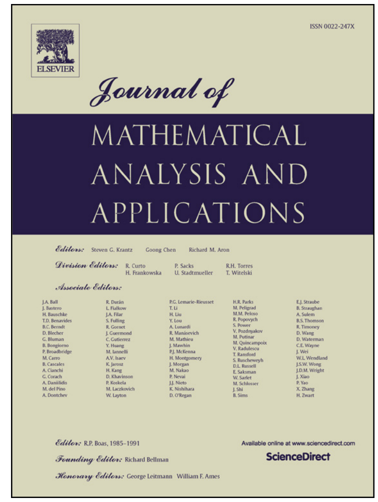
Accepted Manuscript



Double layer potentials on three-dimensional wedges and pseudodifferential operators on Lie Groupoids

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PII: S0022-247X(18)30115-X
DOI: https://doi.org/10.1016/j.jmaa.2018.01.077
Reference: YJMAA 22010

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 8 June 2017

Please cite this article in press as: Y. Qiao, Double layer potentials on three-dimensional wedges and pseudodifferential operators on Lie Groupoids, *J. Math. Anal. Appl.* (2018), https://doi.org/10.1016/j.jmaa.2018.01.077

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DOUBLE LAYER POTENTIALS ON THREE-DIMENSIONAL WEDGES AND PSEUDODIFFERENTIAL OPERATORS ON LIE GROUPOIDS

YU QIAO

ABSTRACT. Let $\mathcal{W}$ be a three-dimensional wedge, and K be the double layer potential operator associated to $\mathcal{W}$ and the Laplacian. We show that $\frac{1}{2} \pm K$ are isomorphisms between suitable weighted Sobolev spaces, which implies a solvability result in weighted Sobolev spaces for the Dirichlet problem on $\mathcal{W}$. Furthermore, we show that the double layer potential operator K is an element in $C^*(\mathcal{G}) \otimes M_2(\mathbb{C})$, where $\mathcal{G}$ is the action (transformation) groupoid $M \rtimes G$, with $G = \left\{ \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} : a \in \mathbb{R}, b \in \mathbb{R}^+, \right\}$, which is a Lie group, and M is a kind of compactification of G. This result can be used to prove the Fredholmness of $\frac{1}{2} + K_\Omega$, where Ω is "a domain with edge singularities" and K_Ω the double layer potential operator associated to the Laplacian and Ω.

Contents

1. Introduction

Potential theory can be dated back to the works of Lagrange, Laplace, Poisson, Gauss, and others [38], and plays a fundamental role in physics. Many works are dedicated to the method of layer potentials, such as Courant and Hilbert [16], Folland [22], Hsiao and Wendland [24], Kress [29], McLean [38], and Taylor [57].

2010 *Mathematics Subject Classification.* Primary 45E10, 58H05, Secondary 45P05, 47G40, 47L80, 47C15.

Key words and phrases. the method of layer potentials, double layer potential operator, three dimensional wedge, convolution integral operator, pseudodifferential operator, groupoid C*-algebra, Lie group, weighted Sobolev space.

Qiao was partially supported by the NSF of China (11301317, 11571211), and the Foundational Research Funds for the Central Universities GK201601004.

These results give a complete account of the classical theory on smooth bounded domains.

Let us give a quick review of the method of double layer potentials. Suppose $\Omega \subset \mathbb{R}^n$ is a (regular) open bounded domain. Consider the interior Dirichlet boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega \\ u|_{\partial\Omega} = \phi & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

and the exterior Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega^c \\ u|_{\partial\Omega} = \phi & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

where Ω^c denotes the complement of $\overline{\Omega}$, i.e., $\Omega^c = \mathbb{R}^n \backslash \overline{\Omega}$.

For $\psi \in \mathcal{C}_c^\infty(\partial\Omega)$, define the double layer potential

$$u(x) = -\omega_n \int_{\partial\Omega} \frac{(x-y)\cdot\nu(y)}{|x-y|^n} \psi(y) d\sigma(y), \qquad (x \in \mathbb{R}^n \backslash \partial\Omega),$$

where $\nu(y)$ is the exterior unit normal to a point $y \in \partial\Omega$ and ω_n is the area of the unit sphere in $\mathbb{R}^n$. Let $u_-(x)$ and $u_+(x)$ denote the limits of $u(z)$ as $z \to x \in \partial\Omega$ nontangentially from $z \in \Omega$ and $z \in \mathbb{R}^n \backslash \overline{\Omega}$, respectively. The classical results [15, 22, 57] on double layer potentials state that for (a.e.) $x \in \partial\Omega$, we have

(1) $u_-(x) = \frac{1}{2}\psi(x) + K\psi(x), i.e., u_- = (\frac{1}{2} + K)\psi$;
(2) $u_+(x) = -\frac{1}{2}\psi(x) + K\psi(x), i.e., u_+ = (-\frac{1}{2} + K)\psi$, where

$$K\psi(x) = \int_{\partial\Omega} k(x,y)\psi(y)d\sigma(y),$$

with $k(x,y) = -\omega_n \dfrac{(x-y)\cdot\nu(y)}{|x-y|^n}$.

Hence, the interior and exterior Dirichlet problems are reduced to solving boundary integral equations $(1/2 + K)\psi = \phi$ and $(-1/2 + K)\psi = \phi$, respectively, where ϕ is the given function, and ψ is the unknown function, both on $\partial\Omega$.

In [22, 57], it is shown that if the domain $\Omega \subset \mathbb{R}^n$ has $\mathcal{C}^2$ boundary $\partial\Omega$, then the double layer potential operator K is compact on $L^2(\partial\Omega)$. Hence operators $1/2 \pm K$ are Fredholm of index zero. Therefore, the solvability of the interior and exterior Dirichlet problems is equivalent to injectivity or surjectivity of $1/2 \pm K$. The paper [19] deals with the case of $\mathcal{C}^1$-domains.

By contrast, if the boundary $\partial\Omega$ is not $\mathcal{C}^1$, the operator K may not be compact any more (see [18, 22, 27, 29, 32, 58]). However, we can still expect that $1/2 \pm K$ are Fredholm operators on appropriate function spaces on the boundary. The case of Lipschitz domains is by far the most studied among the class of non-smooth domains, hence is well understood. See Verchota [58] for related results on Lipschitz domains. Costabel's paper [14] gives a good introduction to the method of layer potentials via more elementary methods.

We are concerned with boundary value problems on domains with singularities. There are a plenty of works and several different methods dealing with such problems.

First of all, boundary value problems on domains with conical points were extensively studied by many authors. We would like to mention in this aspect the work of Kondratiev [27], Kapanadze and Schulze [26], Li, Mazzucato and Nistor [33], Mazzeo and Melrose [37] and Melrose [39], and Schrohe and Schulze [49, 50].

Meanwhile, boundary value problems on domains with edge singularities also attract a lot of attention. Ammann, Ionescu, and Nistor use Lie manifold to study Sobolev spaces, elliptic regularity, and mapping properties of pseudodifferential operators on polyhedral domains of $\mathbb{R}^3$ in [3]. Furthermore, we mention the work of Fabes, Jodeit, and Lewis [18], the paper of Mazzeo [36], and the works of Schulze with his collaborators [20, 23, 25, 51, 52, 53, 54]. Schulze and his collaborators have developed an extensive theory of boundary value problems in the framework of Boutet de Monvel pseudodifferential calculus [9]. (A study of the layer potentials complements this theory.) Most of these works are devoted to constructing suitable algebras of pseudodifferential operators on manifolds with singularities. See also the papers [1, 6, 7, 17] using groupoids to construct algebras of pseudodifferential operators on singular spaces, and [4, 44, 55] for some related constructions.

In addition, many works are dedicated to the study of other analysis problems on manifolds with edge singularities. For instance, Albin and Gell-Redman consider index theory of Dirac operators on incomplete edge spaces [2]. Krainer and Mendoza construct a theory of elliptic boundary value problems for wedge operators on general manifolds with edges [28].

Our long term interest lies in the method of layer potentials on domains with singularities. From the pseudodifferential operator point of view, if the boundary $\partial\Omega$ is smooth, then the double layer potential operator K is a pseudodifferential operator of order -1 on the boundary, i.e., $K \in \Psi^{-1}(\partial\Omega)$ [57]. If the boundary has singularities, there is a natural question to ask:

> Does there exist a canonical way to construct a pseudodifferential operator algebra on the boundary such that
> (a) the double layer potential operator K belongs to this pseudodifferential operator algebra;
> (b) the Fredholmness of $1/2 \pm K$ (on certain function spaces on the boundary) can be proved in the context of this pseudodifferential operator algebra?

The survey [35] stresses the importance of understanding the algebra of pseudodifferential operators on spaces with singularities. For the case where Ω is a simply connected polygon on $\mathbb{R}^2$, Lewis and Parenti constructed a pseudodifferential algebra on the boundary to settle the invertibility of $1/2 \pm K$ on the spaces $L^p(\partial\Omega)$ [32]. In the papers [13, 45, 46], motivated by the study of the method of layer potentials on domains with conical points, the author and collaborators have constructed pseudodifferential algebras on the boundary, and investigated the invertibility of operators $1/2 \pm K$ on weighted Sobolev spaces on the boundary, which implies a solvability result in weighted Sobolev spaces for the interior and exterior Dirichlet problems on Ω. It is possible to extend our method to solve interior and exterior Neumann problems.

In general, Bacuta, Mazzucato, Nistor, and Zikatanov presented a general desingularization procedure for polyhedral domains in [8]. The construction of the desingularization of a polyhedron gives us so-called "Lie manifold with boundary". If we confine ourselves on the boundary of a polyhedral domain, we obtain "Lie manifold" (or Lie algebroid). Then integrating this Lie manifold (or Lie algebroid) leads to a Lie groupoid. (Hence, the entire construction is motivated by boundary value problems and comes from the nature of the singularities.) By the works of Nistor-Weinstein-Xu [44] and Monthubert-Pierrot [43], there is a pseudodifferential calculus on a Lie groupoid. Then we can identify the double layer potential operator K (associated to a singular domain) with a pseudodifferential operator on this Lie groupoid. Finally, combining some general results of pseudodifferential calculus on Lie groupoids and groupoid C^*-algebras, we are able to show that $1/2 \pm K$ are Fredholm operators on suitable weighted Sobolev spaces on the boundary, even the invertibility of $1/2 \pm K$.

To be able to handle double layer potentials on "manifolds with edge singularities", we find it necessary to investigate the behavior of the double layer potential operator K near each edge singularity, and to examine the analytic properties of $1/2 \pm K$ on suitable function spaces. It is exactly the purpose of this paper to study the case of three-dimensional wedges and to prove invertibility results.

In the present paper, we focus on the double layer potential operator K associated with the Laplacian on a three-dimensional wedge. Denote such a wedge by $\mathcal{W}$, i.e.,

$$\mathcal{W} := \{(r\cos\theta, r\sin\theta, z) : r > 0,\, 0 < \theta < \alpha,\, z \in \mathbb{R}\}, \tag{3}$$

where $0 < \alpha < 2\pi$, $\alpha \neq \pi$. The double layer potential operator K associated to $\mathcal{W}$ and the Laplace operator is of the form

$$K = \begin{pmatrix} 0 & \widetilde{K} \\ \widetilde{K} & 0 \end{pmatrix},$$

where $\widetilde{K}$ is a convolution operator on the Lie group

$$G := \left\{ \begin{pmatrix} 1 & 0 \\ u & r \end{pmatrix} \;\middle|\; u \in \mathbb{R},\, r > 0 \right\},$$

which is not unimodular.

Denote by $\mathcal{K}_a^m(\partial\mathcal{W})$ the m-th weighted Sobolev space with weight function r and index $a \in \mathbb{R}$ on $\partial\mathcal{W}$ (see Section 2 for definitions). Denote by M_{r^a} the multiplication operator by r^a. Then the operators $\widetilde{K}_a := M_{r^a}\widetilde{K}M_{r^{-a}}$ are still (bounded linear) convolution operators on the Lie group G for an appropriate range of a, so the operators $K_a := M_{r^a}KM_{r^{-a}}$ act on $\mathcal{K}_a^m(\partial\mathcal{W})$ for a suitable range of a. More explicitly, define

$$\Xi := \{a \in \mathbb{R} : \frac{|\sin a(\pi - \alpha)|}{|\sin a\pi|} < 1\}.$$

We have the following theorem.

Theorem 1.1. *For all $m \in \mathbb{Z}$ and $a \in \Xi$, the operators*

$$\frac{1}{2} \pm K = \begin{pmatrix} \frac{1}{2} & \pm\widetilde{K} \\ \pm\widetilde{K} & \frac{1}{2} \end{pmatrix} : \mathcal{K}_{1+a}^m(\partial\mathcal{W}) \to \mathcal{K}_{1+a}^m(\partial\mathcal{W})$$

are isomorphisms.

Applying the general procedure discussed above to the special case $\mathcal{W}$, we finally get a Lie groupoid over the (desingularized) boundary $\partial\mathcal{W}$. Our general strategy is that certain boundary convolution integral operators (such as K_a) are in fact in the groupoid C^*-algebra. Furthermore, from this fact and among other things, we will be able to show that these integral operators are Fredholm between suitable weighted Sobolev spaces for domains with edge singularities.

As an application of the above discussion, let M be a sort of compactification of G such that G acts on M. Then we can form the action groupoid $\mathcal{G} := M \rtimes G$. Thus, we obtain the following theorem.

Theorem 1.2. *For $a \in (-1, 1)$, $K_a \in C^*(\mathcal{G}) \otimes M_2(\mathbb{C})$;*

The paper is structured in the following way. Section 2 recalls desingularization procedures and definitions of weighted Sobolev spaces on $\mathcal{W}$ and $\partial\mathcal{W}$. Then in Section 3, we review some basic knowledge of Lie groupoids, define pseudo-differential operators on a Lie groupoid and, from this, we define the C^*-algebra of a Lie groupoid. In Section 4, we investigate explicitly the properties of the double layer potential operator K associated to $\mathcal{W}$ and the Laplace operator. Lastly, Section 5 investigates the connection between the operator K with a pseudo-differential operator algebra on some Lie groupoid.

Acknowledgments We would like to thank Victor Nistor and Xiang Tang for helpful comments and enlightening discussions, and the anonymous referee for providing a simple proof of Theorem 4.7, and suggestions which really improve the readability of the paper.

2. Desingularization and weighted Sobolev spaces on three-dimensional wedges

Consider a three-dimensional wedge

$$\mathcal{W} := \{(r\cos\theta, r\sin\theta, z) : r > 0,\ 0 < \theta < \alpha,\ z \in \mathbb{R}\},$$

where $0 < \alpha < 2\pi$, $\alpha \neq \pi$, and $x = r\cos\theta$ and $y = r\sin\theta$ define the usual cylindrical coordinates (r, θ, z), with $(r, \theta, z) \in [0, \infty) \times [0, 2\pi) \times \mathbb{R}$. Following the work in [8, Example 2.10], we see that the manifold of generalized cylindrical coordinates is, in this case, just the domain of the cylindrical coordinates on $\overline{\mathcal{W}}$:

$$\Sigma(\mathcal{W}) = [0, \infty] \times [0, \alpha] \times \mathbb{R}.$$

The desingularization map is $\kappa(r, \theta, z) = (r\sin\theta, r\cos\theta, z)$ and the structural Lie algebra $\mathcal{V}(\mathcal{W})$ of vector fields of $\Sigma(\mathcal{W})$ is

$$a_r(r, \theta, z) r\partial_r + a_\theta(r, \theta, z)\partial_\theta + a_z(r, \theta, z) r\partial_z,$$

where a_r, a_θ and a_z are smooth functions on $\Sigma(\mathcal{W})$. Note that the vector fields in $\mathcal{V}(\mathcal{W})$ may not extend to the closure $\overline{\mathcal{W}}$.

Let $m \in \mathbb{Z}_+$, and $\alpha \in \mathbb{Z}_+^n$ be a multi-index. We define the m-th Sobolev space on $\mathcal{W}$ with weight r and index a by

$$\mathcal{K}_a^m(\mathcal{W}) = \{u \in L^2_{\mathrm{loc}}(\mathcal{W}, dx) \mid r^{|\alpha|-a}\partial^\alpha u \in L^2(\mathcal{W}, dx), \text{ for all } |\alpha| \leqslant m\}.$$

The norm on $\mathcal{K}_a^m(\mathcal{W})$ is $\|u\|^2_{\mathcal{K}_a^m(\mathcal{W})} := \sum_{|\alpha|\leqslant m} \|r^{|\alpha|-a}\partial^\alpha u\|^2_{L^2(\mathcal{W},dx)}$. By Theorem 5.6 in [8], this norm is equivalent to

$$\|u\|^2_{m,a} := \sum_{|\alpha|\leqslant m} \|r^{-a}(r\partial)^\alpha u\|^2_{L^2(\mathcal{W},dx)},$$

where $(r\partial)^\alpha = (r\partial_1)^{\alpha_1}(r\partial_2)^{\alpha_2}(r\partial_3)^{\alpha_3}$. Clearly, we have that $r^t\mathcal{K}_a^m(\mathcal{W}) \cong \mathcal{K}_{a+t}^m(\mathcal{W})$. In general, this isomorphism may not be an isometry.

Proposition 2.1. *We have, for all $m \in \mathbb{Z}$,*

$$\mathcal{K}^m_{\frac{3}{2}}(\mathcal{W}) \cong H^m(\mathcal{W}, g), \quad \textit{and} \quad \mathcal{K}^m_1(\partial\mathcal{W}) \cong H^m(\partial\mathcal{W}, g),$$

where the metric $g = r^{-2}g_e$ with g_e the standard Euclidean metric.

Proof. The result essentially follows from Proposition 5.7 in [8]. Here are the details for the benefit of the reader. We only deal with the first case.

Let us consider the vector fields $X = r\partial_r$, $Y = \partial_\theta$, $Z = r\partial_z$. Then X, Y, Z form an orthonormal basis (at every point of $\mathcal{W}$) with respect to the metric g. Moreover, their Lie brackets are bounded, their Levi-Civita covariant derivatives are also bounded, and X, Y, Z are bounded vector fields (again with respect to the metric g!). We thus obtain

$$\begin{aligned} H^m(\mathcal{W}, g) &= \{u \in L^2_{loc}(\mathcal{W}, g) \mid X^i Y^j Z^k u \in L^2(\mathcal{W}, g), i + j + k \leqslant m\} \\ &= \{u \in L^2_{loc}(\mathcal{W}, g) \mid r^{i+j+k}\partial^i_x \partial^j_y \partial^k_z u \in L^2(\mathcal{W}, g), i + j + k \leqslant m\} \\ &= \{u \in L^2_{loc}(\mathcal{W}, g_e) \mid r^{i+j+k-3/2}\partial^i_x \partial^j_y \partial^k_z u \in L^2(\mathcal{W}, g_e), i + j + k \leqslant m\} \\ &= \mathcal{K}^m_{\frac{3}{2}}(\mathcal{W}) \end{aligned}$$

where we have used the expressions of ∂_x and ∂_y in polar coordinates, as well as the fact that X, Y, Z are bounded with respect to the metric g. The fact that the Lie brackets of the vector fields X, Y, Z are bounded and that their covariant (Levi-Civita) connection are bounded, was used in the first equality to express $H^m(\mathcal{W}, g)$ in terms of derivatives with respect to X, Y, and Z. Also, for the last equalities, we used the key observation that the volume element on $(\mathcal{W}, g)$ is $r^{-3}dx$, where dx is the Euclidean volume element. In particular, we have $f \in L^2(\mathcal{W}, g_e) \Leftrightarrow f \in r^{-3/2}L^2(\mathcal{W}, g)$. See [5] for a comprehensive discussion of these issues in the framework of manifolds with bounded geometry. □

3. Pseudodifferential operators on Lie groupoids

3.1. **Lie groupoids and Lie algebroids.** In this subsection, we review some basic facts on Lie groupoids. We begin with the definition of groupoids.

Definition 3.1. A *groupoid* is a small category $\mathcal{G}$ in which each arrow is invertible.

Let us make this definition more precise [10, 31, 40, 48]. A groupoid $\mathcal{G}$ consists of two sets, a set of units $\mathcal{G}_0$ and a set of arrows $\mathcal{G}_1$. We usually denote the space of units of $\mathcal{G}$ by $M := \mathcal{G}_0$, identify $\mathcal{G}$ with $\mathcal{G}_1$, and use the notation $\mathcal{G} \rightrightarrows M$. Each object of $\mathcal{G}$ can be identified with an arrow of $\mathcal{G}$, thus we have an injective map $u : M \to \mathcal{G}$, where $u(x)$ is the identity arrow of an object x. For each $g \in \mathcal{G}$, we have tow maps: d, $r : \mathcal{G} \to M$. The set of composable pairs is defined by

$$\mathcal{G}^{(2)} := \{(g, h) \in \mathcal{G} \times \mathcal{G} \mid d(g) = r(h)\}.$$

The multiplication $\mu : \mathcal{G}^{(2)} \to \mathcal{G}$ is defined by $\mu(g, h) = gh$. The multiplication is associative. The inverse of an arrow is denoted by $g^{-1} = \iota(g)$. The five structural maps fit into the following diagram (in [40])

$$\mathcal{G}^{(2)} \xrightarrow{\mu} \mathcal{G} \xrightarrow{\iota} \mathcal{G} \underset{r}{\overset{d}{\rightrightarrows}} M \xrightarrow{u} \mathcal{G},$$

and satisfy the following properties:

(1) $d(hg) = d(g)$, $r(hg) = r(h)$,
(2) $k(hg) = (kh)g$
(3) $u(r(g))g = g = gu(d(g))$, and
(4) $d(g^{-1}) = r(g)$, $r(g^{-1}) = d(g)$, $g^{-1}g = u(d(g))$, and $gg^{-1} = u(r(g))$

for any $k, h, g \in \mathcal{G}$ with $d(k) = r(h)$ and $d(h) = r(g)$. The following definition is taken from [31].

Definition 3.2. A *Lie groupoid* is a groupoid

$$\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, d, r, \mu, u, \iota)$$

such that $M := \mathcal{G}_0$ and $\mathcal{G}_1$ are smooth manifolds, possibly with corners, with M Hausdorff, the structural maps d, r, μ, u, and ι are smooth and the domain map d is a submersion (of manifolds with corners).

Remark 3.3. In general, the space $\mathcal{G}_1$ may not be Hausdorff. However, since d is a submersion, it follows that each fiber $\mathcal{G}_x$ is a smooth manifold without corners [31], hence it is Hausdorff. Note that the groupoids that we construct in Section 5 will be Hausdorff.

We now recall the definition of a Lie algebroid [31].

Definition 3.4. A *Lie algebroid* A over a manifold M is a vector bundle A over M, together with a Lie algebra structure on the space $\Gamma(A)$ of the smooth sections of A and a bundle map $\rho : A \to TM$, extended to a map between sections of theses bundles, such that

(1) $\rho([X, Y]) = [\rho(X), \rho(Y)]$;
(2) $[X, fY] = f[X, Y] + (\rho(X)f)Y$,

for all smooth sections X and Y of A and any smooth function f on M. The map ρ is called the *anchor*. Usually we shall denote by (A, ρ) such a Lie algebroid.

Consider a Lie groupoid $\mathcal{G}$ with units M. We can associate a Lie algebroid $A(\mathcal{G})$ to $\mathcal{G}$ as follows. (For more details, one can read [34].) The d-vertical subbundle of $T\mathcal{G}$ for $d : \mathcal{G} \to M$ is denoted by $T^d(\mathcal{G})$ and called simply the d-vertical bundle for $\mathcal{G}$. It is an involutive distribution on $\mathcal{G}$ whose leaves are the components of the d-fibers of $\mathcal{G}$. (Here involutive distribution means that $T^d(\mathcal{G})$ is closed under the Lie bracket, i.e. if $X, Y \in \mathfrak{X}(\mathcal{G})$ are sections of $T^d(\mathcal{G})$, then the vector field $[X, Y]$ is also a section $T^d(\mathcal{G})$.) Hence we obtain

$$T^d\mathcal{G} = \ker d_* = \bigcup_{x \in M} T\mathcal{G}_x \subset T\mathcal{G}.$$

The *Lie algebroid* of $\mathcal{G}$, denoted by $A(\mathcal{G})$, is defined to be $T^d(\mathcal{G})|_M$, the restriction of the d-vertical tangent bundle to the set of units M. In this case, we say that $\mathcal{G}$ integrates $A(\mathcal{G})$.

Remark 3.5. In general, the desingularization process (for the boundary) in [8] gives rise to a Lie algebroid. Then integration of this Lie algebroid leads to a Lie groupoid. In particular, the desingularization of the boundary of a three-dimensional wedge in Section 2, give us the Lie algebra generated by the vector fields $r\partial r$, ∂_θ, $r\partial_z$, and smooth functions on the boundary, which is in fact the smooth sections of a Lie algebroid. By integrating this Lie algebroid, we obtain a Lie groupoid.

3.2. **Pseudodifferential operators and groupoid C^*-algebras.** We recall here the construction of the space of pseudodifferential operators associated to a Lie groupoid $\mathcal{G} \rightrightarrows M$ [30, 31, 42, 41, 43, 44].

Let $P = (P_x)$, $x \in M$ be a smooth family of pseudodifferential operators acting on $\mathcal{G}_x := d^{-1}(x)$. We say that P is *right invariant* if $P_{r(g)}U_g = U_g P_{d(g)}$, for all $g \in \mathcal{G}$, where

$$U_g : \mathcal{C}^\infty(\mathcal{G}_{d(g)}) \to \mathcal{C}^\infty(\mathcal{G}_{r(g)}),\ \ (U_g f)(g') = f(g'g).$$

Let k_x be the distributional kernel of P_x, $x \in M$. Note that the support of the P

$$\mathrm{supp}(P) := \overline{\bigcup_{x\in M} \mathrm{supp}(k_x)} \subset \{(g, g'),\ d(g) = d(g')\} \subset \mathcal{G} \times \mathcal{G}$$

since $\mathrm{supp}(k_x) \subset \mathcal{G}_x \times \mathcal{G}_x$. Let $\mu_1(g', g) := g'g^{-1}$. The family $P = (P_x)$ is called *uniformly supported* if its *reduced support* $\mathrm{supp}_\mu(P) := \mu_1(\mathrm{supp}(P))$ is a compact subset of $\mathcal{G}$.

Definition 3.6. The space $\Psi^m(\mathcal{G})$ of *pseudodifferential operators of order m on a Lie groupoid* $\mathcal{G}$ with units M consists of smooth families of pseudodifferential operators $P = (P_x)$, $x \in M$, with $P_x \in \Psi^m(\mathcal{G}_x)$, which are uniformly supported and right invariant.

We also denote $\Psi^\infty(\mathcal{G}) := \bigcup_{m\in\mathbb{R}} \Psi^m(\mathcal{G})$ and $\Psi^{-\infty}(\mathcal{G}) := \bigcap_{m\in\mathbb{R}} \Psi^m(\mathcal{G})$. We then have a representation π of $\Psi^\infty(\mathcal{G})$ on $\mathcal{C}_c^\infty(M)$ (or on $\mathcal{C}^\infty(M)$, on $L^2(M)$, or on Sobolev spaces), called *vector representation* uniquely determined by the equation

$$(\pi(P)f) \circ r := P(f \circ r),$$

where $f \in \mathcal{C}_c^\infty(M)$ and $P = (P_x) \in \Psi^m(\mathcal{G})$.

Recall that k_x denotes the distributional kernel of P_x, $x \in M$. Then the formula

$$k_P(g) := k_{d(g)}(g, d(g))$$

defines a distribution on the groupoid $\mathcal{G}$, with $\mathrm{supp} k_p = \mathrm{supp}_\mu(P)$ compact, smooth outside M and given by an oscillatory integral on a neighborhood of M. If $P \in \Psi^{-\infty}(\mathcal{G})$, then P identifies with the convolution operator with kernel a smooth, compactly supported function and $\Psi^{-\infty}(\mathcal{G})$ identifies with the convolution algebra $\mathcal{C}_c^\infty(\mathcal{G})$. In particular, we can define

$$\|P\|_{L^1(\mathcal{G})} := \sup_{x\in M} \left\{ \int_{\mathcal{G}_x} |k_P(g^{-1})|\, d\mu_x(g),\ \int_{\mathcal{G}_x} |k_P(g)|\, d\mu_x(g) \right\}.$$

For each $x \in M$, there is an interesting representation of $\Psi^\infty(\mathcal{G})$, the *regular representation* π_x on $\mathcal{C}_c^\infty(\mathcal{G}_x)$, defined by $\pi_x(P) = P_x$. It is clear that if $P \in \Psi^{-n-1}(\mathcal{G})$

$$\|\pi_x(P)\|_{L^2(\mathcal{G}_x)} \leqslant \|P\|_{L^1(\mathcal{G})}.$$

The *reduced C^*–norm* of P is defined by

$$\|P\|_r = \sup_{x\in M} \|\pi_x(P)\| = \sup_{x\in M} \|P_x\|,$$

and the *full norm* of P is defined by

$$\|P\| = \sup_\rho \|\rho(P)\|,$$

where ρ varies over all bounded representations of $\Psi^0(\mathcal{G})$ satisfying

$$\|\rho(P)\| \leqslant \|P\|_{L^1(\mathcal{G})} \quad \text{for all} \quad P \in \Psi^{-\infty}(\mathcal{G}).$$

Definition 3.7. Let $\mathcal{G}$ be a Lie groupoid and $\Psi^{\infty}(\mathcal{G})$ be as above. We define $C^*(\mathcal{G})$ (respectively, $C^*_r(\mathcal{G})$) to be the completion of $\Psi^{-\infty}(\mathcal{G})$ in the norm $\|\cdot\|$ (respectively, $\|\cdot\|_r$). If $\|\cdot\|_r = \|\cdot\|$, that is, if $C^*(\mathcal{G}) \cong C^*_r(\mathcal{G})$, we call $\mathcal{G}$ *amenable*.

We give some examples of Lie groupoids below.

Example 3.8 (Manifolds with corners)*.* A manifold (with corners) M may be viewed as a Lie groupoid, by taking both the object and morphism sets to be M, and the domain and range maps to be the identity map $M \to M$. Then we have $A(M) = 0$, the zero bundle on M, and $\Psi^{\infty}(M) = \mathcal{C}^{\infty}_c(M)$.

Example 3.9 (Lie groups)*.* Every Lie group G can be regarded as a Lie groupoid $\mathcal{G} = G$ with only one unit $M = \{e\}$, the unit of G. In this case, the Lie algebroid $A(G)$ is the Lie algebra of G, and $\Psi^m(\mathcal{G})$ is the algebra of properly supported and invariant pseudodifferential operators on G.

Example 3.10 (Pair groupoid)*.* Let M be a smooth manifold. Let

$$\mathcal{G} = M \times M \qquad \mathcal{G}_0 = M,$$

with structure maps $d(m_1, m_2) = m_2$, $r(m_1, m_2) = m_1$, $(m_1, m_2)(m_2, m_3) = (m_1, m_3)$, $u(m) = (m, m)$, and $\iota(m_1, m_2) = (m_2, m_1)$. Then $\mathcal{G}$ is a Lie groupoid, called *the pair groupoid*. We have $A(\mathcal{G}) = TM$. According to the definition, a pseudodifferential operator P belongs to $\Psi^m(\mathcal{G})$ if and only if the family $P = (P_x)_{x\in M}$ is constant. Hence we obtain $\Psi^m(\mathcal{G}) = \Psi^m_{\mathrm{comp}}(M)$. Also, an important result is that $C^*(\mathcal{G}) \cong \mathcal{K}$, the ideal of compact operators, the isomorphism being given by the vector representation or by any of the regular representations (together with $\mathcal{G}_x \cong M$). If M has dimension 0, say, it is a discrete set with k elements, then $C^*(\mathcal{G}) \cong M_k(\mathbb{C})$ and the convolution product becomes matrix multiplication.

Example 3.11 (The fibered pair groupoid)*.* Let $f : M \to N$ be a submersion of manifolds (with corners). The fiber pair groupoid is defined as

$$\mathcal{G} := M \times_N M = \{(m_1, m_2) |\, f(m_1) = f(m_2),\, m_1, m_2 \in M\},$$

with the operation induced from the pair groupoid $M \times M$. The space of units is M. The Lie algebroid $A(\mathcal{G})$ is the kernel of $f_* : TM \to TN$, i.e., the vertical tangent bundle to the submersion $f : M \to N$, and $\Psi^m(\mathcal{G})$ consists of families of pseudodifferential operators along the fibers $M \to N$ so that their reduced kernel are compactly supported.

Example 3.12 (Transformation (or Action) groupoid)*.* Suppose that a Lie group G acts on the smooth manifold M from the right. The *transformation groupoid* over $M \times \{e\} \cong M$, denoted by $M \rtimes G$, is the set $M \times G$ with structure maps $d(m, g) = (m{\cdot}g, e)$, $r(m, g) = (m, e)$, $(m, g)(m{\cdot}g, h) = (m, gh)$, $u(m, e) = (m, e)$, and $\iota(m, g) = (m \cdot g, g^{-1})$. For more on the action groupoid, one may see [34, 40, 48].

Example 3.13 (Vector bundles)*.* Let E be the total space of a smooth vector bundle over a manifold M, then we can view E as a groupoid as follows: the domain and range maps are both equal to the projection from E to the base space M, and composition of morphisms is addition in the fibers of E. We are therefore viewing V as a smooth family of additive Lie groups over M. In this way, E is considered as a Lie groupoid. This is a particular case of bundles of Lie groups in the next example.

Example 3.14 (Bundle of Lie groups). If $\mathcal{G} \to M$ is a *bundle of Lie groups*, i.e, $d = r$ (hence each fiber is a Lie group), then $\Psi^m(\mathcal{G})$ consists of smooth families of invariant and properly supported pseudodifferential operators on the fibers of $\mathcal{G} \to M$.

4. Double layer potentials on three-dimensional wedges

In this section, we study explicitly the method of double layer potentials for solving the Dirichlet problem on the domain

$$\mathcal{W} := \{(r\cos\theta, r\sin\theta, z) : \ r > 0, \ 0 < \theta < \alpha, \ z \in \mathbb{R}\},$$

where $\alpha \in (0, 2\pi)$ and $\alpha \neq \pi$.

We denote the boundary of $\mathcal{W}$ by $\partial\mathcal{W}$. Then we have

$$\begin{aligned} \partial\mathcal{W} &= \{(x, 0, z) : x > 0, z \in \mathbb{R}\} \bigcup \{(r\sin\alpha, r\cos\alpha, z) : r > 0, z \in \mathbb{R}\} \\ &= B_1 \cup B_2. \end{aligned}$$

We agree from now on that $x, r, s > 0$ and $u, z, t \in \mathbb{R}$.

Suppose that $f_1 \in \mathcal{C}_c(B_1)$ and $f_2 \in \mathcal{C}_c(B_2)$. So the double layer potential operator K can be represented as a 2×2 matrix acting on $\mathcal{C}_c(B_1) \bigoplus \mathcal{C}_c(B_2)$:

$$K_{11} = K_{22} = 0,$$

$$\begin{aligned} (K_{21}f_1)(r, u) &= \frac{1}{4\pi}\int_0^\infty \int_{-\infty}^\infty \frac{-r\sin\alpha}{[(r\cos\alpha - x)^2 + (r\sin\alpha)^2 + (u - z)^2]^{\frac{3}{2}}} f_1(x, z)dzdx \\ &= \frac{1}{4\pi}\int_0^\infty \int_{-\infty}^\infty \frac{-r\sin\alpha}{[r^2 - 2rx\cos\alpha + x^2 + (u - z)^2]^{\frac{3}{2}}} f_1(x, z)dzdx. \end{aligned}$$

$$\begin{aligned} (K_{12}f_2)(x, u) &= \frac{1}{4\pi}\int_0^\infty \int_{-\infty}^\infty \frac{-x\sin\alpha}{[(r\cos\alpha - x)^2 + (r\sin\alpha)^2 + (u - z)^2]^{\frac{3}{2}}} f_2(r, z)dzdr \\ &= \frac{1}{4\pi}\int_0^\infty \int_{-\infty}^\infty \frac{-x\sin\alpha}{[r^2 - 2rx\cos\alpha + x^2 + (u - z)^2]^{\frac{3}{2}}} f_1(x, z)dzdr. \end{aligned}$$

Let us consider $K_{21}f_1$. Making the changes of variables

$$\begin{cases} x &= r/s \\ z &= u - tr/s. \end{cases}$$

gives

$$\frac{\partial(x, z)}{\partial(s, t)} = \begin{vmatrix} -r/s^2 & 0 \\ (tr)/s^2 & -r/s \end{vmatrix} = \frac{r^2}{s^3}.$$

We can write

$$
\begin{aligned}
(4)\qquad & (K_{21}f_1)(r,u) \\
= & \frac{1}{4\pi}\int_0^\infty\int_{-\infty}^\infty \frac{-r\sin\alpha}{[r^2-2rx\cos\alpha+x^2+(u-z)^2]^{\frac{3}{2}}} f_1(x,z)dzdx \\
= & \frac{1}{4\pi}\int_0^\infty\int_{-\infty}^\infty \frac{-r\sin\alpha}{[r^2-2r(\frac{r}{s})\cos\alpha+(\frac{r}{s})^2+(\frac{rt}{s})^2]^{\frac{3}{2}}}\left(\frac{r^2}{s^3}\right) f_1\left(\frac{r}{s},u-\frac{rt}{s}\right)dt\,ds \\
= & \frac{1}{4\pi}\int_0^\infty\int_{-\infty}^\infty \frac{-r\sin\alpha}{(\frac{r}{s})^3(s^2-2s\cos\alpha+1+t^2)^{\frac{3}{2}}}\left(\frac{r^2}{s^3}\right) f_1\left(\frac{r}{s},u-\frac{rt}{s}\right)dt\,ds \\
= & \frac{1}{4\pi}\int_0^\infty\int_{-\infty}^\infty \frac{-s\sin\alpha}{(s^2-2s\cos\alpha+1+t^2)^{\frac{3}{2}}} f_1\left(\frac{r}{s},u-\frac{rt}{s}\right)dt\frac{ds}{s}.
\end{aligned}
$$

Making similar changes of variables, we can rewrite

$$K_{12}f_2(x,u) = \frac{1}{4\pi}\int_0^\infty\int_{-\infty}^\infty \frac{-s\sin\alpha}{(s^2-2s\cos\alpha+1+t^2)^{\frac{3}{2}}} f_2\left(\frac{x}{s},u-\frac{xt}{s}\right)dt\frac{ds}{s}.$$

Now, let us consider the group G of 2×2 matrices

$$G=\left\{\begin{pmatrix}1 & 0\\ r & u\end{pmatrix}\ \middle|\ u\in\mathbb{R},\quad r>0\right\}. \tag{5}$$

Throughout the rest of the paper, we always use G to denote this group.

Then we have

$$\begin{pmatrix}1 & 0\\ u & r\end{pmatrix}\begin{pmatrix}1 & 0\\ t & s\end{pmatrix}^{-1}=\begin{pmatrix}1 & 0\\ u & r\end{pmatrix}\begin{pmatrix}1 & 0\\ -t/s & 1/s\end{pmatrix}=\begin{pmatrix}1 & 0\\ u-(tr)/s & r/s\end{pmatrix}.$$

A right-invariant Haar measure on G is $du\,(dr/r)$ and a left-variant Haar measure is $du\,(dr/r^2)$. So the group G is not unimodular. We need harmonic analysis on this group. Let us recall that the group G can be identified with the "$ax+b$ group" of transformations of the form (called affine transformations)

$$x\to\ ax+b$$

on the real line. The "$ax+b$ group" can be viewed as a semi-product

$$\mathbb{R}\rtimes_\varphi\mathbb{R}^+,$$

where $\mathbb{R}$ is additive, $\mathbb{R}^+$ is multiplicative, and

$$\varphi(a)b=ab,\qquad a\in\mathbb{R}^+,\quad b\in\mathbb{R}.$$

Therefore, we have $G=\mathbb{R}\rtimes_\varphi\mathbb{R}^+$ [21, 56]. The group $G=\mathbb{R}\rtimes_\varphi\mathbb{R}^+$ is amenable since it is solvable. In fact, let $G^{(0)}=G$ and denote by $G^{(i)}$ the commutator subgroup of $G^{(i-1)}$, so we have

$$G^{(1)}=\left\{\begin{pmatrix}1 & 0\\ z & 1\end{pmatrix}: z\in\mathbb{R}\right\},$$

and

$$G^{(2)}=\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}.$$

These imply that G is solvable [21].

Suppose $g, f \in \mathcal{C}_c(G)$. The convolution of f and g (with respect to left-invariant Haar measure $d\mu = dt(ds/s^2)$ is defined by (for instance, [21, 47])

$$\begin{aligned} f * g(r,u) &:= \int_0^\infty \int_{-\infty}^\infty f(s,t)\, g\left((s,t)^{-1}(r,u)\right) \frac{ds}{s^2}\, dt \\ &= \int_0^\infty \int_{-\infty}^\infty f(s,t)\, g\left((s,t)^{-1}(r,u)\right) \left(\frac{r}{s^3}\right)\left(\frac{s}{r}\right) ds\, dt \\ &= \int_0^\infty \int_{-\infty}^\infty f\left((r,u)(s,t)^{-1}\right)\, g(s,t)\, dt \frac{ds}{s}, \end{aligned}$$

where we identify (r,u) with the matrix $\begin{pmatrix} 1 & 0 \\ u & r \end{pmatrix}$.

Therefore, we can realize K_{21} and K_{12} as right convolution operators on G on $L^2(G, d\nu)$, where $d\nu := du\,(dr/r)$ is the right invariant Haar measure on G, with the same convolution kernel:

$$k(r,u) = -\frac{1}{4\pi} \frac{r \sin\alpha}{(r^2 - 2r\cos\alpha + 1 + u^2)^{\frac{3}{2}}},$$

More precisely, $K_{12} f_1 = f_1 * k$ and $K_{21} f_2 = f_2 * k$ with respect to the right Haar measure $d\nu = du\,(dr/r)$. We summarize what we have obtained in the following proposition.

Proposition 4.1. *Let G be as above and $d\nu = du\,(dr/r)$ be the right Haar measure of G. Then the operators K_{12} and K_{21} both identify with right convolution operators (with respect to $d\nu$) with the same convolution kernel*

$$k(r,u) = -\frac{1}{4\pi} \frac{r \sin\alpha}{(r^2 - 2r\cos\alpha + 1 + u^2)^{\frac{3}{2}}}.$$

For simplicity, let $\widetilde{K} := K_{12} = K_{21}$.

Lemma 4.2. *The kernel $k(r,u)$ of the convolution operator $\widetilde{K}$ on G is smooth, and*

$$||k(r,u)||_{L^1(G,d\nu)} = \frac{|\pi - \alpha|}{2\pi} < \frac{1}{2}.$$

Proof. It is clear that the kernel $k(r,u)$ is a smooth function on G. We need the following facts

$$\int_{-\infty}^\infty \frac{1}{(a^2 + u^2)^{\frac{3}{2}}}\, du = \frac{1}{a^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, d\theta = \frac{2}{a^2}, \tag{6}$$

and

$$\int_0^\infty \frac{r}{r^2 - 2s\cos\alpha + 1} \frac{dr}{r} = \frac{\pi - \alpha}{\sin(\pi - \alpha)} = \frac{|\pi - \alpha|}{|\sin\alpha|}.$$

In particular, if $\alpha = \pi$, then we have

$$\int_0^\infty \frac{r}{r^2 + 2r + 1} \frac{dr}{r} = 1 = \lim_{\alpha \to \pi} \frac{|\pi - \alpha|}{|\sin\alpha|}.$$

Thus, we obtain

$$\begin{aligned}
\|k(r,u)\|_{L^1(G,d\nu)} &= \frac{1}{4\pi}\int_0^\infty\int_{-\infty}^\infty \frac{r|\sin\alpha|}{(r^2-2r\cos\alpha+1+u^2)^{\frac{3}{2}}}\,du\,\frac{dr}{r}\\
&= \frac{1}{2\pi}\int_0^\infty \frac{r|\sin\alpha|}{r^2-2r\cos\alpha+1}\,\frac{dr}{r}\\
&= \frac{|\sin\alpha|}{2\pi}\cdot\frac{|\pi-\alpha|}{|\sin\alpha|}\\
&= \frac{|\pi-\alpha|}{2\pi} < \frac{1}{2}.
\end{aligned}$$

□

Remark 4.3. If $f \in L^2(G, d\nu)$, then by Generalized Young's inequality ([22]), we obtain the following estimate (see [47]):

$$\|f * k\|_{L^2(G,d\nu)} \leqslant \|f\|_{L^2(G,d\nu)}\,\|k\|_{L^1(G,d\nu)} < \frac{1}{2}\,\|f\|_{L^2(G,d\nu)},$$

which implies that the operator $\pm\frac{1}{2}+\widetilde{K}$ is invertible on $L^2(G, d\nu)$.

Denote by M_h the multiplication operator by h. By Equation (4), we see that

$$\begin{aligned}
&(M_{r^a_{\mathcal{W}}}\widetilde{K}M_{r^{-a}_{\mathcal{W}}}f)(r,u)\\
=\;& \frac{r^a}{4\pi}\int_0^\infty\int_{-\infty}^\infty \frac{-r\sin\alpha}{[r^2-2rx\cos\alpha+x^2+(u-z)^2]^{\frac{3}{2}}}\,x^{-a}f(x,z)\,dz\,dx\\
=\;& \frac{r^a}{4\pi}\int_0^\infty\int_{-\infty}^\infty \frac{-r\sin\alpha}{[r^2-2r(\frac{r}{s})\cos\alpha+(\frac{r}{s})^2+(\frac{rt}{s})^2]^{\frac{3}{2}}}\left(\frac{r^2}{s^3}\right)\left(\frac{r}{s}\right)^{-a} f\left(\frac{r}{s}, u-\frac{rt}{s}\right)dt\,ds\\
=\;& \frac{1}{4\pi}\int_0^\infty\int_{-\infty}^\infty \frac{-r\sin\alpha}{(\frac{r}{s})^3(s^2-2s\cos\alpha+1+t^2)^{\frac{3}{2}}}\left(\frac{r^2}{s^3}\right)\left(\frac{1}{s}\right)^{-a} f\left(\frac{r}{s}, u-\frac{rt}{s}\right)dt\,ds\\
=\;& \frac{1}{4\pi}\int_0^\infty\int_{-\infty}^\infty \frac{-s^{a+1}\sin\alpha}{(s^2-2s\cos\alpha+1+t^2)^{\frac{3}{2}}}\,f\left(\frac{r}{s}, u-\frac{rt}{s}\right)dt\,\frac{ds}{s}.
\end{aligned}$$

Hence the operator $M_{r^a}\widetilde{K}M_{r^{-a}}$ has convolution kernel

$$k_a(r,u) := -\frac{1}{4\pi}\frac{r^{a+1}\sin\alpha}{(r^2-2r\cos\alpha+1+u^2)^{\frac{3}{2}}}.$$

Next we want to calculate the L^1-norm of $k_a(r,u)$. By Equation (6), we have

$$\begin{aligned}
\int_{-\infty}^\infty |k_a(r,u)|dt &= \frac{1}{4\pi}\int_{-\infty}^\infty \frac{r^{a+1}|\sin\alpha|}{(r^2-2r\cos\alpha+1+u^2)^{\frac{3}{2}}}\,du\\
&= \frac{1}{2\pi}\frac{r^{a+1}|\sin\alpha|}{r^2-2r\cos\alpha+1}.
\end{aligned}$$

Lemma 4.4. *For $a \in (-1,1)$ and $\alpha \in (0, 2\pi)$, we have*

$$\int_0^\infty \frac{r^a}{r^2-2r\cos\alpha+1}\,ds = \frac{\pi}{\sin a\pi}\cdot\frac{\sin a(\alpha-\pi)}{\sin(\alpha-\pi)}.$$

Proof. This is a consequence of the Residual Theorem in complex analysis. We sketch the calculation as follows. Let

$$f(z) = \frac{z^a}{z^2 - 2z\cos z + 1} = \frac{e^{a\log z}}{(z - e^{\alpha i})(z - e^{(2\pi-\alpha)i})}.$$

It is clear that $f(z)$ has two simple poles: $e^{\alpha i}$ and $e^{(2\pi-\alpha)i}$. So we have

$$\begin{aligned}
& 2\pi i\Big(Res(f, e^{\alpha i}) + Res(f, e^{(2\pi-\alpha)i})\Big) \\
= \;& 2\pi i\Big(\frac{e^{a\alpha i}}{e^{\alpha i} - e^{-\alpha i}} + \frac{e^{a(2\pi-\alpha)i}}{e^{(2\pi-\alpha)i} - e^{\alpha i}}\Big) \\
= \;& 2\pi i\Big(\frac{e^{a\alpha i}}{2i\sin\alpha} + \frac{e^{a(2\pi-\alpha)i}}{-2i\sin\alpha}\Big) \\
= \;& \frac{\pi}{\sin\alpha}\Big(e^{a\alpha i} - e^{a(2\pi-\alpha)i}\Big) \\
= \;& \frac{\pi}{\sin\alpha}\Big(-2\sin a\pi\,\sin a(\alpha-\pi) + i\,2\cos a\pi\,\sin a(\alpha-\pi)\Big).
\end{aligned}$$

Moreover, we need the following fact:

$$\frac{1}{1 - e^{2a\pi i}} = \frac{1}{2} + i\,\frac{\cos a\pi}{2\sin a\pi}.$$

Because the numerator of $f(z)$ contains *log* function, by choosing an appropriate contour, we obtain

$$\begin{aligned}
& \int_0^\infty \frac{s^a}{s^2 - 2s\cos\alpha + 1}\,ds \\
= \;& \mathrm{Re}\Big(\frac{1}{1 - e^{2a\pi i}}\;2\pi i\Big(Res(f, e^{\alpha i}) + Res(f, e^{(2\pi-\alpha)i})\Big)\Big) \\
= \;& \frac{\pi}{\sin\alpha}\Big(-\sin a\pi\,\sin a(\alpha-\pi) - \cos a\pi\,\sin a(\alpha-\pi)\,\frac{\cos a\pi}{\sin a\pi}\Big) \\
= \;& \frac{\pi}{\sin\alpha}\,\frac{\sin a(\pi-\alpha)}{\sin a\pi} \\
= \;& \frac{\pi}{\sin a\pi}\,\frac{\sin a(\alpha-\pi)}{\sin(\alpha-\pi)}.
\end{aligned}$$

The proof is now complete. □

Proposition 4.5. *For $a \in (-1, 1)$ and $\alpha \in (0, 2\pi)$, we have*

$$||k_a(r,u)||_{L^1(G,d\nu)} = \frac{1}{2}\,\frac{|\sin a(\pi-\alpha)|}{|\sin a\pi|}.$$

In particular, if $a = 0$, then

$$||k_0(r,u)||_{L^1(G,d\nu)} = ||k(r,u)||_{L^1(G,d\nu)} = \frac{|\pi-\alpha|}{2\pi}.$$

Proof. By the preceding Lemma 4.4, we calculate

$$\begin{aligned}
||k_a(r,u)||_{L^1(G,d\nu)} &= \frac{1}{4\pi}\int_0^\infty \int_{-\infty}^\infty \frac{r^{a+1}|\sin\alpha|}{(r^2-2r\cos\alpha+1+u^2)^{\frac{3}{2}}}\,du\,\frac{dr}{r} \\
&= \frac{1}{2\pi}\int_0^\infty \frac{r^{a+1}|\sin\alpha|}{r^2-2r\cos\alpha+1}\,\frac{dr}{r} \\
&= \frac{|\sin\alpha|}{2\pi}\int_0^\infty \frac{r^a}{r^2-2r\cos\alpha+1}\,dr \\
&= \frac{1}{2}\frac{|\sin a(\pi-\alpha)|}{|\sin a\pi|}.
\end{aligned}$$

This completes the proof. □

Recall that $H^s(G,d\nu)$ be the Sobolev spaces defined by the right-invariant measure ν. Therefore we have the following mapping property for the operator $\widetilde{K}_a$.

Proposition 4.6. *For all $m,l \in \mathbb{Z}$ and $a \in (-1,1)$, the convolution operator $\widetilde{K}_a$ defines a continuous map*

$$\widetilde{K}_a : H^m(G,d\nu) \to H^l(G,d\nu).$$

Proof. It is clear that

$$H^m(G,d\nu) = \{f \mid (r\partial_r)^i\,\partial_u^j f \in L^2(G,d\nu), i+j \leqslant m\}.$$

It is enough to show that $\widetilde{K}_a$ maps $L^2(G,d\nu)$ to $H^m(G,d\nu)$, $m>0$.

By Proposition 4.5, the convolution kernel of $\widetilde{K}_a$

$$k_a(r,u) = -\frac{1}{4\pi}\frac{r^{a+1}\sin\alpha}{(r^2-2r\cos\alpha+1+u^2)^{\frac{3}{2}}}$$

is smooth on G and its L^1-norm is finite for $a \in (-1,1)$. Therefore, it suffices to show that

$$(r\partial_r)^i\,\partial_u^i\,k_a \in L^1(G,d\nu).$$

In fact, in r-direction, after taking derivatives (with respect to $r\partial_r$), the decay remains exponential (making the change of variables $r=e^t$ if necessary), and the derivative in u-direction improves the integrability of k_a at infinity. Hence we complete the proof. □

Let us define

$$\Xi := \{a : \frac{|\sin a(\pi-\alpha)|}{|\sin a\pi|} < 1\}.$$

Clearly, we have $(-\frac{1}{2},\frac{1}{2}) \subset \Xi$. To make the L^1-norm of $k_a(r,u)$ less than $\frac{1}{2}$, it is necessary to assume that $a \in \Xi$. Recall the identifications ([3, 8])

$$\mathcal{K}_1^m(G) \cong H^m(G,d\nu) = H^m(G,du\,(dr/r)),$$

and

$$r^t\mathcal{K}_1^m(G) \cong \mathcal{K}_{1+t}^m(G).$$

We summarize the above results in the following theorem.

Theorem 4.7. *Let $G = \mathbb{R} \rtimes \mathbb{R}^+$ be as above, $m \in \mathbb{Z}$, and $a \in \Xi$. The operators*

$$\frac{1}{2} \pm \widetilde{K} : \mathcal{K}^m_{1+a}(G) \to \mathcal{K}^m_{1+a}(G)$$

are both isomorphisms.

Proof. Under the assumption of a, we have $||k_a(r,u)||_{L^1(G,d\nu)} < \frac{1}{2}$. Then we conclude that $\frac{1}{2} \pm \widetilde{K}$ are invertible on $\mathcal{K}^0_{1+a}(G)$. Therefore, it suffices to show that the inverse of $\frac{1}{2} \pm \widetilde{K}$ maps $\mathcal{K}^m_{1+a}(G)$ to itself.

For simplicity, let $R := 2\widetilde{K}$. By Proposition 4.6, we have the following sequence

$$\mathcal{K}^m_{1+a}(G) \xrightarrow{R} \mathcal{K}^0_{1+a}(G) \xrightarrow{(I \pm R)^{-1}} \mathcal{K}^0_{1+a}(G) \xrightarrow{R} \mathcal{K}^l_{1+a}(G),$$

for all $m, l \in \mathbb{Z}$. As a consequence, we get

$$\mp R + R\,(I \pm R)^{-1}\,R : \mathcal{K}^m_{1+a}(G) \to \mathcal{K}^l_{1+a}(G).$$

From the equality

$$(I \pm R)^{-1} = I \mp R + R\,(I \pm R)^{-1}\,R,$$

we obtain an inverse of $I \pm R$, hence an inverse of $\frac{1}{2} \pm \widetilde{K}$ on $\mathcal{K}^m_{1+a}(G)$ for all $m \in \mathbb{Z}$, which completes the proof. $\square$

For the double layer potential operator K on $\mathcal{W}$, we have the following commutative diagram

$$\begin{tikzcd} \mathcal{K}^m_1(\partial\mathcal{W}) \arrow[r, "K"] & \mathcal{K}^m_1(\partial\mathcal{W}) \arrow[d, "M_{r^a}"] \\ \mathcal{K}^m_{1+a}(\partial\mathcal{W}) \arrow[u, "M_{r^{-a}}"] \arrow[r, "K_a"] & \mathcal{K}^m_{1+a}(\partial\mathcal{W}), \end{tikzcd} \tag{7}$$

where $K_a = M_{r^a} K M_{r^{-a}} = \begin{pmatrix} 0 & M_{r^a}\widetilde{K}M_{r^{-a}} \\ M_{r^a}\widetilde{K}M_{r^{-a}} & 0 \end{pmatrix}$, i.e., the 2×2 matrix with diagonal zero and off-diagonal $M_{r^a}\widetilde{K}M_{r^{-a}}$.

Theorem 4.8. *For all $m \in \mathbb{Z}$ and $a \in \Xi$, the operators*

$$\frac{1}{2} \pm K_a = \begin{pmatrix} \frac{1}{2} & \pm M_{r^a}\widetilde{K}M_{r^{-a}} \\ \pm M_{r^a}\widetilde{K}M_{r^{-a}} & \frac{1}{2} \end{pmatrix} : \mathcal{K}^m_1(\partial\mathcal{W}) \to \mathcal{K}^m_1(\partial\mathcal{W})$$

are isomorphisms. In other words, the operators

$$\frac{1}{2} \pm K = \begin{pmatrix} \frac{1}{2} & \pm\widetilde{K} \\ \pm\widetilde{K} & \frac{1}{2} \end{pmatrix} : \mathcal{K}^m_{1+a}(\partial\mathcal{W}) \to \mathcal{K}^m_{1+a}(\partial\mathcal{W})$$

are isomorphisms.

Proof. The invertibility of the matrix $\frac{1}{2} \pm K$ on $\mathcal{K}^m_{1+a}(\partial\mathcal{W})$ is equivalent to the invertibility of $\frac{1}{4} - \widetilde{K}^2$ on $\mathcal{K}^m_{1+a}(G)$. By Theorem 4.7, the result follows. $\square$

5. Relations to Lie groupoids

We are in position to identify the double layer potential operator K with a smooth invariant family of operators on some Lie groupoid.

Then we compactify the space G in the following way: first of all, we define

$$M := \left\{ \begin{pmatrix} 1 & 0 \\ z & x \end{pmatrix} \mid z \in \mathbb{R},\, x \in [0, \infty) \right\} \bigcup \{\infty\}$$

Then we define $\mathcal{G} := M \rtimes G$. The advantage of this compactification is that the algebraic operation of product of two matrices is preserved and the unit space M of the groupoid $\mathcal{G}$ is compact.

Let us make $\mathcal{G}$ more explicit. The interior of the space M of units of $\mathcal{G}$ is

$$M_0 = \left\{ \begin{pmatrix} 1 & 0 \\ u & r \end{pmatrix} \mid u \in \mathbb{R},\, r \in (0, \infty) \right\},$$

which is an open dense invariant subset of M. Moreover, we have

$$\partial M = \{(z, x) \mid z \in \mathbb{R}, x = 0\} \bigcup \{\infty\},$$

and

$$\mathcal{G}|_{M_0} = G \rtimes G \cong M_0 \times M_0,$$

where $M_0 \times M_0$ is the pair groupoid of M_0.

For any $(z, 0) \in \partial M$, d-fiber $\mathcal{G}_{(z,0)}$ is diffeomorphic to G. More precisely, if $z \in (-\infty, \infty)$, then

$$\mathcal{G}_{(z,0)} = d^{-1} \begin{pmatrix} 1 & 0 \\ z & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 1 & 0 \\ u & r \end{pmatrix} \mid u \in \mathbb{R}, r \in \mathbb{R}^+ \right\} = G,$$

and

$$\mathcal{G}_{\{\infty\}} = G.$$

From Section 4, we can think of $\widetilde{K} = K_{12} = K_{21}$ as right convolution operators on G. Hence we can construct an invariant family $P = (P_m)$, $m \in M_0$, of (pseudodifferential) operators. Since we can extend kernels of P_m to the boundary of M, we obtain a family of (pseudodifferential) operator on $\mathcal{G}$, namely, $P = (P_m)$, $m \in M$. In this way, we can identify $P = (P_m)$ with $\widetilde{K}$. Therefore, we obtain the following theorem:

Proposition 5.1. *If $a \in (-1, 1)$, then $k_a \in L^1(\mathcal{G})$. As a result, we have*

$$M_{r^a} \widetilde{K} M_{r^{-a}} \in C^*(\mathcal{G}).$$

Proof. Because $M_{r^a} \widetilde{K} M_{r^{-a}}$ is a smooth convolution operator on G, it belongs to the (reduced) group C^*-algebra $C^*(G)$. Since the unit space M of $\mathcal{G}$ is compact, we have

$$M_{r^a} \widetilde{K} M_{r^{-a}} \in C^*(G) \subset C^*(\mathcal{G})$$

with the inclusion induced by $\mathbb{C} \to C(M)$.

□

Recall that $K_a := M_{r^a} K M_{r^{-a}} = \begin{pmatrix} 0 & M_{r^a} \widetilde{K} M_{r^{-a}} \\ M_{r^a} \widetilde{K} M_{r^{-a}} & 0 \end{pmatrix}$. We have the following theorem:

Theorem 5.2. *For $a \in (-1, 1)$, $K_a \in C^*(\mathcal{G}) \otimes M_2(\mathbb{C})$;*

Proof. This result follows from Proposition 5.1. □

Remark 5.3. If Ω is "a domain with wedge singularities", we can associate to Ω certain Lie groupoid in the spirit of [4, 13, 45, 46], in particular, in the framework of Fredholm groupoids [11, 12]. More precisely, denote $\mathcal{W}_i$ the i-th wedge singularity and let

$$M_i := \left\{ \begin{pmatrix} 1 & 0 \\ z & x \end{pmatrix} \mid z \in \mathbb{R},\, x \in [0, \infty) \right\}.$$

Near each wedge singularity $\mathcal{W}_i$, we form $\mathcal{J}_i = M_i \rtimes G$. It is clear that G is an invariant subset of $\mathcal{J}_i$ and $\mathcal{J}_i|_G \cong G \times G$. Denote by $\mathring{\Omega}$ the interior of Ω. Then we glue the pair groupoid $\mathring{\Omega} \times \mathring{\Omega}$ with $\mathcal{J}_i \times M_2(\mathbb{C})$ for each $\mathcal{W}_i$. In this way, we obtain a Lie groupoid and denote it by $\mathcal{H}$. Indeed, $\mathcal{H}$ is a *Fredholm groupoid* (see [11] for more details). Then we are able to show that the double layer potential operator K_Ω associated to Ω and the Laplace operator belongs to the C^*-algebra of $\mathcal{H}$. Moreover, we could obtain that the operators $\frac{1}{2} \pm K_\Omega$ are Fredholm between appropriate weighted Sololev spaces on the boundary $\partial\Omega$ by some theorems in [11, 31].

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