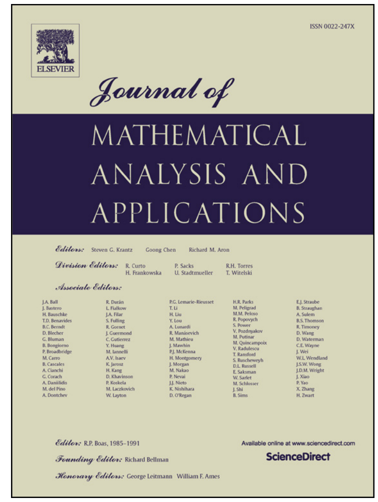
Accepted Manuscript



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PII: S0022-247X(16)30743-0
DOI: http://dx.doi.org/10.1016/j.jmaa.2016.11.053
Reference: YJMAA 20906

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 15 May 2016

Please cite this article in press as: M. Mesk, M.B. Zahaf, A new characterization of ultraspherical, Hermite, and Chebyshev polynomials of the first kind, *J. Math. Anal. Appl.* (2017), http://dx.doi.org/10.1016/j.jmaa.2016.11.053

A new characterization of ultraspherical, Hermite, and Chebyshev polynomials of the first kind

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Abstract

We show that the only polynomial sets with a generating function of the form $F(xt - R(t))$ and satisfying a three-term recursion relation are the monomial set and the rescaled ultraspherical, Hermite, and Chebyshev polynomials of the first kind.

Keywords: Orthogonal polynomials; generating functions; recurrence relations; ultraspherical polynomials; Chebyshev polynomials; Hermite polynomials.
2010 MSC: 33C45, 42C05

1. Introduction and main result

The problem of describing all or just orthogonal polynomials generated by a specific generating function has been investigated by many authors (see for example [1, 2, 3, 4, 5, 6, 7, 8, 9]). For the special case, where the generating function has the form $F(xt - \alpha t^2)$, the authors in [2], [5] and [10] used different methods to show that the orthogonal polynomials are Hermite and ultraspherical polynomials. Recently in [4], the author gave a motivation of this question

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and found, even if F is a formal power series, that the orthogonal polynomials are the ultraspherical, Hermite and Chebychev polynomials of the first kind. Moreover, for F corresponding to Chebychev polynomials of the first kind, he showed that these polynomials remain the only orthogonal polynomials with generating function of the form $F(xU(t) - R(t))$, where $U(t)$ and $R(t)$ are formal power series. A natural question, as mentioned in [4], is to describe (all or just orthogonal) polynomials with generating functions

$$F(xU(t) - R(t)).$$

In this paper, we consider the subclass case $F(xt - R(t)) = \sum_{n\geq 0} \alpha_n P_n(x)t^n$ where the polynomial set (not necessary orthogonal) $\{P_n\}_{n\geq 0}$ satisfies a three-term recursion relation. The main result obtained here is the following:

Theorem 1. *Let $F(t) = \sum_{n\geq 0} \alpha_n t^n$ and $R(t) = \sum_{n\geq 1} R_n t^n/n$ be formal power series where $\{\alpha_n\}$ and $\{R_n\}$ are complex sequences with $\alpha_0 = 1$ and $R_1 = 0$. Define the polynomial set $\{P_n\}_{n\geq 0}$ by*

$$F(xt - R(t)) = \sum_{n\geq 0} \alpha_n P_n(x)t^n. \tag{1}$$

If this polynomial set (which is automatically monic) satisfies the three-term recursion relation

$$\begin{cases} xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \omega_n P_{n-1}(x), & n \geq 0, \\ P_{-1}(x) = 0, \quad P_0(x) = 1 \end{cases} \tag{2}$$

where $\{\beta_n\}$ and $\{\omega_n\}$ are complex sequences, then we have:

a) If $R_2 = 0$ and $\alpha_n \neq 0$ for $n \geq 1$, then $R(t) = 0$, $F(t)$ is arbitrary and $F(xt) = \sum_{n\geq 0} \alpha_n x^n t^n$ generates the monomials $\{x^n\}_{n\geq 0}$.

b) If $\alpha_1 R_2 \neq 0$, then $R(t) = R_2 t^2/2$ and the polynomial sets $\{P_n\}_{n\geq 0}$ are the rescaled ultraspherical, Hermite and Chebychev polynomials of the first kind.

In the above theorem, let us remark that there is no loss of generality in assuming $\alpha_0 = 1$ and $R_1 = 0$. Indeed, we can choose the generating function $\gamma_1 + \gamma_2 F((x+R_1)t - R(t)) = \gamma_1 + \gamma_2 \sum_{n\geq 0} \alpha_n P_n(x+R_1)t^n$ for suitable constants γ_1 and γ_2.

The proof of theorem 1 will be given in section 3. For that purpose some preliminary results must be developed first in section 2. We end the paper by a brief concluding section.

2. Preliminary results

This section contains two propositions and some related corollaries which are the important ingredients used for the proof of Theorem 1.

Proposition 1. *Let $\{P_n\}_{n\geq 0}$ be a monic polynomial set generated by* (1)*. Then we have*

$$\alpha_n x P_n'(x) - \sum_{k=1}^{n} R_{k+1}\alpha_{n-k}P_{n-k}'(x) = n\alpha_n P_n(x), \quad n \geq 1. \tag{3}$$

Proof. By combining the two derivatives $\frac{\partial W}{\partial x}$ and $\frac{\partial W}{\partial t}$ of the generating function $W(x,t) = F(xt - R(t))$, we obtain

$$(x - R'(t))\frac{\partial W}{\partial x} = t\frac{\partial W}{\partial t}. \tag{4}$$

The substitution of the right-hand side of (1) and $R'(t) = \sum_{n\geq 0} R_{n+1}t^n$ in (4) gives

$$\left(x - \sum_{n\geq 0} R_{n+1}t^n\right)\sum_{n\geq 0}\alpha_n P_n'(x)t^n = \sum_{n\geq 0}\alpha_n P_n(x) n t^n. \tag{5}$$

After a resummation procedure in left hand side, namely:

$$\left(\sum_{n\geq 0} R_{n+1}t^n\right)\left(\sum_{n\geq 0}\alpha_n P_n'(x)t^n\right) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} R_{k+1}\alpha_{n-k}P_{n-k}'(x)\right)t^n$$

and a t^n coefficients comparison in (5), the result (3) of proposition 1 follows.

Corollary 1. *Let $\{P_n\}_{n\geq 0}$ be a monic polynomial set generated by* (1)*. If $\alpha_1 R_2 \neq 0$ then $\alpha_n \neq 0$ for $n \geq 2$.*

Proof. *In fact suppose that $\alpha_{n_0} = 0$ for an $n_0 \geq 2$. Then* (3) *implies that $R_{k+1}\alpha_{n_0-k} = 0$ for $k = 1, ..., n_0 - 1$. In particular $R_2\alpha_{n_0-1} = 0$ for $k = 1$ gives $\alpha_{n_0-1} = 0$ since $R_2 \neq 0$. By induction we arrive at $\alpha_1 = 0$ which contradicts the premise $\alpha_1 \neq 0$.*

Corollary 2. *Let $\alpha_1 R_2 \neq 0$. If the polynomial set $\{P_n\}$ generated by (1) is symmetric, then*

$$R_{2l+1} = 0, \quad \textit{for } l \geq 1.$$

Proof. The polynomial set $\{P_n\}$ is symmetric means that $P_n(-x) = (-1)^n P_n(x)$ for $n \geq 0$. The substitution $x \to -x$ in equation (3) minus equation (3) itself left us with

$$(1 - (-1)^{k+1}) R_{k+1} \alpha_{n-k} = 0, \quad \text{for } 1 \leq k \leq n-1.$$

So, by Corollary 1, we have $R_{2l+1} = 0$, for $l \geq 1$, and $R(t) = \sum_{k \geq 1} \frac{R_{2k}}{2k} t^{2k}$.

Proposition 2. *Let $\alpha_1 R_2 \neq 0$ and define*

$$T_k = R_{2k},\ (k \geq 1),\ a_n = \frac{T_1}{2} \frac{\alpha_n}{\alpha_{n+1}},\ (n \geq 0)\ \textit{and}\ c_n = \frac{\alpha_n}{\alpha_{n-1}} \omega_n,\ \ (n \geq 1). \quad (6)$$

For the monic polynomial set generated by (1) and satisfying (2) we have:

a)

$$\beta_n = 0, \quad \textit{for } n \geq 0. \quad (7)$$

b)

$$\omega_n = n a_n - (n-1) a_{n-1}, \quad \textit{for } n \geq 1. \quad (8)$$

c)

$$\frac{4T_2}{T_1^3} \left(1 - \frac{n-3}{n-2} \frac{a_{n-3}}{a_n} \right) = \frac{n+1}{a_n} - \frac{2n}{a_{n-1}} + \frac{n-1}{a_{n-2}}, \quad \textit{for } n \geq 3. \quad (9)$$

d)

$$\frac{2}{T_1} \left(a_n - \frac{n-2k-1}{n-2k} a_{n-2k-1} \right) T_{k+1} + \left(\frac{n+2}{n} c_n - \frac{n-2k+1}{n-2k+2} c_{n-2k+1} \right) T_k =$$
$$= \sum_{l=1}^{k} \frac{T_l T_{k-l+1}}{n-2k+2l}, \quad \textit{for } k \geq 2 \textit{ and } n \geq 2k+1. \quad (10)$$

Proof. By differentiating (2) we get

$$x P_n'(x) + P_n(x) = P_{n+1}'(x) + \beta_n P_n'(x) + \omega_n P_{n-1}'. \quad (11)$$

Then by making the combinations $n\alpha_n Eq(11) + Eq(3)$ and $Eq(3) - \alpha_n Eq(11)$ we obtain, respectively,

$$(n+1)\alpha_n x P_n'(x) = n\alpha_n \left(P_{n+1}'(x) + \beta_n P_n'(x) + \omega_n P_{n-1}'(x)\right) + \sum_{k=1}^{n-1} R_{k+1}\alpha_{n-k}P_{n-k}'(x) \quad (12)$$

and

$$(n+1)\alpha_n P_n(x) = \alpha_n \left(P_{n+1}'(x) + \beta_n P_n'(x) + \omega_n P_{n-1}'(x)\right) - \sum_{k=1}^{n-1} R_{k+1}\alpha_{n-k}P_{n-k}'(x). \quad (13)$$

Multiplying (13) by x and using (2) in the left-hand side gives

$$\begin{aligned}(n+1)\alpha_n \left(P_{n+1}(x) + \beta_n P_n(x) + \omega_n P_{n-1}(x)\right) &= \alpha_n \left(xP_{n+1}'(x) + \beta_n xP_n'(x) + \omega_n xP_{n-1}'(x)\right) \\ &\quad - \sum_{k=1}^{n-1} R_{k+1}\alpha_{n-k} x P_{n-k}'(x). \qquad (14)\end{aligned}$$

For the left-hand side (resp. the right-hand side) of (14) we use (13) (resp. (12)) to get

$$\begin{aligned}
&\frac{n+1}{n+2}\alpha_n \left(P_{n+2}'(x) + \beta_{n+1}P_{n+1}'(x) + \omega_{n+1}P_n'(x)\right) - \frac{n+1}{n+2}\frac{\alpha_n}{\alpha_{n+1}}\sum_{k=1}^{n} R_{k+1}\alpha_{n-k+1}P_{n-k+1}'(x) \\
&+\alpha_n\beta_n \left(P_{n+1}'(x) + \beta_n P_n'(x) + \omega_n P_{n-1}'(x)\right) - \beta_n \sum_{k=1}^{n-1} R_{k+1}\alpha_{n-k}P_{n-k}'(x) \\
&+\frac{n+1}{n}\alpha_n\omega_n \left(P_n'(x) + \beta_{n-1}P_{n-1}'(x) + \omega_{n-1}P_{n-2}'(x)\right) - \frac{n+1}{n}\frac{\alpha_n}{\alpha_{n-1}}\omega_n \sum_{k=1}^{n-2} R_{k+1}\alpha_{n-k-1}P_{n-k-1}'(x) = \\
&= \frac{n+1}{n+2}\alpha_n \left(P_{n+2}'(x) + \beta_{n+1}P_{n+1}'(x) + \omega_{n+1}P_n'(x)\right) + \frac{1}{n+2}\frac{\alpha_n}{\alpha_{n+1}}\sum_{k=1}^{n} R_{k+1}\alpha_{n-k+1}P_{n-k+1}'(x) \\
&+\frac{n}{n+1}\alpha_n\beta_n \left(P_{n+1}' + \beta_n P_n' + \omega_n P_{n-1}'\right) + \frac{1}{n+1}\beta_n \sum_{k=1}^{n-1} R_{k+1}\alpha_{n-k}P_{n-k}'(x) \\
&+\frac{n-1}{n}\alpha_n\omega_n \left(P_n'(x) + \beta_{n-1}P_{n-1}'(x) + \omega_{n-1}P_{n-2}'(x)\right) + \frac{1}{n}\frac{\alpha_n}{\alpha_{n-1}}\omega_n \sum_{k=1}^{n-2} R_{k+1}\alpha_{n-k-1}P_{n-k-1}'(x) \\
&- \sum_{k=1}^{n-1} R_{k+1}\frac{n-k}{n-k+1}\alpha_{n-k}\left(P_{n-k+1}'(x) + \beta_{n-k}P_{n-k}'(x) + \omega_{n-k}P_{n-k-1}'(x)\right) \\
&- \sum_{k=1}^{n-1}\frac{R_{k+1}}{n-k+1}\sum_{l=1}^{n-k-1} R_{l+1}\alpha_{n-k-l}P_{n-k-l}'(x), \qquad (15)
\end{aligned}$$

which can be simplified to

$$-\frac{\alpha_n}{\alpha_{n+1}}\sum_{k=1}^{n} R_{k+1}\alpha_{n-k+1}P'_{n-k+1}(x)+\frac{1}{n+1}\alpha_n\beta_n\left(P'_{n+1}(x)+\beta_nP'_n(x)+\omega_nP'_{n-1}(x)\right)$$
$$-\frac{n+2}{n+1}\beta_n\sum_{k=1}^{n-1}R_{k+1}\alpha_{n-k}P'_{n-k}(x)+\frac{2}{n}\alpha_n\omega_n\left(P'_n(x)+\beta_{n-1}P'_{n-1}(x)+\omega_{n-1}P'_{n-2}(x)\right)$$
$$-\frac{n+2}{n}\frac{\alpha_n}{\alpha_{n-1}}\omega_n\sum_{k=1}^{n-2}R_{k+1}\alpha_{n-k-1}P'_{n-k-1}(x)+\sum_{k=1}^{n-1}R_{k+1}\frac{n-k}{n-k+1}\alpha_{n-k}(P'_{n-k+1}(x)+\beta_{n-k}P'_{n-k}(x)$$
$$+\omega_{n-k}P'_{n-k-1}(x))+\sum_{k=1}^{n-1}\frac{R_{k+1}}{n-k+1}\sum_{l=1}^{n-k-1}R_{l+1}\alpha_{n-k-l}P'_{n-k-l}(x)=0. \tag{16}$$

From (16), the coefficient of $P'_{n+1}(x)$ is null, so we get (7) which means that the polynomial set $\{P_n\}$ is symmetric, see [11, Theorem 4.3]. Therefore, by Corollary 2, the odd part of the R-sequence is null and a computation of the coefficients of $P'_n(x)$, $P'_{n-2}(x)$ and $\{P'_{n+1-k}(x)\}_{n\geq k\geq 4}$ in (16) yields

$$\frac{2}{n}\alpha_n\omega_n = R_2\frac{\alpha_n}{\alpha_{n+1}}\alpha_n - R_2\frac{n-1}{n}\alpha_{n-1},\quad \text{for } n\geq 1, \tag{17}$$

$$\frac{2}{n}\alpha_n\omega_n\omega_{n-1} = R_4\frac{\alpha_n}{\alpha_{n+1}}\alpha_{n-2}+R_2\frac{n+2}{n}\frac{\alpha_n}{\alpha_{n-1}}\alpha_{n-2}\omega_n - R_4\frac{n-3}{n-2}\alpha_{n-3}-R_2\frac{n-1}{n}\alpha_{n-1}\omega_{n-1}-\frac{R_2^2}{n}\alpha_{n-2},\quad \text{for } n\geq 3, \tag{18}$$

and

$$R_{k+1}\left(\frac{\alpha_n}{\alpha_{n+1}}-\frac{n-k}{n-k+1}\frac{\alpha_{n-k}}{\alpha_{n-k+1}}\right)+R_{k-1}\left(\frac{n+2}{n}\frac{\alpha_n}{\alpha_{n-1}}\omega_n-\frac{n-k+2}{n-k+3}\frac{\alpha_{n-k+2}}{\alpha_{n-k+1}}\omega_{n-k+2}\right)$$
$$=\sum_{l=1}^{k-2}\frac{R_{k-l}R_{l+1}}{n-k+l+2},\quad n\geq k\geq 5. \tag{19}$$

respectively.

Finally, by using the notations (6), substituting for ω_n from (17) into (18) and by shifting $(k,l)\to(2k+1,2l-1)$ in (19) we obtain (8), (9) and (10).

In the following corollaries we adopt the same conditions and notations of Proposition 2.

Corollary 3. *If $T_2=0$ then $R(t)=T_1t^2/2$. In this case, the polynomials generated by $F(xt-T_1t^2/2)$ and satisfying (2) reduce to the rescaled ultraspherical, Hermite and Chebychev polynomials of the first kind.*

PROOF. We will use (10) and proceed by induction on k to show that $T_k = 0$ for $k \geq 3$. Indeed $k = 2$ and $n = 5$ in (10) leads to $2a_5T_3/T_1 = 0$ and since $a_n \neq 0$ by Corollary 1 we get $T_3 = 0$. Suppose that $T_3 = T_4 = \ldots = T_k = 0$. Then for $n = 2k+1$ the equation (10) gives $2a_{2k+1}T_{k+1}/T_1 = 0$ and finally $T_{k+1} = 0$. Accordingly, $R(t) = T_1t^2/2$ and the generating function (1) takes the form $F\left(xt - T_1t^2/2\right)$. Now, we make use of (9) (with $T_1 \neq 0$ and $T_2 = 0$) and proceed as in [4] to get the ultraspherical, Hermite and Chebyshev polynomials of the first kind.

Corollary 4. *If $T_\kappa = T_{\kappa+1} = 0$ for some $\kappa \geq 3$, then $T_2 = 0$.*

PROOF. Let $k = \kappa$ in (10). Then for $n \geq 2\kappa + 1$ the fraction $\sum_{l=1}^{\kappa} \frac{T_lT_{\kappa-l+1}}{n-2\kappa+2l}$, as function of integer n, is null even for real n. Multiplying by $n - 2\kappa + 2l$ and tends n to $2\kappa - 2l$ we find $T_lT_{\kappa-l+1} = 0$ for $1 \leq l \leq \kappa$ which is $T_2T_{\kappa-1} = 0$ when $l = 2$. Supposing $T_2 \neq 0$ leads to $T_{\kappa-1} = 0$. So $T_{\kappa-1} = T_\kappa = 0$ and with the same procedure we find $T_{\kappa-2} = 0$. Going so on till we arrive at $T_2 = 0$ which contradicts $T_2 \neq 0$.

Corollary 5. *If $R(t)$ is a polynomial then $R(t) = T_1t^2/2$.*

PROOF. If $R(t)$ is a polynomial then for some $\kappa \geq 2$, $T_k = 0$ whenever $k \geq \kappa$. By Corollary 4, since $T_\kappa = T_{\kappa+1} = 0$, we conclude that $T_2 = 0$ and by Corollary 3 that $T_k = 0$ for $k \geq 3$.

Corollary 6. *If a_n is a rational function of n then $T_2 = 0$.*

PROOF. Observe that $c_n = T_1\left(na_n/a_{n-1} - (n-1)\right)/2$ will also a rational function of n. Then it follows that, in (10), two fractions are equal for natural numbers $n \geq 2k+1$, $k \geq 2$ and consequently will be for real numbers n. If we denote by $N_s(F(x))$ the number of singularities of a rational function $F(x)$ then we can easily verify, for all rational functions F and $\tilde{F}$ of x and a constant $a \neq 0$, that:

a) $N_s(F(x+a)) = N_s(F(x))$,

b) $N_s(aF(x)) = N_s(F(x))$,

c) $N_s(F(x)+\tilde{F}(x)) \leq N_s(F(x)) + N_s(\tilde{F}(x))$.

Using property a) of N_s we have

$$N_s\left(\frac{n-2k-1}{n-2k}a_{n-2k-1}\right) = N_s\left(\frac{n}{n+1}a_n\right) \text{ and } N_s\left(\frac{n-2k+1}{n-2k+2}c_{n-2k+1}\right) = N_s\left(\frac{n}{n+1}c_n\right).$$

According to properties b) and c) of N_s, the N_s of the left-hand side of (10) is finite and independent of k. Thus, the right-hand side of (10) has a finite number of singularities which is independent of k. As consequence there exists a k_0 for which $T_lT_{k-l+1}=0$ for all $k \geq k_0$ and $k_0 \leq l \leq k$. Taking successively $k=k_0=l$ and $k=k_0+1=l$ we get $T_{k_0}=T_{k_0+1}=0$. Then, by Corollary 4 we have $T_2=0$.

Remark 1. *The fact that a_n is a rational function of n means that $F(z) = \sum_{n\geq 0}\alpha_n z^n$ is a series of hypergeometric type.*

Corollary 7. *If $T_\kappa = T_m = 0$ for some $\kappa \neq m \geq 3$, then $T_2 = 0$.*

Proof. If $T_{\kappa+1}=0$ or $T_{m+1}=0$ then by Corollary 4 we have $T_2=0$. Suppose that $T_{\kappa+1}\neq 0$ and $T_{m+1}\neq 0$. Take $k=\kappa$ and $k=m$ in (10) to get, respectively,

$$\frac{2}{T_1}\left(a_n - \frac{n-2\kappa-1}{n-2\kappa}a_{n-2\kappa-1}\right)T_{\kappa+1} = \sum_{l=1}^{\kappa}\frac{T_lT_{\kappa-l+1}}{n-2\kappa+2l}, \text{ for } n \geq 2\kappa+1, \tag{20}$$

and

$$\frac{2}{T_1}\left(a_n - \frac{n-2m-1}{n-2m}a_{n-2m-1}\right)T_{m+1} = \sum_{l=1}^{m}\frac{T_lT_{m-l+1}}{n-2m+2l}, \text{ for } n \geq 2m+1. \tag{21}$$

The operation $Eq(20)/T_{\kappa+1} - Eq(21)/T_{m+1}$ gives

$$\frac{n-2m-1}{n-2m}a_{n-2m-1} - \frac{n-2\kappa-1}{n-2\kappa}a_{n-2\kappa-1} = Q_1(n). \tag{22}$$

Assuming $m>\kappa$ and replacing n by $n+2m+1$ (resp. $n+2m-2\kappa$) in (22) (resp. (21)) leads to

$$\frac{n}{n+1}a_n - \frac{n+2m-2\kappa}{n+2m-2\kappa+1}a_{n+2m-2\kappa} = Q_1(n+2m+1) \tag{23}$$

and

$$a_{n+2m-2\kappa} - \frac{n-2\kappa-1}{n-2\kappa} a_{n-2\kappa-1} = Q_2(n+2m-2\kappa). \quad (24)$$

Now $T_1 Eq(20)/(2T_{\kappa+1}) - Eq(24)$ is the equation

$$a_n - a_{n+2m-2\kappa} = Q_3(n). \quad (25)$$

Multiplying (25) by $\frac{n+2m-2\kappa}{n+2m-2\kappa+1}$ and using (23) we find

$$\left(\frac{n}{n+1} - \frac{n+2m-2\kappa}{n+2m-2\kappa+1}\right) a_n = Q_4(n). \quad (26)$$

Since the $Q_i(n)$ $(i = 1..4)$ functions, the right-hand sides of (22), (23), (24), (25) and (26), are rational functions of n then a_n is also a rational function of n; and by Corollary 6 we deduce $T_2 = 0$.

Corollary 8. *The following equality is true for $k \geq 3$ and $n \geq 2k+3$.*

$$T_{k-1}D_{k+1}(a_n - \tilde{a}_{n-2k-3}) - T_{k+1}D_k(a_{n-2} - \tilde{a}_{n-2k-1}) = \sum_{l=1}^{k-1} \frac{V_{k,l}}{n-2k+2l}, \quad (27)$$

where

- $D_{k,l} = T_k T_{k-l+1} - T_{k+1} T_{k-l}$.
- $D_k = D_{k,1} = T_k^2 - T_{k+1} T_{k-1}$.
- $V_{k,l} = \frac{T_1}{2}\left(T_l T_{k+1} D_{k-1,l-1} - T_{l+1} T_{k-1} D_{k,l}\right)$.
- $\tilde{a}_n = \frac{n}{n+1} a_n$.

Proof. Denoting the equation (10) by $E(k,n)$ then (27) is the result of the operation

$$T_{k+1}\left(T_{k-1}E(k,n) - T_k E(k-1,n-2)\right) - T_{k-1}\left(T_k E(k+1,n) - T_{k+1}E(k,n-2)\right).$$

Now we are in a position to prove Theorem 1.

3. Proof of Theorem 1

The proof of a)

As $R_1 = R_2 = 0$, it is enough to show by induction that $R_n = 0$ for $n \geq 3$. For $n = 1, 2, 3$, the equation (3) gives $P_1(x) = x$, $P_2(x) = x^2$ and $P_3(0) = -\frac{R_3\alpha_1}{3\alpha_3}$. But according to equation (2), for $n = 2$, $P_3(0) = 0$ and then $R_3 = 0$. Now assume that $R_k = 0$ for $2 \leq k \leq n-1$. According to (3) we have, for $2 \leq k \leq n-1$, $P_k(0) = 0$ and $P_n(0) = -\frac{R_n\alpha_1}{n\alpha_n}$. On other hand, by the shift $n \to n-1$ in (2) we have $P_n(0) = 0$ and thus $R_n = 0$. As $R(t) = 0$, the generating function (1) reduces to $F(xt) = \sum_{n\geq 0} \alpha_n x^n t^n$ which generates the monomials with $F(t)$ arbitrary.

The proof of b)

According to Corollary 3, it is sufficient to prove that $T_2 = 0$. In the sequel we will investigate three cases:

Case 1: There exists $k_0 \geq 3$ such that $D_k \neq 0$ for $k \geq k_0$.

Considering Corollary 7 we can choose $\tilde{k} \geq k_0$ such that $T_k \neq 0$ for $k \geq \tilde{k}-1$. Let, for $k \geq \tilde{k}$, $\bar{D}_k = \frac{D_k}{T_{k-1}T_k}$ and $\bar{E}(k,n)$ be the equation (27) divided by $T_{k-1}T_kT_{k+1}$.

By making the operations

$$\bar{D}_{k-1}\bar{E}(k,n+2) - \bar{D}_k\bar{E}(k-1,n) - \bar{D}_k\bar{E}(k,n) + \bar{D}_{k+1}\bar{E}(k-1,n-2),$$

we can eliminate $\tilde{a}_{n-2k-3}$ and $\tilde{a}_{n-2k-1}$ and keeping only, for $k \geq \tilde{k}+1$, the following equation

$$a_{n+2} - a_{n-4} - \tilde{D}_k(a_n - a_{n-2}) = \sum_{l=1}^{k} \frac{W_{k,l}}{n-2k+2l} := Q_k^{(1)}(n), \qquad (28)$$

where $W_{k,l}$ is independent of n and

$$\tilde{D}_k = \frac{\bar{D}_k^2 + \bar{D}_k\bar{D}_{k-1} + \bar{D}_k\bar{D}_{k+1}}{\bar{D}_{k-1}\bar{D}_{k+1}}.$$

Similarly, after eliminating a_n and a_{n-2} by the operations

$$\bar{D}_{k-1}\bar{E}(k,n+2) - \bar{D}_{k+1}\bar{E}(k-1,n+2) - \bar{D}_{k-1}\bar{E}(k,n) + \bar{D}_k\bar{E}(k-1,n) \qquad (29)$$

and then shifting $n \to n+2k+1$ in (29) we obtain

$$\tilde{a}_{n+2} - \tilde{a}_{n-4} - \tilde{D}_k(\tilde{a}_n - \tilde{a}_{n-2}) = \sum_{l=1}^{k} \frac{\widetilde{W}_{k,l}}{n+2l+1} := \widetilde{Q}_k^{(1)}(n), \tag{30}$$

where $\widetilde{W}_{k,l}$ is independent of n.

Now, for $k \neq \kappa \geq \tilde{k}+1$, the equations (28) and (30) give, respectively,

$$(\tilde{D}_\kappa - \tilde{D}_k)(a_n - a_{n-2}) = Q_k^{(1)}(n) - Q_\kappa^{(1)}(n) \tag{31}$$

and

$$(\tilde{D}_\kappa - \tilde{D}_k)\left(\frac{n}{n+1}a_n - \frac{n-2}{n-1}a_{n-2}\right) = \widetilde{Q}_k^{(1)}(n) - \widetilde{Q}_\kappa^{(1)}(n). \tag{32}$$

If $\tilde{D}_k \neq \tilde{D}_\kappa$ for some $k \neq \kappa \geq \tilde{k}+1$, then by (31) and (32) we can eliminate a_{n-2} to get that a_n is a rational function of n. So, by Corollary 6, we have $T_2 = 0$.

If $\tilde{D}_k = D$ for $k \geq \tilde{k}+1$, then (28) and (30) become, respectively,

$$a_{n+2} - a_{n-4} - D(a_n - a_{n-2}) = Q_k^{(1)}(n) \tag{33}$$

and

$$\tilde{a}_{n+2} - \tilde{a}_{n-4} - D(\tilde{a}_n - \tilde{a}_{n-2}) = \widetilde{Q}_k^{(1)}(n). \tag{34}$$

The subtraction $Eq(33) - Eq(34)$ leads to

$$\frac{a_{n+2}}{n+3} - \frac{a_{n-4}}{n-3} - D\left(\frac{a_n}{n+1} - \frac{a_{n-2}}{n-1}\right) = Q_k^{(2)}(n). \tag{35}$$

Then the combinations $(Eq(33) - (n+3)Eq(35))/2$ and $(Eq(33) - (n-3)Eq(35))/2$ give, respectively,

$$\frac{3a_{n-4}}{n-3} - D\left(-\frac{a_n}{n+1} + \frac{2a_{n-2}}{n-1}\right) = Q_k^{(3)}(n) \tag{36}$$

and

$$\frac{3a_{n+2}}{n+3} - D\left(\frac{2a_n}{n+1} - \frac{a_{n-2}}{n-1}\right) = Q_k^{(4)}(n). \tag{37}$$

By shifting $n \to n+2$ in (36) we obtain

$$\frac{3a_{n-2}}{n-1} - D\left(-\frac{a_{n+2}}{n+3} + \frac{2a_n}{n+1}\right) = Q_k^{(3)}(n+2). \tag{38}$$

The elimination of a_{n+2} and a_{n-2} by the operations $DEq(37)-3Eq(38)$ and $3Eq(37)-DEq(38)$, respectively, yields

$$\frac{6D-2D^2}{n+1}a_n+\frac{D^2-9}{n-1}a_{n-2}=Q_k^{(5)}(n) \tag{39}$$

and

$$\frac{9-D^2}{n+3}a_{n+2}-\frac{6D-2D^2}{n+1}a_n=Q_k^{(6)}(n). \tag{40}$$

Finally, the shifting $n\to n-2$ in (40) leads to

$$\frac{9-D^2}{n+1}a_n-\frac{6D-2D^2}{n-1}a_{n-2}=Q_k^{(6)}(n-2) \tag{41}$$

and the operation $(6D-2D^2)Eq(39)+(D^2-9)Eq(41)$ gives

$$[(6D-2D^2)^2+(D^2-9)^2]a_n=Q_k^{(7)}(n). \tag{42}$$

According to manipulations made above, $Q_k^{(7)}(n)$ is a rational function of n. So, if $D\neq 3$, a_n is a rational function of n and then $T_2=0$.

Now, we explore the case $D=3$. We have from (36) and (37):

$$Q_k^{(3)}(n)=\frac{1}{2}\left(Q_k^{(1)}(n)-(n+3)\,Q_k^{(2)}(n)\right)=\frac{1}{2}\left((n+3)\widetilde{Q}_k^{(1)}(n)-(n+2)Q_k^{(1)}(n)\right) \tag{43}$$

and

$$Q_k^{(4)}(n)=\frac{1}{2}\left(Q_k^{(1)}(n)-(n-3)\,Q_k^{(2)}(n)\right)=\frac{1}{2}\left((n-3)\widetilde{Q}_k^{(1)}(n)-(n-4)Q_k^{(1)}(n)\right). \tag{44}$$

Remark that $Q_k^{(j)}(n)\,, 1\leq j\leq 4$, and $\widetilde{Q}_k^{(1)}(n)$ are independent of k. Observe also that, according to the left-hand sides of (36) and (37) for $D=3$, we have

$$Q_k^{(3)}(n+2)=Q_k^{(4)}(n)\,.$$

From (43) and (44) we get

$$(n+4)Q_k^{(1)}(n+2)-(n-4)Q_k^{(1)}(n)=(n+5)\,\widetilde{Q}_k^{(1)}(n+2)-(n-3)\,\widetilde{Q}_k^{(1)}(n)\,. \tag{45}$$

In (45) two rational functions are equal for natural numbers and are so for real numbers. By using the expressions of $Q_k^{(1)}(n)$ and $\widetilde{Q}_k^{(1)}(n)$ (see (28) and (30)) we find that

$$\begin{aligned}(n+4)Q_k^{(1)}(n+2)-(n-4)Q_k^{(1)}(n) &= \sum_{l=1}^{k}\frac{(n+4)W_{k,l}}{n+2-2k+2l}-\sum_{l=1}^{k}\frac{(n-4)W_{k,l}}{n-2k+2l}\\ &= \frac{2W_{k,k}}{n+2}-\frac{(2k-6)W_{k,1}}{n-2k+2} \qquad (46)\\ &\quad +\sum_{l=2}^{k}\frac{(2k-2l+4)W_{k,l-1}-(2k-2l-4)W_{k,l}}{n-2k+2l}.\end{aligned}$$

and

$$\begin{aligned}(n+5)\widetilde{Q}_k^{(1)}(n+2)-(n-3)\widetilde{Q}_k^{(1)}(n) &= \sum_{l=1}^{k}\frac{(n+5)\widetilde{W}_{k,l}}{n+2l+3}-\sum_{l=1}^{k}\frac{(n-3)\widetilde{W}_{k,l}}{n+2l+1} \qquad (47)\\ &= \frac{6\widetilde{W}_{k,1}}{n+3}-\frac{(2k-2)\widetilde{W}_{k,k}}{n+2k+3}+\sum_{l=2}^{k}\frac{(-2l+4)\widetilde{W}_{k,l-1}+(2l+4)\widetilde{W}_{k,l}}{n+2l+1}.\end{aligned}$$

Observe that the singularities of (46) are even numbers, whereas the singularities of (47) are odd ones. So, we should have

$$W_{k,k}=W_{k,1}=\widetilde{W}_{k,1}=\widetilde{W}_{k,k}=0,$$

$$(2k-2l+4)W_{k,l-1}-(2k-2l-4)W_{k,l}=0$$

and

$$(-2l+4)\widetilde{W}_{k,l-1}+(2l+4)\widetilde{W}_{k,l}=0$$

for $2\le l\le k$. Since $k\ge\tilde{k}+1\ge 4$ and by induction on l all the $W_{k,l}$ and $\widetilde{W}_{k,l}$ are null. Thus, (33) reads

$$a_{n+2}-a_{n-4}-3\left(a_n-a_{n-2}\right)=0. \qquad (48)$$

The solution of (48) has the form

$$a_n=\left(C_1+C_2n+C_3n^2\right)(-1)^n+C_4+C_5n+C_6n^2. \qquad (49)$$

Using (49) for n even, the left-hand side of (10) is a rational function of n with finite number of singularities. So, by the same arguments as in Corollary 6 we obtain $T_2=0$.

Case 2: There exists $k_0 \geq 3$ such that $D_k = 0$ for $k \geq k_0$.

Suppose that $D_k = T_k^2 - T_{k-1}T_{k+1} = 0$ for all $k \geq k_0$. First, notice that if there exists a $k_1 \geq k_0$ such that $T_{k_1} = 0$, then $T_{k_1-1}T_{k_1+1} = 0$. So, $T_{k_1-1} = 0$ or $T_{k_1+1} = 0$ and by Corollary 4, $T_2 = 0$. We have also $T_{k_0-1} \neq 0$, otherwise $T_{k_0} = 0$ and by Corollary 4, $T_2 = 0$.

Now, for $T_k \neq 0$ $(k \geq k_0 - 1)$, we have

$$\frac{T_{k+1}}{T_k} = \frac{T_k}{T_{k-1}} = \frac{T_{k_0}}{T_{k_0-1}}. \tag{50}$$

This means that

$$T_k = \left(\frac{T_{k_0}}{T_{k_0-1}}\right)^{k-k_0} T_{k_0} = ab^k \tag{51}$$

where $a = T_{k_0-1}^{k_0}/T_{k_0}^{k_0-1}$ and $b = T_{k_0}/T_{k_0-1}$.

The substitution of T_k by ab^k in (10) for $k \geq k_0$ leads to the equation

$$\frac{2}{T_1}b\left(a_n - \frac{n-2k-1}{n-2k}a_{n-2k-1}\right) + \frac{n+2}{n}c_n - \frac{n-2k+1}{n-2k+2}c_{n-2k+1} = \frac{b^{-k}}{a}\sum_{l=1}^{k}\frac{T_lT_{k-l+1}}{n-2k+2l} = Q_k(n). \tag{52}$$

Let denote (52) by $\widetilde{E}(k,n)$ and make the subtraction $\widetilde{E}(k+1,n+2) - \widetilde{E}(k,n)$ to get

$$\frac{2}{T_1}b\left(a_{n+2} - a_n\right) + \frac{n+4}{n+2}c_{n+2} - \frac{n+2}{n}c_n = Q_{k+1}(n+2) - Q_k(n). \tag{53}$$

On the right hand side of (53) we have, for $k \geq k_0$, the expression

$$\begin{aligned}
\widetilde{Q}_k(n) &:= Q_{k+1}(n+2) - Q_k(n) = \frac{b^{-k-1}}{a}\sum_{l=1}^{k+1}\frac{T_lT_{k-l+2}}{n-2k+2l} - \frac{b^{-k}}{a}\sum_{l=1}^{k}\frac{T_lT_{k-l+1}}{n-2k+2l} \\
&= \frac{b^{-k-1}}{a}\frac{T_{k+1}T_1}{n+2} + \frac{b^{-k-1}}{a}\frac{T_k\left(T_2 - bT_1\right)}{n} + \frac{b^{-k-1}}{a}\sum_{l=1}^{k-1}\frac{T_l\left(T_{k-l+2} - bT_{k-l+1}\right)}{n-2k+2l} \\
&= \frac{T_1}{n+2} + \frac{T_2 - bT_1}{bn} + \frac{b^{-k-1}}{a}\sum_{l=1}^{k-1}\frac{T_l\left(T_{k-l+2} - bT_{k-l+1}\right)}{n-2k+2l}
\end{aligned} \tag{54}$$

from which we deduce

$$\begin{aligned}
\widetilde{Q}_{k+1}(n) &= \frac{T_1}{n+2} + \frac{T_2 - bT_1}{bn} + \frac{b^{-k-2}}{a}\sum_{l=1}^{k}\frac{T_l\left(T_{k-l+3} - bT_{k-l+2}\right)}{n-2k-2+2l} \\
&= \frac{T_1}{n+2} + \frac{T_2 - bT_1}{bn} + \frac{b^{-k-2}}{a}\sum_{l=1}^{k-1}\frac{T_{l+1}\left(T_{k-l+2} - bT_{k-l+1}\right)}{n-2k+2l}.
\end{aligned} \tag{55}$$

Now since the left hand side of equation (53) is independent of k, it follows

$$\widetilde{Q}_{k+1}(n) - \widetilde{Q}_k(n) = \frac{b^{-k-2}}{a} \sum_{l=1}^{k-1} \frac{(T_{l+1} - bT_l)(T_{k-l+2} - bT_{k-l+1})}{n - 2k + 2l} = 0. \quad (56)$$

As a result, for $1 \leq l \leq k-1$ and $k \geq k_0$, we have

$$(T_{l+1} - bT_l)(T_{k-l+2} - bT_{k-l+1}) = 0. \quad (57)$$

Let take $k = 2(k_0 - 2) - 1$ and $l = k_0 - 2$ to get $(T_{k_0-1} - bT_{k_0-2})^2 = 0$ and then $T_{k_0-1} = bT_{k_0-2}$, (or equivalently $D_{k_0-1} = 0$). Thus, the equations (50) and (51) are valid for $k = k_0 - 1$ and by induction we arrive at $T_4 = bT_3$, (or equivalently $D_4 = 0$). For $k = 4$, the right-hand side of (27) is null. So, $V_{4,2} = 0$ and using $T_5 = T_4^2/T_3$ (from $D_4 = 0$) we get $D_3 = 0$.

On the other side suppose that $T_2 \neq 0$, then we can write

$$T_k = \left(\frac{T_3}{T_2}\right)^{k-2} T_2 = ab^k, \text{ for } k \geq 2,$$

where $b = T_3/T_2$ and $a = T_2^3/T_3^2$. Therefore, the equation (52) reads

$$\frac{2}{T_1} b\left(a_n - \frac{n-2k-1}{n-2k} a_{n-2k-1}\right) + \frac{n+2}{n} c_n - \frac{n-2k+1}{n-2k+2} c_{n-2k+1} =$$
$$= \frac{T_1}{n-2k+2} + \frac{T_1}{n} + \sum_{l=2}^{k-1} \frac{ab}{n-2k+2l}, \quad \text{for } k \geq 2 \text{ and } n \geq 2k+1. \quad (58)$$

When $n = 2k+1$ and $n = 2k+2$, the equation (58) gives

$$\frac{2}{T_1} ba_{2k+1} + \frac{2k+3}{2k+1} c_{2k+1} = \frac{2}{3} c_2 + \frac{T_1}{3} + \frac{T_1}{2k+1} + \sum_{l=2}^{k-1} \frac{ab}{2l+1} \quad (59)$$

and

$$\frac{2}{T_1} ba_{2k+2} + \frac{k+2}{k+1} c_{2k+2} = \frac{1}{T_1} a_1 b + \frac{3}{4} c_3 + \frac{T_1}{4} + \frac{T_1}{2k+2} + \sum_{l=2}^{k-1} \frac{ab}{2l+2} \quad (60)$$

respectively. Let take $n = 2N+1$ in (58) and use (59) to obtain the expression

$$\frac{2}{3} c_2 + \frac{T_1}{3} + \frac{T_1}{2N+1} + \sum_{l=2}^{N-1} \frac{ab}{1+2l} - \frac{2(N-k)b}{2(N-k)+1} \frac{2}{T_1} a_{2(N-k)} - \frac{2(N-k)+2}{2(N-k)+3} c_{2(N-k)+2}$$
$$= \frac{T_1}{2(N-k)+3} + \frac{T_1}{2N+1} + \sum_{l=2}^{k-1} \frac{ab}{2(N-k)+2l+1}.$$

In this last equality let put $N-k$ instead of k to get

$$
\begin{aligned}
-\frac{2bk}{2k+1}\frac{2}{T_1}a_{2k}-\frac{2k+2}{2k+3}c_{2k+2} &= -\frac{2}{3}c_2-\frac{T_1}{3}-\sum_{l=2}^{N-1}\frac{ab}{1+2l}+\frac{T_1}{2k+3}+\sum_{l=2}^{N-k-1}\frac{ab}{2k+2l+1}\\
&= -\frac{2}{3}c_2-\frac{T_1}{3}+\frac{T_1}{2k+3}-\sum_{l=1}^{k}\frac{ab}{2l+3}. \qquad (61)
\end{aligned}
$$

After defining $A_1=\frac{a_1}{T_1}+\frac{3}{4}\frac{c_3}{b}+\frac{T_1}{4b}$, $A_2=-\frac{2}{3}\frac{c_2}{b}-\frac{T_1}{3b}$ and $A_3=\frac{T_1}{b}$ and making the operation

$$
\frac{1}{k+2}\left(\frac{2k+2}{2k+3}\,Eq(60)+\frac{k+2}{k+1}\,Eq(61)\right)
$$

we have

$$
\begin{aligned}
&\frac{2(k+1)}{(k+2)(2k+3)}\frac{2}{T_1}\,a_{2(k+1)}-\frac{2k}{(k+1)(2k+1)}\frac{2}{T_1}\,a_{2k}=\\
&= \frac{A_1(2k+2)+A_3}{(k+2)(2k+3)}+\frac{A_2}{k+1}+\frac{A_3}{(k+1)(2k+3)}+\frac{2k+2}{(k+2)(2k+3)}\sum_{l=2}^{k-1}\frac{a}{2+2l}-\frac{1}{k+1}\sum_{l=2}^{k+1}\frac{a}{2l+1}\\
&= -\frac{2A_1}{2\,k+3}+\frac{2\,A_2-A_3}{k+2}+\frac{A_2+A_3}{k+1}+\left(-\frac{1}{2\,k+3}+\frac{1}{k+2}\right)\sum_{l=2}^{k-1}\frac{a}{l+1}-\frac{1}{k+1}\sum_{l=2}^{k+1}\frac{a}{2l+1}\\
&= \frac{B_1}{k+2}+\frac{B_2}{k+\frac{3}{2}}+\frac{B_3}{k+1}+a\left(\frac{1}{k+2}-\frac{1}{2}\frac{1}{k+\frac{3}{2}}\right)\Psi\left(k+1\right)-\frac{1}{2}\frac{a}{k+1}\Psi\left(k+\frac{5}{2}\right), \qquad (62)
\end{aligned}
$$

where short notations $B_1=(-3/2+\gamma)\,a+2A_1-A_3$, $B_2=(3/4-\gamma/2)\,a-A_1$, $B_3=(-\gamma/2-\ln(2)+4/3)\,a+A_2+A_3$, ($\gamma$ is Euler's constant) are introduced as well as $\Psi(x)$ which stands for the Digamma function.

Taking

$$
U_k=\frac{2k}{(k+1)(2k+1)}\frac{2}{T_1}\,a_{2k}
$$

and

$$
G(k{+}1)=\frac{B_1}{k+2}+\frac{B_2}{k+\frac{3}{2}}+\frac{B_3}{k+1}+a\left(\frac{1}{k+2}-\frac{1}{2}\frac{1}{k+\frac{3}{2}}\right)\Psi\left(k+1\right)-\frac{1}{2}\frac{a}{k+1}\Psi\left(k+\frac{5}{2}\right), \qquad (63)
$$

then (62) can be written in compact form as

$$
U_{k+1}-U_k=G(k+1).
$$

The later recurrence is easily solved to give

$$U_k = U_3 + \sum_{j=4}^{k} G(j).$$

By using the formula $\Psi(j+1) = \Psi(j) + 1/j$ and the relations [12, Theorems 3.1 and 3.2]

$$\sum_{l=0}^{k} \frac{\Psi(l+\alpha)}{l+\beta} + \sum_{l=0}^{k} \frac{\Psi(l+\beta+1)}{l+\alpha} = \Psi(k+\alpha+1)\Psi(k+\beta+1) - \Psi(\alpha)\Psi(\beta), \quad (64)$$

$$\sum_{j=0}^{k} \frac{\Psi(j+\beta)}{j+\beta} = \frac{1}{2}\left[\Psi'(k+\beta+1) - \Psi'(\beta) + \Psi(k+\beta+1)^2 - \Psi(\beta)^2\right], \quad (65)$$

we obtain

$$\begin{aligned} U_k = \frac{2k}{(k+1)(2k+1)} \frac{2}{T_1}\, a_{2k} \quad &= \quad \frac{a}{2}\left(\Psi\left(k+2\right)\right)^2 + B_1 \Psi\left(k+2\right) + \frac{a}{2}\Psi'\left(k+2\right) + B_2 \Psi\left(k+\frac{3}{2}\right) + \\ &\left(-\frac{a}{2}\Psi\left(k+\frac{3}{2}\right) + B_3\right)\Psi\left(k+1\right) + \frac{a}{k+1} + \delta_2. \end{aligned} \quad (66)$$

From (66) we deduce the asymptotic behaviour of a_{2k} as $k \to \infty$:

$$\frac{2}{T_1}\, a_{2k} = \left(\delta_1\left(k + \frac{3}{2} + \frac{1}{2k}\right) + \frac{3a}{4} + \frac{5a}{16k} + \frac{a}{32k^2} - \frac{3a}{128k^3} + \ldots\right)\ln(k) + \delta_2 k + \delta_3 + \frac{\delta_4}{k} + \frac{\delta_5}{k^2} + \frac{\delta_6}{k^3} + \cdots \quad (67)$$

where coefficients δ_i are defined by (higher terms are omitted)

$$\begin{aligned} \delta_1 &= B_1 + B_2 + B_3, \\ \delta_2 &= \left(\gamma - \frac{25}{12}\right) B_1 + \left(\gamma + 2\ln(2) - \frac{352}{105}\right) B_2 + \left(\gamma - \frac{11}{6}\right) B_3 \\ &\quad + \left(\left(\ln(2) - \frac{107}{210}\right)\gamma + \frac{3439}{2520} - \frac{11}{6}\,\ln\left(2\right) - \frac{\pi^2}{12}\right) a + \frac{3}{7\,T_1}\, a_6, \\ \delta_3 &= \frac{3}{2}\, B_1 + B_2 + \frac{1}{2}\, B_3 + \frac{3}{2}\, a + \frac{3}{2}\, \delta_2, \\ \delta_4 &= \frac{7}{6}\, B_1 + \frac{25}{24}\, B_2 + \frac{2}{3}\, B_3 + \frac{11}{8}\, a + \frac{1}{2}\, \delta_2, \\ \delta_5 &= \frac{1}{8}\, B_1 + \frac{1}{16}\, B_2 + \frac{1}{8}\, B_3 + \frac{5}{96}\, a, \\ \delta_6 &= -\frac{1}{30}\, B_1 + \frac{13}{960}\, B_2 - \frac{1}{30}\, B_3 - \frac{1}{96}\, a, \\ &\vdots \end{aligned}$$

At this step we should remark that $\lim_{k\to\infty} a_{2k} = \infty$ for all δ_i, $i = 1, 2, 3, ...$, since $a \neq 0$.

Recall that $c_n = T_1 \left(n a_n / a_{n-1} - (n-1)\right)/2$, then the equation (59) can be written as

$$a_{2k+1}\left(b\frac{2}{T_1} + \frac{T_1}{2}\frac{2k+3}{a_{2k}}\right) = \phi(k), \tag{68}$$

where

$$\phi\left(k\right) = \frac{T_1}{2}\frac{2k(2k+3)}{2k+1} - bA_2 + \frac{bA_3}{2k+1} + \sum_{l=2}^{k-1}\frac{ab}{2l+1}.$$

- If we suppose $\lim_{k\to\infty} \frac{a_{2k}}{2k} = \infty$, then from (68) we deduce on one side

$$\lim_{k\to\infty}\frac{a_{2k+1}}{2k+1} = \lim_{k\to\infty}\frac{\frac{\phi(k)}{2k+1}}{\frac{2b}{T_1} + \frac{T_1}{2}\frac{2k+3}{a_{2k}}} = \frac{T_1^2}{4b}. \tag{69}$$

On the other side, for $n = 2k+3$, (9) reads

$$\frac{4T_2}{T_1^3}\left(1 - \frac{2k}{2k+1}\frac{a_{2k}}{a_{2k+3}}\right) = \frac{2k+4}{a_{2k+3}} - 2\frac{2k+3}{a_{2k+2}} + \frac{2k+2}{a_{2k+1}}. \tag{70}$$

Under the assumption $T_2 \neq 0$, (70) admits the limit $\infty = 8b/T_1^2$, as $k \to \infty$, which exhibit a contradiction.

- Now if $\lim_{k\to\infty} \frac{a_{2k}}{2k} = \eta_1 \neq 0$, then from (68) we have

$$\lim_{k\to\infty}\frac{a_{2k+1}}{2k+1} = \lim_{k\to\infty}\frac{\frac{\phi(k)}{2k+1}}{\frac{2b}{T_1} + \frac{T_1}{2}\frac{2k+3}{a_{2k}}} = \frac{\frac{T_1}{2}}{\frac{2b}{T_1} + \frac{T_1}{2\eta_1}} := \eta_2. \tag{71}$$

Equation (9) becomes, for $n = 2k+2$,

$$\frac{4T_2}{T_1^3}\left(1 - \frac{2k-1}{2k}\frac{a_{2k-1}}{a_{2k+2}}\right) = \frac{2k+3}{a_{2k+2}} - 2\frac{2k+2}{a_{2k+1}} + \frac{2k+1}{a_{2k}}. \tag{72}$$

And if we assume that $T_2 \neq 0$ and $\eta_2 = \infty$, then the limit process $k \to \infty$ in (72) left us with the contradiction $\infty = 2/\eta_1$. But if $\eta_2 \neq \infty$, then by taking the limit in (70) and (72) we obtain, respectively,

$$\frac{4T_2}{T_1^3}\left(1 - \frac{\eta_1}{\eta_2}\right) = \frac{2}{\eta_2} - \frac{2}{\eta_1}$$

and

$$\frac{4T_2}{T_1^3}\left(1 - \frac{\eta_2}{\eta_1}\right) = \frac{2}{\eta_1} - \frac{2}{\eta_2}.$$

Adding the two later we get $2 - \eta_1/\eta_2 - \eta_2/\eta_1 = 0$ and therefore $\eta_1 = \eta_2$. According to (71) $\eta_1 = 0$ which is in contradiction with the initial hypothesis $\eta_1 \neq 0$.

- Finally if $\lim_{k\to\infty} \frac{a_{2k}}{2k} = 0$, then from (71) and (68) we have respectively $\lim_{k\to\infty} \frac{a_{2k+1}}{2k+1} = 0$ and

$$\lim_{k\to\infty} \frac{a_{2k+1}}{a_{2k}} = \lim_{k\to\infty} \frac{\frac{\phi(k)}{2k}}{\frac{2b}{T_1}\frac{a_{2k}}{2k} + \frac{T_1}{2}\frac{2k+3}{2k}} = 1.$$

Similarly, from (60) we obtain $\lim_{k\to\infty} \frac{a_{2k+2}}{a_{2k+1}} = 1$. Now, since $\lim_{n\to\infty} \frac{a_n}{n} = 0$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, the left-hand side of (72) tends to 0 as $k \to \infty$. In the other hand, according to (67), $\lim_{k\to\infty} \frac{a_{2k}}{2k} = 0$ implies that $\delta_1 = \delta_2 = 0$ and

$$\begin{aligned} a_{2k} &= \frac{T_1}{2}\frac{(k+1)(2k+1)}{2k}\left(\frac{a}{2}\left(\Psi\left(k+2\right)\right)^2 + B_1\Psi\left(k+2\right) + \frac{a}{2}\Psi'\left(k+2\right) + B_2\Psi\left(k+\frac{3}{2}\right) + \right. \\ &\quad \left.\left(-\frac{a}{2}\Psi\left(k+\frac{3}{2}\right) - B_1 - B_2\right)\Psi\left(k+1\right) + \frac{a}{k+1}\right), \\ &= \frac{T_1}{2}\left(\left(\frac{3a}{4} + \frac{5a}{16k} + \frac{a}{32k^2} - \frac{3a}{128k^3} + \ldots\right)\ln(k) + \delta_3 + \frac{\delta_4}{k} + \frac{\delta_5}{k^2} + \frac{\delta_6}{k^3} + \cdots\right). \end{aligned} \tag{73}$$

From (68) we have

$$a_{2k+1} = \frac{\phi(k)}{b\frac{2}{T_1} + \frac{T_1}{2}\frac{2k+3}{a_{2k}}}, \tag{74}$$

which gives an explicit formula for a_{2k+1}. Using (74), the right-hand side of (72) reads

$$-8\,\frac{b}{T_1}\frac{k+1}{\phi\left(k\right)} + \frac{2\,k+3}{a_{2k+2}} + \left(-2\,T_1\frac{(2k+3)\,\left(k+1\right)}{\phi\left(k\right)} + 2\,k+1\right)\frac{1}{a_{2k}}. \tag{75}$$

By virtue of (73), the limit of the both sides of (72), as $k \to \infty$, gives $-\frac{8b}{3T_1^2} = 0$. So, $b = 0$ and $T_k = 0$ for $k \geq 3$. Therefore, by corollary 4 we have $T_2 = 0$ which contradicts $T_2 \neq 0$.

Case 3: For every $k_0 \geq 3$, there exists $k \geq k_0$ such that $D_k = 0$ or $D_k \neq 0$.

To exclude *Case 1* and *Case 2*, there exists a mixed case with infinitely many k and κ such that: $D_k = 0$ and $D_\kappa \neq 0$. Now, it suffices to take k_1 and k_2, $k_1 \neq k_2$, with $D_{k_1} = 0$, $D_{k_1+1} \neq 0$, $D_{k_2} = 0$ and $D_{k_2+1} \neq 0$ to get two equations similar to (20) and (21). Consequently, a reasoning analogous to that of Corollary 7 completes the proof.

4. Concluding remarks

We have shown that the only polynomial sets (besides the monomial set) $\{P_n\}$ generated by $F(xt - R(t)) = \sum_{n\geq 0} \alpha_n P_n(x) t^n$ and satisfying the three-term recursion $xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \omega_n P_{n-1}(x)$, are the rescaled ultraspherical, Hermite and Chebychev polynomials of the first kind. In [10], the authors generalized the results obtained in [2] and [5] in the context of d-orthogonality by considering the polynomials (which fulfils a $(d+1)$-order difference equation) generated by $F((d+1)xt - t^{d+1})$, where d is a positive integer. Recently in [13], the author characterized the Shefer d-orthogonal polynomials. These polynomials have the generating function $A(t)\exp(xH(t))$ which has the alternative form $\exp(xH(t) + \ln(A(t))) = F(xU(t) - R(t))$. So, a natural extension is to look at polynomial sets generated by $F(xU(t) - R(t))$ and satisfying the $(d+1)$-order recursion

$$xP_n(x) = P_{n+1}(x) + \sum_{l=0}^{d} \gamma_n^l P_{n-l}(x), \tag{76}$$

where $\{\gamma_n^l\}$, $0 \leq l \leq d$, are complex sequences.
Currently, we are attempting to generalize the results given here by investigating polynomial sets satisfying the recursion (76) and generated by $F(xt - R(t))$. This also provides generalizations of the results given in [10] and [13].

Acknowledgements: One of us (M. B. Z.) would like to thank Prof. Dominique Manchon for precious help and useful discussions and for his high hospitality at "Laboratoire de Mathématiques, CNRS-UMR 6620 " (Clermont-Ferrand).

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