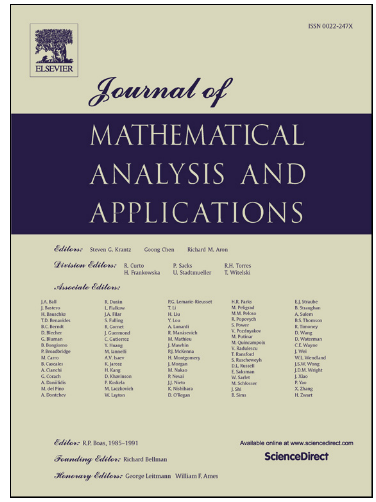


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PII: S0022-247X(17)30895-8
DOI: https://doi.org/10.1016/j.jmaa.2017.10.006
Reference: YJMAA 21724

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 23 January 2017

# Upper and lower conditional probabilities induced by a multivalued mapping

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**Abstract**

Given a (finitely additive) full conditional probability space $(X, \mathcal{F} \times \mathcal{F}^0, \mu)$ and a conditional measurable space $(Y, \mathcal{G} \times \mathcal{G}^0)$, a multivalued mapping $\Gamma$ from $X$ to $Y$ induces a class of full conditional probabilities on $(Y, \mathcal{G} \times \mathcal{G}^0)$. A closed form expression for the lower and upper envelopes $\mu_*$ and $\mu^*$ of such class is provided: the envelopes can be expressed through a generalized Bayesian conditioning rule, relying on two linearly ordered classes of (possibly unbounded) inner and outer measures. For every $B \in \mathcal{G}^0$, $\mu_*(\cdot|B)$ is a normalized totally monotone capacity which is continuous from above if $(X, \mathcal{F} \times \mathcal{F}^0, \mu)$ is a countably additive full conditional probability space and $\mathcal{F}$ is a $\sigma$-algebra. Moreover, the full conditional prevision functional $\mathbf{M}$ induced by $\mu$ on the set of $\mathcal{F}$-continuous conditional gambles is shown to give rise through $\Gamma$ to the lower and upper full conditional prevision functionals $\mathbf{M}_*$ and $\mathbf{M}^*$ on the set of $\mathcal{G}$-continuous conditional gambles. For every $B \in \mathcal{G}^0$, $\mathbf{M}_*(\cdot|B)$ is a totally monotone functional having a Choquet integral expression involving $\mu_*$. Finally, by considering another conditional measurable space $(Z, \mathcal{H} \times \mathcal{H}^0)$ and a multivalued mapping from $Y$ to $Z$, it is shown that the conditional measures $\mu_{**}$, $\mu^{**}$ and functionals $\mathbf{M}_{**}$, $\mathbf{M}^{**}$ induced by $\mu_*$ preserve the same properties of $\mu_*, \mu^*$ and $\mathbf{M}_*$, $\mathbf{M}^*$.

*Keywords:* Multivalued mapping, totally monotone capacity, probability envelopes, finite additivity, lower conditional prevision
*2010 MSC:* 60A05, 62C10, 60A10

## 1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a finitely additive probability space and $(Y, \mathcal{G})$ a measurable space, where both $\mathcal{F}$ and $\mathcal{G}$ are algebras (not necessarily $\sigma$-algebras) of subsets of $X$ and $Y$, respectively. Throughout the paper we consider arbitrary non-empty sets without requiring any topological structure.

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A *multivalued mapping* $\Gamma$ from $X$ to $Y$ is a function $\Gamma : X \to \wp(Y)$, which expresses a set of (logical) relations between $X$ and $Y$ [18]. Every multivalued mapping $\Gamma$ has associated a *lower* and an *upper inverse* [11, 28] defined, for every $B \in \mathcal{G}$, as

$$\begin{aligned} B_* &:= \Gamma_*(B) = \{x \in X \,:\, \emptyset \neq \Gamma(x) \subseteq B\}, \\ B^* &:= \Gamma^*(B) = \{x \in X \,:\, \Gamma(x) \cap B \neq \emptyset\}. \end{aligned}$$

If $\mathcal{F}$ and $\mathcal{G}$ are $\sigma$-algebras, $\Gamma$ is usually referred to as *random set* (see, e.g., [28]).

In the literature there are different notions of measurability for a multivalued mapping [18, 28], in particular, $\Gamma$ is said $(\mathcal{F}, \mathcal{G})$*-strongly measurable* if, for every $B \in \mathcal{G}$, both $B_*$ and $B^*$ belong to $\mathcal{F}$. Weaker notions of measurability essentially rely on topological assumptions on the space $Y$ and on the facts that $\mathcal{F}$ and $\mathcal{G}$ are (suitable) $\sigma$-algebras and $\mu$ is countably additive (see, e.g., [25]).

If $\mu(Y^*) > 0$, a $(\mathcal{F}, \mathcal{G})$-strongly measurable $\Gamma$ induces a *lower* and an *upper probability* on $\mathcal{G}$ (see, e.g., [11]), setting, for every $B \in \mathcal{G}$,

$$\mu_*(B) = \frac{\mu(B_*)}{\mu(Y^*)} \quad \text{and} \quad \mu^*(B) = \frac{\mu(B^*)}{\mu(Y^*)}.$$

In this construction, no restriction besides $(\mathcal{F}, \mathcal{G})$-strong measurability has been imposed on the multivalued mapping $\Gamma$. Note that the renormalization by $\mu(Y^*)$ is necessary when $\{x \in X \,:\, \Gamma(x) = \emptyset\} \neq \emptyset$, in order to obtain normalized capacities $\mu_*$ and $\mu^*$ on $(Y, \mathcal{G})$, i.e., $\mu_*(\emptyset) = \mu^*(\emptyset) = 0$ and $\mu_*(Y) = \mu^*(Y) = 1$.

In the rest of the paper we assume that $\Gamma$ satisfies the following natural requirements:

**(A1)** $\{x \in X \,:\, \Gamma(x) = \emptyset\} = \emptyset$;

**(A2)** $\{y\}^* \neq \emptyset$ for every $y \in Y$.

Notice that $(\mathcal{F}, \mathcal{G})$-strong measurability can be a stringent assumption that cannot be relaxed working in a countably additive setting without topological assumptions since, as is well-known, a countably additive $\mu$ on $\mathcal{F}$ cannot be generally extended to $\wp(X)$ by preserving countable additivity. Then, it is necessary to work with finitely additive probabilities. Finite additivity allows to overcome any measurability and topological restriction: a consequence is that the induced capacities could not satisfy some regularity conditions (such as continuity).

In a finitely additive setting, $\mu$ can be extended to the whole $\wp(X)$ giving rise to the convex compact set of finitely additive probability measures $\mathcal{M}$, whose envelopes $\underline{\mu} = \min \mathcal{M}$ and $\overline{\mu} = \max \mathcal{M}$ coincide with the inner and outer measures generated by $\mu$ on $\wp(X)$ and are defined, for every $A \in \mathcal{F}$, as

$$\begin{aligned} \underline{\mu}(A) &= \sup\{\mu(K) \,:\, K \subseteq A, K \in \mathcal{F}\}, \\ \overline{\mu}(A) &= \inf\{\mu(K) \,:\, A \subseteq K, K \in \mathcal{F}\}. \end{aligned}$$

For a $\Gamma$ satisfying conditions **(A1)** and **(A2)**, the lower and upper probabilities on $\mathcal{G}$ are defined, for every $B \in \mathcal{G}$, as

$$\mu_*(B) = \underline{\mu}(B_*) \quad \text{and} \quad \mu^*(B) = \overline{\mu}(B^*).$$

In game theory and dynamic programming (see, e.g., [15, 22, 35]) it is natural to consider finitely additive conditional probability assessments, moreover, the possibility to condition to events of "zero probability" reveals to be crucial as it deeply impacts on the analysis of a game [21]. Therefore, for such class of problems (see, e.g., [27]), a suitable framework to model uncertainty is that of *full conditional probability spaces* [14, 30].

Leaving apart measurability restrictions, the multivalued mapping $\Gamma$ can be considered as an "imprecise" random quantity used to transport probabilistic information from one space to another [23]. Hence, the aim of the paper is to consider the above transportation problem starting from a full conditional probability space $(X, \mathcal{F} \times \mathcal{F}^0, \mu)$ and provide a characterization of the lower and upper conditional probabilities induced on a conditional measurable space $(Y, \mathcal{G} \times \mathcal{G}^0)$ by the multivalued mapping $\Gamma$. This issue seems to be particularly interesting for partially identified models that are objects gathering more and more interest in economics and statistics [26].

We prove (see Theorem 2) that the lower and upper conditional probabilities on $\mathcal{G} \times \mathcal{G}^0$ induced by $(X, \mathcal{F} \times \mathcal{F}^0, \mu)$ and $\Gamma$ are defined, for every $A|B \in \mathcal{G} \times \mathcal{G}^0$, as $\mu_*(A|B) = 1$ when $A \cap B = B$, and otherwise

$$\mu_*(A|B) = \begin{cases} \frac{\underline{\nu}_\alpha((A\cap B)_*)}{\underline{\nu}_\alpha((A\cap B)_*)+\overline{\nu}_\alpha((A^c\cap B)^*)} & \text{if there is } \alpha \in I \text{ such that} \\ & \underline{\nu}_\alpha((A \cap B)_*) + \overline{\nu}_\alpha((A^c \cap B)^*) \in (0, +\infty), \\ 0 & \text{otherwise,} \end{cases}$$

where $\{\underline{\nu}_\alpha : \alpha \in I\}$ and $\{\overline{\nu}_\alpha : \alpha \in I\}$ are two suitable linearly ordered classes of (possibly unbounded) inner and outer measures on $\wp(X)$, and the complements are taken in $Y$. The dual conditional measure is defined, for every $A|B \in \mathcal{G} \times \mathcal{G}^0$, as

$$\mu^*(A|B) = 1 - \mu_*(A^c|B).$$

Furthermore, it is proved that, for every $B \in \mathcal{G}^0$, $\mu_*(\cdot|B)$ and $\mu^*(\cdot|B)$ are normalized totally monotone and totally alternating capacities on $\mathcal{G}$ which are continuous, respectively, from above and from below if $\mu(\cdot|\cdot)$ is a countably additive full conditional probability and $\mathcal{F}$ is a $\sigma$-algebra.

We show that the full conditional prevision functional $\mathbf{M}$ induced by $\mu$ on the set of $\mathcal{F}$-continuous conditional gambles gives rise through $\Gamma$ to the lower and upper full conditional prevision functionals $\mathbf{M}_*$ and $\mathbf{M}^*$ on the set of $\mathcal{G}$-continuous conditional gambles which are, respectively, totally monotone and totally alternating and have a Choquet integral expression.

Finally, starting from $(Y, \mathcal{G} \times \mathcal{G}^0, \mu_*)$, we consider its transportation to another conditional measurable space $(Z, \mathcal{H} \times \mathcal{H}^0)$ through a multivalued mapping $\Gamma'$ from $Y$ to $Z$. Also in this case we obtain conditional measures $\mu_{**}$, $\mu^{**}$ and functionals $\mathbf{M}_{**}$, $\mathbf{M}^{**}$ preserving the same quoted properties of $\mu_*$, $\mu^*$ and $\mathbf{M}_*$, $\mathbf{M}^*$. Moreover, we show that the conditional measures $\mu_{**}$, $\mu^{**}$ and functionals $\mathbf{M}_{**}$, $\mathbf{M}^{**}$ can be directly obtained transporting $(X, \mathcal{F} \times \mathcal{F}^0, \mu)$ to $(Z, \mathcal{H} \times \mathcal{H}^0)$ through the composition $\Gamma' \circ \Gamma$.

The above results concerning $\mu_*$ and $\mu^*$ (as well as $\mu_{**}$ and $\mu^{**}$) are obtained translating the transportation of $(X, \mathcal{F}\times\mathcal{F}^0, \mu)$ to $(Y, \mathcal{G}\times\mathcal{G}^0)$ through $\Gamma$ in terms of extensions of a suitable full conditional probability space $(\Omega, \mathcal{A} \times \mathcal{A}^0, P)$. For that, the problem of extending an arbitrary full conditional probability space reveals to be fundamental, so a complete characterization is given.

The paper is structured as follows. In Section 2, we first focus on the unconditional case starting from $(X, \mathcal{F}, \mu)$ and $\Gamma$, showing that the lower and upper probabilities $\mu_*(\cdot)$ and $\mu^*(\cdot)$ on $(Y, \mathcal{G})$ can be characterized in terms of extensions of a suitable finitely additive probability space $(\Omega, \mathcal{A}, P)$. Section 3 recalls some preliminaries on full conditional probability spaces. Section 4 provides a closed form expression for the envelopes of the class of full conditional probabilities extending an arbitrary full conditional probability space, together with an investigation of their properties. In Section 5, starting from $(X, \mathcal{F} \times \mathcal{F}, \mu)$ and $\Gamma$, it is shown that the lower and upper conditional probabilities $\mu_*(\cdot|\cdot)$ and $\mu^*(\cdot|\cdot)$ on $(Y, \mathcal{G} \times \mathcal{G}^0)$ can be characterized in terms of extensions of a suitable full conditional probability space $(\Omega, \mathcal{A} \times \mathcal{A}^0, P)$. Finally, in Section 6 we consider the transportation of $(Y, \mathcal{G} \times \mathcal{G}, \mu_*)$ to $(Z, \mathcal{H} \times \mathcal{H}^0)$ through a multivalued mapping $\Gamma'$.

## 2. Unconditional case: extensions of a finitely additive probability

The lower and upper probabilities on $(Y, \mathcal{G})$ induced by $(X, \mathcal{F}, \mu)$ and $\Gamma$ can be interpreted in terms of extensions of a suitable finitely additive probability space.

Define $\Omega = (X \times Y) \setminus \bigcup_{x\in X}(\{x\} \times (Y \setminus \Gamma(x)))$ and consider the algebras of its subsets

$$\mathcal{A} = \{(A \times Y) \cap \Omega \,:\, A \in \mathcal{F}\} \quad \text{and} \quad \mathcal{A}' = \{(X \times B) \cap \Omega \,:\, B \in \mathcal{G}\},$$

that under conditions **(A1)** and **(A2)** are isomorphic to $\mathcal{F}$ and $\mathcal{G}$, respectively.

The finitely additive probability $\mu$ induces a finitely additive probability space $(\Omega, \mathcal{A}, P)$ such that, for every $A \in \mathcal{F}$, $P((A \times Y) \cap \Omega) = \mu(A)$.

In turn, the probability $P$ can be extended, generally not in a unique way, to the whole $\wp(\Omega)$: the set $\mathcal{P}$ of finitely additive probabilities extending $P$ is a convex compact subset of $[0,1]^{\wp(\Omega)}$ endowed with the product topology, whose envelopes $\underline{P} = \min \mathcal{P}$ and $\overline{P} = \max \mathcal{P}$ coincide with the inner and outer measures generated by $P$ on $\wp(\Omega)$. Such functions satisfy the *duality* relation, for every $F \in \wp(\Omega)$, $\overline{P}(F) = 1 - \underline{P}(F^c)$. For every $B \in \mathcal{G}$, it follows

$$\mu_*(B) = \underline{\mu}(B_*) = \underline{P}((X \times B) \cap \Omega) \quad \text{and} \quad \mu^*(B) = \overline{\mu}(B^*) = \overline{P}((X \times B) \cap \Omega),$$

i.e., $\mu_*$ and $\mu^*$ on $(Y, \mathcal{G})$ coincide, respectively, with the restrictions of $\underline{P}$ and $\overline{P}$ on $\mathcal{A}'$.

The envelopes $\underline{P}$ and $\overline{P}$ are, respectively, *totally monotone* and *totally alternating capacities* [3, 4, 5], i.e., they satisfy, for every $n \geq 2$ and for every $A_1, \ldots, A_n \in \wp(\Omega)$,

**(TM)** $\underline{P}\left(\bigcup_{i=1}^n A_i\right) + \sum_{\substack{\emptyset \neq I \subseteq \{1,\ldots,n\} \\ |I| \text{ is even}}} \underline{P}\left(\bigcap_{i \in I} A_i\right) \geq \sum_{\substack{\emptyset \neq I \subseteq \{1,\ldots,n\} \\ |I| \text{ is odd}}} \underline{P}\left(\bigcap_{i \in I} A_i\right)$;

**(TA)** $\overline{P}\left(\bigcap_{i=1}^n A_i\right) + \sum_{\substack{\emptyset \neq I \subseteq \{1,\ldots,n\} \\ |I| \text{ is even}}} \overline{P}\left(\bigcup_{i \in I} A_i\right) \leq \sum_{\substack{\emptyset \neq I \subseteq \{1,\ldots,n\} \\ |I| \text{ is odd}}} \overline{P}\left(\bigcup_{i \in I} A_i\right)$.

In particular, since $\underline{P}$ and $\overline{P}$ are normalized, they are also referred to as *belief* and *plausibility functions*, respectively [33, 34].

In [34] belief and plausibility functions are said *continuous* if they are continuous, respectively, from above and from below, i.e., if $\underline{P}(\bigcap_{n\in\mathbb{N}} A_n) = \lim_{n\in\mathbb{N}} \underline{P}(A_n)$ for every decreasing sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\wp(\Omega)$ and $\overline{P}(\bigcup_{n\in\mathbb{N}} B_n) = \lim_{n\in\mathbb{N}} \overline{P}(B_n)$ for every increasing sequence $\{B_n\}_{n\in\mathbb{N}}$ in $\wp(\Omega)$.

It is well-known that if $(\Omega, \mathcal{A}, P)$ is a countably additive probability space where $\mathcal{A}$ is a $\sigma$-algebra, then continuity of $\underline{P}$ and $\overline{P}$ easily follows, while starting from a finitely additive probability space $(\Omega, \mathcal{A}, P)$ or in case $\mathcal{A}$ is only an algebra, the envelopes $\underline{P}$ and $\overline{P}$ are generally not continuous.

The following example shows that the hypothesis of countable additivity for $P$ is not sufficient to guarantee continuity of $\underline{P}$ and $\overline{P}$ when $\mathcal{A}$ is not a $\sigma$-algebra.

**Example 1.** *Identify $\Omega$ with $[0,1]$, and let $\mathcal{A}$ and $\mathcal{A}'$ be, respectively, the algebra of finite unions of subintervals of $[0,1]$ and the Borel $\sigma$-algebra on $[0,1]$. Let $P$ be the restriction of the Lebesgue measure on $\mathcal{A}$.*

*As is well-known, $P$ is countably additive on $\mathcal{A}$ and has a unique countably additive extension on $\mathcal{A}'$ which is determined by the Carathéodory outer measure $\overline{P}^{\mathbf{c}}$, defined, for every $A \in \wp(\Omega)$, as*

$$\overline{P}^{\mathbf{c}}(A) = \inf\left\{\sum_{i\in I} P(B_i) \,:\, \{B_i\}_{i\in I} \subseteq \mathcal{A}, \operatorname{card} I \leq \aleph_0, A \subseteq \bigcup_{i\in I} B_i\right\}.$$

*The function $\overline{P}^{\mathbf{c}}$ turns out to be countably additive on $\mathcal{A}'$ and so the unique countably additive extension $Q$ is obtained setting $Q(A) = \overline{P}^{\mathbf{c}}(A)$ for every $A \in \mathcal{A}'$.*

*Nevertheless, $Q$ is not the only finitely additive probability extending $P$ on $\mathcal{A}'$, but there is a class $\mathcal{P}$ of such extensions whose pointwise envelopes $\underline{P} = \min \mathcal{P}$ and $\overline{P} = \max \mathcal{P}$ are defined for $A \in \mathcal{A}'$ as*

$$\underline{P}(A) = \sup\{P(B) \,:\, B \subseteq A, B \in \mathcal{A}\} \quad \textit{and} \quad \overline{P}(A) = \inf\{P(B) \,:\, A \subseteq B, B \in \mathcal{A}\}.$$

*In particular, it holds $Q(\mathbb{Q} \cap [0,1]) = 0$ so $\underline{P}(\mathbb{Q} \cap [0,1]) = 0$, while the $\leq$-denseness of $\mathbb{Q} \cap [0,1]$ in $[0,1]$ implies $\overline{P}(\mathbb{Q} \cap [0,1]) = 1$, i.e., there are two finitely additive extensions $\tilde{P}_1$, $\tilde{P}_2$ of $P$ on $\mathcal{A}'$ such that $\tilde{P}_1(\mathbb{Q} \cap [0,1]) = 0$ and $\tilde{P}_2(\mathbb{Q} \cap [0,1]) = 1$.*

*The envelopes $\underline{P}$ and $\overline{P}$ are, respectively, a belief and a plausibility function on $\mathcal{A}'$ since they are normalized and, respectively, totally monotone and totally* *alternating. Nevertheless,* $\underline{P}$ *and* $\overline{P}$ *are not continuous. To prove the claim, let* $q_1, q_2, \ldots$ *be any enumeration of* $\mathbb{Q} \cap [0,1]$ *and denote* $A_n = \{q_1, \ldots, q_n\}$*, for* $n \in \mathbb{N}$*. It holds that* $A_n \uparrow \mathbb{Q} \cap [0,1]$ *but* $\overline{P}(A_n) = 0$*, for* $n \in \mathbb{N}$*, thus* $\lim_{n\in\mathbb{N}} \overline{P}(A_n) = 0 \neq 1 = \overline{P}(\mathbb{Q} \cap [0,1])$*, which implies* $\overline{P}$ *is not continuous from below, and by duality* $\underline{P}$ *is not continuous from above.*

*On the other hand, by considering the unique countably additive extension* $Q$ *of* $P$ *on* $\mathcal{A}'$*, it is well-known, that there is no countably additive extension of* $Q$ *on* $\wp(\Omega)$ *even though there are (infinite) finitely additive extensions forming the set* $\mathcal{Q}$ *whose pointwise envelopes are* $\underline{Q} = \min \mathcal{Q}$ *and* $\overline{Q} = \max \mathcal{Q}$ *which are defined, for* $A \in \wp(\Omega)$*, as*

$$\underline{Q}(A) = \sup\{Q(B) \,:\, B \subseteq A, B \in \mathcal{A}'\} \quad \text{and} \quad \overline{Q}(A) = \inf\{Q(B) \,:\, A \subseteq B, B \in \mathcal{A}'\}.$$

*It actually holds* $\overline{Q}(A) = \overline{P}^{\mathbf{c}}(A)$ *and* $\underline{Q}(A) = 1 - \overline{P}^{\mathbf{c}}(A^c)$*. Also in this case* $\underline{Q}$ *and* $\overline{Q}$ *turn out to be, respectively, a normalized totally monotone capacity and a normalized totally alternating capacity on* $\wp(\Omega)$*, but now their continuity, respectively, from above and from below is easily established.* ■

The previous discussion highlights that the total monotonicity (alternance) of a finitely additive probability $\mu$ on a space $(X, \mathcal{F})$ is preserved both when we take the corresponding inner (outer) measure $\underline{\mu}$ ($\overline{\mu}$) defined on $(X, \wp(X))$ and when we consider the corresponding lower (upper) probability $\mu_*$ ($\mu^*$) defined on a different space $(Y, \mathcal{G})$ through a multivalued mapping $\Gamma : X \to \wp(Y)$ satisfying **(A1)** and **(A2)**. The last statement is in line with [9, 24] where $\Gamma$ is classified as a $n$-monotonicity preserving transformation, with $n \geq 2$.

Actually, a more general result holds by considering a possibly unbounded $[0, +\infty]$-valued $n$-monotone capacity $\varphi$ on $(X, \mathcal{F})$, with $n \geq 2$. Denote again with $\underline{\varphi}$ the corresponding inner measure on $(X, \wp(X))$ and with $\varphi_*$ the corresponding lower capacity defined on a different space $(Y, \mathcal{G})$ through a multivalued mapping $\Gamma : X \to \wp(Y)$ satisfying **(A1)** and **(A2)**, obtained, for every $A \in \wp(X)$ and every $B \in \mathcal{G}$, as

$$\underline{\varphi}(A) = \sup\{K \in \mathcal{F} \,:\, K \subseteq A\} \quad \text{and} \quad \varphi_*(B) = \underline{\varphi}(B_*).$$

The proof of the following proposition follows by essentially known results present in the literature and is reported for completeness.

**Proposition 1.** *If* $\varphi$ *is a possibly unbounded* $[0, +\infty]$*-valued* $n$*-monotone capacity on* $(X, \mathcal{F})$*, with* $n \geq 2$*, then*

*(i)* $\underline{\varphi}$ *is a possibly unbounded* $[0, +\infty]$*-valued* $n$*-monotone capacity on* $(X, \wp(X))$*;*

*(ii)* $\varphi_*$ *is a possibly unbounded* $[0, +\infty]$*-valued* $n$*-monotone capacity on* $(Y, \mathcal{G})$*.*

*Proof.* Both $\underline{\varphi}$ and $\varphi_*$ are trivially monotone with respect to set inclusion and vanish on $\emptyset$. For every $A_1, \ldots, A_n$ belonging either to $\wp(X)$ or to $\mathcal{G}$, it holds

$$\varphi'\left(\bigcup_{i=1}^n A_i\right) + \sum_{\substack{\emptyset \neq I \subseteq \{1,\ldots,n\} \\ |I| \text{ is even}}} \varphi'\left(\bigcap_{i\in I} A_i\right) \geq \sum_{\substack{\emptyset \neq I \subseteq \{1,\ldots,n\} \\ |I| \text{ is odd}}} \varphi'\left(\bigcap_{i\in I} A_i\right),$$

where $\varphi'$ stands either for $\underline{\varphi}$ (and so $A_i \in \wp(X)$) or $\varphi_*$ (and so $A_i \in \mathcal{G}$).

The above inequality is well-defined in case some evaluations of $\underline{\varphi}$ or $\varphi_*$ are equal to $+\infty$. This follows by the monotonicity of $\underline{\varphi}$ and $\varphi_*$, and by the usual convention $+\infty \geq +\infty$. Thus assume that all the evaluations of $\underline{\varphi}$ and $\varphi_*$ in the above inequalities are finite.

The proof of the inequality for $\underline{\varphi}$ can be traced back to Lemma 18.3 in the original work by Choquet [4] where only a short proof is given. Limiting to normalized capacities, it has been explicitly proved for the case $n = 2$ in [38] and for any $n \geq 2$ in [3]. Hence, a proof can be obtained in analogy to the proof of Proposition 1 in [3]. The proof of the inequality for $\varphi_*$ can be carried on in analogy to the proof of Theorem 1 in [24] noticing that $(\bigcup_{i=1}^n A_i)_* \supseteq \bigcup_{i=1}^n A_{i*}$ and $\left(\bigcap_{i\in I} A_i\right)_* = \bigcap_{i\in I} A_{i*}$, and using the $n$-monotonicity of $\underline{\varphi}$. $\square$

Analogously, starting from a possibly unbounded $[0, +\infty]$-valued $n$-alternating capacity $\psi$ on $(X, \mathcal{F})$, with $n \geq 2$, we can prove the following proposition.

**Proposition 2.** *If $\psi$ is a possibly unbounded $[0, +\infty]$-valued $n$-alternating capacity on $(X, \mathcal{F})$, with $n \geq 2$, then:*

*(i)* $\overline{\psi}$ *is a possibly unbounded* $[0, +\infty]$*-valued $n$-alternating capacity on* $(X, \wp(X))$*;*

*(ii)* $\psi^*$ *is a possibly unbounded* $[0, +\infty]$*-valued $n$-alternating capacity on* $(Y, \mathcal{G})$*.*

## 3. Full conditional probability spaces

Let $\Omega$ be a non-empty set and $\mathcal{A}$ an arbitrary algebra of its subsets. For every algebra $\mathcal{A}$ of subsets of $\Omega$, denote $\mathcal{A}^0 = \mathcal{A} \setminus \{\emptyset\}$.

A *full conditional probability on* $\mathcal{A}$ (see, e.g., [14, 29]) is a function $P : \mathcal{A} \times \mathcal{A}^0 \to [0, 1]$ satisfying the following conditions:

**(C1)** $P(E|H) = P(E \cap H|H)$, for every $E \in \mathcal{A}$ and $H \in \mathcal{A}^0$;

**(C2)** $P(\cdot|H)$ is a finitely additive probability on $\mathcal{A}$, for every $H \in \mathcal{A}^0$;

**(C3)** $P(E \cap F|H) = P(E|H) \cdot P(F|E \cap H)$, for every $H, E \cap H \in \mathcal{A}^0$ and $E, F \in \mathcal{A}$.

A full conditional probability on $\mathcal{A}$ is said *countably additive* if condition **(C2)** is reinforced with countable additivity [30]. The pair $(\Omega, \mathcal{A}\times\mathcal{A}^0)$ is said *conditional measurable space*, while $(\Omega, \mathcal{A}\times\mathcal{A}^0, P)$ is referred to as *full conditional probability space* following the terminology of [30], and is said *countably additive* if $P(\cdot|\cdot)$ is.

It is known (see [1, 20, 31]) that any full conditional probability $P(\cdot|\cdot)$ on $\mathcal{A}$ induces a *dimensionally ordered class* $\{m_\alpha : \alpha \in I\}$ of $[0, +\infty]$-valued finitely additive measures on $\mathcal{A}$ which is unique up to the choice of a positive multiplicative constant for each $m_\alpha$. Denoting with $\mathcal{I}_\alpha$ the ideal of $\mathcal{A}$ where $m_\alpha$ is finite, the family $\{m_\alpha : \alpha \in I\}$ must satisfy the following properties:

*(i)* $\{\mathcal{I}_\alpha : \alpha \in I\}$ is linearly ordered by a relation $\leq$ on $I$ defined as $\alpha \leq \beta \Longleftrightarrow \mathcal{I}_\alpha \supseteq \mathcal{I}_\beta$;

*(ii)* for every $E \in \mathcal{A}^0$, there exists a unique $\alpha \in I$ such that $E \in \mathcal{I}_\alpha \setminus \bigcup_{\alpha<\beta} \mathcal{I}_\beta$;

*(iii)* for every $\alpha \in I$, the restriction of $m_\alpha$ on $\mathcal{I}_\alpha$ is a (possibly unbounded) non-trivial finitely additive measure ranging in $[0, +\infty)$ such that for every $E \in \mathcal{I}_\alpha$, $m_\alpha(E) = 0$ if and only if $E \in \bigcup_{\alpha<\beta} \mathcal{I}_\beta \cup \{\emptyset\}$;

*(iv)* for every $\alpha \in I$ and $E, H \in \mathcal{I}_\alpha$, $m_\alpha(E \cap H) = P(E|H) \cdot m_\alpha(H)$.

On the converse, any dimensionally ordered class $\{m_\alpha : \alpha \in I\}$ on an algebra $\mathcal{A}$ uniquely determines a full conditional probability $P(\cdot|\cdot)$ on $\mathcal{A}$, indeed, for every $E|H \in \mathcal{A} \times \mathcal{A}^0$ there exists a unique index $\alpha \in I$ such that $m_\alpha(H) \in (0, +\infty)$ and so

$$P(E|H) = \frac{m_\alpha(E \cap H)}{m_\alpha(H)}.$$

Every dimensionally ordered class $\{m_\alpha : \alpha \in I\}$ on $\mathcal{A}$ induces two classes $\{\underline{m}_\alpha : \alpha \in I\}$ and $\{\overline{m}_\alpha : \alpha \in I\}$ of $[0, +\infty]$-valued functions on $\wp(\Omega)$ defined for every $E \in \wp(\Omega)$ as

$$\begin{aligned} \underline{m}_\alpha(E) &= \sup\{m_\alpha(K) : K \subseteq E, K \in \mathcal{A}\}, \\ \overline{m}_\alpha(E) &= \inf\{m_\alpha(K) : E \subseteq K, K \in \mathcal{A}\}, \end{aligned}$$

where the infimum and the supremum are taken in $[0, +\infty]$, which are, for $\alpha \in I$, the *inner* and *outer measures* induced by $m_\alpha$. It is easy to verify that, for $\alpha \in I$, the functions $\underline{m}_\alpha$ and $\overline{m}_\alpha$ satisfy the following properties:

**(P1)** $\underline{m}_\alpha(E) \leq \overline{m}_\alpha(E)$, for every $E \in \wp(\Omega)$;

**(P2)** $\underline{m}_\alpha(E) \leq \underline{m}_\alpha(F)$ and $\overline{m}_\alpha(E) \leq \overline{m}_\alpha(F)$, for every $E, F \in \wp(\Omega)$ such that $E \subseteq F$;

**(P3)** $\underline{m}_\alpha$ is totally monotone and $\overline{m}_\alpha$ is totally alternating.

## 4. Extensions of a full conditional probability

By Corollary 2 in [14], every full conditional probability $P(\cdot|\cdot)$ on $\mathcal{A}$ can be extended to a full conditional probability on $\wp(\Omega)$: in general, countable additivity is not preserved in the extension process. The extension is generally not unique but we have a class of possible extensions which is a compact subset of $[0, 1]^{\wp(\Omega)\times\wp(\Omega)^0}$ endowed with the product topology.

Hence, given a full conditional probability $P(\cdot|\cdot)$ on $\mathcal{A}$ consider the set

$$\mathcal{P} = \left\{\tilde{P} : \tilde{P} \text{ is a full conditional probability on } \wp(\Omega) \text{ extending } P\right\},$$

whose lower and upper envelopes are denoted as $\underline{P} = \min \mathcal{P}$ and $\overline{P} = \max \mathcal{P}$. Such functions satisfy the *duality* relation, for every $F|K \in \wp(\Omega) \times \wp(\Omega)^0$,

$\overline{P}(F|K) = 1 - \underline{P}(F^c|K)$. By Theorem 4 in [29], for every $F|K \in \wp(\Omega) \times \wp(\Omega)^0$, it holds

$$\begin{aligned}\underline{P}(E|H) &= \sup\left\{\underline{P}^{\mathcal{D}}(E|H) \,:\, \mathcal{D} \subseteq \mathcal{A}, \text{ finite sub-algebra}\right\},\\ \overline{P}(E|H) &= \inf\left\{\overline{P}^{\mathcal{D}}(E|H) \,:\, \mathcal{D} \subseteq \mathcal{A}, \text{ finite sub-algebra}\right\},\end{aligned}$$

where $\underline{P}^{\mathcal{D}}(E|H)$ and $\overline{P}^{\mathcal{D}}(E|H)$ are the bounds obtained extending the full conditional probability $P_{|\mathcal{D}\times\mathcal{D}^0}$ on $\mathcal{D}$ to the whole $\wp(\Omega) \times \wp(\Omega)^0$.

The following theorems generalize some results given in [6] in case of finite domains. The proposed generalizations are not trivial since their proofs rely on different mathematical techniques. The proofs in [6] are essentially based on the representation of a full conditional probability $P(\cdot|\cdot)$ on a finite algebra $\mathcal{A}$ by a finite linearly ordered class of (unconditional) probability measures $\{P_0, \ldots, P_k\}$ defined on $\mathcal{A}$ and having disjoint supports. The above representation is generally not possible with infinite algebras (see, e.g., Example 2) where it is necessary to consider possibly infinite classes of possibly unbounded $[0, +\infty]$-valued finitely additive measures [20].

The following Theorem 1 is new with respect to the results given in [6] and provides a direct characterization of the lower envelope $\underline{P}(\cdot|\cdot)$ only relying on the initial full conditional probability $P(\cdot|\cdot)$. Actually, Theorem 1 is the basis of the proof of Theorem 2 which is the analogue of Theorem 2 in [6] and characterizes $\underline{P}(\cdot|\cdot)$ in terms of the classes of inner and outer measures induced by the dimensionally ordered class $\{m_\alpha \,:\, \alpha \in I\}$ representing $P(\cdot|\cdot)$. In a sense, the direct characterization obtained in Theorem 1 allows to overcome the difficulties in dealing with unbounded measures.

**Theorem 1.** *The lower envelope $\underline{P}(\cdot|\cdot)$ is such that, for every $F|K \in \wp(\Omega) \times \wp(\Omega)^0$, $\underline{P}(F|K) = 1$ when $F \cap K = K$, and otherwise*

*(i) if $K \in \mathcal{A}^0$, then*

$$\underline{P}(F|K) = \sup\{P(B|K) \,:\, B \subseteq F, B \in \mathcal{A}\};$$

*(ii) if $K \in \wp(\Omega)^0 \setminus \mathcal{A}^0$, then if there exists $A \in \mathcal{A}^0$ such that $K \subseteq A$ and $\underline{P}(K|A) > 0$ we have that*

$$\underline{P}(F|K) = \frac{\underline{P}(F \cap K|A)}{\underline{P}(F \cap K|A) + \overline{P}(F^c \cap K|A)},$$

*otherwise $\underline{P}(F|K) = 0$.*

*Proof.* Condition *(i)*. For every $F|K \in \wp(\Omega) \times \mathcal{A}^0$, we can restrict to those finite sub-algebras $\mathcal{D}$ of $\mathcal{A}$ containing $K$, since every finite sub-algebra can be suitably enlarged in order to meet this form. Denote with $B$ the maximal element of $\mathcal{D}$ with respect to inclusion relation such that $B \subseteq F$. In turn, this implies

$$\underline{P}(F|K) = \sup\{P(B|K) \,:\, B \subseteq F, B \in \mathcal{A}\},$$

so $\underline{P}(\cdot|K)$ coincides with the inner measure on $\wp(\Omega)$ generated by $P(\cdot|K)$ and is normalized and totally monotone by Proposition 1.

Condition *(ii)*. For $F|K \in \wp(\Omega) \times (\wp(\Omega)^0 \setminus \mathcal{A}^0)$, if there exists $A \in \mathcal{A}^0$ such that $K \subseteq A$ and $\underline{P}(K|A) > 0$, denote with $\mathcal{P}_{\underline{P}(\cdot|A)}$ the core of $\underline{P}(\cdot|A)$ on $\wp(\Omega)$, i.e., the convex compact set of finitely additive probabilities on $\wp(\Omega)$ dominating $\underline{P}(\cdot|A)$ [13]. Considering the Choquet integral [4], it holds

$$\underline{P}(F \cap K|A) = \oint \mathbf{1}_{F\cap K}(\omega)\underline{P}(\mathrm{d}\omega|A) \quad \text{and} \quad \overline{P}(F^c \cap K|A) = \oint \mathbf{1}_{F^c\cap K}(\omega)\overline{P}(\mathrm{d}\omega|A).$$

It is immediately seen that $\min\{\mathbf{1}_{F\cap H}(\omega), \mathbf{1}_{F^c\cap H}(\omega)\} = 0$ for $\omega \in \Omega$, moreover, $\mathbf{1}_{F\cap H}$ and $(1 - \mathbf{1}_{F^c\cap H})$ are comonotonic, i.e., for every $\omega, \omega' \in \Omega$,

$$[\mathbf{1}_{F\cap H}(\omega) - \mathbf{1}_{F\cap H}(\omega')]\,[(1 - \mathbf{1}_{F^c\cap H}(\omega)) - (1 - \mathbf{1}_{F^c\cap H}(\omega'))] \geq 0.$$

Hence, Proposition 6.26 in [37] implies the existence of a finitely additive probability $\tilde{\pi}_A$ in $\mathcal{P}_{\underline{P}(\cdot|A)}$ such that

$$\begin{aligned}\int \mathbf{1}_{F\cap H}(\omega)\tilde{\pi}_A(\mathrm{d}\omega) &= \oint \mathbf{1}_{F\cap H}(\omega)\underline{P}(\mathrm{d}\omega|A),\\ \int (1 - \mathbf{1}_{F^c\cap H}(\omega))\tilde{\pi}_A(\mathrm{d}\omega) &= \oint (1 - \mathbf{1}_{F^c\cap H}(\omega))\underline{P}(\mathrm{d}\omega|A),\end{aligned}$$

where the left-side integrals are usual Stieltjes integrals [2]. Since

$$\begin{aligned}\int (1 - \mathbf{1}_{F^c\cap H}(\omega))\tilde{\pi}_A(\mathrm{d}\omega) &= 1 - \int \mathbf{1}_{F^c\cap H}(\omega))\tilde{\pi}_A(\mathrm{d}\omega),\\ \oint (1 - \mathbf{1}_{F^c\cap H}(\omega))\underline{P}(\mathrm{d}\omega|A) &= 1 - \oint \mathbf{1}_{F^c\cap H}(\omega)\overline{P}(\mathrm{d}\omega|A),\end{aligned}$$

it follows

$$\oint \mathbf{1}_{F^c\cap H}(\omega)\overline{P}(\mathrm{d}\omega|A) = \int \mathbf{1}_{F^c\cap H}(\omega)\tilde{\pi}_A(\mathrm{d}\omega),$$

i.e., both $\underline{P}(F \cap K|A)$ and $\overline{P}(F^c \cap K|A)$ are obtained integrating with respect to $\tilde{\pi}_A$. Finally we have

$$\begin{aligned}\underline{P}(F|K) &= \min\left\{\frac{\tilde{P}(F\cap K|A)}{\tilde{P}(K|A)} \,:\, \tilde{P} \in \mathcal{P}\right\} = \min\left\{\frac{\tilde{\pi}(F\cap K)}{\tilde{\pi}(K)} \,:\, \tilde{\pi} \in \mathcal{P}_{\underline{P}(\cdot|A)}\right\}\\ &= \frac{\tilde{\pi}_A(F\cap K)}{\tilde{\pi}_A(F\cap K) + \tilde{\pi}_A(F^c\cap K)} = \frac{\underline{P}(F\cap K|A)}{\underline{P}(F\cap K|A) + \overline{P}(F^c\cap K|A)},\end{aligned}$$

where the first two equalities are trivial, while the third and the fourth follow from the previous argument.

Otherwise, for all $A \in \mathcal{A}^0$ with $K \subseteq A$ it holds $\underline{P}(K|A) = 0$, which implies for every such $A$ the existence of $\tilde{P}_A \in \mathcal{P}$ such that $\tilde{P}_A(K|A) = 0$ and so $\tilde{P}_A(K|B) = 0$ for every $B \in \mathcal{A}^0$ with $A \subseteq B$. We show the existence of $\tilde{P}_0 \in \mathcal{P}$ such that $\tilde{P}_0(K|A) = 0$ for all $A \in \mathcal{A}^0$ with $K \subseteq A$. The compactness of $\mathcal{P}$ in the product topology of $[0,1]^{\wp(\Omega)\times\wp(\Omega)^0}$ is equivalent to the fact that every family of non-empty closed subsets of $\mathcal{P}$ with the finite intersection property has non-empty intersection.

For an arbitrary finite sub-algebra $\mathcal{D} \subseteq \mathcal{A}$ define

$$K^*_{\mathcal{D}} = \bigcap \{B \in \mathcal{D}^0 \,:\, K \subseteq B\},$$

which belongs to $\mathcal{D}^0$ since $\mathcal{D}$ is finite. Introduce the collection

$$\mathbf{E}_0 = \left\{ \mathcal{P}^{\mathcal{D}}_0 = \left\{ \tilde{P} \in \mathcal{P} \,:\, \tilde{P}(K|K^*_{\mathcal{D}}) = 0 \right\} \,:\, \mathcal{D} \subseteq \mathcal{A}, \operatorname{card} \mathcal{D} < \aleph_0 \right\},$$

which is a family of non-empty closed subsets of $\mathcal{P}$.

We show that $\mathbf{E}_0$ has the finite intersection property. For any $\mathcal{D}_1, \ldots, \mathcal{D}_n$ finite sub-algebras of $\mathcal{A}$, the algebra $\mathcal{D}'$ generated by $\bigcup_{i=1}^n \mathcal{D}_i$ is still a finite sub-algebra of $\mathcal{A}$, moreover, $K^*_{\mathcal{D}'} \subseteq K^*_{\mathcal{D}_i}$ for $i = 1, \ldots, n$. It follows that, for $i = 1, \ldots, n$, $K \cap K^*_{\mathcal{D}_{\mathcal{L}i}} \subseteq K \cap K^*_{\mathcal{D}'_{\mathcal{L}}}$ and $K^c \cap K_{\mathcal{D}_{\mathcal{L}i}} \supseteq K^c \cap K^*_{\mathcal{D}'_{\mathcal{L}}}$, and so $K|K^*_{\mathcal{D}_i} \subseteq_{GN} K|K^*_{\mathcal{D}'}$ for $i = 1, \ldots, n$, according to the definition of inclusion relation for conditional events $\subseteq_{GN}$ given in [17]. Hence, for every $\tilde{P} \in \mathcal{P}^{\mathcal{D}'}_0$ we have $\tilde{P}(K|K^*_{\mathcal{D}'}) = 0$ and by the monotonicity of $\tilde{P}$ with respect to $\subseteq_{GN}$ relation [7, 17], it follows $\tilde{P}(K|K^*_{\mathcal{D}_i}) = 0$ for $i = 1, \ldots, n$, and so $\tilde{P} \in \mathcal{P}^{\mathcal{D}_i}_0$ for $i = 1, \ldots, n$. This implies $\bigcap_{i=1}^n \mathcal{P}^{\mathcal{D}_i}_0 \neq \emptyset$ and so $\mathbf{E}_0$ satisfies the finite intersection property which, in turn, implies $\bigcap \mathbf{E}_0 \neq \emptyset$, i.e., there exists $\tilde{P}_0 \in \bigcap \mathbf{E}_0$ such that $\tilde{P}_0(K|A) = 0$ for every $A \in \mathcal{A}^0$ with $K \subseteq A$. The restriction $\tilde{Q}_0$ of $\tilde{P}_0$ on $\wp(\Omega) \times \mathcal{A}^0$ can be extended to the whole $\wp(\Omega) \times \wp(\Omega)^0$ obtaining a compact subset $\mathcal{Q}$ of $\mathcal{P}$ whose lower envelope $\underline{Q} = \min \mathcal{Q}$ is such that $\underline{Q} \geq \underline{P}$. Applying Corollary 2 in [6] to the restriction of $\tilde{Q}_0$ to every finite set of the form $\mathcal{E} \times \mathcal{D}^0$, where $\mathcal{E} \subseteq \wp(\Omega)$ and $\mathcal{D} \subseteq \mathcal{A} \cap \mathcal{E}$ are finite sub-algebras with $\{F, K\} \subseteq \mathcal{E}$, we derive $\underline{Q}(F|K) = 0$, which implies $\underline{P}(F|K) = 0$. $\square$

The following proposition states that $\underline{P}(\cdot|K)$ is a normalized totally monotone capacity on $\wp(\Omega)$, for every $K \in \wp(\Omega)^0$.

**Proposition 3.** *The lower envelope $\underline{P}(\cdot|\cdot)$ is such that $\underline{P}(\cdot|K)$ is a normalized totally monotone capacity on $\wp(\Omega)$, for every $K \in \wp(\Omega)^0$.*

*Proof.* For every $K \in \wp(\Omega)^0$, we have $\underline{P}(\emptyset|K) = 0$, $\underline{P}(\Omega|K) = 1$, thus it remains to prove that $\underline{P}(\cdot|K)$ is totally monotone. For condition *(i)* of Theorem 1 the proof follows by Proposition 1, since $\underline{P}(\cdot|K)$ turns out to be an inner measure. For condition *(ii)* of Theorem 1, if there exists $A \in \mathcal{A}^0$ such that $K \subseteq A$ and $\underline{P}(K|A) > 0$, let $E_1, \ldots, E_n \in \wp(\Omega)$ and $\mathcal{D}$ be the algebra generated by $\{E_1, \ldots, E_n, K\}$. Then, the proof follows considering the restriction of $\underline{P}(\cdot|A)$ to $\mathcal{D}$ and applying Theorem 1 in [19]. Finally, if $\underline{P}(K|A) = 0$ for all $A \in \mathcal{A}^0$ such that $K \subseteq A$, $\underline{P}(\cdot|K)$ turns out to be a normalized totally monotone capacity vacuous at $K$, i.e., such that $\underline{P}(F|K) = 1$ if $F \cap K = K$ and 0 otherwise. $\square$

We next investigate the continuity from above of $\underline{P}(\cdot|\cdot)$ in the case the starting full conditional probability space is countably additive and is related to a $\sigma$-algebra.

**Proposition 4.** *If $(\Omega, \mathcal{A} \times \mathcal{A}^0, P)$ is a countably additive full conditional probability space where $\mathcal{A}$ is a $\sigma$-algebra, then the lower envelope $\underline{P}(\cdot|K)$ is a normalized totally monotone capacity continuous from above on $\wp(\Omega)$, for every $K \in \wp(\Omega)^0$.*

*Proof.* If $K \in \mathcal{A}^0$, then the statement immediately follows by condition *(i)* of Theorem 1 since $\underline{P}(\cdot|K)$ turns out to be an inner measure induced by a countably additive probability on a $\sigma$-algebra. If $K \in \wp(\Omega)^0 \setminus \mathcal{A}^0$ and there exists $A \in \mathcal{A}^0$ such that $K \subseteq A$ and $\underline{P}(K|A) > 0$, then the statement follows by the previous point and Proposition 2.3 in [12]. Finally, if $\underline{P}(K|A) = 0$ for all $A \in \mathcal{A}^0$ such that $K \subseteq A$, $\underline{P}(\cdot|K)$ turns out to be a normalized totally monotone capacity vacuous at $K$, which is easily seen to be continuous from above. □

The next theorem characterizes $\underline{P}(\cdot|\cdot)$ in terms of the classes of inner and outer measures $\{\underline{m}_\alpha : \alpha \in I\}$ and $\{\overline{m}_\alpha : \alpha \in I\}$ on $\wp(\Omega)$ induced by the dimensionally ordered class $\{m_\alpha : \alpha \in I\}$ representing $P(\cdot|\cdot)$ on $\mathcal{A}$.

**Theorem 2.** *The lower envelope $\underline{P}(\cdot|\cdot)$ is such that, for every $F|K \in \wp(\Omega) \times \wp(\Omega)^0$, $\underline{P}(F|K) = 1$ when $F \cap K = K$, and otherwise*

$$\underline{P}(F|K) = \begin{cases} \frac{\underline{m}_\alpha(F\cap K)}{\underline{m}_\alpha(F\cap K)+\overline{m}_\alpha(F^c\cap K)} & \text{if there is } \alpha \in I \text{ such that} \\ & \underline{m}_\alpha(F \cap K) + \overline{m}_\alpha(F^c \cap K) \in (0, +\infty), \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* For $K \in \mathcal{A}^0$, there is a unique index $\alpha \in I$ such that $m_\alpha(K) \in (0, +\infty)$. By condition *(i)* of Theorem 1,

$$\begin{aligned} \underline{P}(F|K) &= \sup\{P(B|K) : B \subseteq F, B \in \mathcal{A}\} \\ &= \sup\left\{\frac{m_\alpha(B \cap K)}{m_\alpha(K)} : B \subseteq F, B \in \mathcal{A}\right\} = \frac{\underline{m}_\alpha(F \cap K)}{m_\alpha(K)}. \end{aligned}$$

Consider the rings $\mathcal{A}^K = \{E \cap K : E \in \mathcal{A}\}$ and $\wp(\Omega)^K = \{E \cap K : E \in \wp(\Omega)\}$ both having $K$ as top element. The restriction $n_\alpha = m_{\alpha|\mathcal{A}^K}$ turns out to be a bounded finitely additive measure on $\mathcal{A}^K$ giving rise to the bounded inner and outer measures $\underline{n}_\alpha$ and $\overline{n}_\alpha$ on $\wp(\Omega)^K$, for which it holds $n_\alpha(K) = \underline{n}_\alpha(K) = \overline{n}_\alpha(K) = m_\alpha(K)$. In analogy with the proof of condition *(ii)* of Theorem 1, it is easily proven the existence of a bounded finitely additive measure $\tilde{\pi}_\alpha$ on $\wp(\Omega)^K$ such that $\underline{n}_\alpha \le \tilde{\pi}_\alpha \le \overline{n}_\alpha$, and $\tilde{\pi}_\alpha(F \cap K) = \underline{n}_\alpha(F \cap K) = \underline{m}_\alpha(F \cap K)$ and $\tilde{\pi}_\alpha(F^c \cap K) = \overline{n}_\alpha(F^c \cap K) = \overline{m}_\alpha(F^c \cap K)$. In turn, this implies $\underline{m}_\alpha(F \cap K) + \overline{m}_\alpha(F^c \cap K) = \tilde{\pi}_\alpha(K) = m_\alpha(K)$ and so $\underline{m}_\alpha(F \cap K) + \overline{m}_\alpha(F^c \cap K) \in (0, +\infty)$ and

$$\underline{P}(F|K) = \frac{\underline{m}_\alpha(F \cap K)}{\underline{m}_\alpha(F \cap K) + \overline{m}_\alpha(F^c \cap K)}.$$

For $K \in \wp(\Omega)^0 \setminus \mathcal{A}^0$, by condition *(ii)* of Theorem 1, if there exists $A \in \mathcal{A}^0$ such that $K \subseteq A$ and $\underline{P}(K|A) > 0$, let $\alpha \in I$ be the unique index such that

$m_\alpha(A) \in (0, +\infty)$. By the previous step, it holds

$$\begin{aligned}\underline{P}(F|K) &= \frac{\underline{P}(F \cap K|A)}{\underline{P}(F \cap K|A) + \overline{P}(F^c \cap K|A)} = \frac{\frac{\underline{m}_\alpha(F\cap K)}{m_\alpha(A)}}{\frac{\underline{m}_\alpha(F\cap K)}{m_\alpha(A)} + \frac{\overline{m}_\alpha(F^c\cap K)}{m_\alpha(A)}} \\ &= \frac{\underline{m}_\alpha(F \cap K)}{\underline{m}_\alpha(F \cap K) + \overline{m}_\alpha(F^c \cap K)} \in [0, 1],\end{aligned}$$

and so it must be $\underline{m}_\alpha(F \cap K) + \overline{m}_\alpha(F^c \cap K) \in (0, +\infty)$.

Finally, if $\underline{P}(K|A) = 0$ for every $A \in \mathcal{A}^0$ such that $K \subseteq A$, we have to distinguish two cases. If there is no $\alpha \in I$ such that $\underline{m}_\alpha(F \cap K) + \overline{m}_\alpha(F^c \cap K) \in (0, +\infty)$, then $\underline{P}(F|K) = 0$ and this is consistent with Theorem 1. On the contrary, suppose there is $\alpha \in I$ such that $\underline{m}_\alpha(F \cap K) + \overline{m}_\alpha(F^c \cap K) \in (0, +\infty)$. For every $A \in \mathcal{A}^0$ such that $K \subseteq A$ let $\beta \in I$ be the unique index such that $m_\beta(A) \in (0, +\infty)$. We have $\underline{P}(K|A) = \frac{\underline{m}_\beta(K)}{m_\beta(A)} = 0$, so it follows $\underline{m}_\beta(F \cap K) \leq \underline{m}_\beta(K) = 0 < m_\beta(A) = \underline{m}_\beta(A)$. In turn, this implies $\underline{m}_\beta(F \cap K) = 0$ for every $\beta \in I$ and so, in particular, also for $\alpha$, thus it must be $\overline{m}_\alpha(F^c \cap K) \in (0, +\infty)$. Hence, the value

$$\underline{P}(F|K) = \frac{\underline{m}_\alpha(F \cap K)}{\underline{m}_\alpha(F \cap K) + \overline{m}_\alpha(F^c \cap K)} = 0$$

is consistent with Theorem 1. □

The characterization given in Theorem 2 provides a *generalized Bayesian conditioning rule* corresponding to the one originally introduced in [38] for 2-monotone capacities, and is a generalization of Theorem 2 in [6], the latter holding for finite spaces. The Bayesian conditioning rule has been discussed for belief functions in [11, 16, 19] and for $n$-monotone capacities, with $n \geq 2$, in [12, 36]. In the quoted papers, the starting point is a bounded $n$-monotone capacity $\varphi$ with dual capacity $\psi$, for which the Bayesian conditioning rule produces a conditional bounded $n$-monotone capacity $\varphi(\cdot|K)$, provided the denominator is positive.

The rule given in Theorem 2 covers also the case in which the denominator is zero, but relies on two linearly ordered classes of possibly unbounded totally monotone and totally alternating capacities, respectively.

Figure 1 shows the relationships between the classes $\{m_\alpha : \alpha \in I\}$, $\{\underline{m}_\alpha : \alpha \in I\}$ and $\{\overline{m}_\alpha : \alpha \in I\}$, and the conditional measures $P(\cdot|\cdot)$ and $\underline{P}(\cdot|\cdot)$. Notice that the diagram commutes in the sense that we can arrive to $\underline{P}(\cdot|\cdot)$ either by generating the classes $\{\underline{m}_\alpha : \alpha \in I\}$ and $\{\overline{m}_\alpha : \alpha \in I\}$ from $\{m_\alpha : \alpha \in I\}$ and then applying the generalized Bayesian conditioning rule (Theorem 2), or directly by the conditional inner and outer measures determined by $P(\cdot|\cdot)$ (Theorem 1).

The following example shows a conditional event $F|K \in \wp(\Omega) \times \wp(\Omega)^0$ for which there is no $\alpha \in I$ such that $\underline{m}_\alpha(F \cap K) + \overline{m}_\alpha(F^c \cap K) \in (0, +\infty)$.

$$
\begin{array}{ccc}
\{m_\alpha \,:\, \alpha \in I\} \text{ additive} & \longrightarrow & \begin{array}{l}\{\underline{m}_\alpha \,:\, \alpha \in I\} \text{ totally monotone}\\ \{\overline{m}_\alpha \,:\, \alpha \in I\} \text{ totally alternating}\end{array} \\
\downarrow & & \downarrow \\
P(\cdot|\cdot) \text{ additive} & \longrightarrow & \underline{P}(\cdot|\cdot) \text{ totally monotone}
\end{array}
$$

Figure 1: Diagram showing the relations determined by Theorems 1 and 2

**Example 2.** *Identify $\Omega$ with $\mathbb{N}$ and let $\mathcal{A}$ be the algebra of finite-cofinite subsets of $\mathbb{N}$ and $P(\cdot|\cdot)$ the full conditional probability on $\mathcal{A}$ defined for every $E|H \in \mathcal{A} \times \mathcal{A}^0$ as*

$$
P(E|H) = \begin{cases} \frac{\operatorname{card} E\cap H}{\operatorname{card} H} & \text{if } H \text{ is finite,} \\ 0 & \text{if } E \text{ and } H^c \text{ are finite,} \\ 1 & \text{if } E \text{ and } H \text{ are cofinite.} \end{cases}
$$

*A dimensionally ordered class on $\mathcal{A}$ representing $P(\cdot|\cdot)$ is $\{m_0, m_1\}$ where $m_0$ and $m_1$ are defined for every $E \in \mathcal{A}$ as*

$$
m_0(E) = \begin{cases} 0 & \text{if } E \text{ is finite,} \\ 1 & \text{if } E \text{ is cofinite,} \end{cases} \quad \text{and} \quad m_1(E) = \begin{cases} \operatorname{card} E & \text{if } E \text{ is finite,} \\ +\infty & \text{if } E \text{ is cofinite.} \end{cases}
$$

*Let $\wp(\Omega)$ be the power set of $\mathbb{N}$, and $\{\underline{m}_0, \underline{m}_1\}$ and $\{\overline{m}_0, \overline{m}_1\}$ the corresponding classes of inner and outer measures on $\wp(\Omega)$. Take $F = \{2n \,:\, n \in \mathbb{N}\}$ and $K = \{2n \,:\, n \in \mathbb{N}\} \cup \{1, 3\}$.*

*We have $\underline{m}_0(F \cap K) = \overline{m}_0(F^c \cap K) = 0$, $\underline{m}_1(F \cap K) = +\infty$ and $\overline{m}_1(F^c \cap K) = 2$, so there is no $\alpha \in I$ such that $\underline{m}_\alpha(F\cap K) + \overline{m}_\alpha(F^c \cap K) \in (0, +\infty)$. Theorem 2 implies that $\underline{P}(F|K) = 0$, indeed every $A \in \mathcal{A}^0$ such that $K \subseteq A$ must be a cofinite set of $\mathbb{N}$, for which $m_0(A) = 1$. Moreover, we have*

$$
\begin{aligned}
\underline{P}(K|A) &= \sup\{P(B|A) \,:\, B \subseteq K, B \in \mathcal{A}\} \\
&= \sup\left\{\frac{m_0(B)}{m_0(A)} \,:\, B \subseteq K, B \in \mathcal{A}\right\} = \underline{m}_0(K) = 0.
\end{aligned}
$$

*Hence, we have $\underline{P}(K|A) = 0$ for every $A \in \mathcal{A}^0$ such that $K \subseteq A$ and Theorem 1 implies $\underline{P}(F|K) = 0$.* ■

The following example shows a conditional event $F|K \in \wp(\Omega) \times \wp(\Omega)^0$ for which there is $\alpha \in I$ such that $\underline{m}_\alpha(F \cap K) + \overline{m}_\alpha(F^c \cap K) \in (0, +\infty)$, in the case $\underline{P}(K|A) = 0$ for every $A \in \mathcal{A}^0$ such that $K \subseteq A$.

**Example 3.** *Let $\Omega = \{\omega_1, \ldots, \omega_8\}$ and $\mathcal{A}$ be the finite sub-algebra of $\wp(\Omega)$ with set of atoms $\mathcal{C}_\mathcal{A} = \{C_1, \ldots, C_4\}$, where $C_i = \{\omega_{2i-1}, \omega_{2i}\}$, for $i = 1, \ldots, 4$. Let $P(\cdot|\cdot)$ be the full conditional probability on $\mathcal{A}$ represented by the dimensionally ordered class $\{m_0, m_1\}$ on $\mathcal{A}$ whose distributions on $\mathcal{C}_\mathcal{A}$ are*

| $\mathcal{C}_\mathcal{A}$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|---|---|---|---|---|
| $m_0$ | 1 | 2 | 0 | 0 |
| $m_1$ | $+\infty$ | $+\infty$ | 3 | 1 |

*Taking* $E = \{\omega_6, \omega_7, \omega_8\}$ *and* $H = C_3 \cup C_4$*, simple computations show that*

$$\underline{P}(E|H) = \frac{\underline{m}_1(E \cap H)}{\underline{m}_1(E \cap H) + \overline{m}_1(E^c \cap H)} = \sup\{P(B|H) \ : \ B \subseteq E, B \in \mathcal{A}\} = \frac{1}{4}.$$

*Now, let* $F = \{\omega_3\}$ *and* $K = \{\omega_3, \omega_5\}$*, it holds* $\underline{m}_0(F \cap K) = \overline{m}_0(F^c \cap K) = \underline{m}_1(F \cap K) = 0$ *and* $\overline{m}_1(F^c \cap K) = 3$*, thus* $\underline{m}_1(F \cap K) + \overline{m}_1(F^c \cap K) \in (0, +\infty)$ *and Theorem 2 implies*

$$\underline{P}(F|K) = \frac{\underline{m}_1(F \cap K)}{\underline{m}_1(F \cap K) + \overline{m}_1(F^c \cap K)} = 0.$$

*Every* $A \in \mathcal{A}^0$ *such that* $K \subseteq A$ *must be such that* $A \supseteq C_2 \cup C_3 = \{\omega_3, \omega_4, \omega_5, \omega_6\}$*, so* $m_0(A) \in (0, +\infty)$ *and*

$$\begin{aligned} \underline{P}(K|A) &= \sup\{P(B|A) \ : \ B \subseteq K, B \in \mathcal{A}\} \\ &= \sup\left\{\frac{m_0(B)}{m_0(A)} \ : \ B \subseteq K, B \in \mathcal{A}\right\} = \frac{m_0(K)}{m_0(A)} = 0, \end{aligned}$$

*which by Theorem 1 implies* $\underline{P}(F|K) = 0$. ■

The following example shows that: *(i)* countable additivity cannot be generally preserved in the extension of a full conditional probability; *(ii)* the conditional lower and upper envelopes are generally not continuous totally monotone and totally alternating capacities when $\mathcal{A}$ is not a $\sigma$-algebra.

**Example 4.** *Identify* $\Omega$ *with* $[0, 1]$*, and let* $\mathcal{A}$ *and* $\mathcal{A}'$ *be, respectively, the algebra of finite unions of subintervals of* $[0, 1]$ *and the Borel* $\sigma$*-algebra on* $[0, 1]$.

*Let* $P(\cdot|\cdot)$ *be the full conditional probability on* $\mathcal{A}$ *generated by the dimensionally ordered class* $\{m_\alpha \ : \ \alpha \in \{-1\} \cup [0, 1]\}$ *on* $\mathcal{A}$*, where:*

- $m_{-1}$ *coincides with the restriction of the Lebesgue measure on* $\mathcal{A} = \mathcal{I}_{-1}$*;*
- $\mathcal{I}_0$ *is the ideal of finite subsets of* $[0, 1]$ *and* $m_0$ *coincides with the restriction of the Dirac measure* $\delta_0$ *on* $\mathcal{I}_0$ *and is* $+\infty$ *on* $\mathcal{A} \setminus \mathcal{I}_0$*;*
- *for* $\alpha \in (0, 1]$*,* $\mathcal{I}_\alpha = \mathcal{I}_0 \setminus \{E \in \mathcal{A} \ : \ \beta \in E, 0 \leq \beta < \alpha\}$ *and* $m_\alpha$ *coincides with the restriction of the Dirac measure* $\delta_\alpha$ *on* $\mathcal{I}_\alpha$ *and is* $+\infty$ *on* $\mathcal{A} \setminus \mathcal{I}_\alpha$.

*Since, for* $\alpha \in \{-1\} \cup [0, 1]$*,* $m_\alpha$ *is bounded and countably additive on* $\mathcal{I}_\alpha$*, it follows (see [1, 8, 31]) that* $P(\cdot|H)$ *is countably additive on* $\mathcal{A}$*, for every* $H \in \mathcal{A}^0$.

*In the case only finite additivity is required,* $P(\cdot|\cdot)$ *can be extended to a full conditional probability on* $\mathcal{A}'$*, even though this extension is not unique. To see this, consider the classes of inner and outer measures* $\{\underline{m}_\alpha \ : \ \alpha \in \{-1\} \cup [0, 1]\}$ *and* $\{\overline{m}_\alpha \ : \ \alpha \in \{-1\} \cup [0, 1]\}$ *induced on* $\wp(\Omega)$ *by* $\{m_\alpha \ : \ \alpha \in \{-1\} \cup [0, 1]\}$*. For* $k \in [0, 1)$*, let* $E_k = [0, k] \cup (\mathbb{Q} \cap (k, 1])$ *for which it holds*

$$\underline{m}_{-1}(E_k) = k, \quad \overline{m}_{-1}(E_k) = 1, \quad \underline{m}_{-1}(E_k^c) = 0, \quad \overline{m}_{-1}(E_k^c) = 1 - k,$$

*that implies*

$$\underline{P}(E_k|\Omega) = \underline{m}_{-1}(E_k) = k \quad \textit{and} \quad \overline{P}(E_k|\Omega) = \overline{m}_{-1}(E_k) = 1.$$

*Thus, by Example 1 we have that* $\underline{P}(\cdot|\Omega)$ *and* $\overline{P}(\cdot|\Omega)$ *are, respectively, a normalized totally monotone and a normalized totally alternating capacity on* $\mathcal{A}'$ *but are not continuous. In particular, each extension* $\tilde{P}$ *of* $P$ *on* $\mathcal{A}' \times \mathcal{A}'^0$ *can be further extended (not in a unique way) to a full conditional probability on the whole* $\wp(\Omega) \times \wp(\Omega)^0$.

*On the other hand, the countably additive full conditional probability* $P(\cdot|\cdot)$ *on* $\mathcal{A}$ *can be extended to a countably additive full conditional probability* $Q(\cdot|\cdot)$ *on* $\mathcal{A}'$*. In turn, such an extension is represented by the dimensionally ordered class* $\{n_\alpha : \alpha \in \{-1\} \cup [0,1]\}$ *on* $\mathcal{A}'$*, where:*

- $n_{-1}$ *coincides with the Lebesgue measure on* $\mathcal{A}' = \mathcal{J}_{-1}$*;*
- $\mathcal{J}_0$ *is the* $\sigma$*-ideal of* $n_{-1}$*-null sets and* $n_0$ *coincides with the restriction of the Dirac measure* $\delta_0$ *on* $\mathcal{J}_0$ *and is* $+\infty$ *on* $\mathcal{A}' \setminus \mathcal{J}_0$*;*
- *for* $\alpha \in (0,1]$, $\mathcal{J}_\alpha = \mathcal{J}_0 \setminus \{E \in \mathcal{A}' : \beta \in E, 0 \leq \beta < \alpha\}$ *and* $n_\alpha$ *coincides with the restriction of the Dirac measure* $\delta_\alpha$ *on* $\mathcal{J}_\alpha$ *and is* $+\infty$ *on* $\mathcal{A} \setminus \mathcal{J}_\alpha$*.*

*Notice that such extension is such that* $Q(\mathbb{Q} \cap [0,1]|\Omega) = \frac{n_{-1}(\mathbb{Q}\cap[0,1])}{n_{-1}(\Omega)} = 0$*, moreover, if* $C$ *is the Cantor set in* $[0,1]$*, it holds* $n_{-1}(C) = 0$ *and* $n_0(C) = 1$*, thus* $Q(\mathbb{Q} \cap [0,1]|C) = \frac{n_0(\mathbb{Q}\cap[0,1]\cap C)}{n_0(C)} = 1$*. Nevertheless,* $Q(\cdot|\cdot)$ *cannot be further extended to a countably additive full conditional probability on* $\wp(\Omega)$*, while a (not unique) finitely additive full conditional probability on* $\wp(\Omega)$ *extending* $Q(\cdot|\cdot)$ *exists.*

*In particular, if* $\mathcal{Q}$ *is the set of (finitely additive) full conditional probabilities on* $\wp(\Omega)$ *extending* $Q(\cdot|\cdot)$ *with* $\underline{Q} = \min \mathcal{Q}$ *and* $\overline{Q} = \max \mathcal{Q}$*, by Propositions 3 and 4 we have that, for every* $K \in \wp(\Omega)^0$*,* $\underline{Q}(\cdot|K)$ *and* $\overline{Q}(\cdot|K)$ *are, respectively, a normalized totally monotone capacity continuous from above and a normalized totally alternating capacity continuous from below on* $\wp(\Omega)$*.* ■

Let $\mathbb{L}(\Omega, \mathcal{A})$ be the linear space of $\mathcal{A}$-*continuous* real-valued functions on $\Omega$ [2] and $\mathbb{L}(\Omega) := \mathbb{L}(\Omega, \wp(\Omega))$ the linear super-space of bounded real-valued functions on $\Omega$. Both $\mathbb{L}(\Omega)$ and $\mathbb{L}(\Omega, \mathcal{A})$ are *lattices* (see [10, 37]), i.e., they are closed under pointwise minimum $\wedge$ and maximum $\vee$. The elements of $\mathbb{L}(\Omega, \mathcal{A}) \times \mathcal{A}^0$, denoted as $f|H$'s, are usually called *conditional gambles* (see, e.g., [37]).

Following the terminology of [14], every full conditional probability $P$ on the algebra $\mathcal{A}$ is in bijection with a *full conditional prevision functional on* $\mathbb{L}(\Omega, \mathcal{A})$ (see, e.g., [29, 40]) defined, for every $f|H \in \mathbb{L}(\Omega, \mathcal{A}) \times \mathcal{A}^0$, as

$$\mathbf{P}(f|H) = \int f(\omega) P(\mathrm{d}\omega|H),$$

where the integral is of Stieltjes type [2]. By the same argument, every full conditional probability $\tilde{P}$ on $\wp(\Omega)$ extending $P$ is in bijection with a full conditional prevision functional $\tilde{\mathbf{P}}$ on $\mathbb{L}(\Omega)$. Hence, the class $\mathcal{P}$ of extensions of $P$

determines a class $\mathfrak{P}$ of full conditional previsions on $\mathbb{L}(\Omega)$ extending $\mathbf{P}$ which is a compact subset of $\mathbb{R}^{\mathbb{L}(\Omega)\times\wp(\Omega)^0}$ endowed with the product topology, with envelopes $\underline{\mathbf{P}} = \min \mathfrak{P}$ and $\overline{\mathbf{P}} = \max \mathfrak{P}$, which are *lower* and *upper full conditional prevision functionals on* $\mathbb{L}(\Omega)$ [40].

For every $K \in \wp(\Omega)^0$, the envelopes $\underline{\mathbf{P}}(\cdot|K)$ and $\overline{\mathbf{P}}(\cdot|K)$ reveal to be a totally monotone functional and a totally alternating functional on $\mathbb{L}(\Omega)$, respectively, i.e., they satisfy (see [10]), for every $n \geq 2$ and for every $f_1, \ldots, f_n \in \mathbb{L}(\Omega)$,

**(TM')** $\underline{\mathbf{P}}\left(\bigvee_{i=1}^n f_i|K\right) \geq \sum_{\emptyset\neq I\subseteq\{1,\ldots,n\}}(-1)^{|I|+1}\underline{\mathbf{P}}\left(\bigwedge_{i\in I} f_i|K\right)$;

**(TA')** $\overline{\mathbf{P}}\left(\bigwedge_{i=1}^n f_i|K\right) \leq \sum_{\emptyset\neq I\subseteq\{1,\ldots,n\}}(-1)^{|I|+1}\overline{\mathbf{P}}\left(\bigvee_{i\in I} f_i|K\right)$.

In particular, both conditional functionals have a Choquet integral expression as stated in the following proposition: we focus on $\underline{\mathbf{P}}$ since the characterization of $\overline{\mathbf{P}}$ follows by *duality* being, for every $f|K \in \mathbb{L}(\Omega)\times\wp(\Omega)^0$, $\overline{\mathbf{P}}(f|K) = -\underline{\mathbf{P}}(-f|K)$.

**Proposition 5.** *The lower envelope $\underline{\mathbf{P}}(\cdot|\cdot)$ is such that $\underline{\mathbf{P}}(\cdot|K)$ is a totally monotone functional on $\mathbb{L}(\Omega)$, for every $K \in \wp(\Omega)^0$, moreover, for every $f \in \mathbb{L}(\Omega)$ it holds*

$$\underline{\mathbf{P}}(f|K) = \oint f(\omega)\underline{P}(\mathrm{d}\omega|K).$$

*Proof.* Let $\mathcal{P}$ the set of extensions of $P$ on $\wp(\Omega)\times\wp(\Omega)^0$ and $\mathfrak{P}$ the corresponding set of extensions of $\mathbf{P}$ on $\mathbb{L}(\Omega)\times\wp(\Omega)^0$. For every $f|K \in \mathbb{L}(\Omega)\times\wp(\Omega)^0$, it holds

$$\underline{\mathbf{P}}(f|K) = \min_{\tilde{\mathbf{P}}\in\mathfrak{P}} \tilde{\mathbf{P}}(f|H) = \min_{\tilde{P}\in\mathcal{P}} \int f(\omega)\tilde{P}(\mathrm{d}\omega|K) = \oint f(\omega)\underline{P}(\mathrm{d}\omega|K),$$

where the last equality follows by our Proposition 3 and Proposition 3 in [32]. Finally, the total monotonicity of $\underline{\mathbf{P}}(\cdot|K)$ follows by our Proposition 3 and Corollary 6.17 in [37]. □

**Remark 1.** *The function $\underline{\mathbf{P}}$ defined above is a coherent lower conditional prevision in the sense of Williams since it is the lower envelope of a class $\mathfrak{P}$ of full conditional previsions on $\mathbb{L}(\Omega)$, which are (trivially) coherent conditional previsions in the sense of Williams [40]. Notice that the function $\underline{\mathbf{P}}$ can be interpreted as the natural extension of $\mathbf{P}$ in the jargon of [39], i.e., the pointwise minimal Williams-coherent lower conditional prevision extending the full conditional prevision $\mathbf{P}$, the latter being (trivially) a coherent conditional prevision in the sense of Williams.*

## 5. Interpretation of lower and upper full conditional probabilities

The results proven in Section 4 allow to interpret the lower and upper conditional probabilities induced by a multivalued mapping in terms of extensions of a suitable full conditional probability space, in analogy to the construction carried on in Section 2.

At this aim, consider a full conditional probability space $(X, \mathcal{F}\times\mathcal{F}^0, \mu)$ and a conditional measurable space $(Y, \mathcal{G}\times\mathcal{G}^0)$, where both $\mathcal{F}$ and $\mathcal{G}$ are algebras of subsets of $X$ and $Y$, respectively, and let $\Gamma$ be a multivalued mapping from $X$ to $Y$ satisfying **(A1)** and **(A2)**.

Let $\{\nu_\alpha : \alpha \in I\}$ be a dimensionally ordered class on $\mathcal{F}$ representing $\mu(\cdot|\cdot)$ and denote with $\{\underline{\nu}_\alpha : \alpha \in I\}$ and $\{\overline{\nu}_\alpha : \alpha \in I\}$ the corresponding classes of inner and outer measures on $\wp(X)$.

We define, for every $A|B \in \mathcal{G} \times \mathcal{G}^0$, $\mu_*(A|B) = 1$ when $A \cap B = B$, and otherwise

$$\mu_*(A|B) = \begin{cases} \frac{\underline{\nu}_\alpha((A\cap B)_*)}{\underline{\nu}_\alpha((A\cap B)_*)+\overline{\nu}_\alpha((A^c\cap B)^*)} & \text{if there is } \alpha \in I \text{ such that} \\ & \underline{\nu}_\alpha((A\cap B)_*) + \overline{\nu}_\alpha((A^c \cap B)^*) \in (0,+\infty), \\ 0 & \text{otherwise,} \end{cases}$$

and $\mu^*(A|B) = 1 - \mu_*(A^c|B)$, where the complements are taken in $Y$.

The following proposition holds.

**Proposition 6.** *There exists a class $\widetilde{\mathcal{M}}$ of full conditional probabilities on $\mathcal{G}$ such that $\mu_* = \min \widetilde{\mathcal{M}}$, moreover, for every $B \in \mathcal{G}^0$, $\mu_*(\cdot|B)$ is a normalized totally monotone capacity on $\mathcal{G}$.*

*Proof.* Let $\Omega$, $\mathcal{A}$ and $\mathcal{A}'$ be defined as in Section 2, and set, for every $E|H \in \mathcal{F} \times \mathcal{F}^0$, $P((E\times Y)\cap\Omega|(H\times Y)\cap\Omega) = \mu(E|H)$, obtaining the spaces $(\Omega, \mathcal{A}\times\mathcal{A}^0, P)$ and $(\Omega, \mathcal{A}' \times \mathcal{A}'^0)$ which are isomorphic to $(X, \mathcal{F} \times \mathcal{F}^0, \mu)$ and $(Y, \mathcal{G} \times \mathcal{G}^0)$, respectively.

Thus $\mu(\cdot|\cdot)$ is in bijection with $P(\cdot|\cdot)$, and the dimensionally ordered class $\{\nu_\alpha : \alpha \in I\}$ on $\mathcal{F}$ representing $\mu(\cdot|\cdot)$ is in bijection with a dimensionally ordered class $\{m_\alpha : \alpha \in I\}$ on $\mathcal{A}$ representing $P(\cdot|\cdot)$. Moreover, for every $A|B \in \mathcal{G} \times \mathcal{G}^0$ and every $\alpha \in I$, it holds $\underline{\nu}_\alpha((A \cap B)_*) = \underline{m}_\alpha((X \times (A \cap B)) \cap \Omega)$ and

$$\underline{\nu}_\alpha((A\cap B)_*)+\overline{\nu}_\alpha((A^c\cap B)^*) = \underline{m}_\alpha((X\times(A\cap B))\cap\Omega)+\overline{m}_\alpha((X\times(A^c\cap B))\cap\Omega),$$

where the complements are taken in $Y$.

Now, let $\mathcal{M}$ and $\mathcal{P}$ be, respectively, the set of full conditional probabilities on $\wp(X)$ extending $\mu$ and the set of full conditional probabilities on $\wp(\Omega)$ extending $P$, whose envelopes are $\underline{\mu} = \min \mathcal{M}$, $\overline{\mu} = \max \mathcal{M}$, $\underline{P} = \min \mathcal{P}$ and $\overline{P} = \max \mathcal{P}$. Simple computations show that, for every $A|B \in \mathcal{G} \times \mathcal{G}^0$,

$$\mu_*(A|B) = \underline{\mu}((A \cap B)_*|(A \cap B)_* \cup (A^c \cap B)^*) = \underline{P}((X \times A) \cap \Omega|(X \times B) \cap \Omega),$$

so $\mu_*(\cdot|\cdot)$ defined on $(Y, \mathcal{G}\times\mathcal{G}^0)$ coincides with the restriction of $\underline{P}(\cdot|\cdot)$ on $\mathcal{A}'\times\mathcal{A}'^0$ by Theorem 2.

Hence, for every $B \in \mathcal{G}^0$, Proposition 3 implies that $\mu_*(\cdot|B)$ is a normalized totally monotone capacity on $\mathcal{G}$. Finally, for every $\tilde{P} \in \mathcal{P}$, define the full conditional probability $\tilde{\mu}$ on $\mathcal{G}$ setting, for every $A|B \in \mathcal{G} \times \mathcal{G}^0$, $\tilde{\mu}(A|B) = \tilde{P}((X \times A) \cap \Omega|(X \times B) \cap \Omega)$ and denote with $\widetilde{\mathcal{M}} = \{\tilde{\mu} : \tilde{P} \in \mathcal{P}\}$, for which it holds that $\mu_* = \min \widetilde{\mathcal{M}}$. $\square$

Let us notice that if $(X, \mathcal{F} \times \mathcal{F}^0, \mu)$ is a countably additive full conditional probability space and $\mathcal{F}$ is a $\sigma$-algebra, then $\mu_*(\cdot|B)$ is continuous from above by Proposition 4. Since $\mathcal{G}$ is only required to be an algebra of subsets of $Y$, then continuity from above is understood to hold only on decreasing sequences of elements of $\mathcal{G}$ whose limit belongs to $\mathcal{G}$. Actually, by the proof of Proposition 6 we have that $\mu_*(\cdot|\cdot)$ can be extended to the whole $\wp(Y) \times \wp(Y)^0$ by preserving total monotonicity and continuity from above (as is asked in [34] to unconditional belief functions to be continuous).

The following example shows the construction of $\mu_*$ and $\mu^*$ staring from $(X, \mathcal{F} \times \mathcal{F}^0, \mu)$ and $\Gamma : X \to \wp(Y)$.

**Example 5.** *Let $X = Y = \mathbb{N}$ and $\mathcal{F} = \mathcal{G}$ be the algebra of finite-cofinite subsets of $\mathbb{N}$. Take $(X, \mathcal{F} \times \mathcal{F}, \mu)$ and $(Y, \mathcal{G} \times \mathcal{G}^0)$, where $\mu$ and $\{\nu_0, \nu_1\}$ coincide with the full conditional probability $P$ and the dimensionally ordered class $\{m_0, m_1\}$ of Example 2. Let $\Gamma$ be the multivalued mapping defined, for every $n \in \mathbb{N}$, as $\Gamma(n) = \{k \in \mathbb{N} : k \geq n\}$.*

*For every $A \in \mathcal{G}$ we have that*

$$\begin{aligned} A_* &= \begin{cases} \emptyset & \text{if } A \text{ is finite,} \\ \{k \in \mathbb{N} : k > \max A^c\} & \text{if } A \text{ is cofinite,} \end{cases} \\ A^* &= \begin{cases} \{k \in \mathbb{N} : k \leq \max A\} & \text{if } A \text{ is finite,} \\ \mathbb{N} & \text{if } A \text{ is cofinite,} \end{cases} \end{aligned}$$

*so, it holds*

$$\begin{aligned} \underline{\nu}_0(A_*) = \overline{\nu}_0(A^*) &= \begin{cases} 0 & \text{if } A \text{ is finite,} \\ 1 & \text{if } A \text{ is cofinite,} \end{cases} \\ \underline{\nu}_1(A_*) &= \begin{cases} 0 & \text{if } A \text{ is finite,} \\ +\infty & \text{if } A \text{ is cofinite,} \end{cases} \\ \overline{\nu}_1(A^*) &= \begin{cases} \operatorname{card}\{k \in \mathbb{N} : k \leq \max A\} & \text{if } A \text{ is finite,} \\ +\infty & \text{if } A \text{ is cofinite.} \end{cases} \end{aligned}$$

*This implies that, for every $A|B \in \mathcal{G} \times \mathcal{G}^0$, we have*

$$\mu_*(A|B) = \begin{cases} 1 & \text{if } A \cap B = B \text{ or } A \cap B \text{ is cofinite,} \\ 0 & \text{otherwise,} \end{cases}$$

*and $\mu^*(A|B) = 1 - \mu_*(A^c|B)$, where the complements are taken in $Y$.*

The full conditional probability space $(X, \mathcal{F} \times \mathcal{F}^0, \mu)$ determines a full conditional prevision functional $\mathbf{M}$ defined, for every $f|A \in \mathbb{L}(X, \mathcal{F}) \times \mathcal{F}^0$, as

$$\mathbf{M}(f|A) = \int f(x)\mu(\mathrm{d}x|A).$$

In turn, $\mathbf{M}$ can be transported through $\Gamma$ to $\mathbb{L}(Y, \mathcal{G}) \times \mathcal{G}^0$ obtaining the lower and upper full conditional prevision functionals $\mathbf{M}_*$ and $\mathbf{M}^*$. The above interpretation of $\mu_*$ and $\mu^*$ in terms of extensions of the full conditional probability space

$(\Omega, \mathcal{A} \times \mathcal{A}^0, P)$ allows to apply Proposition 5 so, for every $B \in \mathcal{G}^0$, $\mathbf{M}_*(\cdot|B)$ is a totally monotone functional on $\mathbb{L}(Y, \mathcal{G})$ that can be expressed, for every $g \in \mathbb{L}(Y, \mathcal{G})$, as

$$\mathbf{M}_*(g|B) = \oint g(y)\mu_*(\mathrm{d}y|B).$$

An analogous Choquet integral expression holds for $\mathbf{M}^*(\cdot|B)$ with respect to $\mu^*(\cdot|B)$.

Figure 2 shows the relationships between the classes $\{\nu_\alpha : \alpha \in I\}$, $\{\underline{\nu}_\alpha : \alpha \in I\}$ and $\{\overline{\nu}_\alpha : \alpha \in I\}$, the full conditional probability $\mu(\cdot|\cdot)$, the lower full conditional probability $\mu_*(\cdot|\cdot)$, the full conditional prevision $\mathbf{M}(\cdot|\cdot)$ and the lower full conditional prevision $\mathbf{M}_*(\cdot|\cdot)$.

$$\begin{tikzcd}
\{\nu_\alpha : \alpha \in I\} \text{ additive} \arrow[r] \arrow[d] & \begin{array}{l}\{\underline{\nu}_\alpha : \alpha \in I\} \text{ totally monotone} \\ \{\overline{\nu}_\alpha : \alpha \in I\} \text{ totally alternating}\end{array} \arrow[d, "\Gamma"'] \\
\mu(\cdot|\cdot) \text{ additive} \arrow[r, "\Gamma"'] \arrow[d, leftrightarrow] & \mu_*(\cdot|\cdot) \text{ totally monotone} \arrow[d, leftrightarrow] \\
\mathbf{M}(\cdot|\cdot) \text{ linear} \arrow[r, "\Gamma"'] & \mathbf{M}_*(\cdot|\cdot) \text{ totally monotone}
\end{tikzcd}$$

Figure 2: Relationships between the classes $\{\nu_\alpha : \alpha \in I\}$, $\{\underline{\nu}_\alpha : \alpha \in I\}$ and $\{\overline{\nu}_\alpha : \alpha \in I\}$, $\mu(\cdot|\cdot)$, $\mu_*(\cdot|\cdot)$, $\mathbf{M}(\cdot|\cdot)$ and $\mathbf{M}_*(\cdot|\cdot)$

The following proposition holds.

**Proposition 7.** *There exists a class $\widetilde{\mathfrak{M}}$ of full conditional previsions on $\mathbb{L}(Y, \mathcal{G})$ such that $\mathbf{M}_* = \min \widetilde{\mathfrak{M}}$, moreover, for every $B \in \mathcal{G}^0$, $\mathbf{M}_*(\cdot|B)$ is a totally monotone lower prevision on $\mathbb{L}(Y, \mathcal{G})$.*

*Proof.* Consider the full conditional probability space $(\Omega, \mathcal{A} \times \mathcal{A}, P)$ and the conditional measurable space $(\Omega, \mathcal{A}' \times \mathcal{A}'^0)$ built in the proof of Proposition 6. We have that, if $\mathfrak{P}$ is the set of full conditional previsions on $\mathbb{L}(\Omega)$ extending the full conditional prevision $\mathbf{P}$ on $\mathbb{L}(\Omega, \mathcal{A})$ determined by $(\Omega, \mathcal{A} \times \mathcal{A}^0, P)$, then $\mathbf{M}_*$ coincides with the restriction of $\underline{\mathbf{P}} = \min \mathfrak{P}$ on $\mathbb{L}(\Omega, \mathcal{A}') \times \mathcal{A}'^0$, so, its total monotonicity follows by Proposition 5. Moreover, the class $\widetilde{\mathfrak{M}}$ of full conditional previsions on $\mathbb{L}(Y, \mathcal{G})$ is determined through the Stieltjes integral from the class $\widetilde{\mathcal{M}}$ of full conditional probabilities on $\mathcal{G}$ defined in the proof of Proposition 6. $\square$

The above results generalize to conditional measurable spaces the well-known construction due to Dempster [11] and reveal to be particularly relevant in Bayesian statistics, in random set theory and their applied fields.

## 6. Preservation of total monotonicity

The lower full conditional probability $\mu_*(\cdot|\cdot)$ on $\mathcal{G} \times \mathcal{G}^0$ of Section 5 has been obtained through a multivalued mapping, starting from the full conditional probability $\mu(\cdot|\cdot)$ on $\mathcal{F} \times \mathcal{F}^0$ which is, in particular, a totally monotone lower full conditional probability. Hence, the discussion carried on in the previous section highlights that a multivalued mapping satisfying assumptions **(A1)** and **(A2)** is a transformation that preserves total monotonicity in a conditional framework: this is in line with results presented in [9, 24] for the unconditional case.

In this section we study the above transportation problem starting from a totally monotone lower full conditional probability $\mu_*(\cdot|\cdot)$, obtained, in turn, by a previous transportation of a full conditional probability $\mu(\cdot|\cdot)$ defined on a different space.

At this aim, consider three conditional measurable spaces $(X, \mathcal{F} \times \mathcal{F}^0)$, $(Y, \mathcal{G} \times \mathcal{G}^0)$ and $(Z, \mathcal{H} \times \mathcal{H}^0)$, where $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are algebras of subsets of $X$, $Y$ and $Z$, respectively. Consider two multivalued mappings $\Gamma_1 : X \to \wp(Y)$ and $\Gamma_2 : Y \to \wp(Z)$ satisfying **(A1)** and **(A2)**. The *composition* of the multivalued mappings $\Gamma_1$ and $\Gamma_2$ is the multivalued mapping $\Gamma_2 \circ \Gamma_1 : X \to \wp(Z)$ defined, for every $x \in X$, as

$$(\Gamma_2 \circ \Gamma_1)(x) := \bigcup_{y \in \Gamma_1(x)} \Gamma_2(y).$$

For every $A \in \wp(Z)$ and $B \in \wp(Y)$, denote the lower and upper inverses determined by $\Gamma_1$, $\Gamma_2$ and $\Gamma_2 \circ \Gamma_1$ as

$$\begin{aligned} A_{*,2} := {\Gamma_2}_*(A) \quad &\text{and} \quad A^{*,2} := {\Gamma_2}^*(A), \\ B_{*,1} := {\Gamma_1}_*(B) \quad &\text{and} \quad B^{*,1} := {\Gamma_1}^*(B), \\ A_\circ := (\Gamma_2 \circ \Gamma_1)_*(A) \quad &\text{and} \quad A^\circ := (\Gamma_2 \circ \Gamma_1)^*(A). \end{aligned}$$

The following proposition has an immediate proof that is omitted.

**Proposition 8.** *The composition $\Gamma_2 \circ \Gamma_1$ is a multivalued mapping such that:*

*(i)* $\Gamma_2 \circ \Gamma_1$ *satisfies* **(A1)** *and* **(A2)***;*

*(ii)* $A_\circ = (A_{*,2})_{*,1}$ *and* $A^\circ = (A^{*,2})^{*,1}$*, for every* $A \in \wp(Z)$*.*

Now, consider a full conditional probability $\mu(\cdot|\cdot)$ on $(X, \mathcal{F} \times \mathcal{F}^0)$ represented by the dimensionally ordered class $\{\nu_\alpha : \alpha \in I\}$ and denote with $\{\underline{\nu}_\alpha : \alpha \in I\}$ and $\{\overline{\nu}_\alpha : \alpha \in I\}$ the corresponding classes of inner and outer measures on $\wp(X)$. The multivalued mapping $\Gamma_1$ induces the totally monotone lower full conditional probability $\mu_*(\cdot|\cdot)$ on $(Y, \mathcal{G} \times \mathcal{G}^0)$ that can be further transported to $(Z, \mathcal{H} \times \mathcal{H}^0)$ through the multivalued mapping $\Gamma_2$.

Denote with $\{\underline{\underline{\nu}}_\alpha : \alpha \in I\}$ and $\{\overline{\overline{\nu}}_\alpha : \alpha \in I\}$ the linearly ordered classes of possibly unbounded $[0, +\infty]$-valued functions defined on $\wp(Y)$ obtained by setting, for every $B \in \wp(Y)$,

$$\underline{\underline{\nu}}_\alpha(B) = \underline{\nu}_\alpha(B_{*,1}) \quad \text{and} \quad \overline{\overline{\nu}}_\alpha(B) = \overline{\nu}_\alpha(B^{*,1}).$$

Propositions 1 and 2 imply that, for every $\alpha \in I$, $\underline{\underline{\nu}}_\alpha$ and $\overline{\overline{\nu}}_\alpha$ are, respectively, totally monotone and totally alternating.

Notice that the totally monotone lower full conditional probability $\mu_*(\cdot|\cdot)$ can be expressed in terms of $\{\underline{\underline{\nu}}_\alpha : \alpha \in I\}$ and $\{\overline{\overline{\nu}}_\alpha : \alpha \in I\}$ setting, for every $A|B \in \mathcal{G} \times \mathcal{G}^0$, $\mu_*(A|B) = 1$ when $A \cap B = B$, and otherwise

$$\mu_*(A|B) = \begin{cases} \frac{\underline{\underline{\nu}}_\alpha(A\cap B)}{\underline{\underline{\nu}}_\alpha(A\cap B)+\overline{\overline{\nu}}_\alpha(A^c\cap B)} & \text{if there is } \alpha \in I \text{ such that} \\ & \underline{\underline{\nu}}_\alpha(A \cap B) + \overline{\overline{\nu}}_\alpha(A^c \cap B) \in (0, +\infty), \\ 0 & \text{otherwise.} \end{cases}$$

Again we set $\mu^*(A|B) = 1 - \mu_*(A^c|B)$, where the complements are taken in $Y$.

We define, for every $C|D \in \mathcal{H} \times \mathcal{H}^0$, $\mu_{**}(C|D) = 1$ when $C \cap D = D$, and otherwise

$$\mu_{**}(C|D) = \begin{cases} \frac{\underline{\underline{\nu}}_\alpha((C\cap D)_{*,2})}{\underline{\underline{\nu}}_\alpha((C\cap D)_{*,2})+\overline{\overline{\nu}}_\alpha((C^c\cap D)^{*,2})} & \text{if there is } \alpha \in I \text{ such that} \\ & \underline{\underline{\nu}}_\alpha((C \cap D)_{*,2}) + \overline{\overline{\nu}}_\alpha((C^c \cap D)^{*,2}) \in (0, +\infty), \\ 0 & \text{otherwise,} \end{cases}$$

and $\mu^{**}(C|D) = 1 - \mu_{**}(C^c|D)$, where the complements are taken in $Z$.

The following proposition holds.

**Proposition 9.** *There exists a class $\widetilde{\widetilde{\mathcal{M}}}$ of full conditional probabilities on $\mathcal{H}$ such that $\mu_{**} = \min \widetilde{\widetilde{\mathcal{M}}}$, moreover, for every $D \in \mathcal{H}^0$, $\mu_{**}(\cdot|D)$ is a normalized totally monotone capacity on $\mathcal{H}$.*

*Proof.* Consider the set

$$\Omega = (X \times Y \times Z) \setminus \left[ \bigcup_{x \in X} (\{x\} \times (Y \setminus \Gamma_1(x)) \times Z) \cup \bigcup_{y \in Y} (X \times \{y\} \times (Z \setminus \Gamma_2(y))) \right],$$

and take the algebras of its subsets $\mathcal{A} = \{(A \times Y \times Z) \cap \Omega : A \in \mathcal{F}\}$, $\mathcal{A}' = \{(X \times B \times Z) \cap \Omega : B \in \mathcal{G}\}$ and $\mathcal{A}'' = \{(X \times Y \times C) \cap \Omega : C \in \mathcal{H}\}$, that under conditions **(A1)** and **(A2)** on $\Gamma_1$ and $\Gamma_2$ are isomorphic to $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$, respectively.

Setting, for every $E|H \in \mathcal{F} \times \mathcal{F}^0$, $P((E \times Y \times Z) \cap \Omega | (H \times Y \times Z) \cap \Omega) = \mu(E|H)$, we obtain the spaces $(\Omega, \mathcal{A} \times \mathcal{A}^0, P)$, $(\Omega, \mathcal{A}' \times \mathcal{A}'^0)$ and $(\Omega, \mathcal{A}'' \times \mathcal{A}''^0)$ which are isomorphic to $(X, \mathcal{F} \times \mathcal{F}^0, \mu)$, $(Y, \mathcal{G} \times \mathcal{G}^0)$ and $(Z, \mathcal{F} \times \mathcal{F}^0)$, respectively.

Thus $\mu(\cdot|\cdot)$ is in bijection with $P(\cdot|\cdot)$, and the dimensionally ordered class $\{\nu_\alpha : \alpha \in I\}$ on $\mathcal{F}$ representing $\mu(\cdot|\cdot)$ is in bijection with a dimensionally ordered class $\{m_\alpha : \alpha \in I\}$ on $\mathcal{A}$ representing $P(\cdot|\cdot)$. Moreover, for every $C|D \in \mathcal{H} \times \mathcal{H}^0$ and every $\alpha \in I$, it holds

$$\begin{aligned} \underline{\underline{\nu}}_\alpha((C \cap D)_{*,2}) + \overline{\overline{\nu}}_\alpha((C^c \cap D)^{*,2}) &= \underline{\nu}_\alpha(((C \cap D)_{*,2})_{*,1}) + \overline{\nu}_\alpha(((C^c \cap D)^{*,2})^{*,1}) \\ &= \underline{\nu}_\alpha((C \cap D)_\circ) + \overline{\nu}_\alpha((C^c \cap D)^\circ) \\ &= \underline{m}_\alpha((X \times Y \times (C \cap D)) \cap \Omega) + \overline{m}_\alpha((X \times Y \times (C^c \cap D)) \cap \Omega), \end{aligned}$$

where the complements are taken in $Z$, and $\underline{\nu}_\alpha((C \cap D)_{*,2}) = \underline{m}_\alpha((X \times Y \times (C \cap D)) \cap \Omega)$ follows analogously.

Now, let $\mathcal{M}$ and $\mathcal{P}$ be, respectively, the set of full conditional probabilities on $\wp(X)$ extending $\mu$ and the set of full conditional probabilities on $\wp(\Omega)$ extending $P$, whose envelopes are $\underline{\mu} = \min \mathcal{M}$, $\overline{\mu} = \max \mathcal{M}$, $\underline{P} = \min \mathcal{P}$ and $\overline{P} = \max \mathcal{P}$. Simple computations show that, for every $C|D \in \mathcal{H} \times \mathcal{H}^0$,

$$\mu_{**}(C|D) = \underline{\mu}((C \cap D)_\circ | (C \cap D)_\circ \cup (C^c \cap D)^\circ) = \underline{P}((X \times Y \times C) \cap \Omega | (X \times Y \times D) \cap \Omega),$$

so $\mu_{**}(\cdot|\cdot)$ defined on $(Z, \mathcal{H} \times \mathcal{H}^0)$ coincides with the restriction of $\underline{P}(\cdot|\cdot)$ on $\mathcal{A}'' \times \mathcal{A}''^0$ by Theorem 2.

Hence, for every $D \in \mathcal{H}^0$, Proposition 3 implies that $\mu_{**}(\cdot|D)$ is a normalized totally monotone capacity on $\mathcal{H}$.

Finally, for every $\tilde{P} \in \mathcal{P}$, define the full conditional probability $\tilde{\mu}$ on $\mathcal{H}$ setting, for every $C|D \in \mathcal{H} \times \mathcal{H}^0$, $\tilde{\mu}(C|D) = \tilde{P}((X \times Y \times C) \cap \Omega | (X \times Y \times D) \cap \Omega)$ and denote with $\widetilde{\widetilde{\mathcal{M}}} = \{\tilde{\mu} \,:\, \tilde{P} \in \mathcal{P}\}$, for which it holds that $\mu_{**} = \min \widetilde{\widetilde{\mathcal{M}}}$. $\square$

Figure 3 shows the relations between $\mu(\cdot|\cdot)$, $\mu_*(\cdot|\cdot)$, $\mu_{**}(\cdot|\cdot)$ singled out by $\Gamma_1$, $\Gamma_2$ and $\Gamma_2 \circ \Gamma_1$.

$$\begin{tikzcd} \mu(\cdot|\cdot) \text{ additive} \arrow[r, "\Gamma_1"'] \arrow[dr, "\Gamma_2\circ\Gamma_1"'] & \mu_*(\cdot|\cdot) \text{ totally monotone} \arrow[d, "\Gamma_2"'] \\ & \mu_{**}(\cdot|\cdot) \text{ totally monotone} \end{tikzcd}$$

Figure 3: Relationships between $\mu(\cdot|\cdot)$, $\mu_*(\cdot|\cdot)$ and $\mu_{**}(\cdot|\cdot)$

Since, for every $D \in \mathcal{H}^0$, $\mu_{**}(\cdot|D)$ is a normalized totally monotone capacity on $\mathcal{H}$ then we can define a corresponding conditional functional on $\mathbb{L}(Z, \mathcal{H}) \times \mathcal{H}^0$ setting, for every $h \in \mathbb{L}(Z, \mathcal{H})$,

$$\mathbf{M}_{**}(h|D) = \oint h(z) \mu_{**}(\mathrm{d}z|D).$$

An analogous Choquet integral expression holds for $\mathbf{M}^{**}(\cdot|D)$ with respect to $\mu^{**}(\cdot|D)$.

The following proposition holds.

**Proposition 10.** *There exists a class $\widetilde{\widetilde{\widetilde{\mathfrak{M}}}}$ of full conditional previsions on $\mathbb{L}(Z, \mathcal{H})$ such that $\mathbf{M}_{**} = \min \widetilde{\widetilde{\widetilde{\mathfrak{M}}}}$, moreover, for every $D \in \mathcal{H}^0$, $\mathbf{M}_{**}(\cdot|D)$ is a totally monotone lower prevision on $\mathbb{L}(Z, \mathcal{H})$.*

*Proof.* Consider the full conditional probability space $(\Omega, \mathcal{A} \times \mathcal{A}, P)$ and the conditional measurable space $(\Omega, \mathcal{A}'' \times \mathcal{A}''^0)$ built in the proof of Proposition 9. We have that, if $\mathfrak{P}$ is the set of full conditional previsions on $\mathbb{L}(\Omega)$ extending the full conditional prevision $\mathbf{P}$ on $\mathbb{L}(\Omega, \mathcal{A})$ determined by $(\Omega, \mathcal{A} \times \mathcal{A}^0, P)$, then $\mathbf{M}_{**}$ coincides with the restriction of $\underline{\mathbf{P}} = \min \mathfrak{P}$ on $\mathbb{L}(\Omega, \mathcal{A}'') \times \mathcal{A}''^0$, so, its total monotonicity follows by Proposition 5. Moreover, the class $\widetilde{\widetilde{\mathfrak{M}}}$ of full conditional previsions on $\mathbb{L}(Z,\mathcal{H})$ is determined through the Stieltjes integral by the class $\widetilde{\widetilde{\mathcal{M}}}$ of full conditional probabilities on $\mathcal{H}$ defined in the proof of Proposition 9. □

The following example shows the construction of $\mu_{**}(\cdot|\cdot)$ transporting $\mu_*(\cdot|\cdot)$ through $\Gamma_2$, the latter being obtained by transporting $\mu(\cdot|\cdot)$ through $\Gamma_1$. The same example also shows that $\mu_{**}(\cdot|\cdot)$ can be directly obtained transporting $\mu(\cdot|\cdot)$ through $\Gamma_2 \circ \Gamma_1$.

**Example 6.** *Take $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$, $Z = \{z_1, z_2, z_3\}$, $\mathcal{F} = \wp(X)$, $\mathcal{G} = \wp(Y)$ and $\mathcal{H} = \wp(Z)$, and consider the multivalued mappings $\Gamma_1 : X \to \wp(Y)$ and $\Gamma_2 : Y \to \wp(Z)$ such that*

$$\begin{array}{lll} \Gamma_1(x_1) = \{y_1, y_2\}, & \Gamma_1(x_2) = \{y_1\}, & \Gamma_1(x_3) = \{y_3\}, \\ \Gamma_2(y_1) = \{z_1, z_2\}, & \Gamma_2(y_2) = \{z_1, z_3\}, & \Gamma_2(y_3) = \{z_2\}, \end{array}$$

*that trivially satisfy* **(A1)** *and* **(A2)**. *The composition $\Gamma_2 \circ \Gamma_1$ is defined as*

$$(\Gamma_2 \circ \Gamma_1)(x_1) = Z, \quad (\Gamma_2 \circ \Gamma_1)(x_2) = \{z_1, z_2\}, \quad (\Gamma_2 \circ \Gamma_1)(x_3) = \{z_2\}.$$

*The lower and upper inverses determined by $\Gamma_1$, $\Gamma_2$ and $\Gamma_2 \circ \Gamma_1$ are*

| $\wp(Z)$ | $\emptyset$ | $\{z_1\}$ | $\{z_2\}$ | $\{z_3\}$ | $\{z_1, z_2\}$ | $\{z_1, z_3\}$ | $\{z_2, z_3\}$ | $Z$ |
|---|---|---|---|---|---|---|---|---|
| $(\cdot)_{*,2}$ | $\emptyset$ | $\emptyset$ | $\{y_3\}$ | $\emptyset$ | $\{y_1, y_3\}$ | $\{y_2\}$ | $\{y_3\}$ | $Y$ |
| $(\cdot)^{*,2}$ | $\emptyset$ | $\{y_1, y_2\}$ | $\{y_1, y_3\}$ | $\{y_2\}$ | $Y$ | $\{y_1, y_2\}$ | $Y$ | $Y$ |
| $(\cdot)_\circ$ | $\emptyset$ | $\emptyset$ | $\{x_3\}$ | $\emptyset$ | $\{x_2, x_3\}$ | $\emptyset$ | $\{x_3\}$ | $X$ |
| $(\cdot)^\circ$ | $\emptyset$ | $\{x_1, x_2\}$ | $X$ | $\{x_1\}$ | $X$ | $\{x_1, x_2\}$ | $X$ | $X$ |
| $\wp(Y)$ | $\emptyset$ | $\{y_1\}$ | $\{y_2\}$ | $\{y_3\}$ | $\{y_1, y_2\}$ | $\{y_1, y_3\}$ | $\{y_2, y_3\}$ | $Y$ |
| $(\cdot)_{*,1}$ | $\emptyset$ | $\{x_2\}$ | $\emptyset$ | $\{x_3\}$ | $\{x_1, x_2\}$ | $\{x_2, x_3\}$ | $\{x_3\}$ | $X$ |
| $(\cdot)^{*,1}$ | $\emptyset$ | $\{x_1, x_2\}$ | $\{x_1\}$ | $\{x_3\}$ | $\{x_1, x_2\}$ | $X$ | $\{x_1, x_3\}$ | $X$ |

*Consider the full conditional probability $\mu(\cdot|\cdot)$ on $\mathcal{F} \times \mathcal{F}^0$ determined by the dimensionally ordered class $\{\nu_0, \nu_1\}$ whose distributions on the set of atoms $\mathcal{C}_\mathcal{F}$ of $\mathcal{F}$ are*

| $\mathcal{C}_\mathcal{F}$ | $\{x_1\}$ | $\{x_2\}$ | $\{x_3\}$ |
|---|---|---|---|
| $\nu_0$ | 5 | 0 | 0 |
| $\nu_1$ | $+\infty$ | 3 | 1 |

*Since $\mathcal{F} = \wp(X)$ we have that $\underline{\nu}_\alpha = \overline{\nu}_\alpha = \nu_\alpha$, for $\alpha = 0, 1$. The totally monotone lower full conditional probability $\mu_*(\cdot|\cdot)$ on $\mathcal{G} \times \mathcal{G}^0$ determined by $\mu(\cdot|\cdot)$ is*

| $\mathcal{G}$ | $\emptyset$ | $\{y_1\}$ | $\{y_2\}$ | $\{y_3\}$ | $\{y_1, y_2\}$ | $\{y_1, y_3\}$ | $\{y_2, y_3\}$ | $Y$ |
|---|---|---|---|---|---|---|---|---|
| $\mu_*(\cdot\|\{y_1\})$ | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 |
| $\mu_*(\cdot\|\{y_2\})$ | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| $\mu_*(\cdot\|\{y_3\})$ | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| $\mu_*(\cdot\|\{y_1, y_2\})$ | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| $\mu_*(\cdot\|\{y_1, y_3\})$ | 0 | $\frac{3}{4}$ | 0 | 0 | $\frac{3}{4}$ | 1 | 0 | 1 |
| $\mu_*(\cdot\|\{y_2, y_3\})$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| $\mu_*(\cdot\|Y)$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |

*The classes of totally monotone and totally alternating measures* $\{\underline{\underline{\nu}}_0, \underline{\underline{\nu}}_1\}$ *and* $\{\overline{\overline{\nu}}_0, \overline{\overline{\nu}}_1\}$ *defined on* $\wp(Y)$ *determined by* $\{\nu_0, \nu_1\}$ *are*

| $\wp(Y)$ | $\emptyset$ | $\{y_1\}$ | $\{y_2\}$ | $\{y_3\}$ | $\{y_1, y_2\}$ | $\{y_1, y_3\}$ | $\{y_2, y_3\}$ | $Y$ |
|---|---|---|---|---|---|---|---|---|
| $\underline{\underline{\nu}}_0$ | 0 | 0 | 0 | 0 | 5 | 0 | 0 | 5 |
| $\overline{\overline{\nu}}_0$ | 0 | 5 | 5 | 0 | 5 | 5 | 5 | 5 |
| $\underline{\underline{\nu}}_1$ | 0 | 3 | 0 | 1 | $+\infty$ | 3 | 1 | $+\infty$ |
| $\overline{\overline{\nu}}_1$ | 0 | $+\infty$ | $+\infty$ | 1 | $+\infty$ | $+\infty$ | $+\infty$ | $+\infty$ |

*The classes* $\{\underline{\underline{\nu}}_0, \underline{\underline{\nu}}_1\}$ *and* $\{\overline{\overline{\nu}}_0, \overline{\overline{\nu}}_1\}$ *induce the totally monotone lower full conditional probability* $\mu_{**}(\cdot|\cdot)$ *on* $\mathcal{H} \times \mathcal{H}^0$ *below*

| $\mathcal{H}$ | $\emptyset$ | $\{z_1\}$ | $\{z_2\}$ | $\{z_3\}$ | $\{z_1, z_2\}$ | $\{z_1, z_3\}$ | $\{z_2, z_3\}$ | $Z$ |
|---|---|---|---|---|---|---|---|---|
| $\mu_{**}(\cdot|\{z_1\})$ | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 |
| $\mu_{**}(\cdot|\{z_2\})$ | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| $\mu_{**}(\cdot|\{z_3\})$ | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| $\mu_{**}(\cdot|\{z_1, z_2\})$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| $\mu_{**}(\cdot|\{z_1, z_3\})$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| $\mu_{**}(\cdot|\{z_2, z_3\})$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| $\mu_{**}(\cdot|Z)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

*Simple computations show that the same conditional measure* $\mu_{**}$ *is obtained transporting* $\mu$ *through* $\Gamma_2 \circ \Gamma_1$

## 7. Conclusions

The theory of random sets is usually based on topological and measurability assumptions and is used to transport unconditional probability measures. However, in many applications related to economics and statistics such as game theory, dynamic programming and econometrics, conditional probability spaces need to be considered.

In this paper, avoiding any measurability and topological assumption, multivalued mappings are used to transport a full conditional probability space $(X, \mathcal{F} \times \mathcal{F}^0, \mu)$ to another conditional measurable space $(Y, \mathcal{G} \times \mathcal{G}^0)$. The resulting lower conditional measure $\mu_*(\cdot|B)$ is a normalized totally monotone capacity on $\mathcal{G}$, for every $B \in \mathcal{G}^0$, which is continuous from above if $\mu$ is countably additive and $\mathcal{F}$ is a $\sigma$-algebra. The conditional measure $\mu_*$ has a generalized Bayesian conditioning rule representation in terms of two linearly ordered classes of possibly unbounded totally monotone and totally alternating capacities on $\wp(X)$. Moreover, $\mu_*$ determines a lower conditional prevision functional $\mathbf{M}_*$ on the set of $\mathcal{G}$-continuous conditional gambles $\mathbb{L}(Y, \mathcal{G}) \times \mathcal{G}^0$, which is totally monotone and has a Choquet integral expression in terms of $\mu_*$.

Next we consider the above transportation problem where the starting point is $(Y, \mathcal{G} \times \mathcal{G}^0, \mu_*)$ and the target space is $(Z, \mathcal{H} \times \mathcal{H}^0)$, with $\mu_*$ obtained by a previous transportation of the full conditional probability space $(X, \mathcal{F} \times \mathcal{F}^0, \mu)$. We show that also in this case a multivalued mapping from $Y$ to $Z$ determines a lower conditional measure $\mu_{**}$ that has a generalized Bayesian conditioning rule representation in terms of two linearly ordered classes of possibly unbounded totally monotone and totally alternating capacities on $\wp(Y)$. Moreover, $\mu_{**}$ determines a lower conditional prevision functional $\mathbf{M}_{**}$ on the set of $\mathcal{H}$-continuous conditional gambles $\mathbb{L}(Z,\mathcal{H})\times\mathcal{H}^0$, which is totally monotone and has a Choquet integral expression in terms of $\mu_{**}$.

Both transportation problems can be translated in an extension problem related to a suitable full conditional probability space $(\Omega,\mathcal{A}\times\mathcal{A}^0,P)$, for which a complete characterization is given.

Both $\mu_*$ and $\mu_{**}$ are the lower envelope of a class of full conditional probabilities, therefore, they are totally monotone Williams-coherent lower conditional probabilities. This shows that multivalued mappings are transformations that preserve both Williams-coherence and total monotonicity in a conditional setting: this is in line with results proved in [9, 24] in the unconditional case.

Both $\mu_*$ and $\mu_{**}$ can be obtained by a transportation of a full conditional probability $\mu$ through a multivalued mapping and their properties are essentially determined by the representation of $\mu$ through a dimensionally ordered class of possibly unbounded finitely additive measures $\{\nu_\alpha : \alpha\in I\}$. An open problem is to prove if for any totally monotone lower full conditional probability measure $\mu_*$ on a conditional measurable space $(Y,\mathcal{G}\times\mathcal{G}^0)$ there exist a full conditional probability space $(X,\mathcal{F}\times\mathcal{F}^0,\mu)$ and a multivalued mapping $\Gamma : X\to\wp(Y)$ inducing $\mu_*$.

## Acknowledgements

This work was partially supported by INdAM-GNAMPA through the Project 2016 U2016/000391. The authors would like to thank the anonymous referee for his valuable suggestions concerning Section 6.

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