



Ergodicity of stochastic Magneto-Hydrodynamic equations driven by α -stable noise [☆]



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ABSTRACT

The current paper is devoted to the ergodicity of stochastic Magneto-Hydrodynamic equations driven by α -stable noise with $\alpha \in (\frac{3}{2}, 2)$. By the maximal inequality for the stochastic α -stable convolution and vorticity transformation, the well-posedness of the mild solution for stochastic Magneto-Hydrodynamic equation is established. Due to the discontinuous trajectories, the existence and uniqueness of the invariant measure for stochastic Magneto-Hydrodynamic equation are obtained by the strong Feller property and the accessibility to zero instead of the irreducibility.

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1. Introduction

In recent years, stochastic partial differential equations (SPDEs) driven by Lévy noise have attracted a lot of attention, see [1,7,19,22,26] and references therein. But in these works, Lévy noises are assumed to be square integrable which clearly rules out the interesting α -stable noises. This restriction should be relaxed since α -stable noises have been deeply studied and widely applied to physics, queueing theory, mathematical finances and others. There are a few papers on stochastic partial differential equation driven by α -stable noises (see for instance [9,22,21,20,29–31]). The authors in [22] investigated the structural properties of solutions to the nonlinear stochastic equations with bounded and Lipschitz nonlinearities driven by cylindrical stable processes. While [25] studied the ergodicity of the stochastic equation with unbounded and non-Lipschitz dissipative function driven by α -stable noises with $\alpha \in (1, 2)$. The exponential mixing of the SPDEs driven by α -stable noises has been established in [21,20,31]. Dong, Xu & Zhang in [9] proved the exponential ergodicity and strong Feller of the stochastic Burgers equations driven by $\frac{\alpha}{2}$ -subordinated cylindrical Brownian motions with $\alpha \in (1, 2)$. The existence of the invariant measure has been shown for 2D stochastic Navier–Stokes equation forced by α -stable noises with $\alpha \in (1, 2)$, see [7] for details. Recently,

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Xu in [29] studied the ergodicity of the stochastic real Ginzburg–Landau equation driven by α -stable noises with $\alpha \in (\frac{3}{2}, 2)$ and established a maximal inequality for the stochastic α -stable convolution which is useful for studying other SPDEs forced by α -stable noises.

The dynamics of the velocity and the magnetic field in electrically conducting fluids and some basic physics conservation laws can be described by the magneto-hydrodynamic (MHD) equations. More details of the related background can be found in [4,6,16]. There has been extensive study of the following MHD equations

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu_1 \partial_{x_1}^2 u + \nu_2 \partial_{x_2}^2 u + b \cdot \nabla b, \\ \partial_t b + u \cdot \nabla b = \eta_1 \partial_{x_1}^2 b + \eta_2 \partial_{x_2}^2 b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(0, x) = u_0, \quad b(0, x) = b_0, \end{cases} \quad (1.1)$$

where $(x_1, x_2) \in \mathbb{R}^2$, $t \geq 0$, $u = (u_1, u_2)$ and $b = (b_1, b_2)$ denote the velocity field and magnetic field respectively, p is a scalar pressure, $\nu_1, \nu_2 \geq 0$ is the kinematic viscosity, $\eta_1, \eta_2 \geq 0$ is the magnetic diffusion. Sermange and Temam in [23], and Duvant and Lions in [10] showed the existence and uniqueness of the global solution corresponding to the sufficiently smooth initial data for (1.1) for all parameters $\nu_1, \nu_2, \eta_1, \eta_2 > 0$, see Theorem 6 in [10]. Also, when some of the parameters are positive, the well-posedness of (1.1) was obtained in [5,17,35]. For more details of the regularity for MHD systems, we refer the reader to [11,15,28,34,35,33,36]. For the two dimensional stochastic MHD equations

$$\begin{cases} dX = (\nu \Delta X - (X \cdot \nabla)X + S(B \cdot \nabla)B - \nabla(P + \frac{1}{2}S|B|^2))dt + \sqrt{Q_1}dW_1(t), \\ dB = (\nu_1 \Delta B - (X \cdot \nabla)B + (B \cdot \nabla)X)dt + \sqrt{Q_2}dW_2(t), \\ \nabla \cdot X = 0, \quad \nabla \cdot B = 0, \quad B \cdot n = 0, \quad \text{in } (0, +\infty) \times \mathbb{O}, \\ X = 0, \quad \text{curl} B = 0, \quad \text{on } (0, +\infty) \times \partial\mathbb{O}, \\ X(0, \xi) = x_0(\xi), \quad B(0, \xi) = b_0(\xi), \quad \text{on } \mathbb{O}, \end{cases} \quad (1.2)$$

Barbu and Da Prato in [3] showed the existence of the solution to the stochastic MHD equations (1.2), and proved the existence and uniqueness of an invariant measure by the coupling methods. Recently, Huang and Shen in [14] studied the well-posedness and dynamics of the stochastic 2D incompressible fractional MHD equation driven by Gaussian white noise. Manna and Mohan in [18] studied the incompressible, viscous and non-resistive MHD equations with Levy noise, and proved local in time existence and pathwise uniqueness of strong solution, and obtained the invariant measures. Very recently, Wang et al. in [27] established the existence of the martingale solution for the stochastic compressible Navier–Stokes equation with Brownian motions, Sun and Li in [24] established the existence of weak solutions for stochastic compressible MHD equations. There are other relative works on stochastic compressible fluid flows, see for instance [12,32].

Motivated by the work in [3,14,29], we consider the following stochastic 2D MHD equation in tours $\mathbb{T}^2 = (0, 1]^2$

$$\begin{cases} du = [\Delta u + b \cdot \nabla b - u \cdot \nabla u - \nabla p]dt + dL_t, \quad t \geq 0, \\ db = [\Delta b + b \cdot \nabla u - u \cdot \nabla b]dt + dL_t, \quad t \geq 0, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(0, x) = u_0, \quad b(0, x) = b_0, \end{cases} \quad (1.3)$$

where L_t is cylindrical α -stable noise (be specialized later).

In this paper, we consider the ergodicity of equation (1.3) with $\alpha \in (\frac{3}{2}, 2)$. The maximal inequality for the stochastic α -stable convolution is applied, which is developed by Xu in [29], vorticity transformation and

Banach fixed point theorem to show the existence and uniqueness of the global mild solution. By a priori estimates for the Galerkin approximation equation and classical Bogoliubov–Krylov theorem, the existence of the invariant measures is proved. Due to the discontinuous trajectories, we truncate the bilinearity term to prove the strong Feller property by a gradient estimate developed by Priola and Zabczyk in [22]. Finally, the uniqueness of the invariant measure for the stochastic MHD equation is obtained by the strong Feller property and the accessibility to zero instead of the irreducibility.

The rest of the paper is organized as follows. Section 2 is devoted to the introduction of functional setting, maximal inequality for the stochastic α -stable convolution. In section 3, we apply the Banach fixed point theorem and vorticity transformation to show the well-posedness of the mild solution for stochastic MHD equation. In section 4, some a priori estimates for the Galerkin approximation equation are presented and the existence of the invariant measures for equation (1.3) is obtained. Finally, the strong Feller property and the accessibility of the associated Markov semigroup is established to obtain the uniqueness of the invariant measure for stochastic equation (1.3).

2. Preliminaries

In this section, we will introduce some notations and functional setting, and present the regularity for the Ornstein–Uhlenbeck process with α -stable noise.

Denote

$$H = \{u \in (L^2(\mathbb{T}))^2 : \nabla \cdot u = 0, \int_{\mathbb{T}} u(s) ds = 0\}, \quad V = \{u \in (H_0^1(\mathbb{T}))^2 : \nabla \cdot u = 0\},$$

endowed with the norms

$$\|u\|_H^2 = \langle u, u \rangle = \int_D |u|^2 dx, \quad \|u\|_V^2 = \langle \nabla u, \nabla u \rangle = \int_D |\nabla u|^2 dx.$$

The bilinear operator $B(\cdot, \cdot)$ can be defined by

$$\langle B(u, v), w \rangle = B(u, v, w) = \int_D (u \cdot \nabla v) w dx = \sum_{i,j=1,2} \int_D u_i \partial_i v_j w_j dx, \quad u, v, w \in H \cap V.$$

Let $A = -\Delta$, then $D(A) = H^{2,2}(\mathbb{T}) \cap H$ and

$$Ae_k = \gamma_k e_k, \quad \gamma_k = 4\pi^2 |k|^2,$$

where $D(A) = \{x \in H; x = \sum_{k \in \mathbb{Z}_*} x_k e_k, \sum_{k \in \mathbb{Z}_*} |\gamma_k|^2 |x_k|^2 < \infty\}$, e_k is an orthonormal basis of H and $\mathbb{Z}_* = \mathbb{Z} \setminus \{0\}$.

Define the operator A^σ with $\sigma \geq 0$ by

$$A^\sigma x = \sum_{k \in \mathbb{Z}_*} \gamma_k^\sigma x_k, \quad x \in D(A^\sigma),$$

where $D(A^\sigma) = \{x \in H : x = \sum_{k \in \mathbb{Z}_*} x_k e_k, \sum_{k \in \mathbb{Z}_*} |\gamma_k|^{2\sigma} |x_k|^2 < \infty\}$.

For $x \in D(A^\sigma)$, define

$$\|A^\sigma x\|_H = \left(\sum_{k \in \mathbb{Z}_*} |\gamma_k|^{2\sigma} |x_k|^2 \right)^{\frac{1}{2}}.$$

Then $V = D(A^{\frac{1}{2}}) \cap H$. Let $C_\sigma > 0$ be some constant depending on σ . It follows that

$$\|A^\sigma e^{-At}\| \leq C_\sigma t^{-\sigma}, \quad \forall \sigma > 0, t > 0. \quad (2.1)$$

Thus, the stochastic MHD equation (1.3) can be rewritten as the following

$$\begin{cases} du + [Au + B(u, u) - B(b, b) + \nabla p]dt = dL_t & t \geq 0, \\ db + [Ab + B(u, b) - B(b, u)]dt = dL_t, & t \geq 0, \\ u(0, x, \omega) = u_0, \quad b(0, x, \omega) = b_0, \end{cases} \quad (2.2)$$

where $L_t = \sum_{k \in \mathbb{Z}_*} \beta_k l_k(t) e_k$ is a cylindrical α -stable process on H with $\{l_k(t)\}_{k \in \mathbb{Z}_*}$ being 1 dimensional symmetric α -stable process sequence with $\alpha > 1$. Moreover, there exist $C_1, C_2 > 0$ such that

$$C_1 \gamma_k^{-\beta} \leq |\beta_k| \leq C_2 \gamma_k^{-\beta}, \quad \frac{1}{2} + \frac{1}{2\alpha} < \beta < \frac{3}{2} - \frac{1}{\alpha}. \quad (2.3)$$

Consider the following Ornstein–Uhlenbeck process

$$Z_t = \int_0^t e^{-A(t-s)} dL_s = \sum_{k \in \mathbb{Z}_*} z_k(t) e_k, \quad (2.4)$$

where

$$z_k(t) = \int_0^t e^{-\gamma_k(t-s)} \beta_k dl_k(s), \quad \gamma_k = 4\pi^2 |k|^2. \quad (2.5)$$

The following lemmas play a crucial role in proving the well-posedness, strong Feller and accessibility for the solution of equation (1.3), which are taken from [29].

Lemma 2.1 ([7,29]). *Let $T > 0$ be arbitrary, for all $0 \leq \theta < \beta - \frac{1}{2\alpha}$ and $0 < p < \alpha$, then*

$$\mathbb{E} \sup_{0 \leq t \leq T} \|A^\theta Z_t\|_H^p \leq CT^{p/\alpha},$$

where C depends on α, θ, β, p .

Lemma 2.2 ([29]). *Let $\theta \in [0, \beta - \frac{1}{2\alpha})$ be arbitrary. For all $T > 0$ and $\epsilon > 0$, we have*

$$\mathbb{P}(\sup_{0 \leq t \leq T} \|A^\theta Z_t\|_H \leq \epsilon) > 0. \quad (2.6)$$

3. Well-posedness of the mild solution

In this section, we will apply the Banach fixed point theorem to show the well-posedness of the mild solution for stochastic MHD equations.

Definition 3.1. Let $I = [a, b]$ be an interval in \mathbb{R}^+ . A mapping $g : I \rightarrow \mathbb{R}^d$ is said to be càdlàg if, for all $t \in [a, b]$, g has a left limit and its right is continuous at t . Let $D([0, T], H)$ be the space of all càdlàg paths from $[0, T]$ into H .

For all $\omega \in \Omega$, define $v(\omega) = u(\omega) - Z(\omega)$ and $c(\omega) = b(\omega) - Z(\omega)$, then

$$\begin{cases} dv + [Av + B(v + Z_t, v + Z_t) - B(c + Z_t, c + Z_t) + \nabla p]dt = 0 & t \geq 0, \\ dc + [Ac + B(v + Z_t, c + Z_t) - B(c + Z_t, v + Z_t)]dt = 0, & t \geq 0, \\ v(0, x, \omega) = u_0, \quad c(0, x, \omega) = b_0. \end{cases} \quad (3.1)$$

For each $T > 0$, define

$$K_T(\omega) := \sup_{0 \leq t \leq T} \|Z_t(\omega)\|_V, \quad \omega \in \Omega. \quad (3.2)$$

Lemma 2.2 yields that for every $k \in \mathbb{N}$, there exists some set $N_k \subset \Omega$ such that $\mathbb{P}(N_k) = 0$ and

$$K_k(\omega) < \infty, \quad \omega \notin N_k. \quad (3.3)$$

Letting $N = \bigcup_{k \geq 1} N_k$, it is easy to see $\mathbb{P}(N) = 0$ and that for all $T > 0$

$$K_T(\omega) < \infty, \quad \omega \notin N. \quad (3.4)$$

We introduce the working function space S by

$$S = \{\psi = (v, c) \in C([0, T], H \times H) : \psi_0 = (u_0, b_0), \psi(t) \in V \times V, t \in (0, T], \\ \sup_{0 \leq t \leq T} t^{\frac{1}{2}} \|A^{\frac{1}{2}} v\|_H + \sup_{0 \leq t \leq T} \|v\|_H + \sup_{0 \leq t \leq T} t^{\frac{1}{2}} \|A^{\frac{1}{2}} c\|_H + \sup_{0 \leq t \leq T} \|c\|_H \leq B\},$$

endowed with the metric $d(\cdot, \cdot)$ by

$$d(\psi, \psi') = \sup_{0 \leq t \leq T} t^{\frac{1}{2}} \|A^{\frac{1}{2}}(v - v')\|_H + \sup_{0 \leq t \leq T} \|(v - v')\|_H + \sup_{0 \leq t \leq T} t^{\frac{1}{2}} \|A^{\frac{1}{2}}(c - c')\|_H + \sup_{0 \leq t \leq T} \|(c - c')\|_H,$$

for any $\psi, \psi' \in S$. Then (S, d) is a closed metric space.

Define a map $\mathcal{F} : S \rightarrow C([0, T]; H \times H)$ as the following: for any $\psi \in S$,

$$\mathcal{F}\psi = \begin{pmatrix} e^{-At}u_0 + \int_0^t e^{-A(t-s)}B(v + Z_s, v + Z_s)ds - \int_0^t e^{-A(t-s)}B(c + Z_s, c + Z_s)ds \\ e^{-At}b_0 + \int_0^t e^{-A(t-s)}B(v + Z_s, c + Z_s)ds - \int_0^t e^{-A(t-s)}B(c + Z_s, v + Z_s)ds \end{pmatrix}.$$

Lemma 3.1. *If the condition (2.3) holds, then we have:*

(B1) *For any $u_0, b_0 \in H$, and $\omega \notin N$, there exists some $0 < T(\omega) \leq 1$, depending on $\|u_0\|_H$, $\|b_0\|_H$ and $K_1(\omega)$ such that equation (3.1) has a unique mild solution $\psi(\omega) = (v(\omega), c(\omega)) \in C([0, T]; H \times H)$ satisfying*

$$\|A^{\frac{1}{2}}\psi(\omega)\|_H \leq C(t^{-\frac{1}{2}} + 1), \quad t \in (0, T(\omega)], \quad (3.5)$$

where C is some constant depending on $\|u_0\|_H$, $\|b_0\|_H$ and $K_1(\omega)$.

(B2) *For $u_0, b_0 \in V$, and $\omega \notin N$, there exists some $0 < \widehat{T}(\omega) \leq 1$, depending on $\|u_0\|_H$, $\|b_0\|_H$ and $K_1(\omega)$ such that equation (3.1) has a unique mild solution $\psi(\omega) = (v(\omega), c(\omega)) \in C([0, \widehat{T}]; V \times V)$ satisfying*

$$\sup_{0 \leq t \leq \widehat{T}(\omega)} \|A^{\frac{1}{2}}\psi(\omega)\|_H \leq 1 + \|A^{\frac{1}{2}}\psi_0\|_H, \quad t \in [0, T(\omega)]. \quad (3.6)$$

Proof. For the sake of simplicity, we will omit the variable ω . Let $0 < T \leq 1$ and $B > 0$ be some constant to be determined later.

We claim that there exist $T_0 > 0$ and $B_0 > 0$ such that the following (a) and (b) hold for $t \in (0, T_0]$ and $B \geq B_0$:

- (a) $\mathcal{F}\psi \in S$ for $\psi \in S$.
- (b) $d(\mathcal{F}\psi, \mathcal{F}\psi') \leq \frac{1}{2}d(u, v)$ for $\psi, \psi' \in S$.

In fact, it follows from (2.1) and Hölder inequality that

$$\begin{aligned}
 \|A^{\frac{1}{2}}\mathcal{F}v\|_H &\leq Ct^{-\frac{1}{2}}\|u_0\|_H + \int_0^t \|A^{\frac{1}{2}}e^{-A(t-s)}\| \|B(v + Z_s, v + Z_s)\|_H ds \\
 &\quad + \int_0^t \|A^{\frac{1}{2}}e^{-A(t-s)}\| \|B(c + Z_s, c + Z_s)\|_H ds \\
 &\leq Ct^{-\frac{1}{2}}\|u_0\|_H + C \int_0^t (t-s)^{-\frac{1}{2}} (\|v\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}v\|_H^{\frac{3}{2}} + \|v\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}Z\|_H^{\frac{3}{2}} \\
 &\quad + \|A^{\frac{1}{2}}Z\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}v\|_H^{\frac{3}{2}} + \|A^{\frac{1}{2}}Z\|_H^2) ds \\
 &\quad + C \int_0^t (t-s)^{-\frac{1}{2}} (\|c\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}c\|_H^{\frac{3}{2}} + \|c\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}Z\|_H^{\frac{3}{2}} \\
 &\quad + \|A^{\frac{1}{2}}Z\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}c\|_H^{\frac{3}{2}} + \|A^{\frac{1}{2}}Z\|_H^2) ds \\
 &\leq Ct^{-\frac{1}{2}}\|u_0\|_H + C \int_0^t (t-s)^{-\frac{1}{2}} (B^2s^{-\frac{3}{4}} + K_1^{\frac{3}{2}}B^{\frac{1}{2}} + B^{\frac{3}{2}}K_1^{\frac{1}{2}}s^{-\frac{3}{4}} + K_1^2) ds,
 \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
 \|A^{\frac{1}{2}}\mathcal{F}c\|_H &\leq Ct^{-\frac{1}{2}}\|b_0\|_H + \int_0^t \|A^{\frac{1}{2}}e^{-A(t-s)}\| \|B(v + Z_s, c + Z_s)\|_H ds \\
 &\quad + \int_0^t \|A^{\frac{1}{2}}e^{-A(t-s)}\| \|B(c + Z_s, v + Z_s)\|_H ds \\
 &\leq Ct^{-\frac{1}{2}}\|b_0\|_H + C \int_0^t (t-s)^{-\frac{1}{2}} (\|v\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}v\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}c\|_H + \|v\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}v\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}Z\|_H \\
 &\quad + \|v\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}Z\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}c\|_H + \|v\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}Z\|_H^{\frac{3}{2}} + \|A^{\frac{1}{2}}Z\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}v\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}c\|_H \\
 &\quad + \|A^{\frac{1}{2}}Z\|_H \|A^{\frac{1}{2}}v\|_H^{\frac{1}{2}} + \|A^{\frac{1}{2}}Z\|_H \|A^{\frac{1}{2}}c\|_H + \|A^{\frac{1}{2}}Z\|_H^2) ds \\
 &\quad + C \int_0^t (t-s)^{-\frac{1}{2}} (\|c\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}c\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}v\|_H + \|c\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}c\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}Z\|_H \\
 &\quad + \|c\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}Z\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}v\|_H + \|c\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}Z\|_H^{\frac{3}{2}} + \|A^{\frac{1}{2}}Z\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}c\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}Z\|_H \\
 &\quad + \|c\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}Z\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}c\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}Z\|_H) ds
 \end{aligned} \tag{3.8}$$

$$\begin{aligned}
& + \|A^{\frac{1}{2}}Z\|_H \|A^{\frac{1}{2}}v\|_H^{\frac{1}{2}} + \|A^{\frac{1}{2}}Z\|_H \|A^{\frac{1}{2}}v\|_H + \|A^{\frac{1}{2}}Z\|_H^2 ds \\
& \leq Ct^{-\frac{1}{2}}\|b_0\|_H + C \int_0^t (t-s)^{-\frac{1}{2}} (B^2 s^{-\frac{3}{4}} + BK_1 s^{-\frac{1}{4}} + B^{\frac{3}{2}} K_1^{\frac{1}{2}} s^{-\frac{1}{4}} + B^{\frac{1}{2}} K_1^{\frac{3}{2}} \\
& + K_1^{\frac{1}{2}} B^{\frac{3}{2}} s^{-\frac{3}{4}} + K_1^{\frac{3}{2}} B^{\frac{1}{2}} s^{-\frac{1}{4}} + K_1 s^{-\frac{1}{2}} B + K_1^2) ds.
\end{aligned}$$

Combining (3.7) and (3.8), we have

$$\begin{aligned}
t^{\frac{1}{2}} \|A^{\frac{1}{2}}\mathcal{F}\psi\|_H & \leq C(\|u_0\|_H + \|b_0\|_H) + Ct^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} (B^2 s^{-\frac{3}{4}} + BK_1 s^{-\frac{1}{4}} + B^{\frac{3}{2}} K_1^{\frac{1}{2}} s^{-\frac{1}{4}} + B^{\frac{1}{2}} K_1^{\frac{3}{2}} \\
& + K_1^{\frac{1}{2}} B^{\frac{3}{2}} s^{-\frac{3}{4}} + K_1^{\frac{3}{2}} B^{\frac{1}{2}} s^{-\frac{1}{4}} + K_1 s^{-\frac{1}{2}} B + K_1^2) ds.
\end{aligned} \tag{3.9}$$

Similarly,

$$\begin{aligned}
\|\mathcal{F}\psi\|_H & \leq \|u_0\|_H + \|b_0\|_H + C \int_0^t (B^2 s^{-\frac{3}{4}} + BK_1 s^{-\frac{1}{4}} + B^{\frac{3}{2}} K_1^{\frac{1}{2}} s^{-\frac{1}{4}} + B^{\frac{1}{2}} K_1^{\frac{3}{2}} \\
& + K_1^{\frac{1}{2}} B^{\frac{3}{2}} s^{-\frac{3}{4}} + K_1^{\frac{3}{2}} B^{\frac{1}{2}} s^{-\frac{1}{4}} + K_1 s^{-\frac{1}{2}} B + K_1^2) ds,
\end{aligned} \tag{3.10}$$

which implies the continuity of $\mathcal{F}\psi$.

Let $T > 0$ be small enough and B be large enough, then we derive that the claim-(a) holds from (3.9) and (3.10).

Next, we prove the claim-(b) holds. For any $\psi, \psi' \in S$, it follows from (2.1) and Hölder inequality that

$$\begin{aligned}
t^{\frac{1}{2}} \|A^{\frac{1}{2}}\mathcal{F}v - A^{\frac{1}{2}}\mathcal{F}v'\|_H & \leq T^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} \|B(v + Z_s, v + Z_s) - B(v' + Z_s, v' + Z_s)\|_H ds \\
& + T^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} \|B(c + Z_s, c + Z_s) - B(c' + Z_s, c' + Z_s)\|_H ds \\
& \leq CT^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} \|v + Z_s\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}(v + Z_s)\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}(v - v')\|_H ds \\
& + CT^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} \|A^{\frac{1}{2}}(v' + Z_s)\|_H^{\frac{1}{3}} \|A^{\frac{1}{2}}(v - v')\|_H ds \\
& + CT^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} \|A^{\frac{1}{2}}(v' + Z_s)\|_H^{\frac{1}{3}} \|(v - v')\|_H ds \\
& + CT^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} \|c + Z_s\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}(c + Z_s)\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}(c - c')\|_H ds \\
& + CT^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} \|A^{\frac{1}{2}}(c' + Z_s)\|_H^{\frac{1}{3}} \|A^{\frac{1}{2}}(c - c')\|_H ds
\end{aligned}$$

$$\begin{aligned}
& + CT^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} \|A^{\frac{1}{2}}(c' + Z_s)\|_{\frac{3}{H}}^{\frac{1}{3}} \|(c - c')\|_H ds \\
& \leq CT^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} (Bs^{-\frac{1}{4}} + B^{\frac{1}{2}}K_1^{\frac{1}{2}} + B^{\frac{1}{2}}K_1^{\frac{1}{2}}s^{-\frac{1}{4}} + K_1)s^{-\frac{1}{2}} [s^{\frac{1}{2}}\|A^{\frac{1}{2}}(v - v')\|_H] ds \\
& + CT^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} (s^{-\frac{1}{6}}B^{\frac{1}{3}} + K_1^{\frac{1}{3}})s^{-\frac{1}{2}} [s^{\frac{1}{2}}\|A^{\frac{1}{2}}(v - v')\|_H] ds \\
& + CT^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} (s^{-\frac{5}{6}}B^{\frac{5}{3}} + K_1^{\frac{5}{3}})\|v - v'\|_H ds \\
& + CT^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} (Bs^{-\frac{1}{4}} + B^{\frac{1}{2}}K_1^{\frac{1}{2}} + B^{\frac{1}{2}}K_1^{\frac{1}{2}}s^{-\frac{1}{4}} + K_1)s^{-\frac{1}{2}} [s^{\frac{1}{2}}\|A^{\frac{1}{2}}(c - c')\|_H] ds \\
& + CT^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} (s^{-\frac{1}{6}}B^{\frac{1}{3}} + K_1^{\frac{1}{3}})s^{-\frac{1}{2}} [s^{\frac{1}{2}}\|A^{\frac{1}{2}}(c - c')\|_H] ds \\
& + CT^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} (s^{-\frac{5}{6}}B^{\frac{5}{3}} + K_1^{\frac{5}{3}})\|c - c'\|_H ds \\
& \leq C[BT^{\frac{1}{4}} + B^{\frac{1}{2}}K_1^{\frac{1}{2}}T^{\frac{1}{2}} + B^{\frac{1}{2}}K_1^{\frac{1}{2}}T^{\frac{1}{4}} + K_1T^{\frac{1}{2}} + B^{\frac{1}{3}}T^{\frac{1}{3}} \\
& + K_1^{\frac{1}{3}}T^{\frac{1}{2}}] \sup_{0 \leq t \leq T} [t^{\frac{1}{2}}\|A^{\frac{1}{2}}(\psi - \psi')\|_H] + C[T^{\frac{1}{6}}B^{\frac{1}{3}} + K_1^{\frac{5}{3}}T^{\frac{1}{2}}]\|\psi - \psi'\|_H. \tag{3.11}
\end{aligned}$$

Similarly,

$$\begin{aligned}
t^{\frac{1}{2}}\|A^{\frac{1}{2}}\mathcal{F}c - A^{\frac{1}{2}}\mathcal{F}c'\|_H & \leq C[BT^{\frac{1}{4}} + B^{\frac{1}{2}}K_1^{\frac{1}{2}}T^{\frac{1}{2}} + B^{\frac{1}{2}}K_1^{\frac{1}{2}}T^{\frac{1}{4}} + K_1T^{\frac{1}{2}} + B^{\frac{1}{3}}T^{\frac{1}{3}} \\
& + K_1^{\frac{1}{3}}T^{\frac{1}{2}}] \sup_{0 \leq t \leq T} [t^{\frac{1}{2}}\|A^{\frac{1}{2}}(\psi - \psi')\|_H] + C[T^{\frac{1}{6}}B^{\frac{1}{3}} + K_1^{\frac{5}{3}}T^{\frac{1}{2}}]\|\psi - \psi'\|_H, \tag{3.12}
\end{aligned}$$

$$\begin{aligned}
\|\mathcal{F}v - \mathcal{F}v'\|_H & \leq C[BT^{\frac{1}{4}} + B^{\frac{1}{2}}K_1^{\frac{1}{2}}T^{\frac{1}{2}} + B^{\frac{1}{2}}K_1^{\frac{1}{2}}T^{\frac{1}{4}} + K_1T^{\frac{1}{2}} + B^{\frac{1}{3}}T^{\frac{1}{3}} \\
& + K_1^{\frac{1}{3}}T^{\frac{1}{2}}] \sup_{0 \leq t \leq T} [t^{\frac{1}{2}}\|A^{\frac{1}{2}}(\psi - \psi')\|_H] + C[T^{\frac{1}{6}}B^{\frac{1}{3}} + K_1^{\frac{5}{3}}T^{\frac{1}{2}}]\|\psi - \psi'\|_H, \tag{3.13}
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{F}c - \mathcal{F}c'\|_H & \leq C[BT^{\frac{1}{4}} + B^{\frac{1}{2}}K_1^{\frac{1}{2}}T^{\frac{1}{2}} + B^{\frac{1}{2}}K_1^{\frac{1}{2}}T^{\frac{1}{4}} + K_1T^{\frac{1}{2}} + B^{\frac{1}{3}}T^{\frac{1}{3}} \\
& + K_1^{\frac{1}{3}}T^{\frac{1}{2}}] \sup_{0 \leq t \leq T} [t^{\frac{1}{2}}\|A^{\frac{1}{2}}(\psi - \psi')\|_H] + C[T^{\frac{1}{6}}B^{\frac{1}{3}} + K_1^{\frac{5}{3}}T^{\frac{1}{2}}]\|\psi - \psi'\|_H. \tag{3.14}
\end{aligned}$$

Combining (3.11)–(3.14) gives

$$\begin{aligned}
d(\mathcal{F}\psi, \mathcal{F}\psi') & \leq C[BT^{\frac{1}{4}} + B^{\frac{1}{2}}K_1^{\frac{1}{2}}T^{\frac{1}{2}} + B^{\frac{1}{2}}K_1^{\frac{1}{2}}T^{\frac{1}{4}} + K_1T^{\frac{1}{2}} + B^{\frac{1}{3}}T^{\frac{1}{3}} \\
& + K_1^{\frac{1}{3}}T^{\frac{1}{2}} + T^{\frac{1}{6}}B^{\frac{1}{3}} + K_1^{\frac{5}{3}}T^{\frac{1}{2}}]d(\psi, \psi'). \tag{3.15}
\end{aligned}$$

Choosing T small enough, we can get (b). It follows from (a) and (b) that there exists a unique solution in S for equation (3.1) by the Banach fixed point theorem.

Finally, we prove (B2) holds. Let $0 < \widehat{T} \leq 1$. Define

$$\begin{aligned} \tilde{S} = \{ \psi = (v, c) \in C([0, T], V \times V) : \phi_0 = (u_0, b_0), \sup_{0 \leq t \leq T} \|A^{\frac{1}{2}} v\|_H \\ + \sup_{0 \leq t \leq T} \|A^{\frac{1}{2}} c\|_H \leq 1 + \|A^{\frac{1}{2}} u_0\|_H + \|A^{\frac{1}{2}} b_0\|_H \}, \end{aligned} \quad (3.16)$$

endowed with the metric

$$\tilde{d}(\psi, \psi') = \sup_{0 \leq t \leq T} \|A^{\frac{1}{2}}(v - v')\|_H + \sup_{0 \leq t \leq T} \|A^{\frac{1}{2}}(c - c')\|_H, \quad \forall \psi, \psi' \in \tilde{S}. \quad (3.17)$$

Then (\tilde{S}, \tilde{d}) is a closed metric space.

Define a map $\mathcal{F}_1 : \tilde{S} \rightarrow C([0, T], V \times V)$ by

$$\mathcal{F}_1 \psi = \begin{pmatrix} e^{-At} u_0 + \int_0^t e^{-A(t-s)} B(v + Z_s, v + Z_s) ds - \int_0^t e^{-A(t-s)} B(c + Z_s, c + Z_s) ds \\ e^{-At} b_0 + \int_0^t e^{-A(t-s)} B(v + Z_s, c + Z_s) ds - \int_0^t e^{-A(t-s)} B(c + Z_s, v + Z_s) ds \end{pmatrix}, \quad \text{for any } \psi \in \tilde{S}.$$

It follows from (2.1) and Hölder inequality that

$$\begin{aligned} \|A^{\frac{1}{2}} \mathcal{F}_1 v\| &\leq \|A^{\frac{1}{2}} u_0\|_H + C \int_0^t (t-s)^{-\frac{1}{2}} (\|A^{\frac{1}{2}} v\|_H^2 + K_1^{\frac{3}{2}} \|A^{\frac{1}{2}} v\|_H^{\frac{1}{2}} + K_1 \|A^{\frac{1}{2}} v\|_H^{\frac{3}{2}} + K_1^2) ds \\ &\quad + C \int_0^t (t-s)^{-\frac{1}{2}} (\|A^{\frac{1}{2}} c\|_H^2 + K_1^{\frac{3}{2}} \|A^{\frac{1}{2}} c\|_H^{\frac{1}{2}} + K_1 \|A^{\frac{1}{2}} c\|_H^{\frac{3}{2}} + K_1^2) ds \\ &\leq \|A^{\frac{1}{2}} u_0\|_H + C \int_0^t (t-s)^{-\frac{1}{2}} (1 + \|A^{\frac{1}{2}} u_0\|_H^2 + K_1^2) ds \\ &\quad + C \int_0^t (t-s)^{-\frac{1}{2}} (1 + \|A^{\frac{1}{2}} c_0\|_H^2 + K_1^2) ds, \end{aligned} \quad (3.18)$$

and

$$\begin{aligned} \|A^{\frac{1}{2}} \mathcal{F}_1 c\| &\leq \|A^{\frac{1}{2}} c_0\|_H + C \int_0^t (t-s)^{-\frac{1}{2}} (1 + \|A^{\frac{1}{2}} u_0\|_H^2 + K_1^2) ds \\ &\quad + C \int_0^t (t-s)^{-\frac{1}{2}} (1 + \|A^{\frac{1}{2}} c_0\|_H^2 + K_1^2) ds. \end{aligned} \quad (3.19)$$

Combining (3.18) and (3.19), we have

$$\begin{aligned} \|A^{\frac{1}{2}} \mathcal{F}_1 \psi\|_H &\leq (\|A^{\frac{1}{2}} u_0\|_H + \|A^{\frac{1}{2}} b_0\|_H) + C \int_0^t (t-s)^{-\frac{1}{2}} (1 + \|A^{\frac{1}{2}} u_0\|_H^2 + K_1^2) ds \\ &\quad + C \int_0^t (t-s)^{-\frac{1}{2}} (1 + \|A^{\frac{1}{2}} c_0\|_H^2 + K_1^2) ds, \end{aligned} \quad (3.20)$$

which implies that $\mathcal{F}_1 : \tilde{S} \rightarrow \tilde{S}$ provided $\widehat{T} > 0$ is small enough.

Similarly, we can obtain

$$\sup_{0 \leq t \leq T} \|A^{\frac{1}{2}}(\mathcal{F}_1 v - \mathcal{F}_1 v')\|_H + \sup_{0 \leq t \leq T} \|A^{\frac{1}{2}}(\mathcal{F}_1 c - \mathcal{F}_1 c')\|_H \leq \frac{1}{2} \left(\sup_{0 \leq t \leq T} \|A^{\frac{1}{2}}(v - v')\|_H + \sup_{0 \leq t \leq T} \|A^{\frac{1}{2}}(c - c')\|_H \right),$$

which means that $\tilde{d}(\mathcal{F}_1 \psi, \mathcal{F}_1 \psi') \leq \frac{1}{2} \tilde{d}(\psi, \psi')$. The conclusion follows from the Banach Fixed Point theorem. \square

Lemma 3.2. *It the assumption (2.3) holds, then the following statements hold.*

(C1) For $u_0, b_0 \in H$, and $\omega \notin N$, equation (3.1) has a unique global solution $\psi(\omega) = (v(\omega), c(\omega)) \in C([0, \infty); H \times H)$.

(C2) For $u_0, b_0 \in V$, and $\omega \notin N$, $\psi(\omega) = (v(\omega), c(\omega)) \in C([0, \infty); V \times V)$.

Proof. Multiplying the first term and second term of equation (3.1) with v and c , then integrating over \mathbb{T} leads to

$$\frac{1}{2} \frac{d}{dt} \|v\|_H^2 + B(v + Z_t, v + Z_t, v) + \|v\|_V^2 = B(c + Z_t, c + Z_t, v), \quad (3.21)$$

$$\frac{1}{2} \frac{d}{dt} \|c\|_H^2 + B(v + Z_t, c + Z_t, c) + \|c\|_V^2 = B(c + Z_t, v + Z_t, c). \quad (3.22)$$

We deduce from Hölder inequality that

$$\begin{aligned} |B(v + Z_t, Z_t, v)| &\leq \|v + Z_t\|_H \|A^{\frac{1}{2}} Z_t\|_H \|v\|_V \\ &\leq \|v\|_H^2 \|A^{\frac{1}{2}} Z_t\|_H^2 + \|Z_t\|_H^2 \|A^{\frac{1}{2}} Z_t\|_H^2 + \frac{1}{4} \|v\|_V^2, \end{aligned} \quad (3.23)$$

$$|B(v + Z_t, Z_t, c)| \leq \|v\|_H^2 \|A^{\frac{1}{2}} Z_t\|_H^2 + \|Z_t\|_H^2 \|A^{\frac{1}{2}} Z_t\|_H^2 + \frac{1}{4} \|c\|_V^2, \quad (3.24)$$

$$|B(c + Z_t, Z_t, c)| \leq \|c\|_H^2 \|A^{\frac{1}{2}} Z_t\|_H^2 + \|Z_t\|_H^2 \|A^{\frac{1}{2}} Z_t\|_H^2 + \frac{1}{4} \|v\|_V^2, \quad (3.25)$$

$$|B(c + Z_t, Z_t, v)| \leq \|c\|_H^2 \|A^{\frac{1}{2}} Z_t\|_H^2 + \|Z_t\|_H^2 \|A^{\frac{1}{2}} Z_t\|_H^2 + \frac{1}{4} \|c\|_V^2. \quad (3.26)$$

Combining (3.23)–(3.26) gives

$$\frac{1}{2} \frac{d}{dt} (\|v\|_H^2 + \|c\|_H^2) + \frac{1}{2} (\|v\|_V^2 + \|c\|_V^2) \leq 2 \|A^{\frac{1}{2}} Z_t\|_H^2 (\|v\|_H^2 + \|c\|_H^2) + 4 \|Z_t\|_H^2 \|A^{\frac{1}{2}} Z_t\|_H^2. \quad (3.27)$$

Lemma 2.2 yields that for $T > 0$

$$\mathbb{P} \left(\sup_{0 \leq t \leq T} \|A^{\frac{1}{2}} Z_t\|_H^2 \leq \frac{\pi}{4} \right) > 0. \quad (3.28)$$

Then for $\omega \in \Omega_{\frac{\pi}{4}, T}$, it follows from Poincaré inequality $2\pi \|x\|_H^2 \leq \|x\|_V^2$ and Gronwall inequality that

$$(\|v\|_H^2 + \|c\|_H^2) \leq e^{-\pi t} (\|u_0\|_H^2 + \|b_0\|_H^2) + c\pi^4. \quad (3.29)$$

Integrating (3.27) over $[0, T]$ gives

$$\int_0^T (\|v\|_V^2 + \|c\|_V^2) ds < \infty. \quad (3.30)$$

It follows from (B1) of [Lemma 3.1](#) that equation [\(3.1\)](#) has a unique local solution $\psi(\omega) = (v(\omega), c(\omega)) \in C([0, T]; H \times H)$ for some $T > 0$. By [\(3.29\)](#), we can extend this solution to be $\psi \in C([0, \infty); H \times H)$.

Taking the curl of equation [\(3.1\)](#), and denoting $w = \nabla \times u = \partial_{x_1} u_2 - \partial_{x_2} u_1$, $j = \nabla \times b = \partial_{x_1} b_2 - \partial_{x_2} b_1$ and $L = \nabla \times Z_t = \partial_{x_1} Z_t - \partial_{x_2} Z_t$, we obtain

$$\begin{cases} \partial_t w + B(v + Z, w + L) = \Delta w + B(c + Z, j + L) + j(\partial_{x_1} Z_t + \partial_{x_2} Z_t) - w(\partial_{x_1} Z_t + \partial_{x_2} Z_t), \\ \partial_t j + B(v + Z, j + L) = \Delta j + B(c + Z, w + L) \\ + 2\partial_{x_1}(c_1 + Z_t)(\partial_{x_2} v_1 + \partial_{x_1} v_2) - 2\partial_{x_1}(v_1 + Z_t)(\partial_{x_2} c_1 + \partial_{x_1} c_2). \end{cases} \quad (3.31)$$

Multiplying the first term and second term of [\(3.31\)](#) with w and j respectively and integrating over \mathbb{T} leads to

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w\|_H^2 + \|w\|_V^2 + B(v + Z_t, L, w) &= B(c + Z_t, j + L, w) + \langle j(\partial_{x_1} Z_t + \partial_{x_2} Z_t), w \rangle \\ &\quad - \langle w(\partial_{x_1} Z_t + \partial_{x_2} Z_t), w \rangle, \end{aligned} \quad (3.32)$$

and

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|j\|_H^2 + \|j\|_V^2 + B(v + Z_t, L, j) &= B(c + Z_t, w + L, j) + \langle 2\partial_{x_1}(c_1 + Z_t)(w + 2\partial_{x_2} v_1), j \rangle \\ &\quad - \langle 2\partial_{x_1}(v_1 + Z_t)(j + 2\partial_{x_2} c_1), j \rangle. \end{aligned} \quad (3.33)$$

We derive from Hölder inequality that

$$\begin{aligned} |B(v + Z_t, L, w)| &\leq 4\|v\|_V^2 \|L\|_H^2 + 4\|A^{\frac{1}{2}} Z_t\|_H^2 \|L\|_H^2 + \frac{1}{8}\|w\|_V^2, \\ |B(c + Z_t, L, w)| &\leq 4\|c\|_V^2 \|L\|_H^2 + 4\|A^{\frac{1}{2}} Z_t\|_H^2 \|L\|_H^2 + \frac{1}{8}\|w\|_V^2, \\ |B(v + Z_t, L, j)| &\leq 4\|v\|_V^2 \|L\|_H^2 + 4\|A^{\frac{1}{2}} Z_t\|_H^2 \|L\|_H^2 + \frac{1}{8}\|j\|_V^2, \\ |B(c + Z_t, L, j)| &\leq 4\|c\|_V^2 \|L\|_H^2 + 4\|A^{\frac{1}{2}} Z_t\|_H^2 \|L\|_H^2 + \frac{1}{8}\|j\|_V^2, \end{aligned}$$

and

$$\begin{aligned} \int \partial_{x_2} Z_t j w &\leq 2\|\partial_{x_2} Z_t\|_H^2 \|j\|_H^2 + \frac{1}{8}\|w\|_V^2, & \int \partial_{x_1} Z_t j w &\leq 2\|\partial_{x_1} Z_t\|_H^2 \|j\|_H^2 + \frac{1}{8}\|w\|_V^2, \\ \int \partial_{x_1} Z_t w w &\leq 2\|\partial_{x_1} Z_t\|_H^2 \|w\|_H^2 + \frac{1}{8}\|w\|_V^2, & \int \partial_{x_2} Z_t w w &\leq 2\|\partial_{x_2} Z_t\|_H^2 \|w\|_H^2 + \frac{1}{8}\|w\|_V^2, \\ 2 \int \partial_{x_1} c_1 w j &\leq 8\|\partial_{x_1} c_1\|_H^2 \|j\|_H^2 + \frac{1}{8}\|w\|_V^2, & 2 \int \partial_{x_1} Z_t w j &\leq 8\|\partial_{x_1} Z_t\|_H^2 \|j\|_H^2 + \frac{1}{8}\|w\|_V^2, \\ 4 \int \partial_{x_1} c_1 \partial_{x_2} v_1 j &\leq 16\|\partial_{x_1} c_1\|_H^2 \|\partial_{x_2} v_1\|_H^2 + \frac{1}{8}\|j\|_V^2, & 4 \int \partial_{x_1} Z_t \partial_{x_2} v_1 j &\leq 16\|\partial_{x_1} Z_t\|_H^2 \|\partial_{x_2} v_1\|_H^2 + \frac{1}{8}\|j\|_V^2, \\ 2 \int \partial_{x_1} v_1 j j &\leq 8\|\partial_{x_1} v_1\|_H^2 \|j\|_H^2 + \frac{1}{8}\|j\|_V^2, & 2 \int \partial_{x_1} Z_t j j &\leq 8\|\partial_{x_1} Z_t\|_H^2 \|j\|_H^2 + \frac{1}{8}\|j\|_V^2, \\ 4 \int \partial_{x_1} v_1 \partial_{x_2} c_1 j &\leq 16\|\partial_{x_1} v_1\|_H^2 \|\partial_{x_2} c_1\|_H^2 + \frac{1}{8}\|j\|_V^2, & 4 \int \partial_{x_1} Z_t \partial_{x_2} c_1 j &\leq 16\|\partial_{x_1} Z_t\|_H^2 \|\partial_{x_2} c_1\|_H^2 + \frac{1}{8}\|j\|_V^2. \end{aligned}$$

Combining with the above estimates, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|w\|_H^2 + \|j\|_H^2) + \frac{1}{4} (\|w\|_V^2 + \|j\|_V^2) \\ & \leq C(\|\partial_{x_1} Z_t\|_H^2 + \|\partial_{x_2} Z_t\|_H^2 + \|\partial_{x_1} c_1\|_H^2 + \|\partial_{x_1} v_1\|_H^2)(\|w\|_H^2 + \|j\|_H^2) \\ & \quad + C(\|\partial_{x_1} c_1\|_H^2 \|\partial_{x_2} v_1\|_H^2 + \|\partial_{x_1} Z_t\|_H^2 \|\partial_{x_2} v_1\|_H^2 + \|\partial_{x_1} v_1\|_H^2 \|\partial_{x_2} c_1\|_H^2 + \|\partial_{x_1} Z_t\|_H^2 \|\partial_{x_2} c_1\|_H^2). \end{aligned} \quad (3.34)$$

Due to (3.29), (3.30) and Gronwall inequality, we have

$$\|w\|_H^2 + \|j\|_H^2 < \infty.$$

Similarly, thanks to (B2) of Lemma 3.1, we can extend this solution to be $\psi \in C([0, \infty); V \times V)$. Thus, the proof of Lemma 3.2 is completed. \square

Theorem 3.1. Assume the condition (2.3) holds, then the following results hold.

(A1) For $u_0, b_0 \in H$, and $\omega \in \Omega$, equation (2.2) possesses a unique mild solution $\phi(\omega) = (u(\omega), b(\omega)) \in D([0, \infty); H \times H) \cap D([0, \infty); V \times V)$. Moreover, $\phi(\omega)$ has the following form:

$$\begin{aligned} \phi(\omega) = & \left(e^{-At} u_0 + \int_0^t e^{-A(t-s)} B(u, u) ds - \int_0^t e^{-A(t-s)} B(b, b) ds + \int_0^t e^{-A(t-s)} dL_s(\omega), \right. \\ & \left. e^{-At} b_0 + \int_0^t e^{-A(t-s)} B(u, b) ds - \int_0^t e^{-A(t-s)} B(b, u) ds + \int_0^t e^{-A(t-s)} dL_s(\omega) \right). \end{aligned} \quad (3.35)$$

(A2) ϕ is a Markov process.

(A3) For every $u_0, b_0 \in V$ and $\omega \in \Omega$ a.s., we have $\phi(\omega) \in D([0, \infty); V \times V)$. For every $T > 0$,

$$\sup_{0 \leq t \leq T} \|\phi(\omega)\|_{V \times V} \leq C,$$

where C is some constant depending on T , α , β and ω .

Proof. It follows from Lemma 3.3 of [29] that $Z(t) \in D([0, \infty); V)$. By Lemma 3.2, $\phi(\omega) = (v + Z(\omega), c + Z(\omega))$ is the unique solution to equation (2.2) in $D([0, \infty); H \times H) \cap D([0, \infty); V \times V)$. The Markov property follows from the uniqueness. (A3) follows from (C2) of Lemma 3.2. \square

4. The existence of the invariant measure

In this section, we follow the method in [8] to prove the existence of the invariant measures.

Let $e_k \{k \in \mathbb{Z}_*\}$ be an orthonormal basis of H and define

$$H_m := \text{span}\{e_k; |k| \leq m\}.$$

Then H_m is a finite dimensional Hilbert space equipped with the norm adopted from H . For any $m > 0$, let $\pi_m : H \rightarrow H_m$ be the projection from H to H_m . The Galerkin approximation of (1.3) has the following form

$$\begin{cases} du^m = [\Delta u^m + B^m(b^m, b^m) - B^m(u^m, u^m) - \nabla p^m]dt + dL_t^m, & t \geq 0, \\ db^m = [\Delta b^m + B^m(b^m, u^m) - B^m(u^m, b^m)]dt + dL_t^m, & t \geq 0, \\ \nabla \cdot u^m = 0, \quad \nabla \cdot b^m = 0, \\ u^m(0, x) = u_0, \quad b^m(0, x) = b_0, \end{cases} \quad (4.1)$$

where $u^m = \pi_m u$, $B^m(b^m, b^m) = \pi_m B(b^m, b^m)$, $L_t^m = \sum_{|k| \leq m} \beta_k l_k(t) e_k$.

Lemma 4.1. *If the assumption (2.3) holds, then we have:*

(D1) *For $u_0, b_0 \in W$ with $W = H, V$ and $\omega \in \Omega$ a.s., there exists a unique mild solution $\phi^m(\omega) \in D([0, \infty); W_m \times W_m)$ satisfying*

$$\sup_{0 \leq t \leq T} \|\phi^m(\omega)\|_{W \times W} \leq C, \quad T > 0, \quad (4.2)$$

where C is some constant depending on $\|u_0\|_W$, $\|b_0\|_W$, T and $K_T(\omega)$.

(D2) *For $u_0, b_0 \in W$ with $W = H, V$ and $\omega \in \Omega$ a.s., we have*

$$\lim_{m \rightarrow \infty} \|\phi^m(\omega) - \phi(\omega)\|_{W \times W} = 0, \quad t \geq 0. \quad (4.3)$$

Proof. (D1) can be immediately shown by the same method as in Theorem 3.1. Hence, we just prove (D2) holds. It suffices to show the case $W = V$ since the case $W = H$ can be proved by the similar arguments.

For $t > 0$, Theorem 3.1 implies that

$$\sup_{0 \leq s \leq t} \|\phi\|_{V \times V} \leq \hat{C}, \quad \sup_{0 \leq s \leq t} \|\phi^m\|_{V \times V} \leq \hat{C},$$

where $\hat{C} > 0$ depends on $\|u_0\|_V$, $\|b_0\|_V$, t and K_t .

Direct calculation shows that

$$\begin{aligned} u^m - u &= e^{-At}(u_0^m - u_0) + Z_t - Z_t^m + \int_0^t e^{-A(t-s)}(I - \pi_m)B(u, u) \\ &\quad + \int_0^t e^{-A(t-s)}[B^m(u_m, u_m) - B^m(u, u)] + \int_0^t e^{-A(t-s)}(I - \pi_m)B(b, b) \\ &\quad + \int_0^t e^{-A(t-s)}(I - \pi_m)B(b, b) = I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \end{aligned}$$

and

$$\begin{aligned} b^m - b &= e^{-At}(b_0^m - b_0) + Z_t - Z_t^m + \int_0^t e^{-A(t-s)}(I - \pi_m)B(u, b) \\ &\quad + \int_0^t e^{-A(t-s)}[B^m(u_m, b_m) - B^m(u, b)] + \int_0^t e^{-A(t-s)}(I - \pi_m)B(b, u) \\ &\quad + \int_0^t e^{-A(t-s)}(I - \pi_m)B(b, u) = J_1 + J_2 + J_3 + J_4 + J_5 + J_6. \end{aligned}$$

Let $m \rightarrow \infty$, then

$$\|I_1\|_V \rightarrow 0, \quad \|I_2\|_V \rightarrow 0, \quad \|J_1\|_V \rightarrow 0, \quad \|J_2\|_V \rightarrow 0.$$

Due to Hölder inequality and [Theorem 3.1](#), we derive that

$$\begin{aligned} \|(I - \pi_m)B(u, u)\|_H &\leq C\|(I - \pi_m)\| \|u\|_V^2 \rightarrow 0, \quad m \rightarrow \infty, \\ \|(I - \pi_m)B(b, b)\|_H &\leq C\|(I - \pi_m)\| \|b\|_V^2 \rightarrow 0, \quad m \rightarrow \infty, \\ \|(I - \pi_m)B(b, u)\|_H &\leq C\|(I - \pi_m)\| \|u\|_V \|b\|_V \rightarrow 0, \quad m \rightarrow \infty, \\ \|(I - \pi_m)B(u, b)\|_H &\leq C\|(I - \pi_m)\| \|u\|_V \|b\|_V \rightarrow 0, \quad m \rightarrow \infty. \end{aligned}$$

Combining [\(2.1\)](#) with the dominated convergence theorem, we obtain as $m \rightarrow \infty$,

$$\|I_3\|_V \rightarrow 0, \quad \|I_5\|_V \rightarrow 0, \quad \|J_3\|_V \rightarrow 0, \quad \|J_5\|_V \rightarrow 0.$$

By Hölder inequality and [\(2.1\)](#), we obtain

$$\begin{aligned} \|I\|_4 &\leq C \int_0^t (t-s)^{-\frac{1}{2}} \|B^m(u^m, u^m) - B^m(u, u)\|_H ds \\ &\leq C \int_0^t (t-s)^{-\frac{1}{2}} \|B(u^m, u^m) - B(u, u)\|_H ds \\ &\leq CK \int_0^t (t-s)^{-\frac{1}{2}} \|u_m - u\|_V ds \\ &\leq CK \left(\int_0^t (t-s)^{-\frac{p}{2}} ds \right)^{\frac{1}{p}} \left(\int_0^t \|u^m - u\|_V^q ds \right)^{\frac{1}{q}}, \end{aligned} \tag{4.4}$$

where $K = \sup_{0 \leq s \leq t, m} (\|u\|_V + \|u^m\|_V) \leq 2\hat{C}$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $1 \leq p \leq 2$.

Direct calculation implies

$$\begin{aligned} \|I_5\| &\leq C \int_0^t (t-s)^{-\frac{1}{2}} \|B^m(b^m, b^m) - B^m(b, b)\|_H ds \\ &\leq C\tilde{K} \left(\int_0^t (t-s)^{-\frac{p}{2}} ds \right)^{\frac{1}{p}} \left(\int_0^t \|b^m - b\|_V^q ds \right)^{\frac{1}{q}}, \end{aligned}$$

and

$$\begin{aligned} \|J_4\| + \|J_5\| &\leq C \int_0^t (t-s)^{-\frac{1}{2}} \|B^m(u^m, b^m) - B^m(u, b)\|_H ds \\ &\leq C(K + \tilde{K}) \left[\left(\int_0^t (t-s)^{-\frac{p}{2}} ds \right)^{\frac{1}{p}} \left(\int_0^t \|b^m - b\|_V^q ds \right)^{\frac{1}{q}} \right] \end{aligned}$$

$$+ \left(\int_0^t (t-s)^{-\frac{p}{2}} ds \right)^{\frac{1}{p}} \left(\int_0^t \|u^m - u\|_V^q ds \right)^{\frac{1}{q}}, \quad (4.5)$$

where $\tilde{K} = \sup_{0 \leq s \leq t, m} (\|b\|_V + \|b^m\|_V) \leq 2\hat{C}$.

Due to $\sup_{0 \leq s \leq t} (\|u - u^m\|_V + \|b - b^m\|_V) \leq 4\hat{C}$ and Fatou's theorem, we derive

$$\begin{aligned} & \lim_{m \rightarrow \infty} \sup_m (\|u - u^m\|_V + \|b - b^m\|_V) \\ & \leq C(K + \tilde{K})t^{\frac{1}{p}-\frac{1}{2}} \lim_{m \rightarrow \infty} \sup_m \left[\left(\int_0^t \|u^m - u\|_V^q ds \right)^{\frac{1}{q}} + \left(\int_0^t \|b^m - b\|_V^q ds \right)^{\frac{1}{q}} \right] \end{aligned} \quad (4.6)$$

$$\begin{aligned} & \leq C(K + \tilde{K})t^{\frac{1}{p}-\frac{1}{2}} \left[\left(\int_0^t \lim_{m \rightarrow \infty} \sup_m \|u^m - u\|_V^q ds \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^t \lim_{m \rightarrow \infty} \sup_m \|b^m - b\|_V^q ds \right)^{\frac{1}{q}} \right], \end{aligned} \quad (4.7)$$

which implies

$$\lim_{m \rightarrow \infty} \sup_m (\|u - u^m\|_V + \|b - b^m\|_V) \rightarrow 0. \quad (4.8)$$

Thus, the proof of Lemma 4.1 is completed. \square

Theorem 4.1. *The solution ϕ of equation (2.2) admits at least one invariant measure. The invariant measures are supported on $V \times V$.*

Proof. We follow the method in [8]. Define

$$f(\phi) := (\|\phi\|_{H \times H}^2 + 1)^{\frac{1}{2}}, \quad \phi \in H_m \times H_m. \quad (4.9)$$

Equation (4.1) can be written as

$$\begin{cases} d\phi^m + B_1(\phi^m)dt = \begin{pmatrix} -\nabla p^m \\ 0 \end{pmatrix} dt + \Delta \phi^m + B_2(\phi^m)dt + \begin{pmatrix} dL_t^m \\ dL_t^m \end{pmatrix}, \\ \phi_0^m = (u_0^m, b_0^m), \end{cases} \quad (4.10)$$

where $B_1(\phi^m) = (B(u_m, u_m), B(u_m, b_m))$ and $B_2(\phi^m) = (B(b_m, b_m), B(b_m, u_m))$.

Applying Itô formula [2] gives

$$f(\phi^m) =: f(\phi_0^m) - G_1^m(t) + G_2^m(t) + G_3^m(t) + G_4^m(t), \quad (4.11)$$

where

$$G_1^m(t) := \int_0^t \frac{\|\phi^m\|_{V \times V}^2}{(1 + \|\phi^m\|_{H \times H}^2)^{\frac{1}{2}}} ds + \int_0^t \frac{\langle B_1(\phi_m), \phi_m \rangle - \langle B_2(\phi_m), \phi_m \rangle}{(1 + \|\phi^m\|_{H \times H}^2)^{\frac{1}{2}}} ds,$$

$$\begin{aligned}
G_2^m(t) &:= \sum_{|j| < m} \int_0^t \int_{|y| \leq 1} [f(u^m + \gamma \beta_j e_j) - f(u^m) + f(b^m + \gamma \beta_j e_j) - f(b^m)] \tilde{N}^{(j)}(ds, dy), \\
G_3^m(t) &:= \sum_{|j| < m} \int_0^t \int_{|y| \leq 1} [f(u^m + \gamma \beta_j e_j) - f(u^m) + f(b^m + \gamma \beta_j e_j) - f(b^m)] \\
&\quad - \frac{\langle u^m, y \beta_j e_j \rangle + \langle b^m, y \beta_j e_j \rangle}{\|\phi^m\|_{H \times H}^2 + 1} \nu(dy) ds, \\
G_4^m(t) &:= \sum_{|j| < m} \int_0^t \int_{|y| > 1} [f(u^m + \gamma \beta_j e_j) - f(u^m) + f(b^m + \gamma \beta_j e_j) - f(b^m)] \tilde{N}(ds, dy),
\end{aligned}$$

and $\nu(\cdot)$ is the Lévy measure that satisfies $\int_{R \setminus \{0\}} 1 \wedge |y|^2 \nu(dy) < \infty$. For $t > 0$ and $\Gamma \in \mathcal{B}(R \setminus \{0\})$, the Poisson random measure associated with α -stable Lévy noise is defined by $N^{(j)}(t, \Gamma) = \sum_{s \in (0, t]} 1_\Gamma(l_j(s) - l_j(s-))$, and the Compensated Poisson random measure is $\tilde{N}^{(j)}(t, \Gamma) = N^{(j)}(t, \Gamma) - t\nu(\Gamma)$.

Notice that $\langle B_1(\phi^m), \phi^m \rangle = 0$, $\langle B_1(\phi^m), \phi^m \rangle = 0$. Similar to the arguments in [8], we have

$$\mathbb{E}[\sup_{0 < t \leq T} |I_2^m(t)|] \leq CT^{\frac{1}{2}}, \quad \mathbb{E}[\sup_{0 < t \leq T} |I_3^m(t)|] \leq CT, \quad \mathbb{E}[\sup_{0 < t \leq T} |I_4^m(t)|] \leq CT.$$

Direct calculation shows that

$$\mathbb{E}[\sup_{0 < t \leq T} (\|\phi_m\|_{H \times H}^2 + 1)^{\frac{1}{2}}] + \mathbb{E} \int_0^t \frac{\|\phi^m\|_{V \times V}^2}{(1 + \|\phi^m\|_{H \times H}^2)^{\frac{1}{2}}} ds \leq (\|\phi_0\|_{H \times H}^2 + 1)^{\frac{1}{2}} + CT + CT^{\frac{1}{2}}.$$

Lemma 4.1 implies that

$$\lim_{m \rightarrow \infty} \|\phi^m\|_{H \times H} = \|\phi\|_{H \times H}, \quad \lim_{m \rightarrow \infty} \|\phi^m\|_{V \times V} = \|\phi\|_{V \times V}.$$

By Fatou's Lemma, we obtain

$$\mathbb{E}[\sup_{0 < t \leq T} (\|\phi\|_{H \times H}^2 + 1)^{\frac{1}{2}}] + \mathbb{E} \int_0^t \frac{\|\phi\|_{V \times V}^2}{(1 + \|\phi\|_{H \times H}^2)^{\frac{1}{2}}} ds \leq (\|\phi_0\|_{H \times H}^2 + 1)^{\frac{1}{2}} + CT + CT^{\frac{1}{2}}.$$

It follows from Young's inequality that

$$\begin{aligned}
\mathbb{E} \left(\int_0^T \|\phi\|_{V \times V} ds \right) &\leq \mathbb{E} \left(\int_0^T \frac{\|\phi\|_{V \times V} (\|\phi\|_{H \times H} + 1)}{(\|\phi\|_{H \times H}^2 + 1)^{\frac{1}{2}}} ds \right) \\
&\leq C \mathbb{E} \left(\int_0^T \frac{\|\phi\|_{V \times V}^2 + 1}{(\|\phi\|_{H \times H}^2 + 1)^{\frac{1}{2}}} ds \right) \\
&\leq C(1 + \|\phi_0\|_{H \times H} + T),
\end{aligned}$$

which implies the existence of the invariant measures by the classical Bogoliubov–Krylov's argument and the support of the invariant measure is $V \times V$. \square

5. The uniqueness of the invariant measure

In this section, we will follow the idea of [29] to show the uniqueness of the invariant measure. Due to the discontinuous trajectories, the uniqueness of the invariant measure is obtained by the strong Feller property and the accessibility to zero instead of the irreducibility.

5.1. Strong Feller property

Denote by $B_b(E)$ the space of bounded measurable function $f : E \rightarrow \mathbb{R}$. For all $f \in B_b(H \times H)$, define

$$\mathbb{P}_t(f(u_0, b_0)) = \mathbb{E}[f(u, b)],$$

where $t \geq 0$ and $(u_0, b_0) \in H \times H$. It follows from Theorem 3.1 that $(\mathbb{P}_t)_{t \geq 0}$ is a Markov semigroup on $B_b(H \times H)$. It requires to obtain a gradient estimate for the $O - U$ semigroup corresponding to $(Z_t)_{t \geq 0}$ for noise $(L_t)_{t \geq 0}$ in norm $\|\cdot\|_V$, which is devoted to proving the strong Feller property of the semigroup $(\mathbb{P}_t)_{t \geq 0}$ on $B_b(V \times V)$.

Since the bilinear operator B is not bounded, we consider the equation with truncated bi-linear term as follows:

$$\begin{cases} du^\rho = [\Delta u^\rho + B^\rho(b^\rho, b^\rho) - B^\rho(u^\rho, u^\rho) - \nabla p^\rho]dt + dL_t, & t \geq 0, \\ db^\rho = [\Delta b^\rho + B^\rho(b^\rho, u^\rho) - B(u^\rho, b^\rho)]dt + dL_t, & t \geq 0, \end{cases} \quad (5.1)$$

where $\rho > 0$, $B^\rho(x, x) = B(x, x)\chi(\frac{\|x\|_V}{\rho})$ for all $x \in V$ and $\chi : \mathbb{R} \rightarrow [0, 1]$ is a smooth function such that

$$\chi(z) = 1, \quad |z| < 1, \quad \chi(z) = 0, \quad |z| \geq 2.$$

For all $x \in V$, we derive that

$$\|B^\rho(x, x)\|_H \leq C\|x\|_V^2\chi(\frac{\|x\|_V}{\rho}) \leq C\rho^2,$$

and

$$\|B^\rho(x, x) - B^\rho(y, y)\|_H \leq C(1 + \rho^2)\|x - y\|_V,$$

which implies that B^ρ is a Lipschitz function from V to H . Hence, there exists a unique solution $(u^\rho, b^\rho) \in D([0, \infty); V \times V)$ for equation (5.1).

For every $f \in B_b(V \times V)$, define

$$\mathbb{P}_t^\rho(f(u_0, b_0)) = \mathbb{E}[f(u^\rho, b^\rho)], \quad t \geq 0, \quad (u_0, b_0) \in V \times V,$$

$(\mathbb{P}_t^\rho)_{t \geq 0}$ is a Markov semigroup.

Define the derivative of $f \in C_b^1(V \times V)$:

$$D_h f(x, y) := \lim_{\varepsilon \rightarrow 0} \frac{f(x + \varepsilon h_1, y + \varepsilon h_2) - f(x, y)}{\varepsilon}, \quad h = (h_1, h_2).$$

Due to Riesz representation theorem, there exists some $Df(x, y) \in V \times V$ such that

$$D_h f(x, y) = \langle Df(x, y), h \rangle_{V \times V}, \quad h \in V \times V.$$

Define

$$\|D_h f(x, y)\|_\infty = \sup_{x \in V, y \in V} \|Df(x, y)\|_{V \times V}.$$

Noticing that B^ρ is a bounded Lipschitz function. We deduce from Lemma 5.9 of [22] and $\beta < \frac{3}{2} - \frac{1}{\alpha}$ that the following proposition holds.

Proposition 5.1. *For $\alpha \in (\frac{3}{2}, 2)$, there exists some $\theta \in [\frac{1}{\alpha}, 1)$ such that*

$$\|D\mathbb{P}_t^\rho f\|_\infty \leq Ct^{-\theta} \|f\|_\infty, \quad t > 0,$$

where $f \in B_b(V \times V)$ and $C > 0$ depends on ρ , α and θ .

Theorem 5.1. $(\mathbb{P}_t)_{t \geq 0}$ as a semigroup on $B_b(V \times V)$ is strong Feller.

Proof. Assume $\|f\|_\infty = 1$. For $T_0 > 0$, it suffices to show that for all $t \in (0, T_0]$

$$\lim_{\|u'_0 - u_0\|_V = 0, \|b'_0 - b_0\|_V = 0} \mathbb{P}_t(f(u'_0, b'_0)) = \mathbb{P}_t(f(u_0, b_0)).$$

Denote

$$K_{T_0}(\omega) := \sup_{0 \leq t \leq T_0} \|Z_t(\omega)\|_V, \omega \in \Omega.$$

Lemma 2.2 and the Markov inequality give

$$\mathbb{P}[K_{T_0}(\omega) > \frac{\rho}{4}] \leq \frac{C}{\rho}.$$

Let ρ be large enough such that $\|u_0\|_V < \frac{\sqrt{\rho}}{2}$, $\|v_0\|_V < \frac{\sqrt{\rho}}{2}$ and $A := \{K_{T_0} \leq \frac{\rho}{4}\}$. We deduce from Lemma 3.1 that there exists some $0 < t_0 \leq T_0$ depending on ρ such that for all $\omega \in A$,

$$\sup_{0 \leq t \leq t_0} (\|u(\omega)\|_V + \|b(\omega)\|_V) \leq 1 + \|u_0\|_V + \|b_0\|_V. \quad (5.2)$$

Hence, we have

$$\begin{aligned} \mathbb{P}(\sup_{0 \leq t \leq t_0} (\|u\|_V + \|b\|_V) \geq \rho) &\leq \mathbb{P}(\sup_{0 \leq t \leq t_0} (\|v\|_V + \|c\|_V) + 2 \sup_{0 \leq t \leq t_0} \|Z_t\|_V \geq \rho) \\ &\leq \mathbb{P}(K_{T_0} > \frac{\rho}{4}) + \mathbb{P}(\sup_{0 \leq t \leq t_0} (\|v\|_V + \|c\|_V) > \frac{\rho}{2}, A). \end{aligned}$$

For any $\omega \in A$, (5.2) implies that

$$\sup_{0 \leq t \leq t_0} (\|v(\omega)\|_V + \|c(\omega)\|_V) \leq 1 + \|u_0\|_V + \|b_0\|_V \leq 1 + \sqrt{\rho} \leq \frac{\rho}{2}.$$

Hence, we have

$$\mathbb{P}(\sup_{0 \leq t \leq t_0} (\|v\|_V + \|c\|_V) \geq \frac{\rho}{2}, A) = 0,$$

and

$$\mathbb{P}\left(\sup_{0 \leq t \leq t_0} (\|u\|_V + \|b\|_V) \geq \rho\right) \leq \mathbb{P}(K_{T_0} > \frac{\rho}{4}) \leq \frac{C}{\rho}.$$

Define the stopping time by

$$\tau = \inf\{t > 0, \|u\|_V + \|b\|_V \geq \rho\}.$$

Then for $t \in [0, t_0]$,

$$\mathbb{P}(\tau \leq t) = \mathbb{P}\left(\sup_{0 \leq t \leq t_0} (\|u\|_V + \|b\|_V) \geq \rho\right) \leq \frac{C}{\rho}. \quad (5.3)$$

Since for $t \in [0, \tau)$, both equation (1.3) and (5.1) have a unique mild solution,

$$u^\rho = u, \quad b^\rho = b, \quad a.s.$$

Let $u', b' \in V$ such that $\|u - u'\|_V \leq 1$ and $\|b - b'\|_V \leq 1$ and ρ be so large that $\|u\|_V, \|u'\|_V, \|b\|_V, \|b'\|_V \leq \frac{\sqrt{\rho}}{2}$. Then for $t \in (0, t_0]$, we have

$$|\mathbb{P}_t(f(u_0, b_0)) - \mathbb{P}_t(f(u'_0, b'_0))| = |\mathbb{E}[f(u, b)] - \mathbb{E}[f(u', b')]| = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= |\mathbb{E}[(f(u, b))1_{\tau > t}] - \mathbb{E}[(f(u', b'))1_{\tau > t}]|, \\ I_2 &= |\mathbb{E}[(f(u, b))1_{\tau \leq t}]|, \\ I_3 &= |\mathbb{E}[(f(u', b'))1_{\tau \leq t}]|. \end{aligned}$$

It follows from (5.3) that

$$I_2 \leq \frac{C}{\rho}, \quad I_3 \leq \frac{C}{\rho}.$$

Proposition 5.1 and (5.3) give

$$\begin{aligned} I_1 &= |\mathbb{E}[(f(u^\rho, b^\rho))1_{\tau > t}] - \mathbb{E}[(f(u'^\rho, b'^\rho))1_{\tau > t}]| \\ &\leq |\mathbb{E}[(f(u^\rho, b^\rho)) - \mathbb{E}(f(u'^\rho, b'^\rho))]| + |\mathbb{E}[(f(u^\rho, b^\rho))1_{\tau < t}]| + |\mathbb{E}[(f(u'^\rho, b'^\rho))1_{\tau < t}]| \\ &\leq Ct^{-\theta}(\|u_0 - u'_0\|_V + \|b_0 - b'_0\|_V) + \frac{2C}{\rho}. \end{aligned}$$

Thus, we deduce that

$$|\mathbb{P}_t(f(u_0, b_0)) - \mathbb{P}_t(f(u'_0, b'_0))| \leq Ct^{-\theta}(\|u_0 - u'_0\|_V + \|b_0 - b'_0\|_V) + \frac{4C}{\rho}.$$

For all $\varepsilon > 0$, let $\rho \geq \max\{\frac{12C}{\varepsilon}, 2\|u_0\|_V^2 + 2, 2\|b_0\|_V^2 + 2\}$ and $\delta \leq \frac{\varepsilon t^\theta}{2C}$. Then for $\|u_0 - u'_0\|_V + \|b_0 - b'_0\|_V < \delta$, we get

$$|\mathbb{P}_t(f(u_0, b_0)) - \mathbb{P}_t(f(u'_0, b'_0))| \leq \varepsilon, \quad t \in (0, t_0].$$

For $t_0 < t \leq T_0$, we derive from the Markov property and strong Feller property that

$$\mathbb{P}_t(f(u_0, b_0)) - \mathbb{P}_t(f(u'_0, b'_0)) = \mathbb{P}_{t_0}[\mathbb{P}_{t-t_0}f](u_0, b_0) - \mathbb{P}_{t_0}[\mathbb{P}_{t-t_0}f](u'_0, b'_0) \rightarrow 0,$$

as $\|u_0 - u'_0\|_V + \|b_0 - b'_0\|_V \rightarrow 0$. \square

Theorem 5.2. $(\mathbb{P}_t)_{t \geq 0}$ as a semigroup on $B_b(H \times H)$ is strong Feller.

Proof. For any $T_0 > 0$, it suffices to show that for all $t \in (0, T_0]$ and $(u_0, b_0) \in H \times H$

$$\lim_{\|u'_0 - u_0\|_H \rightarrow 0, \|b'_0 - b_0\|_H \rightarrow 0} \mathbb{P}_t(f(u'_0, b'_0)) = \mathbb{P}_t(f(u_0, b_0)).$$

Let $\Omega_N := \{\sup_{0 \leq t_0 \leq T_0} \|Z(t)\|_V \leq N\}$, then Lemma 2.2 and Chebyshev's inequality imply that

$$\mathbb{P}(\Omega_N^c) \leq \frac{c}{N}, \quad (5.4)$$

and

$$u - u' = I_1 + I_2 + I_3, \quad (5.5)$$

$$b - b' = J_1 + J_2 + J_3, \quad (5.6)$$

where

$$\begin{aligned} I_1 &= e^{-At}u_0 - e^{-At}u'_0, & I_2 &= \int_0^t e^{-A(t-s)}[B(u, u) - B(u', u')]ds, \\ J_1 &= e^{-At}b_0 - e^{-At}b'_0, & J_2 &= \int_0^t e^{-A(t-s)}[B(u, b) - B(u', b')]ds, \\ I_3 &= \int_0^t e^{-A(t-s)}[B(b, b) - B(b', b')]ds, & J_3 &= \int_0^t e^{-A(t-s)}[B(b, u) - B(b', u')]ds. \end{aligned}$$

We get the estimates from (2.1) that

$$\|I_1(t)\|_V \leq Ct^{-\frac{1}{2}}\|u_0 - u'_0\|_H, \quad \|J_1(t)\|_V \leq Ct^{-\frac{1}{2}}\|b_0 - b'_0\|_H.$$

Due to the Hölder inequality, we obtain

$$\begin{aligned} \|I_2(t)\|_V &\leq C \int_0^t (t-s)^{-\frac{1}{2}}(\|u\|_V + \|u'\|_V)\|u - u'\|_V, \\ \|J_2(t)\|_V &\leq C \int_0^t (t-s)^{-\frac{1}{2}}(\|u\|_V\|b - b'\|_V + \|b'\|_V\|u - u'\|_V), \\ \|I_3(t)\|_V &\leq C \int_0^t (t-s)^{-\frac{1}{2}}(\|b\|_V + \|b'\|_V)\|b - b'\|_V, \end{aligned}$$

$$\|J_3(t)\|_V \leq C \int_0^t (t-s)^{-\frac{1}{2}} (\|u\|_V \|b-b'\|_V + \|b'\|_V \|u-u'\|_V) ds.$$

Theorem 3.1 implies that

$$\|u\|_V \leq \|v\|_V + \|Z\|_V \leq C, \quad \|b\|_V \leq \|c\|_V + \|Z\|_V \leq C.$$

Similarly, $\|u'\|_V \leq C$ and $\|b'\|_V \leq C$. Hence,

$$\begin{aligned} \|I_2(t)\|_V &\leq C \int_0^t (t-s)^{-\frac{1}{2}} \|u-u'\|_V ds, \\ \|J_2(t)\|_V &\leq C \int_0^t (t-s)^{-\frac{1}{2}} (\|b-b'\|_V + \|u-u'\|_V) ds, \\ \|I_3(t)\|_V &\leq C \int_0^t (t-s)^{-\frac{1}{2}} \|b-b'\|_V ds, \\ \|J_3(t)\|_V &\leq C \int_0^t (t-s)^{-\frac{1}{2}} (\|b-b'\|_V + \|u-u'\|_V) ds. \end{aligned}$$

For $r \in (0, t_0]$, define

$$\Phi_r = \sup_{0 \leq t \leq r} t^{\frac{1}{2}} \|u-u'\|_V, \quad \Psi_r = \sup_{0 \leq t \leq r} t^{\frac{1}{2}} \|b-b'\|_V.$$

Theorem 3.1 gives

$$\Phi_r \leq \sup_{0 \leq t \leq r} t^{\frac{1}{2}} (\|v\|_V + \|v'\|_V) + 2r^{\frac{1}{2}} N < \infty, \quad \Psi_r \leq \sup_{0 \leq t \leq r} t^{\frac{1}{2}} (\|c\|_V + \|c'\|_V) + 2r^{\frac{1}{2}} N < \infty.$$

Due to (5.5) and (5.6), we have

$$\begin{aligned} \Phi_r &\leq C \|u_0 - u'_0\|_H + C \sup_{0 \leq t \leq r} [t^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds] \Phi_r + C \sup_{0 \leq t \leq r} [t^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds] \Psi_r \\ &\leq C \|u_0 - u'_0\|_H + Cr^{\frac{1}{2}} \Phi_r + Cr^{\frac{1}{2}} \Psi_r, \end{aligned}$$

and

$$\Psi_r \leq C \|c_0 - c'_0\|_H + Cr^{\frac{1}{2}} \Phi_r + Cr^{\frac{1}{2}} \Psi_r.$$

Choosing r small enough such that $Cr^{\frac{1}{2}} \leq \frac{1}{4}$, we get

$$\Phi_r + \Psi_r \leq C (\|u_0 - u'_0\|_H + \|b_0 - b'_0\|_H),$$

which implies that

$$\|u-u'\|_V + \|b-b'\|_V \leq Ct^{-\frac{1}{2}} (\|u_0 - u'_0\|_H + \|b_0 - b'_0\|_H). \quad (5.7)$$

For all $0 < t \leq T_0$, by the Markov property, we obtain

$$|\mathbb{P}_t f(u_0, b_0) - \mathbb{P}_t f(u'_0, b'_0)| \leq |\mathbb{E}[\mathbb{P}_{t-s} f(u, b) - \mathbb{P}_{t-s} f(u', b')]| \leq H_1 + H_2,$$

where $s = \frac{t}{2} \wedge r$ and

$$H_1 = |\mathbb{E}[\mathbb{P}_{t-s} f(u, b) - \mathbb{P}_{t-s} f(u', b')]1_{\Omega_N}|, \quad H_2 = |\mathbb{E}[\mathbb{P}_{t-s} f(u, b) - \mathbb{P}_{t-s} f(u', b')]1_{\Omega_N}|.$$

We derive from (5.4) that

$$H_1 \leq 2c \frac{\|f\|_\infty}{N}.$$

Combining (5.7), Theorem 5.1 with the dominated convergence theorem, we obtain that

$$H_2 \rightarrow 0, \quad \|u_0 - u'_0\|_H + \|b_0 - b'_0\|_H \rightarrow 0.$$

Thus, the proof of Theorem 5.2 is completed. \square

5.2. The accessibility

Because we can't get the irreducibility, we fail to apply the classical Doob's Theorem to get the ergodicity. Alternatively, we apply a criterion in [13]. Finally, let's introduce the conception of the accessibility.

Definition 5.1 (*Accessibility*). Let $(X_t)_{t \geq 0}$ be a stochastic process valued on a metric space E and let $(\mathbb{P}_t(x, \cdot))_{x \in E}$ be the transition probability family. $(X_t)_{t \geq 0}$ is said to be accessible to $x_0 \in E$ if the resolvent \mathfrak{R}_λ satisfies

$$\mathfrak{R}_\lambda(x, \mathfrak{U}) := \int_0^\infty e^{-\lambda t} \mathbb{P}_t(x, \mathfrak{U}) dt > 0,$$

for all $x \in E$ and all neighborhoods \mathfrak{U} of x_0 , where $\lambda > 0$ is arbitrary.

The following theorem is the key tool to show the uniqueness of the invariant measure, which is developed by Hairer in [13].

Theorem 5.3 ([13]). *If $(X_t)_{t \geq 0}$ is strong Feller at an accessible point $x \in E$, then it can have at most one invariant measure.*

Theorem 5.4. *Assume that $\alpha \in (\frac{3}{2}, 2)$ and $\frac{1}{2} + \frac{1}{2\alpha} \leq \beta \leq \frac{3}{2} - \frac{1}{\alpha}$. The solution ϕ of the stochastic MHD equation (1.3) has a unique invariant measure.*

Proof. Lemma 2.2 implies that for $t > 0$ and $\varepsilon > 0$

$$\mathbb{P}(\sup_{0 \leq s \leq t} \|Z_s\|_V^2 \leq \varepsilon) > 0. \quad (5.8)$$

We derive from (3.29) that for $\omega \in \Omega_{\varepsilon, t}$

$$(\|v\|_H^2 + \|c\|_H^2) \leq e^{-2(\pi-2\varepsilon)t} (\|u_0\|_H^2 + \|b_0\|_H^2) + c\varepsilon^4. \quad (5.9)$$

Define

$$B_{H \times H}(r) := \{u_0, b_0 \in H; \|u_0\|_H + \|b_0\|_H \leq r\}, \quad \forall r > 0. \quad (5.10)$$

Then for all $R > 0$, let $T := T_{R,\delta}$ be sufficiently large and $\varepsilon := \varepsilon_{R,\delta}$ be sufficiently small. It follows from (5.9) that for all $\delta > 0$,

$$(\|u\|_H^2 + \|b\|_H^2) \leq e^{-2(\pi-2\varepsilon)t}(\|u_0\|_H^2 + \|b_0\|_H^2) + c(\varepsilon^4 + \varepsilon) < \delta, \quad t \geq T, \quad (5.11)$$

for $x \in B_{H \times H}(r)$ and $\omega \in \Omega_{\varepsilon,t}$.

We get from (5.8) that for all $(u_0, b_0) \in B_{H \times H}(r)$,

$$\mathbb{P}(t, x, B_{H \times H}(\delta)) > 0, \quad t \geq T, \quad (5.12)$$

which implies that

$$\mathfrak{R}_\lambda(x, B_{H \times H}(\delta)) > 0. \quad (5.13)$$

Since $R > 0$ is arbitrary, the above inequality is true for all $u_0, b_0 \in H$. Therefore $(u, b)_{t \geq 0}$ is accessible to 0. Theorem [13] guarantees that the stochastic equation (1.3) has a unique invariant measure. Thus, the proof of Theorem 5.4 is completed. \square

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