Z}^{(\mathbb{N})}$ respectively, and the abstract Fourier transform becomes the usual Fourier transform.

(b) c_0 satisfies the hypothesis of both Proposition 3.1 and Proposition 3.2. But, in general, Proposition 3.2 cannot be applied to $C(K)$ for an arbitrary compact space K, because the maximal ideal space $M(C(K))$ of $C(K)$, that coincides with the set $\{\delta_x : x \in K\}$ of the evaluations at points of K, is not a total subset of $C(K)^*$, i.e. they do not separate points in $C(K)^{**}$. Indeed, let K be a perfect Hausdorff compact set and let m be a regular measure on K satisfying that all singletons have measure zero. This happens if $K \subset \mathbb{R}^n$ and m is the Lebesgue measure or if K is an infinite compact group and m is the Haar measure. Let us assume without loss of generality $m(K) = 1$. Under these hypotheses, m cannot be approached by a finite linear combination of evaluations.
Let $1 > \delta > 0$. We denote $m(f) = \int_K f dm$. Let $\{x_1, \ldots, x_n\} \subset K$ and $\{a_1, \ldots, a_n\} \in C$ be arbitrary. Let U be an open neighborhood of $\{x_1, \ldots, x_n\}$ with $m(U) < \delta$. Let V be a closed neighborhood of $\{x_1, \ldots, x_n\}$ contained in U. We apply Uryshon's Lemma to get a positive function f in the closed unit ball of $C(K)$ such that $f|_V = 0$ and $f_{K\setminus U} = 1$. We have now $m(f) > 1 - \delta$ since f is positive and identically 1 in $K \setminus U$ and $m(K \setminus U) > 1 - \delta$. Moreover $\sum a_i f(x_i) = 0$. Thus

$$||m - \sum_{i=1}^{n} a_i \delta_{x_i}||| \geq |m(f) - \sum a_i f(x_i)| > 1 - \delta.$$

Hence the evaluations at points of K is not a total subset of $C(K)^*$. Further, we have proved that the distance of m to the closed span of $\{\delta_x : \ x \in K\}$ is 1.
Reciprocally, the Banach algebras $L_p(G)$ with respect to the convolution $\star$, where G is an abelian compact topological group and $1 < p < \infty$, are not subalgebras of any $C(K)$, since there exists f in $L_p(G)$ such that $\|f \star f\| < \|f\|^2$. Hence $L_p(G)$ fulfills the hypothesis of Proposition 3.2 but not the one of Proposition 3.1.

Acknowledgments

The second author wants to thank M. Filali and J. Galindo for several nice and helpful discussions about concrete algebras satisfying the conditions of Proposition 3.2 during his visit in April of 2015 to the University of Oulu.

References

[1] R.M. Aron, D. Carando, T.W. Gamelin, S. Lassalle, M. Maestre, Cluster values of analytic functions on a Banach space, Math. Ann. 353 (2012) 293–303.

[2] R.M. Aron, D. Carando, S. Lassalle, M. Maestre, Cluster values of holomorphic functions of bounded type, Trans. Amer. Math. Soc. (2015), http://dx.doi.org/10.1090/tran/6407, article electronically published on July 1, 2015.

[3] R.M. Aron, B. Cole, T. Gamelin, Weak-star continuous analytic functions, Canad. J. Math. 47 (1995) 673–683.

[4] Y.S. Choi, D. García, S.K. Kim, M. Maestre, Some geometric properties on disk algebras, J. Math. Anal. Appl. 409 (2014) 147–157, http://dx.doi.org/10.1016/j.jmaa.2013.07.002.

[5] Y. Choi, M.J. Heath, Characterizing derivations from the disk algebra to its dual, Proc. Amer. Math. Soc. 139 (3) (2011) 1073–1080.

[6] N. Dunford, Uniformity in linear spaces, Trans. Amer. Math. Soc. 44 (1938) 305–356.

[7] M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler, Banach Space Theory. The Basis for Linear and Nonlinear Analysis, CMS Books Math./Ouvrages Math. SMC, Springer, New York, 2011.

[8] G.B. Folland, A Course in Abstract Harmonic Analysis, Stud. Adv. Math., CRC Press, 1995.

[9] P. Gorkin, R. Mortini, N. Nikolski, Norm controlled inversions and a corona theorem for H^∞-quotient algebras, J. Funct. Anal. 255 (4) (2008) 854–876.

[10] E. Hille, R.S. Phillips, Functional Analysis and Semi-Groups, Amer. Math. Soc. Colloq. Publ., vol. 31, 1957.

[11] P. Lefèvre, Some characterizations of weakly compact operators in H^∞ and on the disk algebra. Application to composition operators, J. Operator Theory 54 (2) (2005) 229–238.

[12] R. Mortini, A. Sasane, Ideals of denominators in the disk-algebra, Bull. Lond. Math. Soc. 41 (4) (2009) 669–675.

[13] Z.H. Mozhyrovska, A.V. Zagorodnyuk, Analytic hypercyclic operators, Mat. Stud. 29 (2) (2008) 218–220.

[14] V. Nestoridis, An extension of the disc algebra, I, Bull. Lond. Math. Soc. 44 (4) (2012) 775–788.

[15] V.I. Paulsen, D. Singh, Modules over subalgebras of the disk algebra, Indiana Univ. Math. J. 55 (5) (2006) 1751–1766.

Please cite this article in press as: D. García et al., A characterization of the Schur property through the disk algebra, J. Math. Anal. Appl. (2016), http://dx.doi.org/10.1016/j.jmaa.2016.01.028