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# Norming points and critical points

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A B S T R A C T

Using a diffeomorphism between the unit sphere and a closed hyperplane of an infinite dimensional Banach space, we introduce the differentiation of a function defined on the unit sphere, and show that a continuous linear functional attains its norm if and only if it has a critical point on the unit sphere. Furthermore, we provide a strong version of the Bishop–Phelps–Bollobás theorem for a Lipschitz smooth Banach space.



## 1. Main results

Let $X$ be a Banach space and $S_X$ be its unit sphere. A continuous linear functional $f \in X^*$ is said to attain its norm if there exists $x_0 \in S_X$ such that $|f(x_0)| = ||f||$, i.e. $|f|$ has a maximum on $S_X$. The point $x_0$ is called a "norming point" of $f$. The first cornerstone in studying norm-attaining linear functionals is James' characterization [8] of a reflexive Banach space, which says that every continuous linear functional on $X$ attains its norm if and only if $X$ is reflexive. After the celebrated Bishop–Phelps theorem [3], "for a Banach space $X$, the set of all norm-attaining linear functionals is dense in $X^*$" appeared in 1961, a lot of attention has been paid to the study of this property for linear operators between Banach spaces.

In this short paper we want to show that a norming point $x_0 \in S_X$ of $f$ is a critical point of $f$, that is, $f'(x_0) = 0$. However, from the concept of the Frechét differentiation of $f$ we have $f'(x_0) = f$. Hence we introduce a concept of the differentiation of a function $f$ defined on $S_X$, which is compatible with the differentiation on a manifold.

We now assume that $X$ is a Banach space with a $C^p$ smooth norm ($1 \le p \le \infty$). For every $z \in S_X$, we denote by $H_{-z}$ the hyperplane tangent to $S_X$ at $-z$. Let $\pi_z$ be the stereographic projection from $S_X \setminus \{z\}$

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onto $H_{-z}$. The desired manifold structure on $S_X$ is defined by the family $\{\pi_z \mid z \in S_X\}$. It is easily checked that $S_X$ is a $C^p$ submanifold, modelled on a codimension one, closed linear subspace $X_0$ of $X$ [10, II, §2, Example]. Let $\Psi$ be a diffeomorphism from a closed hyperplane $H$ of a Banach space $X$ onto $S_X$. For $f : S_X \to \mathbb{R}$ we define $Df_\Psi : S_X \to H^*$ by $Df_\Psi(u)(h) := D(f \circ \Psi)(\Psi^{-1}(u))(h)$ for $u \in S_X$ and $h \in H$, where $D$ is the Frechét differentiation. We say that $f$ has a "critical point" at $u \in S_X$ if $Df_\Psi(u) = 0$ for some diffeomorphism $\Psi$ from a closed hyperplane $H$ of a Banach space $X$ onto $S_X$.

For its well-definedness, it is easy to check that given diffeomorphisms $\Psi_1 : H_1 \to S_X$ and $\Psi_2 : H_2 \to S_X$, $Df_{\Psi_1}(u) = 0$ if and only if $Df_{\Psi_2}(u) = 0$. Indeed,

$$\begin{aligned} D(f \circ \Psi_1)(\Psi_1^{-1}(u)) &= D(f \circ \Psi_2 \circ \Psi_2^{-1} \circ \Psi_1)(\Psi_1^{-1}(u)) \\ &= D(f \circ \Psi_2)(\Psi_2^{-1}(u)) \circ D(\Psi_2^{-1} \circ \Psi_1)(\Psi_1^{-1}(u)). \end{aligned}$$

Since $\Psi_2^{-1} \circ \Psi_1$ is a diffeomorphism, $D(\Psi_2^{-1} \circ \Psi_1)(\Psi_1^{-1}(u)) \in \mathcal{L}(H_1, H_2)$ is an isomorphism.

In 1966, C. Bessaga [2] proved that every infinite dimensional Hilbert space is $C^\infty$ diffeomorphic to its unit sphere. By improving Bessaga's non-complete technique, H.T. Dobrowolski [5] proved in 1979 that every infinite dimensional Banach space $X$ which is linearly injectable into some $c_0(\Gamma)$ is $C^\infty$ diffeomorphic to $X \setminus \{0\}$. More generally, Azagra [1] showed the following result in 1997.

**Theorem 1.1.** *(See [1, Theorem 1].) Let $X$ be an infinite dimensional Banach space with a $C^p$ smooth norm, where $p \in \mathbb{N} \cup \{\infty\}$. Then for every closed hyperplane $H$ in $X$, there exists a $C^p$ diffeomorphism between $S_X$ and $H$.*

From now on, let $\Psi$ denote the $C^p$ diffeomorphism from $H$ onto $S_X$ given in the above theorem.

**Theorem 1.2.** *Let $X$ be an infinite dimensional Banach space with a $C^p$ smooth norm ($1 \le p \le \infty$). Then $f \in S_{X^*}$ attains its norm at $u \in S_X$ if and only if $f|_{S_X}$ has a critical point at $u \in S_X$.*

**Proof.** Suppose $f$ attains its norm at $u \in S_X$. For each vector $h \in H$, $\|h\| = 1$ we define $g : \mathbb{R} \to \mathbb{R}$ by $g(\lambda) = f \circ \Psi(\lambda h + \Psi^{-1}(u))$. Then it is clear that $g$ is differentiable on $\mathbb{R}$ and also attains either a maximum or a minimum at $\lambda = 0$. Hence $g'(0) = 0 = D(f \circ \Psi)(\Psi^{-1}(u))(h)$, which implies that $Df_\Psi(u) = 0$.

For the proof of the other implication it is enough to show that if $0 \le f(x_1) < 1$ at $x_1 \in S_X$, then $Df_\Psi(x_1) \ne 0$. Choose $x_2 \in S_X$ with $f(x_2) > f(x_1)$ and define $\gamma : [0, 1] \to H$ by

$$\gamma(s) := \Psi^{-1}\left(\frac{x_1 + s(x_2 - x_1)}{\|x_1 + s(x_2 - x_1)\|}\right).$$

Since $s \mapsto \left(\frac{x_1+s(x_2-x_1)}{\|x_1+s(x_2-x_1)\|}\right)$ is $C^p$, so is $\gamma(s)$. We first want to show that

$$0 < \lim_{s \to 0^+} \frac{f(\Psi \circ \gamma(s)) - f(x_1)}{\|\Psi \circ \gamma(s) - x_1\|}.$$

Since $\|x_1 + s(x_2 - x_1)\| \le 1$ for $s \in [0, 1]$, it follows that

$$sf(x_2 - x_1) = f(x_1 + s(x_2 - x_1)) - f(x_1) \le f(\Psi \circ \gamma(s)) - f(x_1).$$

Put $z = s(x_2 - x_1)$, and we have

$$\begin{aligned} \|\Psi \circ \gamma(s) - x_1\| &= \left\|\frac{x_1+z-x_1\|x_1+z\|}{\|x_1+z\|}\right\| = \left\|\frac{(x_1+z)(1-\|x_1+z\|)+z\|x_1+z\|}{\|x_1+z\|}\right\| \\ &\le |1 - \|x_1 + z\|| + \|z\| \le 2\|z\| = 2s\|x_2 - x_1\|. \end{aligned}$$

Combined with the above inequality, for every $s \in [0,1]$ we have

$$0 < \frac{f(x_2 - x_1)}{2\|x_2 - x_1\|} \leq \frac{f(\Psi \circ \gamma(s)) - f(x_1)}{\|\Psi \circ \gamma(s) - x_1\|},$$

hence we are done.

Suppose that $Df_\Psi(x_1) = 0$, i.e. $D(f \circ \Psi)(\Psi^{-1}(x_1)) = 0$. By the chain rule, we have

$$Df(x_1) \circ D\Psi(\Psi^{-1}(x_1)) = 0.$$

Note that

$$D\Psi : H \to \mathcal{L}(H, X) \quad \text{and} \quad Df : X \to X^*.$$

Since $Df(x_1) = f \in X^*$, the above equality implies that

$$D\Psi(\Psi^{-1}(x_1))(h) \subseteq \ker(f)$$

for every $h \in H$. Hence for every $s \in [0,1]$

$$D(\Psi \circ \gamma)(0)(s) = D\Psi(\Psi^{-1}(x_1)) \circ D\gamma(0)(s) \subseteq \ker f.$$

We note that the Taylor expansion of $(\Psi \circ \gamma)(s)$ at 0 gives us

$$(\Psi \circ \gamma)(s) = (\Psi \circ \gamma)(0) + D(\Psi \circ \gamma)(0)(s) + o(s).$$

Now we want to show that $D(\Psi \circ \gamma)(0) > 0$. Indeed,

$$\begin{aligned}
D(\Psi \circ \gamma)(0) &= \lim_{s\to 0^+} \tfrac{\|\Psi\circ\gamma(s) - \Psi\circ\gamma(0)\|}{s} \\
&= \lim_{s\to 0^+} \left\| \tfrac{x_1 + s(x_2 - x_1) - x_1\|x_1 + s(x_2 - x_1)\|}{s\|x_1 + s(x_2 - x_1)\|} \right\| \\
&= \lim_{s\to 0^+} \left\| \tfrac{x_1}{s} + (x_2 - x_1) - \tfrac{x_1}{s}\|x_1 + s(x_2 - x_1)\| \right\| \cdot \tfrac{1}{\|x_1 + s(x_2 - x_1)\|} \\
&= \lim_{s\to 0^+} \left\| x_1 \left( \tfrac{1 - \|x_1 + s(x_2 - x_1)\|}{s} - 1 \right) + x_2 \right\| \\
&> 0,
\end{aligned}$$

where the last inequality follows from the fact that $x_1$ and $x_2$ are linearly independent. Since $D(\Psi\circ\gamma)(0) > 0$, we obtain the following contradiction

$$\begin{aligned}
0 < \lim_{s\to 0^+} & \frac{f(\Psi \circ \gamma(s)) - f(x_1)}{\|\Psi \circ \gamma(s) - x_1\|} \\
&= \lim_{s\to 0^+} \frac{[f(\Psi \circ \gamma(s)) - f(\Psi \circ \gamma(0))]/s}{\|\Psi \circ \gamma(s) - \Psi \circ \gamma(0)\|/s} \\
&= \lim_{s\to 0^+} \frac{f(\frac{o(s)}{s})}{D(\Psi \circ \gamma)(0)} = 0,
\end{aligned}$$

which implies that $Df_\Psi(x_1) \neq 0$ for every $x_1 \in S_X$ with $0 \leq f(x_1) < 1$. $\square$

We recall that S.K. Kim and H.J. Lee [9, Theorem 3, Corollary 4] proved that a Banach space $X$ is uniformly convex if and only if for every $\epsilon > 0$ there is $0 < \eta(\epsilon) < 1$ such that for all $f \in S_{X^*}$ and all $x \in B_X$ satisfying $|f(x)| > 1 - \eta(\epsilon)$, there exists $x_0 \in S_X$ satisfying $|f(x_0)| = 1$ and $\|x - x_0\| < \epsilon$. In particular, a reflexive Banach space $X$ is uniformly smooth if and only if for every $\epsilon > 0$ there is $0 < \eta(\epsilon) < 1$ such that for all $f \in S_{X*}$ and all $x \in B_X$ satisfying $|f(x)| > 1 - \eta(\epsilon)$, there exists $f_0 \in S_{X*}$ satisfying $|f_0(x)| = 1$ and $\|f - f_0\| < \epsilon$. Applying the diffeomorphisms in the proof of [1, Theorem 1] we obtain the following strong version of the Bishop–Phelps–Bollobás theorem for a Lipschitz smooth Banach space. We say that a Banach space $X$ is Lipschitz smooth if its norm $\|\cdot\|$ is Frechét differentiable on $S_X$ and the mapping $x \in S_X \to D\|\cdot\|(x) \in S_{X^*}$ is Lipschitz. Note that a Banach space $X$ is uniformly smooth if and only if its norm $\|\cdot\|$ is Frechét differentiable on $S_X$ and the mapping $x \in S_X \to D\|\cdot\|(x) \in S_{X^*}$ is uniformly continuous (see [7, Fact 9.7]).

**Theorem 1.3.** *Let $X$ be a Lipschitz smooth Banach space and $0 < \epsilon < \frac{1}{8}$. Then there exist a constant $a > 0$ (depending only on $X$) and $\beta(\epsilon)$ with $\lim_{\epsilon\to 0^+} \beta(\epsilon) = 0$ such that for all $f \in S_{X*}$ and all $z \in S_X$ satisfying $f(z) > 1 - a\epsilon^2$, there exists $f_0 \in S_{X*}$ satisfying $|f_0(z)| = 1$ and $\|f - f_0\| < \beta(\epsilon)$.*

Before we give its proof, we first explain some materials we need. When $S_X$ does not contain any line segment passing through $-z \in S_X$, the explicit formula for the stereographic projection $\pi_z$ from $S_X \setminus \{-z\}$ onto $H_z = \{x : D\|\cdot\|(z)(x) = 1\}$, a hyperplane of $X$, is that

$$\pi_z(x) = -z + \frac{2z^*(z)}{z^*(z+x)}(x+z) = -z + \frac{2}{1+z^*(x)}(x+z),$$

where $x \in S_X\setminus\{-z\}$ and $z^* = \|\cdot\|'(z)$ (see [6]). It is also known that its inverse is represented as $\pi_z^{-1}(y) = -z + t(y)\cdot(y+z)$, where the mapping $y \mapsto t(y)$ is $C^1$ [6, Lemma 4]. Then its derivative can be expressed as

$$D\pi_z^{-1}(y)(h) = t(y)h + Dt(y)(h)(y+z). \tag{1.1}$$

We also note that $D\|\cdot\|(x)$ has norm less than or equal to 1 for every $x \neq 0$ in $X$. In fact, for an arbitrary $h \neq 0$ in $X$ and $k > 0$, we have the Taylor expansion

$$\|x + kh\| = \|x\| + D\|\cdot\|(x)(kh) + R(kh) \leq \|x\| + \|kh\|.$$

By dividing by $\|kh\|$, we have

$$D\|\cdot\|(x)\left(\frac{h}{\|h\|}\right) + \frac{R(kh)}{\|kh\|} \leq 1.$$

Since $R(kh) = o(\|kh\|)$, the conclusion follows as $k \to 0$.

As in the proof of [1, Theorem 1], there are a convex body $V$, which is diffeomorphic to $B_X$, and a $C^1$ diffeomorphism $\tilde{g_1}$ from $\partial V \setminus \{-z\}$ onto $S_X \setminus \{-z\}$, where $\partial V$ does not contain any line segment passing through $-z$. Since $\tilde{g_1}(x) = \lambda(x)x$ with $0 \leq \lambda(x) < \infty$, $\lambda(x)$ is $C^1$ (see [1,4]). We can also choose a $C^1$ smooth bump function $\varphi$ whose support is contained in $B(z; \frac{1}{2})$ and $\varphi(x) = 1$ for $x \in B(z; \frac{1}{4})$. Define

$$g_1(x) := (1 - \varphi(x))\lambda(x)x + \varphi(x)x.$$

Then $g_1 : X \to X$ is a $C^1$ smooth diffeomorphism and $g_1(x) = x$ for every $x \in B(z; \frac{1}{4})$. There is also a $C^1$ diffeomorphism from $S_X \setminus \{-z\}$ onto $S_X$, which we denote by $g_2$.

Further, setting $H_z = \{x \in X : D\|\cdot\|(z)(x) = z^*(x) = 1\}$, we can have a $C^1$ smooth stereographic projection

$$\pi_z : \partial V \setminus \{-z\} \to H_z,$$

and denote $\pi_z^{-1}$ by $\psi$. Let $H_0 = H_z - z$, which is a closed subspace of $X$. Define $\iota(h) : H_z \to H_0$ by $\iota(h) = h - z$. We can easily verify that the $C^1$ diffeomorphism $g_2 \circ g_1 \circ \psi \circ \iota^{-1}$ from $H_0$ onto $S_X$ satisfies

$$g_2 \circ g_1 \circ \psi \circ \iota^{-1}(0) = g_2 \circ g_1 \circ \psi(z) = g_2 \circ g_1(z) = g_2(z) = z.$$

More precisely, $g_2 \circ g_1(x) = x$ for every $x \in B(z; \frac{1}{4})$. Furthermore, we can see that

$$g_2 \circ g_1|_{\psi(B(z;\frac{1}{8}) \cap H)} = Id.$$

Indeed, for every $y \in H_0$ with $\|y\| \le \delta < \frac{1}{8}$ it follows from $\|\psi(y+z)\| = 1$ that

$$1 = \|(2t(y+z) - 1)z + t(y+z)y\| \le |2t(y+z) - 1| + \delta t(y+z).$$

Since $0 \le t(y+z) \le 1$, we can deduce $|2t(y+z) - 1| = 2t(y+z) - 1$ and

$$\frac{2}{2+\delta} \le t(y+z) \le 1, \tag{1.2}$$

and

$$\|\psi(y+z) - z\| = \|2(1 - t(y+z))(-z) + t(y+z)y\| \le 2(1 - t(y+z)) + \delta \le 2\delta, \tag{1.3}$$

Equation (1.3) reveals that

$$\psi\left(B\left(z; \frac{1}{8}\right) \cap H\right) \subset B\left(\psi(z); \frac{1}{4}\right) = B\left(z; \frac{1}{4}\right).$$

From the definition of $\varphi$ it follows that $g_1|_{B(z;\frac{1}{4})} = Id$. The proof of the fact that $g_2|_{S_X \cap B(z;\frac{1}{4})} = Id$ can be found in [1] by adapting the construction of the diffeomorphism $g_2$ between $S_X \setminus \{-z\}$ and $S_X$. We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** Note that for every $y \in H_0$ with $\|y\| \le \frac{1}{8}$, $\|g_1 \circ \psi \circ \iota^{-1}(y)\| = 1$. By differentiating both sides, we get for every $h \in H_0$

$$\begin{aligned} 0 &= D\|\cdot\|(g_1(\psi(\iota^{-1}(y)))) \circ D(g_1 \circ \psi \circ \iota^{-1}(y))(h) \\ &= D\|\cdot\|(g_1(\psi(\iota^{-1}(y)))) \circ Dg_1(\psi(\iota^{-1}(y))) \circ D\psi(\iota^{-1}(y)) \circ D\iota^{-1}(y)(h) \\ &= D\|\cdot\|(g_1(\psi(y+z))) \circ Dg_1(\psi(y+z)) \circ (t(y+z)h + Dt(y+z)(h)(y+z+z)) \\ &= t(y+z)(D\|\cdot\|(g_1(\psi(y+z))) \circ Dg_1(\psi(y+z))(h) \\ &\quad + Dt(y+z)(h)(D\|\cdot\|(g_1(\psi(y+z))) \circ Dg_1(\psi(y+z))(y+z+z)), \end{aligned}$$

which yields that

$$Dt(y+z)(h) = -\frac{t(y+z)}{w^*(y+2z)} w^*(h), \tag{1.4}$$

where $w^* = D\|\cdot\|(g_1(\psi(y+z)) \circ Dg_1(\psi(y+z))$.

By assumption we have

$$f(z) = (f \circ g_2 \circ g_1 \circ \psi \circ \iota^{-1})(0) > 1 - a\epsilon^2,$$

where $a > 0$ is the constant in the Deville–Godefroy–Zizler variational principle [11, Theorem 4.10]. By this variational principle there exist $\phi : H_0 \to \mathbb{R}$ and $x_0 \in H_0$ such that $(f \circ g_2 \circ g_1 \circ \psi \circ \iota^{-1} + \phi)$ has strong maximum at $x_0$ and

$$\|\phi\|_\infty \le \epsilon, \quad \|D\phi\|_\infty \le \epsilon \quad \text{and} \quad \|x_0\| \le \epsilon.$$

Clearly we get $D(f \circ g_2 \circ g_1 \circ \psi \circ \iota^{-1} + \phi)(x_0) = 0$, which implies that

$$\|D(f \circ g_2 \circ g_1 \circ \psi \circ \iota^{-1})(x_0)\|_{H_0} \le \epsilon. \tag{1.5}$$

It follows from Equation (1.1) and $Df(x) = f \in X^*$ that

$$\begin{aligned} D(f \circ g_2 \circ g_1 \circ \psi \circ \iota^{-1})(x_0)(h) &= Df(\psi(x_0 + z)) \circ D\psi(x_0 + z)(h) \\ &= t(x_0 + z) \cdot f(h) + Dt(x_0 + z)(h) \cdot f(x_0 + 2z) \end{aligned}$$

Equation (1.4) yields that

$$D(f \circ g_2 \circ g_1 \circ \psi \circ \iota^{-1})(x_0)(h) = t(x_0 + z) \cdot \left\{ f(h) - \frac{f(x_0 + 2z)}{w^*(x_0 + 2z)} w^*(h) \right\}, \tag{1.6}$$

where $w^* = \| \cdot \|'(g_1(\psi(x_0 + z))) \circ Dg_1(\psi(x_0 + z)) = \| \cdot \|'(\psi(x_0 + z))$. Note that $w^*(\psi(x_0 + z)) = 1$. From Equation (1.3) we can check easily that

$$2 - 5\epsilon \le w^*(x_0 + 2z) \le 2 + \epsilon.$$

Since $1 - 2\epsilon < 1 - \epsilon^2$, we also have

$$\frac{2 - 5\epsilon}{2 + \epsilon} \le \frac{f(x_0 + 2z)}{w^*(x_0 + 2z)} \le \frac{2 + \epsilon}{2 - 5\epsilon}. \tag{1.7}$$

Then it follows from Equations (1.2), (1.5), (1.6) and (1.7) that

$$|f(h) - w^*(h)| \le \left| f(h) - \frac{f(x_0 + 2z)}{w^*(x_0 + 2z)} w^*(h) \right| + \left| \frac{f(x_0 + 2z)}{w^*(x_0 + 2z)} - 1 \right| \le \epsilon \left( \frac{2 + \epsilon}{2} \right) + \frac{6\epsilon}{2 - 5\epsilon}.$$

We can easily check that $(f - w^*)(z) \le 2\epsilon$. Since $X = H_0 \oplus [z]$, there is a norm one projection from $X$ onto $[z]$, and we can see that

$$\|f - w^*\| \le 2\epsilon \left[ \left( \frac{2 + \epsilon}{2} \right) + \frac{6}{2 - 5\epsilon} + 1 \right] < 18\epsilon.$$

Let $M > 0$ be the Lipschitz constant such that

$$\|\| \cdot \|'(x) - \| \cdot \|'(y)\| \le M\|x - y\|,$$

for every $x, y \in S_X$. Then it follows from (1.3) that

$$\|f - z^*\| \le \|f - w^*\| + \|w^* - z^*\| \le 2\epsilon(M + 9),$$

where $z^* = \| \cdot \|'(z)$. Set $\beta(\epsilon) = 2\epsilon(M + 9)$, which completes the proof. □

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