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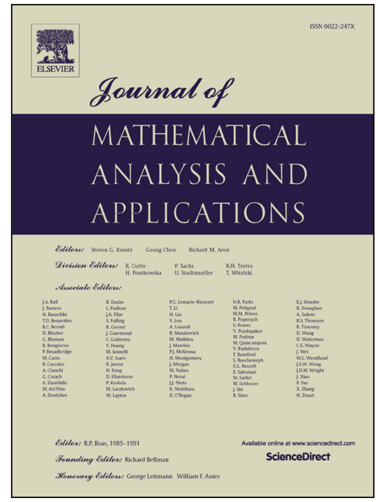

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Reverse Schwarz inequalities and some consequences in inner product spaces

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Abstract

Necessary and sufficient conditions for a general reverse of Schwarz's inequality in inner product spaces are given. Results are applied to the triangle inequality and Grušs' type inequality.

Keywords: Schwarz's inequality, Triangle inequality, Grüss' inequality
2010 MSC: 46C99

1. Introduction and motivation

Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). The inequality

$$|\langle x, y\rangle| \leq \|x\|\|y\|,\ x, y \in V, \tag{1}$$

is known in the literature as Schwarz's (or Cauchy-Schwarz or Cauchy-Bunyakovsky-Schwarz) inequality, where $\|v\|^2 = \langle v, v\rangle,\ v \in V$.

In 1925 Pólya and Szegö [20] proved probably the first reverse of Schwarz's inequality: if $x_i, y_i,\ i = 1, \ldots, n$ are positive numbers such that $0 < a \leq x_i \leq A < \infty$ and $0 < b \leq y_i \leq B < \infty$ for some constants a, A, b, B, then

$$\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 \leq \frac{(ab + AB)^2}{4abAB} \left(\sum_{i=1}^{n} x_i y_i\right)^2.$$

Dragomir [7, Theorem 2.2] obtained the following result

$$\|x\|\|y\| \leq \frac{|\Gamma + \gamma|}{2\sqrt{\mathrm{Re}(\Gamma\overline{\gamma})}} |\langle x, y\rangle|,\ x, y \in V, \tag{2}$$

if scalars Γ, γ satisfy $\mathrm{Re}(\Gamma\overline{\gamma}) > 0$ and

$$\mathrm{Re}\,\langle \Gamma y - x, x - \gamma y\rangle \geq 0 \tag{3}$$

or, equivalently,

$$\left\| x - \frac{\Gamma + \gamma}{2} y \right\| \leq \frac{1}{2}|\Gamma - \gamma|\|y\|. \tag{4}$$

The expressions $\frac{(ab+AB)^2}{4abAB}$ and $\frac{|\Gamma+\gamma|}{2\sqrt{\mathrm{Re}(\Gamma\overline{\gamma})}}$, apparent on the right in two inequalities above, are called Kantorovich type constants. It is because of Kantorovich inequality [17] that relates to both mentioned ones.

In moving towards the trigonometric interpretation of the inequality (2) on the assumption (4), $\frac{|\Gamma+\gamma|}{2\sqrt{\mathrm{Re}(\Gamma\overline{\gamma})}}$ is the secant of an angle in a right triangle with the length of the hypotenuse equals $\frac{|\Gamma+\gamma|}{2}\|y\|$ and the opposite side equal to $\frac{|\Gamma-\gamma|}{2}\|y\|$.

Presently, there is a lot of reverses of the Schwarz inequality under various conditions in the literature. We give some examples. For discrete variants of the inequality see [5]. Counterparts for integrals, isotone functionals and other extensions in the context of inner product spaces are considered in [1],[2],[8],[9],[10] and references therein. Some improvements, complements of Dragomir's result and other classical references the reader can find in [11]. Moreover, there exist many generalizations of the inequality in more abstract structures, see [12],[13],[14],

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[15],[16],[18],[19] and references therein. As a continuation of this research, inspired by the classic Pólya and Szegö inequality, we will obtain a natural generalization of Dragomir's result.

It is easily seen that for fixed $C \geq 1$ and real A and B:

$$A \leq BC \iff A - B \leq \frac{C-1}{C} A \leq (C-1)B. \tag{5}$$

By (5) the equivalent reformulation of (2) is as follows

$$\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \leq \left|\frac{\Gamma-\gamma}{\Gamma+\gamma}\right|^2 \|x\|^2\|y\|^2 \leq \frac{|\Gamma-\gamma|^2}{4\mathrm{Re}(\Gamma\overline{\gamma})}|\langle x, y\rangle|^2,\ x, y \in V. \tag{6}$$

The inequality between the first and last term in the above, see e.g. [7, Corollary 2.4].

Our interest are inequalities of the form (2) with an other constant $\theta \geq 1$ instead of the Kantorovich type constant $\frac{|\Gamma+\gamma|}{2\sqrt{\mathrm{Re}(\Gamma\overline{\gamma})}}$, i.e.

$$\|x\|\|y\| \leq \theta|\langle x, y\rangle|,\ x, y \in V, \tag{7}$$

or, by (5), in the following equivalent form

$$\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{\theta^2-1}{\theta^2}\|x\|^2\|y\|^2 \leq (\theta^2-1)|\langle x, y\rangle|^2,\ x, y \in V. \tag{8}$$

Notice, if $\theta = \sec\alpha$, $\frac{\theta^2-1}{\theta^2} = \sin^2\alpha$ and $\theta^2 - 1 = \tan^2\alpha$. This remark will be useful for the presentation of some results in this paper.

The first inequality in (8) is an additive form while the second one is called a multiplicative form of the reverse Schwarz inequality.

It is known that

$$\left.\begin{array}{l} \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2} = \left\|x - \frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2 \\ \|y\|^2 - \frac{|\langle x,y\rangle|^2}{\|x\|^2} = \left\|y - \frac{\langle y,x\rangle}{\|x\|^2}x\right\|^2 \end{array}\right\} \leq \|x-y\|^2,\ 0 \neq x, y \in V.$$

Hence we obtain the following general additive-multiplicative version of reverse Schwarz's inequality

$$\begin{aligned} \|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 &\leq \min\left\{\frac{\|x-y\|^2}{\|x\|^2},\ \frac{\|x-y\|^2}{\|y\|^2}\right\}\|x\|^2\|y\|^2 \leq \\ &\leq \min\left\{\frac{\|x-y\|^2}{\|x\|^2-\|x-y\|^2},\ \frac{\|x-y\|^2}{\|y\|^2-\|x-y\|^2}\right\}|\langle x, y\rangle|^2. \end{aligned} \tag{9}$$

The multiplicative part of the inequality under the additional assumption $\|x-y\| < \min\{\|x\|, \|y\|\}$. It is worth noting, if (3) or, equivalently, (4) is met, then by the inequality (9) we can directly derive the inequality (6).

A brief presentation of our results is below.

Firstly, in Theorem 1 we show that conditions similar to (3) or (4) are sufficient and with small modifications also necessary for the reverse of Schwarz inequality (7) and (8) to hold. In addition, some new necessary and sufficient conditions of the type are given. The theorem generalizes the important Dragomir's result (3) or (4)⇒(2) and gives the possibility to obtain an extended version of the inequality (9) (see Corollary 2).

Secondly, we present strengthened versions of the both mentioned inequalities (see Theorem 3 and Corollary 6).

Finally, some new versions of Grüss' type inequalities (Theorem 2, Proposition 1) and reverses of the triangle inequality (Theorem 4, Proposition 2) in inner product space settings are obtained. Moreover, known results in the subject are complemented or rediscovered (see Corollaries 1,3,7).

Few adequate examples support the main results in the paper.

2. Reverses of Schwarz inequality

For given $a, b \in V$ with $\mathrm{Re}\langle a, b\rangle > 0$ we define two real quantities

$$\kappa := \kappa_{a,b} = \frac{\|a+b\|}{2\sqrt{\mathrm{Re}\langle a, b\rangle}},\quad \tau := \tau_{a,b} = \frac{\|a-b\|}{2\sqrt{\mathrm{Re}\langle a, b\rangle}}.$$

If $a = \gamma v$ and $b = \Gamma v$ for $\gamma, \Gamma \in \mathbb{F}$ with $\mathrm{Re}(\Gamma\overline{\gamma}) > 0$ and $v \in V \setminus \{0\}$, then we get

$$\kappa := \kappa_{\gamma,\Gamma} = \frac{|\gamma + \Gamma|}{2\sqrt{\mathrm{Re}\Gamma\overline{\gamma}}}, \quad \tau := \tau_{\gamma,\Gamma} = \frac{|\gamma - \Gamma|}{2\sqrt{\mathrm{Re}\Gamma\overline{\gamma}}}.$$

It is clear that $\tau^2 + 1 = \kappa^2$ and $\kappa \geq 1,\ \tau \geq 0$. The quantity κ is known as Kantorovich type constant.

Given a nonzero vector $v \in V$ and $\tau \geq 0$, we define

$$C_{v,\tau} = \{c(v+h) : c \in \mathbb{F} \setminus \{0\}, h \in V, \langle v, h\rangle = 0, \|h\| \leq \tau\|v\|\}.$$

Theorem 1. *Let $\tau_i \geq 0$ and $\kappa_i \geq 1,\ i = 1,2$ be arbitrary with $\kappa_i^2 = \tau_i^2 + 1$ and $\tau_1\tau_2 < 1$.*

For nonzero vectors $x, y \in V$ the below conditions are mutually equivalent:

(i) *the following equivalent inequalities hold true*

$$\|x\|\|y\| \leq \frac{\kappa_1\kappa_2}{1-\tau_1\tau_2}|\langle x, y\rangle|, \tag{10}$$

$$\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{(\tau_1+\tau_2)^2}{\kappa_1^2\kappa_2^2}\|x\|^2\|y\|^2 \leq \frac{(\tau_1+\tau_2)^2}{(1-\tau_1\tau_2)^2}|\langle x, y\rangle|^2, \tag{11}$$

(ii) *there exist nonzero scalars $c_1, c_2 \in \mathbb{F}$ and a nonzero vector $v \in V$ such that*

$$\|c_1x - v\| \leq \frac{\tau_1}{\kappa_1}\|v\| \text{ and } \|c_2y - v\| \leq \frac{\tau_2}{\kappa_2}\|v\|, \tag{12}$$

(iii) *there exist nonzero scalars $c_1, c_2 \in \mathbb{F}$ and a nonzero vector $v \in V$ such that*

$$\|c_1x - v\| \leq \frac{\tau_1}{\kappa_1+1}\|c_1x + v\| \text{ and } \|c_2y - v\| \leq \frac{\tau_2}{\kappa_2+1}\|c_2y + v\|, \tag{13}$$

(iv) *there exist $a, b, d, g \in V$ with $\mathrm{Re}\langle a, b\rangle, \mathrm{Re}\langle d, g\rangle > 0$ and $a + b, d + b \in \mathrm{span}\{v\}$ for a certain nonzero vector $v \in V$ such that*

$$\mathrm{Re}\langle b - x, x - a\rangle \geq 0 \quad \text{and} \quad \mathrm{Re}\langle g - y, y - d\rangle \geq 0, \tag{14}$$

and $\tau_1 = \tau_{a,b},\ \kappa_1 = \kappa_{a,b},\ \tau_2 = \tau_{d,g},\ \kappa_2 = \kappa_{d,g}$,

(v) *there exists a nonzero vector $v \in V$ such that $x \in C_{v,\tau_1},\ y \in C_{v,\tau_2}$.*

To prove the theorem we need three auxiliary lemmas.

Lemma 1. *For $a, b \in V$ and $\alpha \geq 0$,*

$$\|a - b\| \left\{\begin{matrix} \leq \\ = \\ \geq \end{matrix}\right. \alpha\|a + b\| \iff \begin{cases} \|a - \frac{1+\alpha^2}{1-\alpha^2}b\| \left\{\begin{matrix} \leq \\ = \\ \geq \end{matrix}\right. \frac{2\alpha}{1-\alpha^2}\|b\|, & 0 \leq \alpha < 1, \\ \mathrm{Re}\langle a, b\rangle \left\{\begin{matrix} \geq \\ = \\ \leq \end{matrix}\right. 0, & \alpha = 1, \\ \|a - \frac{1+\alpha^2}{1-\alpha^2}b\| \left\{\begin{matrix} \geq \\ = \\ \leq \end{matrix}\right. \frac{2\alpha}{\alpha^2-1}\|b\|, & \alpha > 1. \end{cases} \tag{15}$$

Particularly, for $\kappa \geq 1$ and $\tau \geq 0$ such that $\kappa^2 = \tau^2 + 1$ we have:

$$\|a - b\| \leq \frac{\tau}{\kappa+1}\|a + b\| \iff \|a - \kappa b\| \leq \tau\|b\|. \tag{16}$$

Proof. Taking the square and rearranging the terms, the inequality/equality on the left in (15) is equivalent to

$$(1-\alpha^2)\|a\|^2 + (1-\alpha^2)\|b\|^2 - 2(1+\alpha^2)\mathrm{Re}\langle a, b\rangle \left\{\begin{matrix} \leq \\ = \\ \geq \end{matrix}\right. 0. \tag{17}$$

For $\alpha = 1$ it reduces to the second inequality/equality on the right in (15).

If $0 \le \alpha < 1$, then dividing both of sides of (17) by $1-\alpha^2 > 0$ and adding the same quantity $\frac{4\alpha^2}{(1-\alpha^2)^2}\|b\|^2$, we obtain

$$\|a\|^2 + \left(\frac{1+\alpha^2}{1-\alpha^2}\right)^2 \|b\|^2 - 2\frac{1+\alpha^2}{1-\alpha^2}\mathrm{Re}\,\langle a, b\rangle \left\{ \begin{array}{c} \le \\ = \\ \ge \end{array} \right. \frac{4\alpha^2}{(1-\alpha^2)^2}\|b\|^2,$$

or, equivalently,

$$\|a - \frac{1+\alpha^2}{1-\alpha^2}b\|^2 \left\{ \begin{array}{c} \le \\ = \\ \ge \end{array} \right. \frac{4\alpha^2}{(1-\alpha^2)^2}\|b\|^2.$$

This is the first inequality/equality on the right in (15).

In the similar way, one can obtain the third variant of (15), i.e. if $\alpha > 1$.

The particular case we get by simple computations. □

Lemma 2. *Fix a nonzero vector $s \in V$ and $0 \le \varrho < \|s\|$. If $\|x - s\| \le \varrho$, then there exist vectors $v, h \in V$ such that $v \in \operatorname{span}\{s\} \setminus \{0\}$, $x = v + h$, $\langle v, h\rangle = 0$ and*

$$\|h\|^2 \le \frac{\varrho^2}{\|s\|^2 - \varrho^2}\|v\|^2. \tag{18}$$

Proof. Given $x \in V$, we define $v := \frac{\langle x,s\rangle}{\|s\|^2}s$, $h := x - v$. It is clear that $\langle v, h\rangle = 0$ and $\|v\|^2 = \frac{|\langle x,s\rangle|^2}{\|s\|^2}$, $\|h\|^2 = \|x\|^2 - \frac{|\langle x,s\rangle|^2}{\|s\|^2}$. For such v and h the inequality (18) takes the form

$$\varrho^2\|x\|^2 + |\langle s, x\rangle|^2 - \|x\|^2\|s\|^2 \ge 0 \tag{19}$$

If $\|x - s\| \le \varrho$, then $x = s + r$ for a certain vector r with $\|r\| \le \varrho$ and

$$\|x\|^2 = \|s\|^2 + \|r\|^2 + 2\mathrm{Re}\,\langle s, r\rangle \ \text{ and } |\langle s, x\rangle|^2 = \|s\|^4 + |\langle s, r\rangle|^2 + 2\|s\|^2\mathrm{Re}\,\langle s, r\rangle\,.$$

Substituting the above to the left hand side of (19) and replacing $\langle s, r\rangle$ by $\alpha + i\beta$, $(i^2 = -1)$, for any $\alpha, \beta \in \mathbb{R}$ we have:

$$\begin{array}{rl} |\langle s, r\rangle|^2 + 2\varrho^2\mathrm{Re}\,\langle s, r\rangle + \varrho^2(\|s\|^2 + \|r\|^2) - \|s\|^2\|r\|^2 & = \\ \alpha^2 + \beta^2 + 2\varrho^2\alpha + \varrho^2(\|s\|^2 + \|r\|^2) - \|s\|^2\|r\|^2 & = \\ (\alpha + \varrho^2)^2 + \beta^2 + (\|s\|^2 - \varrho^2)(\varrho^2 - \|r\|^2) & \ge \\ (\|s\|^2 - \varrho^2)(\varrho^2 - \|r\|^2) & \ge 0, \end{array}$$

if $\|r\| \le \varrho < \|s\|$. It proves that (18) is valid.

Moreover, vector v must be nonzero. Otherwise, $\langle x, s\rangle = 0$ and $\|x - s\|^2 = \langle x - s, x - s\rangle = \|x\|^2 + \|s\|^2 - 2\mathrm{Re}\,\langle x, s\rangle = \|x\|^2 + \|s\|^2$. Then $\varrho^2 < \|s\|^2 \le \|x\|^2 + \|s\|^2 = \|x - s\|^2 \le \varrho^2$. This contradiction finishes the proof. □

Lemma 3. *(see [4, Lemma 2.1], c.f. [7, Theorem 2.2]) Let $x, y, a, b \in V$ and $\Gamma, \gamma \in \mathbb{F}$. Then*

$$\mathrm{Re}\,\langle b - x, x - a\rangle \ge 0 \iff \left\|x - \frac{a+b}{2}\right\| \le \frac{1}{2}\|a - b\|, \tag{20}$$

$$\mathrm{Re}\,\langle \Gamma y - x, x - \gamma y\rangle \ge 0 \iff \left\|x - \frac{\Gamma+\gamma}{2}y\right\| \le \frac{1}{2}|\Gamma - \gamma|\|y\|. \tag{21}$$

In particular, the equations on the left are equivalent responsible equations on the right. □

Proof of Theorem 1. The layout of the proof is as follows:

$$\text{(v)} \Rightarrow \text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(v)},\ \text{(iii)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(iv)}.$$

Let us start from the observation, the inequalities (10) and (11) are specifications of (7) and (8), respectively, for $\theta = \frac{\kappa_1\kappa_2}{1-\tau_1\tau_2}$. Therefore they are equivalent, by (5).

(v)⇒(i). Let $c_1x = v + h_1$ and $c_2y = v + h_2$, where $\langle v, h_i\rangle = 0$, $\|h_i\| \le \tau_i\|v\|$, $c_i \in \mathbb{F} \setminus \{0\}$, $i = 1, 2$. Observe, the classic Schwarz inequality (1) gives $|\mathrm{Re}\,\langle h_1, h_2\rangle| \le |\langle h_1, h_2\rangle| \le \|h_1\|\|h_2\| \le \tau_1\tau_2\|v\|^2$ and consequently,

$$\mathrm{Re}\,\langle h_1, h_2\rangle \ge -\tau_1\tau_2\|v\|^2. \tag{22}$$

We have $|c_1|^2\|x\|^2 = \|v\|^2 + \|h_1\|^2 \le (1+\tau_1^2)\|v\|^2$ and $|c_2|^2\|y\|^2 = \|v\|^2 + \|h_2\|^2 \le (1+\tau_2^2)\|v\|^2$. Hence

$$|c_1|^2|c_2|^2\|x\|^2\|y\|^2 \le (1+\tau_1^2)(1+\tau_2^2)\|v\|^4. \tag{23}$$

On the other hand $c_1\overline{c}_2\langle x,y\rangle = \langle v+h_1, v+h_2\rangle = \|v\|^2 + \langle h_1,h_2\rangle$. It implies that $|c_1|^2|c_2|^2|\langle x,y\rangle|^2 = (\|v\|^2 + \operatorname{Re}\langle h_1,h_2\rangle)^2 + \operatorname{Im}^2\langle h_1,h_2\rangle$. Hence, making use of (22) leads to

$$|c_1|^2|c_2|^2|\langle x,y\rangle|^2 \ge (\|v\|^2 + \operatorname{Re}\langle h_1,h_2\rangle)^2 \ge (\|v\|^2 - \tau_1\tau_2\|v\|^2)^2 = (1-\tau_1\tau_2)^2\|v\|^4. \tag{24}$$

Now, combining inequalities (23) and (24) yields (10) and, equivalently, (11).

(i)⇒(ii). The function $(\tau_1,\tau_2) \mapsto \frac{\kappa_1\kappa_2}{1-\tau_1\tau_2}$, $0 \le \tau_1,\tau_2$ and $\tau_1\tau_2 < 1$ is increasing w.r.t. both variables. Thus, for nonzero vectors x, y, fulfilling (7) with $\theta = \frac{\kappa_1\kappa_2}{1-\tau_1\tau_2}$ there exist $0 \le \tau_i' \le \tau_i, i = 1,2$ that

$$\frac{|\langle x,y\rangle|}{\|x\|\|y\|} = \frac{1-\tau_1'\tau_2'}{\kappa_1'\kappa_2'}, \tag{25}$$

where, as before, ${\kappa_i'}^2 = {\tau_1'}^2 + 1, \kappa_i \ge 1, i = 1,2$.

Note $\langle x,y\rangle \ne 0$ and let

$$x' = \frac{\kappa_1'\langle y,x\rangle}{|\langle x,y\rangle|\|x\|}x,\ \ y' = \frac{\kappa_2'}{\|y\|}y,\quad \alpha = \frac{\tau_2'}{\tau_1'+\tau_2'},\ \beta = \frac{\tau_1'}{\tau_1'+\tau_2'}.$$

Since (25), we obtain

$$\langle x',y'\rangle = \langle y',x'\rangle = 1-\tau_1'\tau_2', \tag{26}$$

moreover,

$$\|x'\|^2 = {\kappa_1'}^2 \text{ and } \|y'\|^2 = {\kappa_2'}^2 \tag{27}$$

and, consequently,

$$\|x'-y'\|^2 = (\tau_1'+\tau_2')^2. \tag{28}$$

Now, we are ready to define

$$v := \alpha x' + \beta y',\ \ h_1 := \beta(x'-y'),\ \ h_2 := \alpha(y'-x').$$

By (26), (27) and (28) one can easy verify

$$x' = v + h_1,\ \ y' = v + h_2; \tag{29}$$

$$\langle v,h_1\rangle = 0 = \langle v,h_2\rangle; \tag{30}$$

$$\|v\|^2 = 1,\ \ \|h_1\|^2 = {\tau_1'}^2,\ \ \|h_2\|^2 = {\tau_2'}^2. \tag{31}$$

Regarding (29),(30) and (31) we obtain

$$\left\|\frac{1}{{\kappa_1'}^2}x' - v\right\|^2 = \frac{{\tau_1'}^2}{{\kappa_1'}^2}\|v\|^2,\quad \left\|\frac{1}{{\kappa_2'}^2}y' - v\right\|^2 = \frac{{\tau_2'}^2}{{\kappa_2'}^2}\|v\|^2$$

or, equivalently,

$$\|c_1x - v\|^2 = \frac{{\tau_1'}^2}{{\kappa_1'}^2}\|v\|^2, \text{ and } \|c_2y - v\|^2 = \frac{{\tau_2'}^2}{{\kappa_2'}^2}\|v\|^2,$$

where $c_1 = \langle y,x\rangle/\kappa_1'|\langle x,y\rangle|\|x\|$ and $c_2 = 1/\kappa_2'\|y\|$. To this end, we notice that ${\tau_i'}^2/{\kappa_i'}^2 \le {\tau_i}^2/{\kappa_i}^2$, whenever $\tau_i' \le \tau_i,\ i = 1,2$.

(ii)⇒(v). It suffices making use of Lemma 2.

(ii)⇔(iii). It follows from the particular case of Lemma 1.

(iv)⇒(ii). $\operatorname{Re}\langle a,b\rangle, \operatorname{Re}\langle d,g\rangle > 0$ ensures that $\|a+b\|, \|d+g\| \ne 0$. Since $a+b, d+g \in \operatorname{span}\{v\}$ and $v \ne 0$, there exist nonzero scalars c_1, c_2 such that

$$c_1\frac{a+b}{2} = v = c_2\frac{d+g}{2} \quad\text{and}\quad \|v\| = |c_1|\frac{\|a+b\|}{2} = |c_2|\frac{\|d+g\|}{2}.$$

Now, if (14), then by (20) we have

$$\left\|x - \frac{a+b}{2}\right\| \le \frac{1}{2}\|a-b\| \text{ and } \left\|y - \frac{d+g}{2}\right\| \le \frac{1}{2}\|d-g\|$$

or, equivalently,

$$\|c_1 x - v\| \le \frac{\|a-b\|}{\|a+b\|}\|v\| \quad \text{and} \quad \|c_2 y - v\| \le \frac{\|d-g\|}{\|d+g\|}\|v\|.$$

This is nothing but (12), because $\frac{\|a-b\|}{\|a+b\|} = \frac{\tau_{a,b}}{\kappa_{a,b}}$ and $\frac{\|d-g\|}{\|d+g\|} = \frac{\tau_{d,g}}{\kappa_{d,g}}$.

(ii)⇒(iv). Define $a, b, d, g \in V$ as follows

$$c_1 a = \left(1 + \frac{\tau_1}{\kappa_1}\right) v,\ c_1 b = \left(1 - \frac{\tau_1}{\kappa_1}\right) v, \quad \text{and} \quad c_2 d = \left(1 + \frac{\tau_2}{\kappa_2}\right) v,\ c_2 g = \left(1 - \frac{\tau_2}{\kappa_2}\right) v.$$

It is easy to see that

$$c_1 \frac{a+b}{2} = v,\ \frac{1}{2}|c_1|\|a-b\| = \frac{\tau_1}{\kappa_1}\|v\| \quad \text{and} \quad c_2 \frac{d+g}{2} = v,\ \frac{1}{2}|c_2|\|d-g\| = \frac{\tau_2}{\kappa_2}\|v\|.$$

By the above, (12) can be expressed in the form

$$\left\|c_1 x - c_1 \frac{a+b}{2}\right\| \le \frac{1}{2}|c_1|\|a-b\| \text{ and } \left\|c_2 y - c_2 \frac{d+g}{2}\right\| \le \frac{1}{2}|c_2|\|d-g\|.$$

By (20) it is equivalent to (14). Clearly, $a+b, d+g \in \operatorname{span}\{v\}$ and $v \ne 0$ (by the hypothesis), moreover, $\operatorname{Re}\langle a, b\rangle = \|v\|^2/\kappa_1^2 > 0$ and $\operatorname{Re}\langle d, g\rangle = \|v\|^2/\kappa_2^2 > 0$ and $\tau_{a,b} = \tau_1,\ \tau_{d,g} = \tau_2$.

The proof is completely finished. □

Now, we shall present some consequences of Theorem 1. Let us start from the example that illustrates how our theorem works.

Example 1. *Consider l^2, the space of all complex sequences $z = (z_1, z_2, \ldots)$ such that $\sum |z_i|^2 < \infty$, with the inner product $\langle x, y\rangle = \sum x_i \overline{y_i}$ and norm $\|x\| = \sqrt{\langle x, x\rangle}, x, y \in l^2$.*

Fix $v = (v_1, v_2, \ldots) \in l^2$ with nonzero entries, $z_1, z_2 \in \mathbb{C}\setminus\{0\}$ and choose $\varrho_i \ge 0$ with $\varrho_i < |z_i|$. For the interpretation, $\frac{\varrho_i}{|z_i|}$ is assumed to be equal $\sin\alpha_i$, $i = 1, 2$, where $0 \le \alpha_i < \pi/2$.

Given $x, y \in l^2$, let

$$\left|\frac{x_i}{v_i} - z_1\right| \le \varrho_1 < |z_1|, \quad \text{and} \quad \left|\frac{y_i}{v_i} - z_2\right| \le \varrho_2 < |z_2|,\ i = 1, 2, \ldots.$$

Multiplying the above inequalities by $|v_i|/|z_1| > 0$ and $|v_i|/|z_2| > 0$, respectively, taking the square, summing over i and extracting the square root of both of sides gives

$$\left\|\frac{1}{z_1}x - v\right\| \le \frac{\varrho_1}{|z_1|}\|v\| \quad \text{and} \quad \left\|\frac{1}{z_2}y - v\right\| \le \frac{\varrho_2}{|z_2|}\|v\|.$$

Applying (ii)⇒(i) of Theorem 1 with $\frac{\tau_i}{\kappa_i} = \frac{\varrho_i}{|z_i|}$, $i = 1, 2$ simultaneously assuming $\frac{\varrho_1^2}{|z_1|^2} + \frac{\varrho_2^2}{|z_2|^2} < 1$, we obtain

$$\|x\|^2\|y\|^2 \le \sec^2(\alpha_1 + \alpha_2)|\langle x, y\rangle|^2,$$

or, equivalently,

$$\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \le \sin^2(\alpha_1 + \alpha_2)\|x\|^2\|y\|^2 \le \tan^2(\alpha_1 + \alpha_2)|\langle x, y\rangle|^2. \tag{32}$$

We omit the details. □

Analysing the proof of Theorem 1, especially the part (ii)⇔(iv), we conclude

Remark 1. *Condition* (iv) *in Theorem 1 can be equivalently replaced by the following one:*

(iv') *there exist nonzero scalars $\gamma, \Gamma, \lambda, \Lambda \in \mathbb{F}$ with $\operatorname{Re}(\gamma\overline{\Gamma}) > 0$, $\operatorname{Re}(\lambda\overline{\Lambda}) > 0$ and a nonzero vector $v \in V$ such that*

$$\operatorname{Re}\langle \Gamma v - x, x - \gamma v\rangle \ge 0 \quad \text{and} \quad \operatorname{Re}\langle \Lambda v - y, y - \lambda v\rangle \ge 0 \tag{33}$$

and $\tau_1 = \tau_{\gamma,\Gamma}$, $\kappa_1 = \kappa_{\gamma,\Gamma}$, $\tau_2 = \tau_{\lambda,\Lambda}$, $\kappa_2 = \kappa_{\lambda,\Lambda}$.

Theorem 1 together with Remark 1 lead to the following supplement of Dragomir's result [7, Theorem 2.2].

Corollary 1. *Fix $\tau \geq 0$ and $\kappa \geq 1$ with $\kappa^2 = \tau^2 + 1$. For nonzero vectors $x, y \in V$ the following conditions are mutually equivalent:*

(i) *the following equivalent inequalities hold true*

$$\|x\|\|y\| \leq \kappa |\langle x, y\rangle| \quad \text{and} \quad \|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{\tau^2}{\kappa^2}\|x\|^2\|y\|^2 \leq \tau^2 |\langle x, y\rangle|^2,$$

(ii) *there exists $c \in \mathbb{F} \setminus \{0\}$ such that $\|cx - y\| \leq \frac{\tau}{\kappa}\|y\|$,*

(iii) *there exists $c \in \mathbb{F} \setminus \{0\}$ such that $\|cx - y\| \leq \frac{\tau}{\kappa+1}\|cx + y\|$,*

(iv) *there exist $a, b \in V$ with $\operatorname{Re}\langle a, b\rangle > 0$ and $a + b \in \operatorname{span}\{y\}$ such that $\operatorname{Re}\langle b - x, x - a\rangle \geq 0$, and $\tau_1 = \tau_{a,b}$, $\kappa_1 = \kappa_{a,b}$,*

(iv') *there exist $\Gamma, \gamma \in \mathbb{F}$ with $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$ such that $\operatorname{Re}\langle \Gamma y - cx, cx - \gamma y\rangle \geq 0$ and $\tau = \tau_{\gamma,\Gamma}$ and $\kappa = \kappa_{\gamma,\Gamma}$,*

(v) *there exist $c \in \mathbb{F} \setminus \{0\}$ and $h \in V$ such that $cx = y + h$, $\langle y, h\rangle = 0$ and $\|h\| \leq \tau\|y\|$.*

Proof. Make use Theorem 1 together with Remark 1 for $v = y$. Then you can set $\tau_2 = 0$, $1 = c_2 = \kappa_2 = \lambda = \Lambda$ and $d = g = y$. Observe, in this situation we have $\tau_{1,1} = \tau_{y,y} = 0$ and $\kappa_{1,1} = \kappa_{y,y} = 1$. □

Theorem 1 also allows us to generalize the inequality (9) as follows.

Corollary 2. *Let $x, y, v \in V$, where $v \neq 0$. If $\|x - v\|$, $\|y - v\| < \|v\|$ and $\|x - v\|^2\|y - v\|^2 < (\|v\|^2 - \|x - v\|^2)(\|v\|^2 - \|y - v\|^2)$, then we have*

$$\|x\|\|y\| \leq \frac{\|v\|^2}{\sqrt{\|v\|^2 - \|x - v\|^2}\sqrt{\|v\|^2 - \|y - v\|^2} - \|x - v\|\|y - v\|} |\langle x, y\rangle|, \tag{34}$$

or, equivalently,

$$\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \leq \left(\frac{\|x - v\|\sqrt{\|v\|^2 - \|y - v\|^2} + \|y - v\|\sqrt{\|v\|^2 - \|x - v\|^2}}{\|v\|^2}\right)^2 \|x\|^2\|y\|^2. \tag{35}$$

Proof. Let

$$\frac{\|x - v\|^2}{\|v\|^2} = \frac{\tau_1^2}{\kappa_1^2},\ \kappa_1^2 = \tau_1^2 + 1, \quad \frac{\|y - v\|^2}{\|v\|^2} = \frac{\tau_2^2}{\kappa_2^2},\ \kappa_2^2 = \tau_2^2 + 1.$$

Hence

$$\tau_1 = \frac{\|x - v\|}{\sqrt{\|v\|^2 - \|x - v\|^2}}, \quad \kappa_1 = \frac{\|v\|}{\sqrt{\|v\|^2 - \|x - v\|^2}},$$
$$\tau_2 = \frac{\|y - v\|}{\sqrt{\|v\|^2 - \|y - v\|^2}}, \quad \kappa_2 = \frac{\|v\|}{\sqrt{\|v\|^2 - \|y - v\|^2}}.$$

Now enough to apply Theorem 1, (ii)⇒(i). □

Remark 2. *For $\|x - y\| < \min\{\|x\|, \|y\|\}$, setting consecutively $v = x$ and $v = y$ in (34) and (35) we derive (9).* □

Remark 3. *On the assumption as in Corollary 2, inequalities (34) and (35) can be jointly expressed in the clear form (32), where $\alpha_1 = \arcsin \frac{\|x-v\|}{\|v\|}$ and $\alpha_2 = \arcsin \frac{\|y-v\|}{\|v\|}$.* □

The inequality

$$\|x\|\|y\| \leq \frac{\kappa_{\gamma,\Gamma}\kappa_{\lambda,\Lambda}}{1 - \tau_{\gamma,\Gamma}\tau_{\lambda,\Lambda}} |\langle x, y\rangle|$$

is sharp in the sense that the constant $\frac{\kappa_{\gamma,\Gamma}\kappa_{\lambda,\Lambda}}{1-\tau_{\gamma,\Gamma}\tau_{\lambda,\Lambda}}$ cannot be improved multiplying it by any scalar smaller than 1. In other words, there exist vectors $x, y \in V$ fulfilling (14) (in practice (33)) such that

$$\|x\|\|y\| = \frac{\kappa_{\gamma,\Gamma}\kappa_{\lambda,\Lambda}}{1 - \tau_{\gamma,\Gamma}\tau_{\lambda,\Lambda}} |\langle x, y\rangle|. \tag{36}$$

The suitable vectors of the type are specified in the following example.

Example 2. *Fix a vector $v \in V$ and scalars $\gamma, \Gamma, \lambda, \Lambda \in \mathbb{F}$ as in condition* **(iv')** *(see Remark 1). Denote $\tau_1 := \tau_{\gamma,\Gamma}$, $\tau_2 := \tau_{\lambda,\Lambda}$ and set $\kappa_i^2 = 1 + \tau_i^2$, $i = 1, 2$. Choose arbitrary $w \in V \setminus \{0\}$ with $\langle v, w\rangle = 0$ (in this moment, V is assumed to be at least 2-dimensional).*

We define

$$x := \frac{\gamma+\Gamma}{2\kappa_1^2\|w\|}(\|w\|v + \tau_1\|v\|w), \ y := \frac{\lambda+\Lambda}{2\kappa_2^2\|w\|}(\|w\|v - \tau_2\|v\|w).$$

One can easy verify that:

$$\|x\| = \sqrt{\mathrm{Re}\gamma\overline{\Gamma}}\|v\| \quad \text{and} \quad \|y\| = \sqrt{\mathrm{Re}\lambda\overline{\Lambda}}\|v\|,$$

hence

$$\|x\|\|y\| = \sqrt{\mathrm{Re}\gamma\overline{\Gamma}\mathrm{Re}\lambda\overline{\Lambda}}\|v\|^2, \tag{37}$$

and next

$$|\langle x, y\rangle| = \frac{|\gamma+\Gamma||\lambda+\Lambda|}{4\kappa_1^2\kappa_2^2}(1-\tau_1\tau_2)\|v\|^2, \tag{38}$$

$$\|x - \frac{\gamma+\Gamma}{2}v\| = \frac{|\gamma-\Gamma|}{2}\|v\| \quad \text{and} \quad \|y - \frac{\lambda+\Lambda}{2}v\| = \frac{|\lambda-\Lambda|}{2}\|v\|.$$

By virtue of (21) the above is equivalent to

$$\mathrm{Re}\,\langle \Gamma v - x, x - \gamma v\rangle = 0 \quad \text{and} \quad \mathrm{Re}\,\langle \Lambda v - y, y - \lambda v\rangle = 0$$

and (37) and (38) lead to (36). □

3. Applications to Grüss' inequality

Let x, y and $z \neq 0$ be vectors in an inner vectors space V. Grüss' type inequalities state upper bounds for the quantity $|\langle x, y\rangle - \frac{\langle x,z\rangle\langle z,y\rangle}{\|z\|^2}|$. Usually but not necessarily $\|z\| = 1$.

Applying classic Schwarz's inequality for the vectors $x - \frac{\langle x,z\rangle}{\|z\|^2}z$ and $y - \frac{\langle y,z\rangle}{\|z\|^2}z$ and taking into account that

$$\left\langle x - \frac{\langle x,z\rangle}{\|z\|^2}z, y - \frac{\langle y,z\rangle}{\|z\|^2}z\right\rangle = \langle x, y\rangle - \frac{\langle x,z\rangle\langle z,y\rangle}{\|z\|^2},$$

$$\left\| x - \frac{\langle x,z\rangle}{\|z\|^2}z\right\|^2 = \frac{\|x\|^2\|z\|^2 - |\langle x,z\rangle|^2}{\|z\|^2} \quad \text{and} \quad \left\| y - \frac{\langle y,z\rangle}{\|z\|^2}z\right\|^2 = \frac{\|y\|^2\|z\|^2 - |\langle y,z\rangle|^2}{\|z\|^2}$$

we have the initial estimate

$$\left|\langle x, y\rangle - \frac{\langle x,z\rangle\langle z,y\rangle}{\|z\|^2}\right|^2 \le \frac{(\|x\|^2\|z\|^2 - |\langle x,z\rangle|^2)(\|y\|^2\|z\|^2 - |\langle y,z\rangle|^2)}{\|z\|^4}. \tag{39}$$

Basing on Theorem 1 we obtain the following Grüss' type inequalities.

Theorem 2. *Let $v \in V \setminus \{0\}$ and $\tau_i \ge 0, \kappa_i \ge 1$, $i = 0, 1, 2$, be such scalars that $\tau_i^2 + 1 = \kappa_i^2$ and $\tau_1\tau_0, \tau_2\tau_0 < 1$.*

For $x, y, z \in V$, where $z \neq 0$, the inequality

$$\left|\langle x, y\rangle - \frac{\langle x,z\rangle\langle z,y\rangle}{\|z\|^2}\right| \le \frac{(\tau_1+\tau_0)(\tau_2+\tau_0)}{\kappa_1\kappa_0^2\kappa_2}\|x\|\|y\| \le \frac{(\tau_1+\tau_0)(\tau_2+\tau_0)}{(1-\tau_1\tau_0)(1-\tau_2\tau_0)}\frac{|\langle x,z\rangle\langle z,y\rangle|}{\|z\|^2}$$

holds true whenever

(i) $\|z - v\| \le \frac{\tau_0}{\kappa_0}\|v\|, \ \|x - v\| \le \frac{\tau_1}{\kappa_1}\|v\|, \ \|y - v\| \le \frac{\tau_2}{\kappa_2}\|v\|,$

(ii) $\|z - v\| \le \frac{\tau_0}{\kappa_0+1}\|z + v\|, \ \|x - v\| \le \frac{\tau_1}{\kappa_1+1}\|x + v\|, \ \|y - v\| \le \frac{\tau_2}{\kappa_2+1}\|y + v\|,$

(iii) $\mathrm{Re}\,\langle b_0 - z, z - a_0\rangle \ge 0, \ \mathrm{Re}\,\langle b_1 - x, x - a_1\rangle \ge 0, \ \mathrm{Re}\,\langle b_2 - y, y - a_2\rangle \ge 0,$
where $a_i, b_i \in V$, $i = 0, 1, 2$ with $\mathrm{Re}\,\langle a_i, b_i\rangle > 0$, $a_i + b_i \in \mathrm{span}\{v\}$ and $\tau_i := \tau_{a_i,b_i}$, $\kappa_i = \kappa_{a_i,b_i}$,

(iii') $\mathrm{Re}\,\langle \Gamma_0 v - z, z - \gamma_0 v\rangle \ge 0, \ \mathrm{Re}\,\langle \Gamma_1 v - x, x - \gamma_1 v\rangle \ge 0, \ \mathrm{Re}\,\langle \Gamma_2 v - y, y - \gamma_2 v\rangle \ge 0,$
where $\Gamma_i, \gamma_i \in \mathbb{F}$, $i = 0, 1, 2$ with $\mathrm{Re}(\Gamma_i\overline{\gamma}_i) > 0$ and $\tau_i := \tau_{\gamma_i,\Gamma_i}$, $\kappa_i = \kappa_{\gamma_i,\Gamma_i}$,

(iv) $z \in C_{v,\tau_0},\ x \in C_{v,\tau_1},\ y \in C_{v,\tau_2}$.

Proof. Apply Theorem 1 together with Remark 1 for triples of vectors x, z, v and y, z, v and make use of the estimate (39). $\square$

Analogously, (39) and Corollary 1 used for pairs of vectors x, z and y, z or specification of the above theorem for $v = z$ gives

Corollary 3. *Let $z \in V \setminus \{0\}$ and $\tau_i \geq 0, \kappa_i \geq 1,\ i = 0, 1, 2,$ be such scalars that $\tau_i^2 + 1 = \kappa_i^2$.*
If for $x, y \in V$ one of the following conditions holds:

(i) $\|x - z\| \leq \frac{\tau_1}{\kappa_1}\|z\|$ *and* $\|y - z\| \leq \frac{\tau_2}{\kappa_2}\|z\|$

(ii) $\|x - z\| \leq \frac{\tau_1}{1+\kappa_1}\|x + z\|$ and $\|y - z\| \leq \frac{\tau_2}{1+\kappa_2}\|y + z\|$,

(iii) $\mathrm{Re}\,\langle b_1 - x, x - a_1\rangle \geq 0$ *and* $\mathrm{Re}\,\langle b_2 - y, y - a_2\rangle \geq 0$,
where $a_i, b_i \in V,\ i = 1, 2$ *with* $\mathrm{Re}\,\langle a_i, b_i\rangle > 0,\ a_i + b_i \in \mathrm{span}\{z\}$ *and* $\tau_i := \tau_{a_i,b_i},\ \kappa_i = \kappa_{a_i,b_i}$,

(iii') $\mathrm{Re}\,\langle \Gamma_1 z - x, x - \gamma_1 z\rangle \geq 0$ *and* $\mathrm{Re}\,\langle \Gamma_2 z - y, y - \gamma_2 z\rangle \geq 0$,
where $\Gamma_i, \gamma_i \in \mathbb{F}$ *with* $\mathrm{Re}(\Gamma_i \overline{\gamma}_i) > 0$ *and* $\tau_i := \tau_{\gamma_i,\Gamma_i},\ \kappa_i := \kappa_{\gamma_i,\Gamma_i},\ i = 1, 2$,

(iv) $x \in C_{z,\tau_1},\ y \in C_{z,\tau_2}$,

then

$$\left|\langle x, y\rangle - \frac{\langle x, z\rangle\langle z, y\rangle}{\|z\|^2}\right| \leq \frac{\tau_1\tau_2}{\kappa_1\kappa_2}\|x\|\|y\| \leq \tau_1\tau_2\frac{|\langle x, z\rangle\langle z, y\rangle|}{\|z\|^2}.$$

$\square$

The above corollary covers Dragomir's results [7, Theorem 4.1-4.2].

Similarly, combining inequalities (39) with Corollary 2 and taking into account Remark 3 we can state

Proposition 1. *Let $x, y, z, v \in V$, provided $v, z \neq 0$.*
If

$$\begin{array}{rcl} \|x - v\|, \|y - v\|, \|z - v\| & < & \|v\|, \\ \|x - v\|^2\|z - v\|^2 & < & (\|v\|^2 - \|x - v\|^2)(\|v\|^2 - \|z - v\|^2), \\ \|y - v\|^2\|z - v\|^2 & < & (\|v\|^2 - \|y - v\|^2)(\|v\|^2 - \|z - v\|^2), \end{array}$$

then

$$\left|\langle x, y\rangle - \frac{\langle x, z\rangle\langle z, y\rangle}{\|z\|^2}\right| \leq \sin(\alpha_1 + \alpha_0)\sin(\alpha_2 + \alpha_0)\|x\|\|y\| \leq \tan(\alpha_1 + \alpha_0)\tan(\alpha_2 + \alpha_0)\frac{|\langle x, z\rangle\langle z, y\rangle|}{\|z\|^2},$$

where $\alpha_0 = \arcsin\frac{\|z-v\|}{\|v\|},\ \alpha_1 = \arcsin\frac{\|x-v\|}{\|v\|},\ \alpha_2 = \arcsin\frac{\|y-v\|}{\|v\|}$. $\square$

Substituting $v := z$ and changing of roles $x \leftrightarrow z$ and $y \leftrightarrow z$ in the above gives

Corollary 4. *Given nonzero vectors $x, y, z \in V$. If $\|x - z\| < \min\{\|x\|, \|z\|\}$ and $\|y - z\| < \min\{\|y\|, \|z\|\}$, then*

$$\begin{array}{rl} |\langle x, y\rangle & - \frac{\langle x,z\rangle\langle z,y\rangle}{\|z\|^2}| \leq \frac{\|x-z\|\|y-z\|}{\max\{\|x\|,\|z\|\}\max\{\|y\|,\|z\|\}}\|x\|\|y\| \leq \\ & \leq \frac{\|x-z\|\|y-z\|}{\max\{\sqrt{\|x\|^2-\|x-z\|^2},\sqrt{\|z\|^2-\|x-z\|^2}\}\max\{\sqrt{\|y\|^2-\|y-z\|^2},\sqrt{\|z\|^2-\|y-z\|^2}\}}\frac{|\langle x,z\rangle\langle z,y\rangle|}{\|z\|^2}. \square \end{array}$$

Remark 4. *The above inequality can also be obtained directly by (39) and (9).* $\square$

4. A strengthening of reverse Schwarz's inequality

A straight specification of the conditions (ii)-(v) included in Theorem 1 yields stronger inequalities than (10) and (11).

For $v \in V \setminus \{0\}$ and $\tau \geq 0$ let

$$\begin{array}{rcl} \mathrm{cone}\{v\} & := & \{\alpha v : \alpha > 0\}, \\ C^+_{v,\tau} & := & \{c(v + h) : 0 < c \in \mathbb{R}, h \in V, \langle v, h\rangle = 0, \|h\| \leq \tau\|v\|\}. \end{array}$$

Theorem 3. *Fix $\tau_k \geq 0,\ \kappa_k \geq 1,\ k=1,2$ with $\kappa_k^2 = \tau_k^2 + 1$ and $\tau_1\tau_2 < 1$ and let $x, y, v \in V \setminus \{0\}$.*
The following equivalent inequalities

$$\|x\|\|y\| \leq \frac{\kappa_1\kappa_2}{1-\tau_1\tau_2}\mathrm{Re}\,\langle x, y\rangle\,, \tag{40}$$

$$\|x\|\|y\| - \mathrm{Re}\,\langle x, y\rangle \leq \frac{\tau_1\tau_2 + \kappa_1\kappa_2 - 1}{\kappa_1\kappa_2}\|x\|\|y\| \leq \frac{\tau_1\tau_2 + \kappa_1\kappa_2 - 1}{1-\tau_1\tau_2}\mathrm{Re}\,\langle x, y\rangle\,, \tag{41}$$

hold true, if one of the below conditions is met

(i) *there exist $c_1, c_2 > 0$ such that (12) holds, i.e.*

$$\|c_1 x - v\| \leq \frac{\tau_1}{\kappa_1}\|v\| \text{ and } \|c_2 y - v\| \leq \frac{\tau_2}{\kappa_2}\|v\|,$$

(ii) *there exist $c_1, c_2 > 0$ such that (13) holds, i.e.*

$$\|c_1 x - v\| \leq \frac{\tau_1}{\kappa_1 + 1}\|c_1 x + v\| \text{ and } \|c_2 y - v\| \leq \frac{\tau_2}{\kappa_2 + 1}\|c_2 y + v\|,$$

(iii) *there exist $a, b, d, g \in V$ with $\mathrm{Re}\,\langle a, b\rangle > 0, \mathrm{Re}\,\langle d, g\rangle > 0$ and $a+b, d+g \in \mathrm{cone}\{v\}$ such that (14) holds, i.e.*

$$\mathrm{Re}\,\langle b - x, x - a\rangle \geq 0 \quad \text{and} \quad \mathrm{Re}\,\langle g - y, y - d\rangle \geq 0,$$

and $\tau_1 = \tau_{a,b},\ \kappa_1 = \kappa_{a,b},\ \tau_2 = \tau_{d,g},\ \kappa_2 = \kappa_{d,g}$,

(iii') *there exist $\gamma, \Gamma, \lambda, \Lambda > 0$ such that (33) holds, i.e.*

$$\mathrm{Re}\,\langle \Gamma v - x, x - \gamma v\rangle \geq 0 \quad \text{and} \quad \mathrm{Re}\,\langle \Lambda v - y, y - \lambda v\rangle \geq 0$$

and $\tau_1 = \tau_{\gamma,\Gamma},\ \kappa_1 = \kappa_{\gamma,\Gamma},\ \tau_2 = \tau_{\lambda,\Lambda},\ \kappa_2 = \kappa_{\lambda,\Lambda}$,

(iv) *$x \in C^+_{v,\tau_1},\ y \in C^+_{v,\tau_2}$.*

Proof. It is clear that (iii')⇒(iii) and inequalities (40), (41) are equivalent by virtue of (5). Moreover, the particular case of Lemma 1 and (20) ensure that either (ii) or (iii) implies (i). Thus, it is sufficient to show that (i),(iv) implies (40).

(i)⇒(40). Condition (i) gives

$$c_1^2\|x\|^2 + \|v\|^2 - 2c_1\mathrm{Re}\,\langle x, v\rangle \leq \frac{\tau_1^2}{\kappa_1^2}\|v\|^2 \quad \text{and} \quad c_2^2\|y\|^2 + \|v\|^2 - 2c_2\mathrm{Re}\,\langle v, y\rangle \leq \frac{\tau_2^2}{\kappa_2^2}\|v\|^2.$$

On making use of the elementary inequality $2ab \leq a^2 + b^2, a, b \in \mathbb{R}$ and dividing both sides of the above inequalities by $c_1, c_2 > 0$ we get

$$\frac{1}{\kappa_1}\|x\|\|v\| \leq \mathrm{Re}\,\langle x, v\rangle \quad \text{and} \quad \frac{1}{\kappa_2}\|y\|\|v\| \leq \mathrm{Re}\,\langle v, y\rangle\,.$$

Hence

$$\frac{1}{\kappa_1\kappa_2}\|x\|\|y\| \leq \frac{\mathrm{Re}\,\langle x, v\rangle\,\mathrm{Re}\,\langle v, y\rangle}{\|v\|^2} \tag{42}$$

and

$$\|x\|\|v\| \leq \kappa_1 |\,\langle x, v\rangle\,| \quad \text{and} \quad \|y\|\|v\| \leq \kappa_1 |\,\langle x, v\rangle\,|,$$

or, equivalently (c.f. (7)⇔(8)),

$$\|x\|^2\|v\|^2 - |\,\langle x, v\rangle\,|^2 \leq \frac{\tau_1^2}{\kappa_1^2}\|x\|^2\|v\|^2 \quad \text{and} \quad \|y\|^2\|v\|^2 - |\,\langle y, v\rangle\,|^2 \leq \frac{\tau_2^2}{\kappa_2^2}\|y\|^2\|v\|^2. \tag{43}$$

On the other hand, utilizing the estimate (39) for vectors x, y and v and taking into consideration (43) we deduce

$$\begin{aligned} \left|\mathrm{Re}\,\langle x, y\rangle - \tfrac{\mathrm{Re}\langle x,v\rangle\langle v,y\rangle}{\|v\|^2}\right| &\leq \left|\langle x, y\rangle - \tfrac{\langle x,v\rangle\langle v,y\rangle}{\|v\|^2}\right| \leq \\ &\leq \tfrac{\sqrt{\|x\|^2\|v\|^2 - |\langle x,v\rangle|^2}\sqrt{\|y\|^2\|v\|^2 - |\langle y,v\rangle|^2}}{\|v\|^2} \leq \tfrac{\tau_1\tau_2}{\kappa_1\kappa_2}\|x\|\|y\|. \end{aligned}$$

It leads to

$$\frac{\mathrm{Re}\,\langle x,v\rangle\,\langle v,y\rangle}{\|v\|^2} \leq \mathrm{Re}\,\langle x,y\rangle + \frac{\tau_1\tau_2}{\kappa_1\kappa_2}\|x\|\|y\|. \tag{44}$$

If V is an inner product space over $\mathbb{R}$, i.e. $\mathrm{Re}\,\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle$, then linking (42) and (44) we obtain (40). For the general case, let us note that every (complex) vector space V with a (complex) inner product can be simultaneously considered as a real inner product space, where the (real) inner product is equal to the real part of the previous one. Moreover, the respective norms are equal. Now, it is sufficient to use the real case which has already been proved.

(iv)⇒(40). If (iv), then there exist $0 < c_i \in \mathbb{R}$ and $h_i \in V$, $i = 1, 2$ such that $c_1x = v + h_1$, $c_2y = v + h_2$ and $\|h_i\| \leq \tau_i\|v\|$, $\langle v, h_i\rangle = 0$. Next, proceeding as in the proof of Theorem 1 (part (v)⇒(i)) we easy obtain

$$c_1c_2\|x\|\|y\| \leq \kappa_1\kappa_2\|v\|^2 \text{ and } c_1c_2\mathrm{Re}\,\langle x,y\rangle \geq (1-\tau_1\tau_2)\|v\|^2.$$

Linking the above inequalities gives (40). Thus the proof is completely finished. □

The below usage of Theorem 3 will be useful in the sequel.

Corollary 5. *Fix $\tau \geq 0$ and $\kappa \geq 1$ with $\kappa^2 = \tau^2 + 1$ and let $x, y \in V \setminus \{0\}$.*

The following equivalent inequalities

$$\|x\|\|y\| \leq \kappa\mathrm{Re}\,\langle x,y\rangle\,,$$

$$\|x\|\|y\| - \mathrm{Re}\,\langle x,y\rangle\,| \leq \frac{\kappa-1}{\kappa}\|x\|\|y\| \leq (\kappa-1)\mathrm{Re}\,\langle x,y\rangle\,,$$

hold true, if one of the below conditions is met

(i) *there exists $c > 0$ such that $\|cx - y\| \leq \frac{\tau}{\kappa}\|y\|$,*

(ii) *there exists $c > 0$ such that $\|cx - y\| \leq \frac{\tau}{1+\kappa}\|cx + y\|$,*

(iii) *there exist $a, b \in V$ with $\mathrm{Re}\,\langle a,b\rangle > 0$ and $a + b \in \mathrm{cone}\{y\}$ such that $\mathrm{Re}\,\langle b-x, x-a\rangle \geq 0$ and $\tau = \tau_{a,b}$, $\kappa = \kappa_{a,b}$,*

(iii') *there exist $\gamma, \Gamma > 0$ such that $\mathrm{Re}\,\langle\Gamma y - x, x - \gamma y\rangle \geq 0$ and $\tau = \tau_{\gamma,\Gamma}$, $\kappa = \kappa_{\gamma,\Gamma}$,*

(iv) *there exist $c > 0$ and $h \in V$ such that $cx = y + h$, $\langle y, h\rangle = 0$ and $\|h\| \leq \tau\|y\|$.*

Proof. Apply Theorem 3 for $v = y$. Then one can take $\tau_2 = 0$, $1 = c_2 = \kappa_2 = \lambda = \Lambda$ and $d = g = y$. Moreover we have $\tau_{1,1} = \tau_{y,y} = 0$ and $\kappa_{1,1} = \kappa_{y,y} = 1$. □

Applying Theorem 3, (i)⇒(40),(41) instead of Theorem 1, (ii)⇒(i) (see Corollary 2) we can improve the inequality (34) and (35) as follows.

Corollary 6. *Let $x, y, v \in V \setminus \{0\}$. If $\|x - v\|$, $\|y - v\| < \|v\|$ and $\|x-v\|^2\|y-v\|^2 < (\|v\|^2 - \|x-v\|^2)(\|v\|^2 - \|y-v\|^2)$, then we have*

$$\|x\|\|y\| \leq \frac{\|v\|^2}{\sqrt{\|v\|^2 - \|x-v\|^2}\sqrt{\|v\|^2 - \|y-v\|^2} - \|x-v\|\|y-v\|}\mathrm{Re}\,\langle x,y\rangle\,, \tag{45}$$

or, equivalently,

$$\|x\|\|y\| - \mathrm{Re}\,\langle x,y\rangle \leq \frac{\|x-v\|\|y-v\| + \|v\|^2 - \sqrt{\|v\|^2 - \|x-v\|^2}\sqrt{\|v\|^2 - \|y-v\|^2}}{\|v\|^2}\|x\|\|y\|.\square \tag{46}$$

Remark 5. *On the assumption as in Corollary 6, inequalities (45) and (46) can be presented in the compact trigonometric form, where $\alpha_1 = \arcsin\frac{\|x-v\|}{\|v\|}$ and $\alpha_2 = \arcsin\frac{\|y-v\|}{\|v\|}$, as follows*

$$\|x\|\|y\| - \mathrm{Re}\,\langle x,y\rangle \leq (1 - \cos(\alpha_1+\alpha_2))\|x\|\|y\| \leq \frac{1-\cos(\alpha_1+\alpha_2)}{\cos(\alpha_1+\alpha_2)}\mathrm{Re}\,\langle x,y\rangle\,.\square$$

The inequality (40) is sharp in the sense that the constant $\frac{\kappa_1\kappa_2}{1-\tau_1\tau_2}$ cannot be improved multiplying it by any scalar smaller than 1. To show it we could adjust Example 2 taking scalars $\gamma, \Gamma, \lambda, \Lambda$ positive but more interesting is the following one.

Example 3. *Let $\tau_k, \kappa_k,\ k = 1, 2$ be as in Theorem 3 and $V \ni v = v_1 + v_2$, where $\langle v_1, v_2\rangle = 0$ and $\|v\|^2 = \|v_1\|^2 + \|v_2\|^2 > 0$. Without loss of generality we can assume that $\tau_k = \tan\alpha_k,\ 0 \le \alpha_k < \pi/2$. Then $\kappa_k = \sec\alpha_k,\ \tau_k/\kappa_k = \sin\alpha_k$ and $\frac{\kappa_1\kappa_2}{1-\tau_1\tau_2} = \sec(\alpha_1 + \alpha_2)$. Denote $c_k = \cos\alpha_k + i\sin\alpha_k,\ k = 1, 2,\ i^2 = -1$ and define*

$$x = (c_1 v_1 + \overline{c_1} v_2)\cos\alpha_1, \quad y = (\overline{c_2} v_1 + c_2 v_2)\cos\alpha_2.$$

Elementary calculations give

$$\begin{aligned}
\|x\| &= \|v\|\cos\alpha_1,\\
\langle x, v\rangle &= \cos^2\alpha_1(\|v_1\|^2 + \|v_2\|^2) + i\sin\alpha_1\cos\alpha_1(\|v_1\|^2 - \|v_2\|^2),\\
\mathrm{Re}\,\langle x, v\rangle &= \cos^2\alpha_1(\|v_1\|^2 + \|v_2\|^2) = \|v\|^2\cos^2\alpha_1,\\
\|x - v\|^2 &= \|x\|^2 - 2\mathrm{Re}\,\langle x, v\rangle + \|v\|^2 = \|v\|^2\sin^2\alpha_1,\\
\|y\| &= \|v\|\cos\alpha_2,\\
\langle y, v\rangle &= \cos^2\alpha_2(\|v_1\|^2 + \|v_2\|^2) + i\sin\alpha_2\cos\alpha_2(\|v_1\|^2 - \|v_2\|^2),\\
\mathrm{Re}\,\langle y, v\rangle &= \cos^2\alpha_2(\|v_1\|^2 + \|v_2\|^2) = \|v\|^2\cos^2\alpha_2,\\
\|y - v\|^2 &= \|y\|^2 - 2\mathrm{Re}\,\langle y, v\rangle + \|v\|^2 = \|v\|^2\sin^2\alpha_2,\\
\langle x, y\rangle &= \cos\alpha_1\cos\alpha_2[\cos(\alpha_1+\alpha_2)(\|v_1\|^2 + \|v_2\|^2) + i\sin(\alpha_1+\alpha_2)(\|v_1\|^2 - \|v_2\|^2)],\\
\mathrm{Re}\,\langle x, y\rangle &= \cos\alpha_1\cos\alpha_2\cos(\alpha_1+\alpha_2)(\|v_1\|^2 + \|v_2\|^2) = \|x\|\|y\|\cos(\alpha_1+\alpha_2).
\end{aligned}$$

Therefore

$$\|x\|\|y\| = \frac{\kappa_1\kappa_2}{1 - \tau_1\tau_2}\mathrm{Re}\,\langle x, y\rangle$$

and $x, y \in V$ fulfil condition (i) of Theorem 3. In fact, $\|x - v\| = \frac{\tau_1}{\kappa_1}\|v\|$ and $\|y - v\| = \frac{\tau_2}{\kappa_2}\|v\|$.

It is worth noting, $\|x\|\|y\| < \frac{\kappa_1\kappa_2}{1-\tau_1\tau_2}|\langle x, y\rangle|$ whenever $\|v_1\| \ne \|v_2\|$. □

5. Applications to the triangle inequality

If x and y are vectors in a norm space V, then the triangle inequality $\|x + y\| \le \|x\| + \|y\|$ holds. In this section we are interested in reverses of this inequality in inner product spaces. The key tools for this purpose are reverse Schwarz inequalities of the form

$$\|x\|\|y\| - \mathrm{Re}\,\langle x, y\rangle \le \frac{\theta - 1}{\theta}\|x\|\|y\| \le (\theta - 1)\mathrm{Re}\,\langle x, y\rangle\,, \quad x, y \in V, \tag{47}$$

where $\theta \ge 1$ is a known constant. If (47) holds for fixed $\theta \ge 1$ and vectors x, y, then

$$0 \le (\|x\| + \|y\|)^2 - \|x + y\|^2 = 2(\|x\|\|y\| - \mathrm{Re}\,\langle x, y\rangle) \le 2\frac{\theta - 1}{\theta}\|x\|\|y\| \le 2(\theta - 1)\mathrm{Re}\,\langle x, y\rangle\,.$$

Hence

$$0 \le (\|x\| + \|y\|)^2 - \|x + y\|^2 \le 2\frac{\theta - 1}{\theta}\|x\|\|y\| \le 2(\theta - 1)\mathrm{Re}\,\langle x, y\rangle\,.$$

Applying the elementary inequalities $\sqrt{a} - \sqrt{b} \le \sqrt{a - b} \le \sqrt{c}$, if $0 \le a - b \le c,\ b \ge 0$, leads to

$$0 \le \|x\| + \|y\| - \|x + y\| \le \sqrt{2\frac{\theta - 1}{\theta}\|x\|\|y\|} \le \sqrt{2(\theta - 1)\mathrm{Re}\,\langle x, y\rangle}. \tag{48}$$

In this way, consecutively using Theorem 3, Corollary 5 and Corollary 6, the following three results on reverses of the triangle inequality can be easy established. We omit the details.

The first.

Theorem 4. *Fix $\tau_k \ge 0,\ \kappa_k \ge 1,\ k = 1, 2$ with $\kappa_k^2 = \tau_k^2 + 1$ and $\tau_1\tau_2 < 1$ and let $x, y, v \in V \setminus \{0\}$.*

The following inequalities

$$0 \le \|x\| + \|y\| - \|x + y\| \le \sqrt{2\frac{\kappa_1\kappa_2 + \tau_1\tau_2 - 1}{\kappa_1\kappa_2}\|x\|\|y\|} \le \sqrt{2\frac{\kappa_1\kappa_2 + \tau_1\tau_2 - 1}{1 - \tau_1\tau_2}\mathrm{Re}\,\langle x, y\rangle}.$$

hold true whenever one of the below conditions is met

(i) *there exist $0 < c_1, c_2 \in \mathbb{R}$ such that (12) holds, i.e.*

$$\|c_1 x - v\| \le \frac{\tau_1}{\kappa_1}\|v\| \text{ and } \|c_2 y - v\| \le \frac{\tau_2}{\kappa_2}\|v\|,$$

(ii) *there exist* $0 < c_1, c_2 \in \mathbb{R}$ *such that (13) holds, i.e.*

$$\|c_1 x - v\| \leq \frac{\tau_1}{\kappa_1 + 1}\|c_1 x + v\| \text{ and } \|c_2 y - v\| \leq \frac{\tau_2}{\kappa_2 + 1}\|c_2 y + v\|,$$

(iii) *there exist* $a, b, d, g \in V$ *with* $\mathrm{Re}\,\langle a, b\rangle > 0, \mathrm{Re}\,\langle d, g\rangle > 0$ *and* $a+b, d+g \in \mathrm{cone}\{v\}$ *such that (14) holds, i.e.*

$$\mathrm{Re}\,\langle b - x, x - a\rangle \geq 0 \quad \text{and} \quad \mathrm{Re}\,\langle g - y, y - d\rangle \geq 0,$$

and $\tau_1 = \tau_{a,b}$, $\kappa_1 = \kappa_{a,b}$, $\tau_2 = \tau_{d,g}$, $\kappa_2 = \kappa_{d,g}$,

(iii') *there exist* $\gamma, \Gamma, \lambda, \Lambda > 0$ *such that (33) holds, i.e.*

$$\mathrm{Re}\,\langle \Gamma v - x, x - \gamma v\rangle \geq 0 \quad \text{and} \quad \mathrm{Re}\,\langle \Lambda v - y, y - \lambda v\rangle \geq 0$$

and $\tau_1 = \tau_{\gamma,\Gamma}$, $\kappa_1 = \kappa_{\gamma,\Gamma}$, $\tau_2 = \tau_{\lambda,\Lambda}$, $\kappa_2 = \kappa_{\lambda,\Lambda}$,

(iv) $x \in C^+_{v,\tau_1}$, $y \in C^+_{v,\tau_2}$. □

The second one.

Corollary 7. *Fix* $\tau \geq 0$ *and* $\kappa \geq 1$ *with* $\kappa^2 = \tau^2 + 1$ *and let* $x, y \in V \setminus \{0\}$.
The following inequalities

$$0 \leq \|x\| + \|y\| - \|x+y\| \leq \sqrt{2\frac{\kappa - 1}{\kappa}\|x\|\|y\|} \leq \sqrt{2(\kappa-1)\mathrm{Re}\,\langle x, y\rangle} \tag{49}$$

hold true whenever one of the below conditions is met

(i) *there exists* $c > 0$ *such that* $\|cx - y\| \leq \frac{\tau}{\kappa}\|y\|$,

(ii) *there exists* $c > 0$ *such that* $\|cx - y\| \leq \frac{\tau}{1+\kappa}\|cx + y\|$,

(iii) *there exist* $a, b \in V$ *with* $\mathrm{Re}\,\langle a, b\rangle > 0$ *and* $a + b \in \mathrm{cone}\{y\}$ *such that* $\mathrm{Re}\,\langle b - x, x - a\rangle \geq 0$ *and* $\tau = \tau_{a,b}$, $\kappa = \kappa_{a,b}$,

(iii') *there exist* $\gamma, \Gamma > 0$ *such that* $\mathrm{Re}\,\langle \Gamma y - x, x - \gamma y\rangle \geq 0$ *and* $\tau = \tau_{\gamma,\Gamma}$, $\kappa = \kappa_{\gamma,\Gamma}$,

(iv) *there exist* $c > 0$ *and* $h \in V$ *such that* $cx = y + h$, $\langle y, h\rangle = 0$ *and* $\|h\| \leq \tau\|y\|$. □

Remark 6. *Observe, if (iii') holds for* $\Gamma \geq \gamma > 0$, *then inequalities (49) take the form*

$$0 \leq \|x\| + \|y\| - \|x+y\| \leq \sqrt{2}\frac{\sqrt{\Gamma} - \sqrt{\gamma}}{\sqrt{\Gamma + \gamma}}\sqrt{\|x\|\|y\|} \leq \frac{\sqrt{\Gamma} - \sqrt{\gamma}}{\sqrt[4]{\Gamma\gamma}}\sqrt{\mathrm{Re}\,\langle x, y\rangle}.$$

The implication (iii')⇒(49) is due to S.S. Dragomir [7, Proposition 3.2]. □

And the foretold third result.

Proposition 2. *Let* $x, y, v \in V$ *be as in Corollary 6.*
Then

$$\begin{aligned} 0 \leq \|x\| + \|y\| \quad &- \quad \|x+y\| \leq \sqrt{2\tfrac{\|x-v\|\|y-v\| + \|v\|^2 - \sqrt{\|v\|^2 - \|x-v\|^2}\sqrt{\|v\|^2 - \|y-v\|^2}}{\|v\|^2}\|x\|\|y\|} \leq \\ &\leq \quad \sqrt{2\tfrac{\|x-v\|\|y-v\| + \|v\|^2 - \sqrt{\|v\|^2 - \|x-v\|^2}\sqrt{\|v\|^2 - \|y-v\|^2}}{\sqrt{\|v\|^2 - \|x-v\|^2}\sqrt{\|v\|^2 - \|y-v\|^2} - \|x-v\|\|y-v\|}\mathrm{Re}\,\langle x, y\rangle}, \end{aligned}$$

or, equivalently, setting $\alpha_1 = \arcsin\frac{\|x-v\|}{\|v\|}$ *and* $\alpha_2 = \arcsin\frac{\|y-v\|}{\|v\|}$

$$0 \leq \|x\| + \|y\| - \|x+y\| \leq \sqrt{1 - \cos(\alpha_1 + \alpha_2)\|x\|\|y\|} \leq \sqrt{2\frac{(1 - \cos(\alpha_1 + \alpha_2)}{\cos(\alpha_1 + \alpha_2)}\mathrm{Re}\,\langle x, y\rangle}.$$

□

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