J. Math. Anal. Appl. 461 (2018) 657–675

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Isometric weighted composition operators on weighted Bergman spaces ☆

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ARTICLE INFO

*Article history:*
Received 1 September 2017
Submitted by D. Khavinson

*Keywords:*
Weighted composition operators
Weighted Bergman spaces
Isometry
Wold decomposition
Numerical range

ABSTRACT

We characterize the isometric weighted composition operators on weighted Bergman spaces over the unit disk. We also determine the Wold decomposition of isometric weighted composition operators acting on a class of general reproducing kernel Hilbert spaces in the case when the composition symbol has an interior fixed point, and characterize the numerical range of isometric weighted composition operators on weighted Bergman spaces.



## 1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane, and let $\mathcal{H}(\mathbb{D})$ be the space of holomorphic, complex valued, functions on $\mathbb{D}$. For $u, \phi \in \mathcal{H}(\mathbb{D})$, with a nonconstant $\phi : \mathbb{D} \to \mathbb{D}$, the weighted composition operator (WCO) $W_{u,\phi}$ on $\mathcal{H}(\mathbb{D})$ is defined by

$$W_{u,\phi} f = u(f \circ \phi).$$

Choosing $\phi(z) = z$, the weighted composition operator $W_{u,\phi}$ becomes the multiplication operator $M_u$. In the case when $u \equiv 1$ on $\mathbb{D}$, $W_{u,\phi}$ is the composition operator $C_\phi$ on $\mathcal{H}(\mathbb{D})$.

In this paper we investigate the isometric weighted composition operators on spaces of holomorphic functions, and in particular on the weighted Bergman spaces over the unit disk.

For $\alpha > -1$, the weighted Bergman space $L^2_a(dm_\alpha)$ on the unit disk is defined as

$$L^2_a(dm_\alpha) = \{f \in \mathcal{H}(\mathbb{D}); \|f\|^2_\alpha = \int_{\mathbb{D}} |f(z)|^2 dm_\alpha(z) < \infty\},$$

☆ Research supported in part by NSERC grant 004975.
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https://doi.org/10.1016/j.jmaa.2018.01.036
0022-247X/

where $m$ denotes the normalized Lebesgue area measure on $\mathbb{D}$ and

$$dm_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dm(z).$$

When $\alpha = 0$, we get the classical Bergman space $L_a^2(dm)$.

Weighted Bergman spaces are reproducing kernel Hilbert spaces with a positive definite kernel $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ given by

$$K^\alpha(w,z) = \frac{1}{(1-\overline{z}w)^{\alpha+2}}.$$

We will denote the corresponding point evaluation functions at $z$ by $K_z^\alpha$, and the normalized point evaluation functions by $k_z^\alpha$. Since $\|K_z^\alpha\|^2 = K^\alpha(z,z) = \frac{1}{(1-|z|^2)^{\alpha+2}}$,

$$k_z^\alpha(w) = \frac{(1-|z|^2)^{\frac{\alpha}{2}+1}}{(1-\overline{z}w)^{\alpha+2}}.$$

The class of weighted composition operators is defined by using the natural operations that one can perform on spaces of functions. This class plays a particularly important role when determining the isometric operators on some Banach spaces of holomorphic functions. For example, Forelli determined in [7] that the isometries of the Hardy spaces $H^p$, $p \neq 2$, over the unit disk $\mathbb{D}$, are weighted composition operators. Similarly, Kolaski's results from [12] characterize the isometric operators on the Bergman spaces $L_a^p$, $p \neq 2$, over general Runge domains as weighted composition operators.

The cases when $p = 2$ represent the Hilbert space case, and so there are many other isometries acting on these spaces. Such are, for example, all of the unitaries that are defined by a simple change of basis. Still, the isometries (unitary, or not) that are also weighted composition operators are of particular interest also in the Hilbert space case, and are sometimes referred to as canonical isometries.

There is a vast literature dealing with the properties of weighted composition operators acting on a variety of spaces. Their boundedness and compactness on the Hardy spaces was determined in [3], and on the Bergman spaces in [5], [6] and [13]. The isometric weighted composition operators on the Hardy space $H^2$ were explored in [13] and [16]. The unitary weighted composition operators are of a particular interest, and they have been characterized even for some more general classes of reproducing kernel Hilbert spaces, including the Hardy and the Bergman spaces. See, for example, [14], [15], [18], and the references therein.

As it was shown in [14], a weighted composition operator on the weighted Bergman spaces is a co-isometry if and only if it is unitary. On the other hand, since for a unitary weighted composition operator on the Bergman spaces the composition symbol has to be a disk automorphism, the following simple example shows the existence of non-unitary isometric weighted composition operators.

**Example.** Let $\phi$ be a finite Blaschke product of degree $n$, and let $u(z) = \frac{1}{\sqrt{n}}\phi'(z)$. Then the weighted composition operator $W_{u,\phi}$ is isometric on the Bergman space $L_a^2(dm)$. This is easy to see since, using the change of variable formula and the fact that $\phi$ is of constant multiplicity n, we have that for any $f \in L_a^2(dm)$

$$\|W_{u,\phi}f\|^2 = \int_{\mathbb{D}} |f(\phi(z))|^2 \frac{1}{n}|\phi'(z)|^2 dm(z) = \int_{\mathbb{D}} |f(w)|^2 dm(w) = \|f\|^2.$$

Including the introduction, the paper contains three sections. Section 2 deals with the conditions that determine the isometric weighted composition operators acting on the weighted Bergman spaces over the unit disk.

Section 3 explores the Wold decomposition and some related properties of isometric weighted composition operators on more general reproducing kernel Hilbert spaces, and the characterization of the Wold decomposition of isometric $W_{u,\phi}$ on the weighted Bergman spaces over the unit disk, in the case when $\phi$ has an interior fixed point. This section also contains the description of the numerical range of isometric weighted composition operators acting on weighted Bergman spaces over the unit disk.

## 2. Characterization of isometric WCO

The characterizations of a number of properties of weighted composition operators on the Bergman spaces are usually given through a measure theoretic approach. We will start by exploring this approach also for the characterization of isometric weighted composition operators.

For $\alpha > -1$, $u, \phi \in \mathcal{H}(\mathbb{D})$ with a nonconstant $\phi : \mathbb{D} \to \mathbb{D}$, define Borel measures $\mu_u^\alpha$ and $\mu_{u,\phi}^\alpha$ on $\mathbb{D}$ by

$$\mu_u^\alpha(E) = \int\limits_E |u(z)|^2 dm_\alpha(z),$$

and

$$\mu_{u,\phi}^\alpha(E) = \mu_u^\alpha(\phi^{-1}(E)) = \int\limits_{\phi^{-1}(E)} |u(z)|^2 dm_\alpha(z),$$

for any Borel set $E$ contained in $\mathbb{D}$. Thus, $\mu_{u,\phi}^\alpha$ is a $u$ weighted, $\phi$ pull-back measure of $dm_\alpha$. Since $\phi$ is a nonconstant holomorphic map on $\mathbb{D}$ and $u$ is holomorphic, it is not too hard to see that $\mu_{u,\phi}^\alpha$ is a Borel measure that is absolutely continuous with respect to $m_\alpha$. We denote by $h_{u,\phi}^\alpha$ the Radon–Nikodym derivative of $\mu_{u,\phi}^\alpha$ with respect to $m_\alpha$, i.e.

$$h_{u,\phi}^\alpha(z) = \frac{d\mu_{u,\phi}^\alpha}{dm_\alpha}(z).$$

Recall that for a measurable function $h$ in $L^1(dm_\alpha)$ the Toeplitz operator $T_h$ on $L_a^2(dm_\alpha)$ is defined by

$$T_h f(z) = \int\limits_{\mathbb{D}} h(w) f(w) \frac{1}{(1 - z\overline{w})^{\alpha+2}} dm_\alpha(z).$$

The Berezin transform of $T_h$ on $L_a^2(dm_\alpha)$ is defined by

$$\widetilde{T_h}(z) = \tilde{h}(z) = < T_h k_z^\alpha, k_z^\alpha > = \int\limits_{\mathbb{D}} h(w) \frac{(1 - |z|^2)^{\alpha+2}}{|1 - z\overline{w}|^{2\alpha+4}} dm_\alpha(w).$$

The characterization of the boundedness and compactness of weighted composition operators on weighted Bergman spaces in [5] and [6] was given via a similar Berezin transform formula.

We have the following characterization of isometric weighted composition operators acting on weighted Bergman spaces over the unit disk.

**Theorem 2.1.** *Let $\alpha > -1$, $\phi$ a nonconstant holomorphic self-map of $\mathbb{D}$ and $u \in \mathcal{H}(\mathbb{D})$ are such that $W_{u,\phi} : L_a^2(dm_\alpha) \to L_a^2(dm_\alpha)$ is bounded. Then:*

*(i)* $W_{u,\phi}^* W_{u,\phi} = T_{h_{u,\phi}^\alpha}$.
*(ii)* $W_{u,\phi}$ *is an isometry if and only if* $h_{u,\phi}^\alpha = 1$ *almost everywhere on* $\mathbb{D}$.
*(iii) If* $W_{u,\phi}$ *is an isometry, then* $\phi$ *is a full map, i.e.* $m(\mathbb{D} \setminus \phi(\mathbb{D})) = 0$.

*(iv) $W_{u,\phi}$ is an isometry if and only if for all $z \in \mathbb{D}$,*

$$\widetilde{h^{\alpha}_{u,\phi}}(z) = \int_{\mathbb{D}} |u(w)|^2 \frac{(1-|z|^2)^{\alpha+2}}{|1-z\overline{\phi(w)}|^{2\alpha+4}} dm_\alpha(w) = 1.$$

*(v) $W_{u,\phi}$ is an isometry if and only if $\|W_{u,\phi}k^{\alpha}_z\|_\alpha = 1$ for every $z$ in $\mathbb{D}$.*

**Proof.** (i) Since $h^{\alpha}_{u,\phi}$ is a non-negative function, $T_{h^{\alpha}_{u,\phi}}$ is a positive operator. Thus, we need to show that $\forall f \in L^2_a(dm_\alpha)$, we have $< W^*_{u,\phi}W_{u,\phi}f, f >=< T_{h^{\alpha}_{u,\phi}}f, f >$. This holds since

$$\begin{aligned}\|W_{u,\phi}f\|^2 &= \int_{\mathbb{D}} |u(z)|^2|f(\phi(z))|^2 dm_\alpha(z)\\ &= \int_{\mathbb{D}} |f(w)|^2 d\mu^{\alpha}_{u,\phi}(w)\\ &= \int_{\mathbb{D}} |f(w)|^2 h^{\alpha}_{u,\phi}(w) dm_\alpha(w)\\ &= < T_{h^{\alpha}_{u,\phi}}f, f > .\end{aligned}$$

(ii) The operator $W_{u,\phi}$ is an isometry if and only if $W^*_{u,\phi}W_{u,\phi}$ is the identity operator on $L^2_a(dm_\alpha)$. By (i), this is equivalent to $T_{h^{\alpha}_{u,\phi}} = I$. It is easy to see that then $h^{\alpha}_{u,\phi} - 1$ is orthogonal to every polynomial in $z$ and $\overline{z}$. Since the set of such polynomials is dense in $L^2(dm_\alpha)$, we get that $h^{\alpha}_{u,\phi} = 1$ almost everywhere on $\mathbb{D}$.

(iii) If $W_{u,\phi}$ is an isometry, then it follows from (ii) that $h^{\alpha}_{u,\phi} = 1$ almost everywhere on $\mathbb{D}$, and so $\mu^{\alpha}_{u,\phi}(E) = m_\alpha(E)$, for every Borel set $E$. If $E \subset \mathbb{D} \setminus \phi(\mathbb{D})$, then $m_\alpha(\phi^{-1}(E)) = 0$ and so

$$\mu^{\alpha}_{u,\phi}(E) = \int_{\phi^{-1}(E)} |u(z)|^2 dm_\alpha(z) = 0.$$

But then $m_\alpha(E) = \mu^{\alpha}_{u,\phi}(E) = 0$ and so $m(E) = 0$. Since the Lebesgue measure is inner regular, it follows that $m(E) = 0$ also for every Lebesgue measurable subset of $\mathbb{D} \setminus \phi(\mathbb{D})$.

(iv) This follows from the fact that the Berezin transform is an injective map, and since the Berezin transform of the identity operator is the constant function 1. Furthermore, using that $h^{\alpha}_{u,\phi} = d\mu^{\alpha}_{u,\phi}/dm_\alpha$, we have that for any $z \in \mathbb{D}$

$$\begin{aligned}\widetilde{h^{\alpha}_{u,\phi}}(z) &= \int_{\mathbb{D}} h^{\alpha}_{u,\phi}(w) \frac{(1-|z|^2)^{\alpha+2}}{|1-z\overline{w}|^{2\alpha+4}} dm_\alpha(w)\\ &= \int_{\mathbb{D}} \frac{(1-|z|^2)^{\alpha+2}}{|1-z\overline{w}|^{2\alpha+4}} d\mu^{\alpha}_{u,\phi}(w)\\ &= \int_{\mathbb{D}} |u(w)|^2 \frac{(1-|z|^2)^{\alpha+2}}{|1-z\overline{\phi(w)}|^{2\alpha+4}} dm_\alpha(w) = 1.\end{aligned}$$

Part (v) follows from part (iv) since $\widetilde{h^{\alpha}_{u,\phi}}(z) = \|W_{u,\phi}k^{\alpha}_z\|^2_\alpha$. $\square$

Note that the isometric weighted composition operators on the Hardy space $H^2$ have a slightly more explicit characterization. Namely, as was shown in [16], $W_{u,\phi}$ is an isometry on $H^2$ iff $\phi$ is an inner function and $u \in H^2$ is such that

$$\int_{\phi^{-1}(E)} |u(\xi)|^2 d\sigma(\xi) = \int_{\phi^{-1}(E)} \frac{1}{(P_{\phi(0)} \circ \phi)(\xi)} d\sigma(\xi),$$

for every measurable $E \subset \partial\mathbb{D}$, where $d\sigma$ denotes the normalized Lebesgue measure on the unit circle $\partial\mathbb{D}$, and $P_{\phi(0)}$ is the Poisson kernel function at $\phi(0)$.

It is interesting to see why this happens, and what exactly are the nuances in the Bergman space case. One way to explain these differences is by noting that, first of all, the Radon–Nikodym derivative $d\sigma_{\phi^{-1}}/d\sigma$ in the Hardy space case, where $\sigma_{\phi^{-1}}(E) = \sigma(\phi^{-1}(E))$, is easily describable. Namely, it is the Poisson kernel function at $\phi(0)$. This can be seen directly, by comparing the Fourier coefficients of the measures corresponding to $d\sigma_{\phi^{-1}}$ and to $P_{\phi(0)}d\sigma$.

The rest of the argument follows since in the Hardy space case it is also true that $W_{u,\phi}$ is an isometry if and only if $\phi$ is inner and $d\sigma_{u,\phi}/d\sigma$ is equal to 1, and since furthermore $d\sigma_{u,\phi}/d\sigma_{\phi^{-1}} = |u|^2$. Here, $d\sigma_{u,\phi}$ denotes the $u$ weighted $\phi$ pull-back of $d\sigma$.

The other difference is that in the Hardy space case the Radon–Nikodym derivative $d\sigma_{\phi^{-1}}/d\sigma$, i.e. the Poisson kernel function, is a harmonic function and, in general, a Berezin transform of a harmonic function is the function itself. The explicit formula and the harmonicity of the Radon–Nikodym derivative are non-existent in general in the Bergman space case. Still, using the previous theorem and adjusting the discussion above to the weighted Bergman spaces by replacing the measure $\sigma$ on $\partial\mathbb{D}$ with the measure $m_\alpha$ on $\mathbb{D}$, and the measures $\sigma_{u,\phi}$ with $\mu^\alpha_{u,\phi}$ and $\sigma_{\phi^{-1}}$ with $m_{\alpha,\phi^{-1}}$, we have the following.

**Proposition 2.1.** *Let $W_{u,\phi}$ be bounded on $L^2_a(dm_\alpha)$ and let $g_\phi$ be the Radon–Nikodym derivative $dm_{\alpha,\phi^{-1}}/dm_\alpha$. Then $W_{u,\phi}$ is an isometry on $L^2_a(dm_\alpha)$ if and only if $\phi$ is a full map and*

$$\int_{\phi^{-1}(E)} |u(z)|^2 dm_\alpha(z) = \int_{\phi^{-1}(E)} \frac{1}{(g_\phi \circ \phi)(z)} dm_\alpha(z),$$

*for every measurable $E \subset \mathbb{D}$.*

We can come closer to "a formula" for the Radon–Nikodym derivative $d\mu^\alpha_{u,\phi}/dm_\alpha$ in the case when $\phi$ is of bounded multiplicity. In particular, when $\phi$ is univalent and $W_{u,\phi}$ is an isometry on $L^2_a(dm_\alpha)$, this gives us a characterization of $u$ in terms of $\phi$ and $\phi'$, as described in the next result.

**Theorem 2.2.** *Let $\alpha > -1$, $\phi$ a non-constant holomorphic self-map of $\mathbb{D}$ and $u \in \mathcal{H}(\mathbb{D})$ are such that $W_{u,\phi} : L^2_a(dm_\alpha) \to L^2_a(dm_\alpha)$ is bounded. Then:*

*(i) If $\phi$ is of multiplicity bounded by $N$, then the Radon–Nikodym derivative of $\mu^\alpha_{u,\phi}$ with respect to $m_\alpha$ is given by $h^\alpha_{u,\phi}(z) = 0$, if $z \notin \phi(\mathbb{D})$, and otherwise*

$$h^\alpha_{u,\phi}(z) = \sum_{n=1}^{N} \frac{|u(z_n)|^2(1-|z_n|^2)^\alpha}{|\phi'(z_n)|^2(1-|\phi(z_n)|^2)^\alpha},$$

*where for each $n$, $\phi(z_n) = z$ and $\phi'(z_n) \neq 0$.*

*(ii) If $\phi$ is univalent and $W_{u,\phi}$ is an isometry on $L^2_a(dm_\alpha)$, then $\phi$ is a full map, $u$ is never zero, and*

$$u(z) = \frac{\overline{\phi'(0)}}{\overline{u(0)}} \phi'(z)(1-\overline{\phi(0)}\phi(z))^\alpha,$$

*with* $|u(0)| = |\phi'(0)|(1-|\phi(0)|^2)^{\alpha/2}$. *In particular, if* $\alpha = 0$*, then* $u = c\phi'$, $|c| = 1$. *Furthermore, if* $\exists a \in \mathbb{D}$ *such that* $\phi(a) = 0$*, then*

$$u(z) = c\phi'(z)\frac{(1-|a|^2)^{\alpha/2}}{(1-\overline{a}z)^{\alpha}},\ |c| = 1.$$

*(iii) If* $\phi$ *is a disk automorphism and* $W_{u,\phi}$ *is an isometry on* $L^2_a(dm_\alpha)$*, then*

$$u = ck^{\alpha}_{\phi^{-1}(0)},\ |c| = 1,$$

*and so* $W_{u,\phi}$ *is a unitary.*

**Proof.** (i) First of all, it is easy to see that $h^{\alpha}_{u,\phi} = 0$ on any measurable subset $E$ of $\mathbb{D} \setminus \phi(\mathbb{D})$, using that $m_\alpha(\phi^{-1}(E)) = 0$. Since $\phi$ is non-constant holomorphic function, the set $Z'$ of zeroes of $\phi'$ is at most countable, i.e. $Z'$ is a set of measure zero. Also, $\phi$ maps sets of measure zero into sets of measure zero. Thus, we consider an almost everywhere definition of $h^{\alpha}_{u,\phi}$ on $\phi(\mathbb{D} \setminus Z')$. Using the change of variable formula, we have that for every $f \in L^2_a(dm_\alpha)$

$$\begin{aligned}\|W_{u,\phi}f\|^2 &= \int\limits_{\mathbb{D}} |u(z)|^2|f(\phi(z))|^2 dm_\alpha(z)\\ &= \int\limits_{\mathbb{D}\setminus Z'} |u(z)|^2|f(\phi(z))|^2\frac{|\phi'(z)|^2}{|\phi'(z)|^2}dm_\alpha(z)\\ &= \int\limits_{\phi(\mathbb{D}\setminus Z')} |f(w))|^2\sum_{n=1}^{N}\frac{|u(w_n)|^2(1-|w_n|^2)^\alpha}{|\phi'(w_n)|^2(1-|\phi(w_n)|^2)^\alpha}dm_\alpha(w)\end{aligned}$$

But, as was shown in the proof of Theorem 2.1, $\|W_{u,\phi}f\|^2 =< T_{h^{\alpha}_{u,\phi}}f, f >$, and so the (almost everywhere) definition of $h^{\alpha}_{u,\phi}$ follows.

(ii) If $\phi$ is univalent we get from (i) that $h_{u,\phi}(z) = 0$, if $z \notin \phi(\mathbb{D})$, or otherwise, if also $z \notin \phi(Z')$, we have that

$$h_{u,\phi}(z) = \frac{|u(\phi^{-1}(z))|^2(1-|\phi^{-1}(z)|^2)^\alpha}{|\phi'(\phi^{-1}(z))|^2(1-|z|^2)^\alpha}.$$

But if $W_{u,\phi}$ is an isometry on $L^2_a(dm_\alpha)$, then by parts (ii) and (iii) of Theorem 2.1, $m(\mathbb{D} \setminus \phi(\mathbb{D})) = 0$ and $h_{u,\phi}(z) = 1$ almost everywhere on $\mathbb{D}$. Since $u$, $\phi$ and $\phi'$ are holomorphic on $\mathbb{D}$ and

$$|u(w)|^2 = |\phi'(w)|^2\frac{(1-|\phi(w)|^2)^\alpha}{(1-|w|^2)^\alpha}$$

almost everywhere on $\mathbb{D}$, we can view this equation as a diagonal equality for two functions of two complex variables, both analytic in the first variable, and conjugate analytic in the second variable. By a standard argument, this implies that the two function are equal on $\mathbb{D} \times \mathbb{D}$, i.e. that

$$u(z)\overline{u(w)} = \phi'(z)\overline{\phi'(w)}\frac{(1-\phi(z)\overline{\phi(w)})^\alpha}{(1-z\overline{w})^\alpha}.$$

Since $\phi$ is univalent and maps the unit disk into itself, the right hand side is never zero, and so neither can be the left hand side, i.e. $u$ is never zero. Now, taking $w = 0$, we get the claimed equation.

In the case when $\phi(a) = 0$ for some $a \in \mathbb{D}$, we can instead take $w = a$, which gives us $u(z) = \frac{\overline{\phi'(a)}}{u(a)}\phi'(z)\frac{1}{(1-\overline{a}z)^{\alpha}}$. Since $|u(a)| = |\phi'(a)|\frac{1}{(1-|a|^2)^{\alpha/2}}$, we get that

$$u(z) = c\phi'(z)\frac{(1-|a|^2)^{\alpha/2}}{(1-\overline{a}z)^{\alpha}},\ |c| = 1.$$

(iii) The proof follows by using the last formula from part (ii), and the fact that when $\phi$ is a disk automorphism with $\phi^{-1}(0) = a$, then $|\phi'(z)| = \frac{(1-|a|^2)}{|1-\overline{a}z|^2}$. Thus, we have that $|u(z)| = |\phi'(z)|^{1+\alpha/2}$. Since furthermore $|k_a^{\alpha}(z)| = |\phi'(z)|^{1+\alpha/2}$ and $u$ and $k_a^{\alpha}$ are holomorphic functions, we conclude that $u = ck_a^{\alpha}$, with $|c| = 1$. But then by results proven in [14] (or [18]), it follows that $W_{u,\phi}$ is a unitary weighted composition operator. □

When $\alpha = 0$, i.e. when the space is the classical Bergman space on the unit disk, part (ii) of Theorem 2.2 (or a direct change of variable formula) shows that $W_{u,\phi}$ with univalent $\phi$ is an isometry if and only if $u = c\phi'(z)$, $|c| = 1$. Thus, here is a simple example of a non-unitary isometry $W_{u,\phi}$ on $L_a^2(dm)$ with univalent composition symbol $\phi$.

**Example.** Let $\phi$ be the Riemann map from $\mathbb{D}$ onto $\mathbb{D} \setminus [0, 1)$, and let $u = \phi'$. Then the weighted composition operator $W_{u,\phi}$ is a non-unitary isometry on the classical Bergman space $L^2(dm)$.

If $\phi$ is of unbounded multiplicity, but the series

$$\sum_{n=1}^{\infty} \frac{|u(z_n)|^2(1-|z_n|^2)^{\alpha}}{|\phi'(z_n)|^2(1-|\phi(z_n)|^2)^{\alpha}},$$

with $\phi(z_n) = z$ and $\phi'(z_n) \neq 0$, converges for every $z$ in $\phi(\mathbb{D})$, the formula for the Radon–Nikodym derivative $h_{u,\phi}^{\alpha}$ in part (ii) of Theorem 2.2 can also be extended to this case. It would be nice to know exactly when does this happen, and have an analytic and geometric characterization of such cases.

Similarly, the result from [5] states that the operator $W_{u,\phi}$ is bounded on $L_a^2(dm_{\alpha})$ if and only if the Berezin transform of $h_{u,\phi}^{\alpha}$ is bounded. Nevertheless, it is not clear in general what is the geometric meaning of this condition.

These are two interesting problems for which we would like to add few more remarks and make some connections to the geometric behaviour of the symbols $u$ and $\phi$ of $W_{u,\phi}$.

**Remarks.** Recall that the local hyperbolic distortion of $\phi$, a holomorphic selfmap of $\mathbb{D}$, is defined by

$$\tau_{\phi}(z) = \frac{|\phi'(z)|(1-|z|^2)}{1-|\phi(z)|^2},$$

and that by the Schwarz–Pick lemma $\tau_{\phi}(z) \le 1$, for all $z$ in $\mathbb{D}$. Thus, assuming the convergence of the series,

$$\begin{aligned} h_{u,\phi}^{\alpha}(z) &= \sum_{n=1}^{\infty} \frac{|u(z_n)|^2(1-|z_n|^2)^{\alpha}}{|\phi'(z_n)|^2(1-|\phi(z_n)|^2)^{\alpha}} \\ &= \sum_{n=1}^{\infty} \frac{|u(z_n)|^2}{\tau_{\phi}(z_n)^2}\frac{\|K_{\phi(z_n)}^{\alpha}\|^2}{\|K_{(z_n)}^{\alpha}\|^2} \\ &= \sum_{n=1}^{\infty} \frac{\|W_{u,\phi}^{*}k_{z_n}^{\alpha}\|^2}{\tau_{\phi}(z_n)^2} \\ &\ge \sum_{n=1}^{\infty} \|W_{u,\phi}^{*}k_{z_n}^{\alpha}\|^2. \end{aligned}$$

Hence, if the series converges we have that $\frac{\|W^*_{u,\phi}k^\alpha_{z_n}\|}{\tau_\phi(z_n)} \to 0$, as $n \to \infty$, and also

$$\|W^*_{u,\phi}k^\alpha_{z_n}\|^2 = \frac{|u(z_n)|^2(1-|z_n|^2)^{2+\alpha}}{(1-|\phi(z_n)|^2)^{2+\alpha}} \to 0,$$

as $n \to \infty$. Now the last condition is definitely satisfied if $u$ and $\phi$ are such that

$$\lim_{|z|\to 1} \frac{|u(z)|(1-|z|^2)^{1+\alpha/2}}{(1-|\phi(z)|^2)^{1+\alpha/2}} = 0.$$

Consider the seemingly natural choice of $u = \phi'$. In this case the last condition means that $\phi$ belongs to the so called little, hyperbolic $(1+\alpha/2)$-Bloch space $\mathcal{B}^h_{0,(1+\alpha/2)}$.

An example of this kind for $\alpha = 0$ was used in [5] to support the claim that the condition itself is not sufficient either for compactness, or even for boundedness of $W_{u,\phi}$. Namely, if $\phi$ is an infinite Blaschke product in the little hyperbolic 1-Bloch space $\mathcal{B}^h_0$ and $u = \phi'$, the corresponding weighted composition operator $W_{u,\phi}$ is unbounded on the Bergman space $L^2_a(dm)$, even though $\lim_{|z|\to 1} \frac{|u(z)|(1-|z|^2)}{(1-|\phi(z)|^2)} = 0$.

The example suggests two possible conclusions. One is that for the chosen infinite Blaschke product $\phi$ in the previous example, $u = \phi'$ is a bad choice if we want to have boundedness (or compactness) of the weighted composition operator with composition symbol $\phi$. Moreover, when $\alpha = 0$ and $\phi$ is an infinite Blaschke product, it is clear that $u = \phi'$ is a bad choice also since then $u$ is not in the Bergman space. Hence, $W_{u,\phi}$ can not be bounded on the Bergman space. Note also that in this case $h^\alpha_{u,\phi}(z)$ can not be well defined through the infinite series. The second possible conclusion is that the inequality showing in the above series of formulas is essential and can not be overlooked, i.e. that we have to also include the hyperbolic distortion $\tau_\phi$ in the conditions that guaranty that $W_{u,\phi}$ is bounded, compact, or an isometry.

For example, recall that $\lim_{|z|\to 1} \tau_\phi(z) = 1$ if and only if $\phi$ is a finite Blaschke product, and so the fact that in the example above $\phi$ is an infinite Blaschke product is essential.

The boundary behaviour of the local hyperbolic distortion is also closely related to the geometric properties of the map $\phi$ and its "boundary smoothness", given through the existence (or nonexistence) of angular derivatives of the map $\phi$. Namely, note that the condition

$$\lim_{|z|\to 1} \frac{|u(z)|(1-|z|^2)^{1+\alpha/2}}{(1-|\phi(z)|^2)^{1+\alpha/2}} = 0$$

implies that whenever $\phi$ has an angular derivative at $\xi \in \partial\mathbb{D}$, $u(z)$ converges to 0 as $z \to \xi$ in an angular region at $\xi$, but this is not sufficient for the compactness of $W_{u,\phi}$ on the weighted Bergman spaces. On the other hand, it is easy to see that the condition is necessary for the compactness of $W_{u,\phi}$, since the normalized point evaluations $k^\alpha_z$ converge weakly to zero, as $|z| \to 1$, and since

$$\frac{|u(z)|(1-|z|^2)^{1+\alpha/2}}{(1-|\phi(z)|^2)^{1+\alpha/2}} = \|W^*_{u,\phi}k^\alpha_z\|_\alpha.$$

We can get analogous conclusions about the boundedness of $W_{u,\phi}$ on the weighted Bergman spaces and the condition

$$\sup_{z\in\mathbb{D}} \frac{|u(z)|(1-|z|^2)^{1+\alpha/2}}{(1-|\phi(z)|^2)^{1+\alpha/2}} < \infty,$$

which is equivalent to $\sup_{z\in\mathbb{D}} \|W^*_{u,\phi}k^\alpha_z\|_\alpha < \infty$, and which implies that whenever $\phi$ has an angular derivative at $\xi \in \partial\mathbb{D}$, then $|u(z)|$ is bounded above by $|\phi'(\xi)|^{1+\alpha/2}$ in any angular region with a corner at $\xi$.

Similar conclusions were also derived in [8], while exploring the boundedness and compactness of weighted composition operators on the Hardy spaces. □

We summarize the last part of the remarks in a slightly more general statement by using the hyperbolic distortion $\tau_\phi$. Recall that if $\phi$ has an angular derivative at $\xi \in \partial\mathbb{D}$, then $\tau_\phi(z) \to 1$, as $z \to \xi$ in any angular region with a corner at $\xi$, and that $\phi$ is a disk automorphism if and only if $\exists z_0 \in \mathbb{D}$ such that $\tau_\phi(z_0) = 1$, which is further equivalent to $\tau_\phi(z) = 1$, $\forall z \in \mathbb{D}$. Also, as it was already mentioned before, $\lim_{|z|\to 1} \tau_\phi(z) = 1$ if and only if $\phi$ is a finite Blaschke product.

Using these few facts as a motivation, we also introduce the following notation: for $\delta \in (0, 1]$,

$$\Omega_\delta(\phi) = \{z \in \mathbb{D}; \tau_\phi(z) \geq \delta\}.$$

**Proposition 2.2.** *Let $W_{u,\phi}$ be bounded on $L^2_a(dm_\alpha)$ and $\delta \in (0, 1]$. Then $\forall z \in \Omega_\delta$*

$$|u(z)| \leq \frac{\|W_{u,\phi}\|}{\delta^{1+\alpha/2}} |\phi'(z)|^{1+\alpha/2},$$

*and moreover:*

*(i) If $\phi$ has angular derivative at $\xi \in \partial\mathbb{D}$, then $u$ is bounded in every angular region in $\mathbb{D}$ with a corner at $\xi$.*
*(ii) If $\phi$ is a finite Blaschke product and $W_{u,\phi}$ is a contraction on $L^2_a(dm_\alpha)$, then $\forall \varepsilon > 0$, $\exists r_\varepsilon \in [0, 1)$ such that whenever $|z| \geq r_\varepsilon$, $|u(z)| \leq \frac{1}{(1-\varepsilon)^{1+\alpha/2}} |\phi'(z)|^{1+\alpha/2}$.*
*(iii) If $\phi$ is a disk automorphism and $W_{u,\phi}$ is a contraction on $L^2_a(dm_\alpha)$, then $|u(z)| \leq |\phi'(z)|^{1+\alpha/2}$, $\forall z \in \mathbb{D}$.*

**Proof.** When $W_{u,\phi}$ is bounded, then $W^*_{u,\phi}$ has the same norm and so $\forall z \in \mathbb{D}$

$$\begin{aligned}\|W^*_{u,\phi} k^\alpha_z\|_\alpha &= \|\overline{u(z)} \frac{K^\alpha_{\phi(z)}}{\|K^\alpha_z\|_\alpha}\|_\alpha = \frac{|u(z)|(1-|z|^2)^{1+\alpha/2}}{(1-|\phi(z)|^2)^{1+\alpha/2}} \\ &= \frac{|u(z)|}{|\phi'(z)|^{1+\alpha/2}} \left(\tau_\phi(z)\right)^{1+\alpha/2} \leq \|W_{u,\phi}\|.\end{aligned}$$

Thus, the required inequality follows whenever $z \in \Omega_\delta$.

The parts (i), (ii) and (iii) follow directly by using the main inequality proven above, and using the behaviour of $\tau_\phi$ in each of the three special cases. □

Note that part (iii) of the previous proposition, together with the equations in the proof, gives a short proof of the fact that if $W_{u,\phi}$ is a co-isometry on $L^2_a(dm_\alpha)$ and $\phi$ is a disk automorphism, then $|u(z)| = |\phi'(z)|^{1+\alpha/2}$, which furthermore implies that $W_{u,\phi}$ is a unitary. Actually, it is not too hard to see that if $W_{u,\phi}$ is a co-isometry on $L^2_a(dm_\alpha)$ over the unit disk then $\phi$ must be a disk automorphism, and so $W_{u,\phi}$ is a co-isometry if and only if it is a unitary. This has also been shown in [14] for WCO' on weighted Bergman spaces over the unit ball, by using different methods.

We end this section by also posing two specific questions related to the discussion in the remarks.

**Question 2.1.** Does the condition

$$\lim_{|z|\to 1} \frac{|u(z)|(1-|z|^2)^{1+\alpha/2}}{\tau_\phi(z)(1-|\phi(z)|^2)^{1+\alpha/2}} = 0$$

imply boundedness, or even compactness of the weighted composition operator $W_{u,\phi}$ on $L^2_a(dm_\alpha)$? If not, then what is the largest class of functions $u$ and $\phi$ for which this might be true?

**Question 2.2.** Does the boundedness of $W_{u,\phi}$ on $L_a^2(dm_\alpha)$ imply the convergence of the series

$$\sum_{n=1}^{\infty} \frac{|u(z_n)|^2(1-|z_n|^2)^\alpha}{|\phi'(z_n)|^2(1-|\phi(z_n)|^2)^\alpha},$$

$\phi(z_n) = z$ and $\phi'(z_n) \neq 0$, for almost every $z$ in $\phi(\mathbb{D})$? If not, then what is the largest class of functions $u$ and $\phi$ for which this might be true and what is the geometric meaning of the convergence of this series?

## 3. Wold decomposition and numerical range of isometric WCO

In this section we determine the Wold decomposition of isometric weighted composition operators $W_{u,\phi}$ when $\phi$ has an interior fixed point, and determine the numerical range of isometric weighted composition operators acting on the weighted Bergman spaces.

We start with few results on isometric weighted composition operators in a slightly more general context, namely in the context of general reproducing kernel Hilbert spaces of holomorphic functions, with particular interest in their Wold decomposition. A good reference with more details on the topic of reproducing kernel Hilbert spaces is, for example, [1].

Let $\Omega$ be a domain in $\mathbb{C}^n$, and let $\mathcal{H}_K$ be a reproducing kernel Hilbert space (RKHS) of functions holomorphic on $\Omega$ with a positive definite kernel function $K : \Omega \times \Omega \to \mathbb{C}$. The point evaluation function at $z \in \Omega$ will be denoted by $K_z$, and the corresponding normalized point evaluation function by $k_z$. We assume that $\exists z_0 \in \Omega$ such that $K_{z_0}(z) = 1$, $\forall z \in \Omega$, and that the kernel $K$ is never zero.

The following proposition includes few very general results about WCO on RKHS' of holomorphic functions. We will use them when showing the main results of this section, but they also connect nicely to the characterizations of isometric WCO on the weighted Bergman spaces from the previous section.

**Proposition 3.1.** *Let $\mathcal{H}_K$ be a RKHS' of holomorphic functions on $\Omega$, let $\phi$ be a nonconstant holomorphic self-map of $\Omega$, and let $u \in \mathcal{H}_K$. Then:*

*(i) If $u$ is such that $\|u\| = 1$ and $|u(z_0)| = 1$, then $u(z) \equiv c$, $|c| = 1$.*
*(ii) If $W_{u,\phi}$ is an isometry on $\mathcal{H}_K$ and $|u(z_0)| = 1$, then $u(z) \equiv c$, $|c| = 1$, and $\phi(z_0) = (z_0)$.*
*(iii) Let $\phi(a) = a$ for some $a \in \Omega$. If $W_{u,\phi}$ is an isometry on $\mathcal{H}_K$ then $|u(a)| \leq 1$, and if furthermore $|u(a)| = 1$, then $u(z)$ is never zero and*

$$u = u(a)\frac{k_a}{k_a \circ \phi}.$$

**Proof.** (i) By the assumptions on $u$ and by the Cauchy–Schwarz inequality,

$$1 = |u(z_0)| = |< u, K_{z_0} >| \leq \|u\|\|K_{z_0}\| = 1.$$

Thus, $|< u, K_{z_0} >| = \|u\|\|K_{z_0}\|$, and so $u = cK_{z_0} = c$ with $|c| = 1$.

(ii) If $W_{u,\phi}$ is an isometry on $\mathcal{H}_K$, then $\|u\| = \|W_{u,\phi}K_{z_0}\| = \|K_{z_0}\| = 1$. But then by part (i), $u(z) \equiv c$ with $|c| = 1$, and so $W_{u,\phi} = c\ C_\phi$, i.e. the WCO is a unimodular constant multiple of the composition operator $C_\phi$. Since then $C_\phi$ is also an isometry on $\mathcal{H}_K$, we have that $C_\phi^* C_\phi = I$. But then, using the function $e_1(z) = z$, we get that

$$z_0 =< e_1, K_{z_0} >=< C_\phi^* C_\phi e_1, K_{z_0} >=< C_\phi e_1, C_\phi K_{z_0} >=< \phi, K_{z_0} >= \phi(z_0).$$

(iii) Note that if $a = z_0$, the conclusions on $u$ follow directly from parts (i) and (ii), even without assuming that $\phi(z_0) = z_0$.

In general, if $\phi(a) = a$ for a general $a \in \Omega$, we have that if $W_{u,\phi}$ is an isometry on $\mathcal{H}_K$, then

$$\begin{aligned}|u(a)| &= | < k_a, \overline{u(a)} k_a > | = | < k_a, W^*_{u,\phi} k_a > | \\ &= | < W_{u,\phi} k_a, k_a > | \le \|W_{u,\phi} k_a\| \|k_a\| = \|k_a\|^2 = 1.\end{aligned}$$

Thus, if $|u(a)| = 1$ we get that $| < W_{u,\phi} k_a, k_a > | = \|W_{u,\phi} k_a\| \|k_a\|$, and so $W_{u,\phi} k_a = c k_a$, with $|c| = 1$. But then $u(k_a \circ \phi) = c k_a$, and so $u = u(a)\frac{k_a}{k_a \circ \phi}$, which furthermore implies that $u(z)$ is never zero since $k_a(z)$ is never zero. □

Note that parts (ii) and (iii) from Proposition 3.1 give the following corollary when applied to the weighted Bergman spaces, thus extending the characterizations of special classes of isometric WCO' from Theorem 2.2.

**Corollary 3.1.** *Let $\alpha > -1$, $\phi$ a non-constant holomorphic self-map of $\mathbb{D}$ and $u \in \mathcal{H}(\mathbb{D})$ are such that $W_{u,\phi} : L^2_a(dm_\alpha) \to L^2_a(dm_\alpha)$ is bounded. Then:*

*(i) If $W_{u,\phi}$ is an isometry on $L^2_a(dm_\alpha)$ and $|u(0)| = 1$, then $u(z) \equiv c$, $|c| = 1$, and $\phi(0) = (0)$.*
*(ii) If $W_{u,\phi}$ is an isometry on $L^2_a(dm_\alpha)$ and $\phi(a) = a$ for some $a \in \mathbb{D}$, then $|u(a)| \le 1$. In the case when $|u(a)| = 1$, then $u(z)$ is never zero and*

$$u(z) = u(a)\frac{(1 - \overline{a}\phi(z))^{\alpha+2}}{(1 - \overline{a}z)^{\alpha+2}}.$$

*If furthermore $\phi$ is also univalent, then $a = 0$, and so $u(z) \equiv c$, $|c| = 1$.*

**Proof.** The only part that requires some explanation is the last statement of part (ii): when $\phi$ is univalent, combining the formula for $u$ from part (ii) of Theorem 2.2 and the above formula for $u$ when $|u(a)| = 1$ we get that

$$\phi'(z) = \phi'(a)(1 - \overline{\phi(0)}a)^\alpha \frac{(1 - \overline{a}\phi(z))^{\alpha+2}}{(1 - \overline{a}z)^{\alpha+2}}.$$

Evaluating at $a$ and using that $\phi(a) = a$ gives $(1 - \overline{\phi(0)}a)^\alpha = 1$. But then either $a = 0$, or $\phi(0) = 0$ which again implies that the (unique) interior fixed point $a$ of $\phi$ must be 0. □

Next we turn to the characterization of Wold decomposition of WCO on RKHS' of holomorphic functions. Recall that if the operator $T$ is an isometry on a separable, infinite dimensional Hilbert space $\mathcal{H}$, then the Wold decomposition for $T$ states that there exist orthogonal reducing subspaces $\mathcal{H}^U$ and $\mathcal{H}^S$ of $\mathcal{H}$, such that $\mathcal{H} = \mathcal{H}^U \oplus \mathcal{H}^S$, $U = T/\mathcal{H}^U$ is unitary, and $S = T/\mathcal{H}^S$ is a forward shift. Recall that $S$ is a forward shift on $\mathcal{H}^S$ if and only if $\bigcap_{m=1}^\infty S^m(\mathcal{H}^S) = \{0\}$, or equivalently if $(S^*)^m \to 0$ in the strong operator topology.

The following result generalizes the results from [16] on the Wold decomposition of weighted composition operators on the Hardy space $H^2$, in the case when the composition symbol has an interior fixed point.

**Theorem 3.1.** *Let $\mathcal{H}_K$ be a separable RKHS of holomorphic functions on the unit disk $\mathbb{D}$, let $\phi$ be a nonconstant holomorphic self-map of $\mathbb{D}$ with a fixed point $a$ in $\mathbb{D}$, and let $u \in \mathcal{H}_K$ be such that $W_{u,\phi}$ is an isometry on $\mathcal{H}_K$. Then* *(i)* $W_{u,\phi}$ *is a forward shift if and only if* $|u(a)| < 1$.

*(ii) If* $|u(a)| = 1$ *and* $\phi$ *is not a disk automorphism, then the Wold decomposition for* $W_{u,\phi}$ *is*

$$\mathcal{H}_K = \mathbb{C}k_a \bigoplus \{k_a\}^{\perp}.$$

*(iii) If* $|u(a)| = 1$ *and* $\phi$ *is a disk automorphism, and* $W_{k_a,\psi_a}$ *is a bounded operator on* $\mathcal{H}_K$ *for* $\psi_a$ *the involutive disk automorphism with* $\psi_a(0) = a$*, then* $W_{u,\phi}$ *is a unitary operator on* $\mathcal{H}_K$*.*

**Proof.** (i) If $W_{u,\phi}$ is a forward shift, then $|u(a)| < 1$ even if the RKHS $\mathcal{H}_K$ is over a general domain $\Omega \subset \mathbb{C}^n$. Namely, since $\phi(a) = a$, $W^*_{u,\phi}K_a = \overline{u(a)}K_a$,

$$(W^*_{u,\phi})^m K_a = \overline{u(a)}^m K_a$$

for every $m \geq 1$, and so $\|(W^*_{u,\phi})^m K_a\| \to 0$, as $m \to \infty$, implies that $|u(a)| < 1$.

The other direction follows as in [10], where the result was obtained for $\mathcal{H}_K = H^2$, the classical Hardy space on the unit disk. The proof uses the properties of the pseudo-hyperbolic metric $\rho$ on $\mathbb{D}$, and the Schwarz–Pick lemma for $\phi$. We provide the details here for completeness.

Namely, if $|u(a)| < 1$, there exists $0 < r < 1$ is such that $|u(z)| < \delta < 1$, $\forall z \in D(a, r)$, where $D(a, r)$ is the pseudo-hyperbolic disk centred at $a$, with radius $r$. By the Schwartz–Pick lemma $\phi^{(m)}(D(a, r)) \subset D(a, r)$, for all $m \geq 1$, and since the map $z \to \|K_z\|$ is continuous, $\exists C_r > 0$ such that $\|K_{\phi^{(m)}(z)}\| \leq C_r\|K_{\phi^{(m)}(a)}\| = C_r\|K_a\|$ for all $z \in D(a, r)$. Now if $f \in \bigcap_{m=1}^{\infty} W^m_{u,\phi}(\mathcal{H}_K)$, then for every $m$, $\exists f_m \in \mathcal{H}_K$ such that $f = W^m_{u,\phi} f_m$. But then for every $m \geq 1$ and every $z \in D(a, r)$

$$|f(z)| \leq \delta^m\|f_m\|\|K_{\phi^{(m)}(z)}\| \leq C_r\delta^m\|f\|\|K_a\|.$$

Taking $m \to \infty$, we see that $f(z) = 0$ for all $z \in \mathbb{D}(a, r)$. But then $f \equiv 0$ on $\mathbb{D}$, and so $W_{u,\phi}$ is a forward shift.

(ii) As was shown in the proof of Proposition 3.1, part (iii), if $|u(a)| = 1$, then $W_{u,\phi}k_a = ck_a$, with $|c| = 1$. Using again that $\phi(a) = a$, we also have that $W^*_{u,\phi}k_a = \overline{u(a)}k_a$, and so $k_a \in \mathcal{H}^U$, the unitary part of the Wold decomposition of $\mathcal{H}_K$.

Next, we will show that $W_{u,\phi}$ is a shift on $\mathcal{K} = \{k_a\}^{\perp}$, by showing that

$$\bigcap_{m=1}^{\infty} (W_{u,\phi}/\mathcal{K})^m(\mathcal{K}) = \{0\}.$$

So, assume that for every $m \in \mathbb{N}$, $\exists f_m \in \mathcal{K}$ such that $f = W^m_{u,\phi} f_m$. The goal is to show that then $f \equiv 0$.

For any $z \in \mathbb{D}$,

$$\begin{aligned} f(z) &= W^m_{u,\phi} f_m(z) \\ &= u(z)u(\phi(z))...u(\phi^{(m-1)}(z))f_m(\phi^{(m)}(z)) \\ &= (u(a))^m \frac{k_a(z)}{k_a \circ \phi^{(m)}(z)} f_m(\phi^{(m)}(z)), \end{aligned}$$

since by Proposition 3.1, part (iii), $u = u(a)\frac{k_a}{k_a \circ \phi}$.

By the Denjoy–Wolff Theorem $\phi^{(m)}(z) \to a$ as $m \to \infty$. Since the map $z \to K_z$ is continuous, we also have that $\|K_{\phi^{(m)}(z)} - K_a\| \to 0$, as $m \to \infty$. Also, note that $\|f\| = \|f_m\|$ for all $m \in \mathbb{N}$ since $W_{u,\phi}$ is an isometry, and that $f_m(a) = 0$, $\forall m \in \mathbb{N}$, since each $f_m \in \mathcal{K} = \{k_a\}^{\perp}$. Thus, as $m \to \infty$, $k_a \circ \phi^{(m)}(z) \to k_a(a) = \|K_a\|$, and

$$|f_m(\phi^{(m)}(z))| = | < f_m, K_{\phi^{(m)}(z)} > |$$
$$\leq \|f\|\|K_{\phi^{(m)}(z)} - K_a\| + | < f_m, K_a > | \to 0.$$

Since also $|u(a)^m| = 1$, $\forall m \in \mathbb{N}$ and $|k_a(z)|$ does not depend on $m$, taking $m \to \infty$ gives that $|f(z)| = 0$ for any fixed $z$, i.e. that $f \equiv 0$. Hence, $\mathcal{K} = \mathcal{H}^S$, the forward shift part of the Wold decomposition of $\mathcal{H}_K$ for the isometry $W_{u,\phi}$.

(iii) Under the given assumptions, using also Proposition 3.1, part (iii), we can see that the isometry $W_{u,\phi}$ is unitary equivalent, via the self-adjoint unitary $W_{k_a,\psi_a}$, to the operator $W_{u_a,\phi_a}$, where $u_a = u(a)k_a(k_a \circ \psi_a)$, and $\phi_a = \psi_a \circ \phi \circ \psi_a$.

Since then $|u_a(0)| = 1$, by Proposition 3.1, part (ii), $u_a(z) \equiv c$, $|c| = 1$, and so $W_{u_a,\phi_a} = c\, C_{\phi_a}$. But $\phi_a$ is a disk automorphism with $\phi_a(0) = 0$, and so $\phi_a$ is a rotation and $C_{\phi_a}$ is unitary. Thus, $W_{u_a,\phi_a}$ and $W_{u,\phi}$ are both unitary operators on $\mathcal{H}_K$. □

In the case when the RKHS is a weighted Bergman space over the unit disk we get the following characterization of the Wold decomposition of isometric weighted composition operators whose composition symbol has an interior fixed point.

**Theorem 3.2.** *Let $\phi$ be a holomorphic nonconstant self-map of $\mathbb{D}$ with a fixed point $a$ in $\mathbb{D}$, and let $u \in L^2_a(dm_\alpha)$ be such that $W_{u,\phi}$ is an isometry on $L^2_a(dm_\alpha)$. Then*

*(i) If $\phi$ is univalent, either $W_{u,\phi}$ is unitary in the case when $\phi$ is a disk automorphism, or otherwise $W_{u,\phi}$ is a forward shift.*

*(ii) If $W_{u,\phi}$ is not a forward shift, then $u = u(a)\frac{k_a}{k_a \circ \phi}$. If $\phi$ is univalent then $\phi$ is a rotation and $W_{u,\phi}$ is unitary. If $\phi$ is not univalent, then $W_{u,\phi}$ has nontrivial unitary and forward shift parts, and following the previously determined general Wold decomposition:*

$$L^2_a(dm_\alpha) = \mathbb{C}k^\alpha_a \bigoplus \{k^\alpha_a\}^\perp.$$

*(iii) If $a = 0$ and $W_{u,\phi}$ is not a shift, then $u$ is a unimodular constant and $\phi$ is a rotation. Thus, $W_{u,\phi}$ is unitary.*

**Proof.** (i) If $\phi$ is an automorphism of $\mathbb{D}$ and $W_{u,\phi}$ is an isometry then by Theorem 2.2, part (iii), $W_{u,\phi}$ is a unitary operator. If $\phi$ is not an automorphism, since $\phi$ has a fixed point in $\mathbb{D}$, $\phi$ is an elliptic non-automorphic selfmap of $\mathbb{D}$ and so by the Schwarz–Pick lemma $|\phi'(a)| < 1$. Since $\phi$ is univalent and $W_{u,\phi}$ is an isometry, from the proof of part (ii) of Theorem 2.2 we have that $|u(w)|^2 = |\phi'(w)|^2 \frac{(1-|\phi(w)|^2)^\alpha}{(1-|w|^2)^\alpha}$ for all $w$ in $\phi^{-1}(\mathbb{D})$. Taking $w = a$, we get that $|u(a)| = |\phi'(a)| < 1$ and so by Theorem 3.1, part (i), $W_{u,\phi}$ is a forward shift.

(ii) If $W_{u,\phi}$ is not a shift then by Theorem 3.1, part (i), $|u(a)| = 1$. Hence, by Proposition 3.1, part (iii), $u = u(a)\frac{k_a}{k_a \circ \phi}$.

In case $\phi$ is univalent, as in the proof of part (i), $|u(a)| = |\phi'(a)|$, and since $|u(a)| = 1$, and $\phi(a) = a$, $\phi$ must be an elliptic disk automorphism. Furthermore, $a = 0$ by part (ii) of Corollary 3.1, and so $\phi$ is a rotation. Hence, $W_{u,\phi}$ is unitary.

If $\phi$ is not univalent then $W_{u,\phi}$ can not be unitary. The rest follows since $W_{u,\phi}$ is also not a forward shift.

(iii) If $\phi(0) = 0$ and $W_{u,\phi}$ is not a shift, by Theorem 3.1, part (i), $|u(0)| = 1$. But then, by Proposition 3.1, part (ii), $u(z) \equiv c$, $|c| = 1$, and so the composition operator $C_\phi$ is an isometry on the weighted Bergman space $L^2_a(dm_\alpha)$. Thus, $\phi$ must be a rotation, i.e. $\phi(z) = \lambda z$, $|\lambda| = 1$ (see for example [14]), $C_\phi$ is unitary and so $W_{u,\phi}$ is also a unitary operator on $L^2_a(dm_\alpha)$. □

The Hardy space Wold decomposition of isometric weighted composition operators $W_{u,\phi}$ with $\phi$ a non-elliptic disk automorphism, i.e. when $\phi$ is a disk automorphism with no interior fixed point, was described in [16].

This class of examples is also interesting for pointing out the differences between the Wold decomposition of isometric weighted composition operators in the Hardy, and in the Bergman space case. While in the Hardy space case a parabolic or a hyperbolic disk automorphism can induce a weighted composition operator $W_{u,\phi}$ with a nontrivial shift part, or even be a shift (see [16] for more details), in the Bergman space case, by Theorem 2.2, part (iii), each such isometric operator $W_{u,\phi}$ must be unitary. Note that the main difference here steams from the fact that, while in the Hardy space case a multiplication operator $M_u$ is isometric if and only if $u$ is an inner function, the only isometric multiplication operators on the weighted Bergman spaces are the unimodular constant multiples of the identity operator. This is a known fact that can be shown in many different ways. For example, since $M_u = W_{u,\phi}$ with $\phi(z) = z$ (a disk automorphism with $\phi(0) = 0$), by Theorem 2.2, part (iii), $u = ck_0^\alpha$, $|c| = 1$. But $k_0^\alpha \equiv 1$, and so $u$ is a unimodular constant.

There are also many other (non-unitary) examples of isometric weighted composition operators on the weighted Bergman spaces where the composition symbol has no interior fixed point. For example, as shown before, if $\phi$ is any univalent non-automorphic selfmap of $\mathbb{D}$ with no interior fixed point, and $u = \phi'$, then $W_{u,\phi}$ is an isometry on the Bergman space.

On the other hand, under some additional conditions, an isometric $W_{u,\phi}$ must be such that $\phi$ has an interior fixed point. For example, it was shown in [17] that if $W_{u,\phi}$ is a unitary operator (on a general Bergman space over a bounded simply connected domain in $\mathbb{C}^n$) which has an eigenvector, then $\phi$ must be an automorphism of the domain with an interior fixed point.

We have a similar characterization for general isometric weighted composition operators on weighted Bergman spaces over the unit disk.

**Proposition 3.2.** *Let $\phi$ be a non-constant holomorphic self-map of $\mathbb{D}$ and let $u \in L_a^2(dm_\alpha)$ be such that $W_{u,\phi}$ is an isometry on $L_a^2(dm_\alpha)$. Then if $W_{u,\phi}$ has an eigenvector, $\phi$ must have a fixed point in $\mathbb{D}$.*

**Proof.** In general, if $\phi$ is a self-map of $\mathbb{D}$ that is not an elliptic automorphism, the orbit of $\phi$ at any point $a$ in $\mathbb{D}$ converges to the Denjoy–Wolff point of $\phi$, which is either an interior fixed point of $\phi$, or a point on the unit circle. So, if $\phi$ has no interior fixed point, $|\phi^{(n)}(a)| \to 1$, for any $a$ in $\mathbb{D}$.

Let $f \in L_a^2(dm_\alpha)$ be a non-zero eigenvector of the isometry $W_{u,\phi}$, i.e. $W_{u,\phi} f = \lambda f$, for some unimodular constant $\lambda$. Suppose that $\phi$ has no interior fixed point, and let $a \in \mathbb{D}$, and $0 < r < 1$. For the pseudo-hyperbolic disk $D(a, r)$ we have that

$$\begin{aligned}
\int\limits_{D(a,r)} |f(z)|^2 dm_\alpha(z) &= \int\limits_{D(a,r)} |u(z)|^2 |f(\phi(z))|^2 dm_\alpha(z) \\
&= \int\limits_{\mathbb{D}} |u(z)|^2 |f(\phi(z))|^2 \chi_{D(a,r)}(z) dm_\alpha(z) \\
&\leq \int\limits_{\mathbb{D}} |u(z)|^2 |f(\phi(z))|^2 \chi_{\phi(D(a,r))}(\phi(z)) dm_\alpha(z) \\
&= \int\limits_{\mathbb{D}} |f(w)|^2 \chi_{\phi(D(a,r))}(w) d\mu_{u,\phi}^\alpha(w) \\
&= \int\limits_{\phi(D(a,r))} |f(w)|^2 d\mu_{u,\phi}^\alpha(w)
\end{aligned}$$

$$= \int_{\phi(D(a,r))} |f(w)|^2 dm_\alpha(w)$$
$$\leq \int_{D(\phi(a),r))} |f(w)|^2 dm_\alpha(w),$$

where in the last equality we used that $W_{u,\phi}$ is an isometry, i.e. that $d\mu^\alpha_{u,\phi} = dm_\alpha$. The last inequality follows by the Schwarz–Pick lemma, since the pseudo-hyperbolic distance $\rho$ is such that $\rho(\phi(a), \phi(z)) \leq \rho(a, z)$ and so $\phi(D(a, r)) \subset D(\phi(a), r)$.

Now every power of $W_{u,\phi}$ is also an isometry with the same eigenvector $f$, and $W^n_{u,\phi} = W_{u_n,\phi^{(n)}}$ for appropriately chosen $u_n$. Thus, similarly as above, replacing $\phi$ with $\phi^{(n)}$ and $u$ with $u_n$ we get that for every $n$,

$$\int_{D(a,r)} |f(z)|^2 dm_\alpha(z) \leq \int_{D(\phi^{(n)}(a),r))} |f(w)|^2 dm_\alpha(w).$$

Next, since $|\phi^{(n)}(a)| \to 1$ as $n \to \infty$, there is a subsequence $\{\phi^{(n_k)}(a)\}$ such that the corresponding pseudo-hyperbolic disks $D_k = D(\phi^{(n_k)}(a), r)$ are disjoint. But then

$$\|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 dm_\alpha(z) \geq \int_{\bigcup_{k=1}^\infty D_k} |f(z)|^2 dm_\alpha(z)$$
$$= \sum_{k=1}^\infty \int_{D_k} |f(w)|^2 dm_\alpha(w) \geq \sum_{k=1}^\infty \int_{D(a,r)} |f(w)|^2 dm_\alpha(w) = \infty,$$

and we get a contradiction to $f \in L^2_a(dm_\alpha)$. Hence, $\phi$ must have an interior fixed point. □

Note that the last proposition together with the previous results on the Wold decomposition of isometric $W_{u,\phi}$ implies the following.

**Corollary 3.2.** *Let $W_{u,\phi}$ be an isometry on $L^2_a(dm_\alpha)$ that is neither unitary nor a shift. Then:*

*(i) $W_{u,\phi}$ has a single one dimensional eigenspace if and only if $\phi$ has an interior fixed point.*
*(ii) $W_{u,\phi}$ has no eigenvalues if and only if $\phi$ does not have an interior fixed point.*

The spectrum of weighted composition operators has been investigated in several different settings (see [2], [4], [9], [10], [11]). A closely related topic is also the determination of the numerical range of weighted composition operators, in particular when the operator is an isometry (see [10] and [16] for this type of results on the Hardy space). Using our previous conclusions on the Wold decomposition and eigenvectors of isometric weighted composition operators, we present next few results on the point spectrum, spectrum and numerical range of isometric WCO' on the weighted Bergman spaces.

For a bounded operator $T$ on a Hilbert space $\mathcal{H}$, we denote by $\sigma(T)$ and $\sigma_p(T)$ the spectrum and the point spectrum of $T$. Recall also that

$$W(T) = \{< Tf, f >; f \in \mathcal{H}, \|f\| = 1\}$$

is the numerical range of $T$. The standard well know facts about the spectrum and the numerical range of an operator, some of which will be used in the proof of the next results, can be found in most basic operator theory textbooks.

We will also use the following folk results, also mentioned and/or proven in a slightly different form in [10] and [16]:

- If $\lambda \in W(T)$ is a boundary point of the numerical range of $T$ and is not contained in a closed disk in $W(T)$, or if $|\lambda| = \|T\|$, then $\lambda \in \sigma_p(T)$.
- If $T$ is an isometry on a separable Hilbert space that is a forward shift, then $W(T) = \mathbb{D}$.
- If $T$ is an isometry on a separable Hilbert space that is not unitary, then $W(T) = \mathbb{D} \cup \sigma_p(T)$.
- If $T$ is a unitary operator on a separable Hilbert space, then $\sigma_p(T)$ is at most countable.

We need first the following lemma describing the point spectrum of unitary weighted composition operators on weighted Bergman spaces. The proof uses standard methods in determining the parts of the spectrum of such operators, with the ideas coming mostly from the Hardy space case. We provide the proof for completeness, and cite the relevant references for more details on the main ideas. Note that the general spectrum of weighted composition operators with automorphic composition symbols acting on weighted Bergman spaces was determined in [11].

**Lemma 3.1.** *Let $\phi$ be a holomorphic disk automorphism and let $u \in L^2_a(dm_\alpha)$ be such that $W_{u,\phi}$ is an isometry on $L^2_a(dm_\alpha)$. Then:*

*(i) If $\phi$ is an elliptic disk automorphism with a fixed point $a$, then*

$$\sigma_p(W_{u,\phi}) = \{u(a)\phi'(a)^n; n = 0, 1, 2...\}.$$

*(ii) If $\phi$ is a parabolic or hyperbolic disk automorphism, then $\sigma_p(W_{u,\phi}) = \emptyset$.*

**Proof.** The main idea of the proof is to use the fact that the point spectrum of two unitary equivalent, or two similar operators is the same, thus simplifying the work to a case which is easy to handle directly.

(i) Since $\phi$ is an elliptic automorphism with a fixed point $a$, it is easy to see that $W_{u,\phi}$ is unitary equivalent to $W_{\tilde{u},\tilde{\phi}}$ with $\tilde{\phi}(z) = \phi'(a)z$, and such that $\tilde{u}(0) = u(a)$. The equivalence is achieved via the selfadjoint unitary $W_{k_a^\alpha}, \psi_a$ where $\psi_a$ is the unique involutive disk automorphism with $\psi_a(0) = a$. Hence $\tilde{\phi} = \psi_a \circ \phi \circ \psi_a$, it maps 0 into 0, and so it must be a rotation. But then by Theorem 3.2, $\tilde{u}$ must be the constant function $u(a)$ with $|u(a)| = 1$, and so $W_{\tilde{u},\tilde{\phi}} = u(a)C_{\tilde{\phi}}$.

Since $\tilde{\phi}$ is a rotation and the weighted Bergman spaces contain the functions $e_n(z) = z^n$ for any non-negative integer $n$, a standard proof shows that

$$\sigma_p(C_{\tilde{\phi}}) = \{\phi'(a)^n; n = 0, 1, 2...\}.$$

Now $\sigma_p(W_{u,\phi}) = \sigma_p(W_{\tilde{u},\tilde{\phi}}) = u(a)\sigma_p(C_{\tilde{\phi}})$, which completes the proof of (i).

(ii) We will show first that in the parabolic and the hyperbolic case, $\sigma_p(W_{u,\phi})$ must have circular symmetry. Since the operator is unitary, the point spectrum is either the unit circle, or it is empty. But the point spectrum can not be the unit circle since the weighted Bergman spaces are separable Hilbert spaces, and so it must be empty.

If $\phi$ is a parabolic disk automorphism, we show the circular symmetry of $\sigma_p(W_{u,\phi})$ by first using the fact that $W_{u,\phi}$ is similar to $W_{\tilde{u},\tilde{\phi}}$ with $\tilde{\phi}(z) = ((1+i)z-1)/(z+i-1)$ or $\tilde{\phi}(z) = ((1-i)z-1)/(z-i-1)$ (see [9, Lemma 3.0.6.]).

If $\lambda \in \sigma_p(W_{\tilde{u},\tilde{\phi}})$ then, as was shown in the proof of [10, Theorem 4.7, part (2)], we can use the inner functions

$$f_r(z) = e^{r\frac{z+1}{z-1}},\ r > 0$$

to show that then each $e^{-2ir}\lambda$ is also in the point spectrum of $W_{\tilde{u},\tilde{\phi}}$. Hence, the point spectrum is the unit circle, which is a contradiction. Note that the proof for the Hardy space from [10] works also in this case since each $f_r$ is in $H^\infty$, the multiplier algebra for the Hardy space and also for the weighted Bergman spaces.

If $\phi$ is hyperbolic, then we can use the idea from the proof of [4, Theorem 4.3.]. Namely, for every real $\theta$ there is a function $f_\theta$ such that $f$ and $1/f$ are in $H^\infty$ and such that $M_{f_\theta}^{-1} C_\phi M_{f_\theta} = e^{i\theta} C_\phi$ over the polynomials, and so also on each weighted Bergman space (see [4] for more details). But then it is easy to see that $M_{f_\theta}^{-1} W_{u,\phi} M_{f_\theta} = e^{i\theta} W_{u,\phi}$, and so $\sigma_p(W_{u,\phi})$ has a circular symmetry.

As before, the proof for the Hardy space works also for the weighted Bergman spaces since the multiplier algebra for both types of spaces is $H^\infty$. □

Next we describe the numerical range of isometric weighted composition operators acting on weighted Bergman spaces, and determine the spectrum of isometric $W_{u,\phi}$ when $\phi$ is not automorphic. As mentioned before, the spectrum of $W_{u,\phi}$ with automorphic $\phi$ has been determined in [11].

**Theorem 3.3.** *Let $\phi$ be a holomorphic self-map of $\mathbb{D}$ and let $u \in L^2_a(dm_\alpha)$ be such that $W_{u,\phi}$ is an isometry on $L^2_a(dm_\alpha)$. Then:*

*a) If $\phi$ is a disk automorphism, then $\sigma(W_{u,\phi}) \subseteq \partial\mathbb{D}$ and:*

*(i) If $\phi$ is an elliptic disk automorphism with a fixed point $a \in \mathbb{D}$, then either*

$$(W(W_{u,\phi})) = \mathbb{D} \cup \{u(a)\phi'(a)^n; n = 0, 1, 2...\}$$

*when $\phi'(a)$ is not a root of unity, or $(W(W_{u,\phi}))$ is a polygon with vertex points in $\{u(a)\phi'(a)^n;$ $n = 0, 1, 2..., n-1\}$ if $\phi'(a)$ is an $n$-th root of unity.*

*(ii) If $\phi$ is a parabolic or a hyperbolic disk automorphism then $(W(W_{u,\phi})) = \mathbb{D}$.*

*b) If $\phi$ is not a disk automorphism, then $\sigma(W_{u,\phi}) = \overline{\mathbb{D}}$. Furthermore:*

*(i) If $\exists a \in \mathbb{D}$ such that $\phi(a) = a$, then either $W(W_{u,\phi}) = \mathbb{D}$ if $|u(a)| < 1$, or $W(W_{u,\phi}) = \mathbb{D} \cup \{u(a)\}$ if $|u(a)| = 1$.*

*(ii) If $\phi$ has no interior fixed point then $W(W_{u,\phi}) = \mathbb{D}$.*

**Proof.** a) It is trivial that when $\phi$ is a disk automorphism, i.e. when $W_{u,\phi}$ is unitary, then $\sigma(W_{u,\phi}) \subseteq \partial\mathbb{D}$. To further determine the numerical range of $W_{u,\phi}$ we use the description of the spectrum of $W_{u,\phi}$ from [11] and the characterization of the point spectrum from Lemma 3.1.

(i) If $\phi$ is an elliptic disk automorphism with fixed point $a \in \mathbb{D}$ and $\phi'(a)$ is not a root of unity by Lemma 3.1, part (i), $\sigma_p(W_{u,\phi}) = \{u(a)\phi'(a)^n; n = 0, 1, 2...\}$. The spectrum is a closed set and so $\sigma(W_{u,\phi}) = \partial\mathbb{D}$. Since $W_{u,\phi}$ is unitary, it is also normal and so

$$\overline{\mathbb{D}} = \text{convhull}(\sigma(W_{u,\phi})) \subseteq \text{cl}(W(W_{u,\phi})) \subseteq \overline{\mathbb{D}}.$$

On the other hand, any point on the unit circle is in the boundary of the numerical range, and so it has to be in the point spectrum of $W_{u,\phi}$. Thus, as claimed

$$(W(W_{u,\phi})) = \mathbb{D} \cup \{u(a)\phi'(a)^n; n = 0, 1, 2...\}.$$

If $\phi'(a)$ is an $n$-th root of unity, then by [11] and Lemma 3.1, part (i),

$$\sigma(W_{u,\phi}) = \{u(a)\phi'(a)^n; n = 0, 1, 2..., n-1\} = \sigma_p(W_{u,\phi}).$$

We get the needed conclusion by the same discussion as in the first part, and by using the fact that the numerical range is a convex set.

(ii) If $\phi$ is a parabolic or a hyperbolic disk automorphism by [11], $\sigma(W_{u,\phi}) = \partial\mathbb{D}$. By Lemma 3.1, part (ii), $\sigma_p(W_{u,\phi}) = \emptyset$. Using the same general facts about the spectrum and the numerical range of a normal operator, and the relation between the boundary of the numerical range and the point spectrum as in the first part of the proof of part (i), we get that $\mathrm{cl}(W(W_{u,\phi})) = \overline{\mathbb{D}}$, and so $W(W_{u,\phi}) = \mathbb{D}$.

b) If $\phi$ is not a disk automorphism, then $W_{u,\phi}$ is a non-unitary isometry. Hence, the forward shift part in the Wold decomposition of $W_{u,\phi}$ is non-trivial, $\mathbb{D} \subset \sigma(W_{u,\phi})$ and since the spectrum is a closed set $\sigma(W_{u,\phi}) = \overline{\mathbb{D}}$. Recall also that since $W_{u,\phi}$ is not unitary, $W(W_{u,\phi}) = \mathbb{D} \cup \sigma_p(W_{u,\phi})$. Thus:

(i) If $a$ is an interior fixed point of $\phi$ by Proposition 3.1, part (iii), $|u(a)| \leq 1$. If $|u(a)| < 1$, by Theorem 3.1, part (i), $W_{u,\phi}$ is a shift, and so $W(W_{u,\phi}) = \mathbb{D}$. If $|u(a)| = 1$, since $\phi$ is not a disk automorphism by the Wold decomposition described in Theorem 3.1, part (iii), $\sigma_p(W_{u,\phi}) = \{u(a)\}$ and so $W(W_{u,\phi}) = \mathbb{D} \cup \{u(a)\}$.

(ii) If $\phi$ has no interior fixed point and $\phi$ is not a disk automorphism then either $W_{u,\phi}$ is a forward shift, in which case $W(W_{u,\phi}) = \mathbb{D}$, or $W_{u,\phi}$ is neither unitary nor a forward shift. But in the latter case by Corollary 3.1, $W_{u,\phi}$ has no eigenvalues, i.e. $\sigma_p(W_{u,\phi}) = \emptyset$. Hence, in this case again $W(W_{u,\phi}) = \mathbb{D}$. $\square$

We add the following two general questions for future investigations of isometric weighted composition operators within this context:

**Question 3.1.** Are Theorem 3.1 and Theorem 3.2 true for more general RKHS $\mathcal{H}_K$ over more general domains in $\mathbb{C}^n$?

**Question 3.2.** What more can be said about the Wold decomposition of isometric weighted composition operators on the weighted Bergman spaces, or on the Hardy space, when the composition symbol has no interior fixed point?

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