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A critical point theorem in bounded convex sets and localization of Nash-type equilibria of nonvariational systems

Radu Precup

Department of Mathematics, Babeş-Bolyai University, Cluj, Romania

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A B S T R A C T

The localization of a critical point of minimum type of a smooth functional is obtained in a bounded convex conical set defined by a norm and a concave upper semicontinuous functional. A vector version is also given in order to localize componentwise solutions of variational systems. The technique is then used for the localization and multiplicity of Nash-type positive equilibria of nonvariational systems. Applications are given to periodic problems.

1. Introduction

Many equations and systems arising from mathematical modeling require positive solutions as acceptable states of the investigated real processes. Mathematically, finding positive solutions means to work in the positive cone of the space of all possible states. However, a cone is an unbounded set and in many cases nonlinear problems have several positive solutions. Thus it is important to localize solutions in bounded subsets of a cone. There are known methods for the localization of solutions based on topological fixed point theory [6], [8]; Leray–Schauder degree theory [6]; upper and lower solutions, maximum principles and differential inequalities [2–4], [21]; and critical point theory [1], [5], [7], [12], [15–17], [20], [22], [23]. In case of problems having a variational structure, that is, whose solutions are critical points of an 'energy' functional, the variational techniques are of particular interest since they are able not only to prove the existence of solutions but also to give information about the variational properties of the solutions of a physical relevance, for instance, of being a minimizer, a maximizer or a saddle point of the associated energy

E-mail address: r.precup@math.ubbcluj.ro.

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functional. As known from the classical Fermat's theorem, local extrema of a differentiable functional in a bounded region are not necessarily critical points of that functional. However, this happens if the functional has an appropriate behavior on the boundary of the region (see [12], [15], [22] and [23]).

The problem becomes even more interesting in case of a system which has not a variational structure, but each of its component equations has, i.e., there exist real functionals E_1, E_2 such that the system is equivalent to the equations

$$\begin{cases} E_{11}(u,v) = 0 \\ E_{22}(u,v) = 0 \end{cases}$$

where $E_{11}(u,v)$ is the partial derivative of E_1 with respect to u, and $E_{22}(u,v)$ is the partial derivative of E_2 with respect to v. How the solutions (u,v) of this system are connected to the variational properties of the two functionals? One possible situation, which fits to physical principles, is that a solution (u,v) is a Nash-type equilibrium of the pair of functionals (E_1, E_2) (see, e.g., [9], [13] and [24]), that is

$$E_1(u,v) = \min_w E_1(w,v)$$

$$E_2(u,v) = \min_w E_2(u,w).$$

A result in this direction is given in [18] for the case when $\min_w$ is taken, first over an entire Banach space and then, over a ball. Non-smooth analogues of those results, for Szulkin functionals, are presented in [19].

In the present paper the localization of a Nash-type equilibrium (u,v) is obtained in the Cartesian product of two conical sets, more exactly $u \in K_1$, $v \in K_2$ where $K_i\ (i = 1,2)$ is a cone of a Hilbert space X_i with norm $\|\cdot\|_i$, and

$$r_1 \le l_1(u), \qquad \|u\|_1 \le R_1,$$

$$r_2 \le l_2(v), \qquad \|v\|_2 \le R_2,$$

for some positive numbers r_i and R_i, $i = 1,2$. Here $l_i : K_i \to \mathbb{R}_+$ are two given functionals. Compared to our previous papers on the localization of critical points in annular conical sets (see [15–17] and [20]), where l_i were norms, here they are upper semicontinuous concave functionals. In applications, when working in spaces of functions, such a functional $l(u)$ can be $\inf u$. If in addition, due to some embedding result, the norm $\|u\|$ is comparable with $\sup u$ in the sense that $\sup u \le c\|u\|$ for every nonnegative function u and some constant $c > 0$, then the values of any nonnegative function u satisfying $r \le l(u)$ and $\|u\| \le R$ belong to the interval $[r, cR]$, which is very convenient for finding multiple solutions located in disjoint annular conical sets.

The paper is structured as follows: first in Section 2 we establish the localization of a critical point of minimum type in a convex conical set as above and we explain how this result can be used in order to obtain finitely or infinitely many solutions. The result can be seen as a variational analogue of some Krasnoselskii's type compression–expansion theorems from fixed point theory (see, e.g., [8], [10] and [11]). The vector version of this result for gradient type systems is obtained in Section 3. It allows to localize individually the components of a solution. Section 4 is devoted to the existence and localization of Nash-type equilibria for nonvariational systems of two equations. An iterative algorithm is used and its convergence is established assuming a **local** matricial contraction condition. The local character of the contraction condition makes possible a repeat application of the algorithm to a number of disjoint conical sets and thus the obtainment of multiple Nash-type equilibria. The theory developed in Sections 2, 3 and 4 is illustrated in Section 5 on the periodic problem.

2. A localization critical point theorem

Let X be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$ which is identified with its dual, let $K \subset X$ be a wedge, and $l : K \to \mathbb{R}_+$ be a concave upper semicontinuous function with $l(0) = 0$. Let $E \in C^1(X)$ be a functional and let $N : X \to X$ be the operator $N(u) := u - E'(u)$.

For any two numbers $r, R > 0$ we consider the conical set

$$K_{rR} := \{u \in K : \ r \le l(u) \ \text{and} \ \|u\| \le R\}.$$

This set is convex since l is concave, and closed since l is upper semicontinuous.

Assume that $K_{rR} \neq \emptyset$ and

$$N(K_{rR}) \subset K.$$

Lemma 2.1. *Let the following conditions be satisfied:*

$$m := \inf_{u \in K_{rR}} E(u) > -\infty; \tag{2.1}$$

$$\text{there is } \varepsilon > 0 \text{ such that } \ E(u) \ge m + \varepsilon \ \text{ for} \tag{2.2}$$

$$\text{all } \ u \in K_{rR} \ \text{ which simultaneously satisfy } l(u) = r \text{ and } \|u\| = R;$$

$$l(N(u)) \ge r \ \text{ for every } u \in K_{rR}. \tag{2.3}$$

Then there exists a sequence $(u_n) \subset K_{rR}$ *such that*

$$E(u_n) \le m + \frac{1}{n} \tag{2.4}$$

and

$$\|E'(u_n) + \lambda_n u_n\| \le \frac{1}{n}, \tag{2.5}$$

where

$$\lambda_n = \begin{cases} -\frac{\langle E'(u_n), u_n \rangle}{R^2} & \text{if } \|u_n\| = R \text{ and } \langle E'(u_n), u_n \rangle < 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Proof. Ekeland's variational principle (see, e.g., [21]) guarantees the existence of a sequence $(u_n) \subset K_{rR}$ such that

$$E(u_n) \le m + \frac{1}{n}, \tag{2.7}$$

$$E(u_n) \le E(v) + \frac{1}{n} \|v - u_n\| \tag{2.8}$$

for all $v \in K_{rR}$ and $n \ge 1$. Two cases are possible:

Case (1). There is a subsequence of (u_n) (also denoted by (u_n)) such that $r \le l(u_n)$ and $\|u_n\| < R$ for every n. For a fixed but arbitrary n and $t > 0$, consider the element

$$v = u_n - tE'(u_n).$$

Since $v = (1-t)\, u_n + tN(u_n)$ and both u_n and $N(u_n)$ belong to K, one has that $v \in K$ for every $t \in (0,1)$. Also, the concavity of l and (2.3) yield

$$l(v) \geq (1-t)\, l(u_n) + tl(N(u_n)) \geq r$$

for all $t \in (0,1)$. In addition the continuity of the norm gives $\|v\| \leq R$ for every $t \in (0,1)$ small enough. Hence $v \in K_{rR}$ for all sufficiently small $t > 0$. Replacing v in (2.8) we obtain

$$E(u_n - tE'(u_n)) - E(u_n) \geq -\frac{t}{n}\, \|E'(u_n)\|.$$

Dividing by t and letting t go to zero yields

$$-\langle E'(u_n), E'(u_n)\rangle \geq -\frac{1}{n}\, \|E'(u_n)\|,$$

that is

$$\|E'(u_n)\| \leq \frac{1}{n}.$$

Thus, in this case, relation (2.5) holds with $\lambda_n = 0$.

Case (2). There is a subsequence of (u_n) (also denoted by (u_n)) such that $\|u_n\| = R$ for every n. Passing eventually to a new subsequence, in view of (2.2) and (2.7), we may assume that $l(u_n) > r$ for every n. Two subcases are possible:

(a) For a subsequence (still denoted by (u_n)), $\langle E'(u_n), u_n\rangle > 0$ for every n. Then for any fixed index n, the above choice of v in (2.8) is still possible since

$$\begin{aligned} \|v\|^2 = \|u_n - tE'(u_n)\|^2 &= \|u_n\|^2 + t^2 \|E'(u_n)\|^2 - 2t \langle E'(u_n), u_n\rangle \\ &= R^2 + t^2 \|E'(u_n)\|^2 - 2t \langle E'(u_n), u_n\rangle \leq R^2 \end{aligned}$$

for $0 < t \leq 2 \langle E'(u_n), u_n\rangle / \|E'(u_n)\|^2$.

(b) Assume $\langle E'(u_n), u_n\rangle \leq 0$ for every n. Then for any fixed index n, we use (2.8) with

$$v = u_n - t(E'(u_n) + \lambda_n u_n + \epsilon u_n),$$

where $t, \epsilon > 0$ and $\lambda_n = -\langle E'(u_n), u_n\rangle / R^2 \geq 0$. Since

$$v = (1-t)\, \frac{1 - t - t\lambda_n - t\epsilon}{1-t} u_n + tN(u_n),$$

we immediately see that $v \in K$ for every $t \in (0,1)$ small enough that $1 - t - t\lambda_n - t\epsilon > 0$. Also,

$$\langle E'(u_n) + \lambda_n u_n + \epsilon u_n,\ u_n\rangle = \epsilon R^2 > 0,$$

and as in case (a), we find that $\|v\| \leq R$ for sufficiently small $t > 0$. On the other hand, from $l(u_n) > r$, we have $\delta l(u_n) = r$ for some number $\delta \in (0,1)$. Then, for any $\rho \in [\delta, 1]$, one has

$$\begin{aligned} l(\rho u_n) = l(\rho u_n + (1-\rho)\, 0) &\geq \rho l(u_n) + (1-\rho)\, l(0) \\ &= \rho l(u_n) \geq \delta l(u_n) = r. \end{aligned}$$

In particular, we may take $\rho = (1-t-t\lambda_n - t\epsilon)/(1-t)$ which belongs to $[\delta, 1]$ for sufficiently small t. Consequently,

$$l(v) = l\left((1-t)\frac{1-t-t\lambda_n - t\epsilon}{1-t}u_n + tN(u_n)\right)$$
$$= l((1-t)\rho u_n + tN(u_n)) \geq (1-t)\,l(\rho u_n) + tl(N(u_n)) \geq r.$$

Therefore $v \in K_{rR}$ for every sufficiently small $t > 0$. Replacing v in (2.8) and letting $t \to 0$ yields

$$\langle E'(u_n), -E'(u_n) - \lambda_n u_n - \epsilon u_n\rangle \geq -\frac{1}{n}\|E'(u_n) + \lambda_n u_n + \epsilon u_n\|.$$

Finally, let ϵ tend to zero and use $\langle u_n, E'(u_n) + \lambda_n u_n\rangle = 0$ to deduce

$$\|E'(u_n) + \lambda_n u_n\| \leq \frac{1}{n},$$

that is (2.5). □

Lemma 2.1 yields the following critical point theorem.

Theorem 2.2. *Assume that the assumptions of Lemma 2.1 are satisfied. In addition assume that there is a number ν such that*

$$\langle E'(u), u\rangle \geq \nu \quad \textit{for every } u \in K_{rR} \textit{ with } \|u\| = R, \tag{2.9}$$

$$E'(u) + \lambda u \neq 0 \quad \textit{for all } u \in K_{rR} \textit{ with } \|u\| = R \textit{ and } \lambda > 0, \tag{2.10}$$

and a Palais–Smale type condition holds, more exactly, any sequence as in the conclusion of Lemma 2.1 has a convergent subsequence. Then there exists $u \in K_{rR}$ such that

$$E(u) = m \quad \textit{and} \quad E'(u) = 0.$$

Proof. The sequence (λ_n) given by (2.6) is bounded as a consequence of (2.9). Hence, passing eventually to a subsequence we may suppose that (λ_n) converges to some number λ. Clearly $\lambda \geq 0$. Next using the Palais–Smale type condition we may assume that the sequence (u_n) converges to some element $u \in K_{rR}$. Then letting $n \to \infty$ in (2.4) and (2.5) gives $E(u) = m$ and $E'(u) + \lambda u = 0$. From (2.6) we have that the case $\lambda > 0$ is possible only if $\|u\| = R$, which is excluded by assumption (2.10). Therefore $\lambda = 0$ and so $E'(u) = 0$. □

Remark 2.3. If the functional l is continuous on K_{rR}, then instead of (2.3) we can take the weaker boundary condition

$$l(N(u)) \geq r \quad \text{for every } u \in K_{rR} \text{ with } l(u) = r.$$

Remark 2.4 *(A sufficient condition for (2.3)).* Assume that the space X is continuously embedded in a normed linear space $(Y, \|.\|_Y)$ which is ordered by the cone C inducing on Y the partial ordering $\preccurlyeq$, where

$$u \preccurlyeq v \quad \text{if} \quad v - u \in C.$$

Also assume that K is a subcone of C, $K \subset C \cap X$, and that $N : Y \to X$ and $l : K \to \mathbb{R}_+$ are such that

$$N(C) \subset K \text{ and } 0 \preccurlyeq u \preccurlyeq v \ \text{ implies } \ N(u) \preccurlyeq N(v) \quad (u, v \in Y);$$
$$u \preccurlyeq v \ \text{ implies } \ l(u) \leq l(v) \quad (u, v \in K).$$

If there are two elements $\phi, \psi \in C \setminus \{0\}$ such that

$$l(u)\,\phi \preccurlyeq u \preccurlyeq \|u\|\,\psi$$

for every $u \in K$, then a sufficient condition for (2.3) to hold is

$$l(N(r\phi)) \geq r.$$

Indeed, if $u \in K_{rR}$, then $0 \preccurlyeq r\phi \preccurlyeq l(u)\,\phi \preccurlyeq u$ implies $N(r\phi) \preccurlyeq N(u)$ and next $l(N(r\phi)) \leq l(N(u))$, whence the conclusion.

2.1. Multiple critical points

Assume that there is a constant $c > 0$ such that

$$l(u) \leq c\,\|u\| \tag{2.11}$$

for all $u \in K$. Then from the assumption $K_{rR} \neq \emptyset$, one finds $r \leq cR$. Indeed, if $u \in K_{rR}$, then $r \leq l(u) \leq c\,\|u\| \leq cR$.

Also, if

$$r_1 \leq cR_1, \ r_2 \leq cR_2 \ \text{ and } \ cR_1 < r_2,$$

then the sets $K_{r_1R_1}$ and $K_{r_2R_2}$ are disjoint. Indeed, if $u \in K_{r_1R_1}$, then

$$r_1 \leq l(u) \leq c\,\|u\| \leq cR_1 < r_2.$$

Hence $l(u) < r_2$ which shows that $u \notin K_{r_2R_2}$. The same conclusion holds if

$$r_1 \leq cR_1, \ r_2 \leq cR_2 \ \text{ and } \ r_1 > cR_2.$$

These remarks allow us to state the following multiplicity results.

Theorem 2.5. *Assume that (2.11) holds.*

(1^0) *If there are finite or infinite sequences of numbers $(r_j)_{1\leq j\leq n}$, $(R_j)_{1\leq j\leq n}$ $(1 \leq n \leq +\infty)$ with $r_j \leq cR_j$ for $1 \leq j \leq n$ and $cR_j < r_{j+1}$ for $1 \leq j < n$, such that the assumptions of Theorem 2.2 are satisfied for each of the sets $K_{r_jR_j}$, then for every j, there exists $u_j \in K_{r_jR_j}$ with*

$$E(u_j) = \inf_{K_{r_jR_j}} E \quad \textit{and} \quad E'(u_j) = 0. \tag{2.12}$$

(2^0) *If there are infinite sequences of numbers $(r_j)_{j\geq 1}$, $(R_j)_{j\geq 1}$ with $cR_{j+1} < r_j \leq cR_j$ for all j, such that the assumptions of Theorem 2.2 hold for each of the sets $K_{r_jR_j}$, then for every j, there exists $u_j \in K_{r_jR_j}$ which satisfies (2.12).*

3. A vector version of the localization critical point theorem

Let us now duplicate the objects X, K, and l considered in Section 2. Thus we consider instead, two Hilbert spaces X_i with scalar products $\langle .,.\rangle_i$ and norms $\|.\|_i$ $(i=1,2)$; two wedges $K_i \subset X_i$, and two upper semicontinuous functionals $l_i : K_i \to \mathbb{R}_+$ with $l_i(0)=0$. Also we assume that E is now a C^1 functional on the product space $X_1 \times X_2$. We have $E'(u,v) = (E'_u(u,v), E'_v(u,v))$, for $u \in X_1$, $v \in X_2$, and we denote by N_1, N_2 the operators

$$N_1(u,v) = u - E'_u(u,v), \quad N_2(u,v) = v - E'_v(u,v). \tag{3.1}$$

Here we are interested to find a solution (u,v) of the system

$$\begin{cases} u = N_1(u,v) \\ v = N_2(u,v), \end{cases} \tag{3.2}$$

or equivalently, a critical point of E, that is

$$\begin{cases} E'_u(u,v) = 0 \\ E'_v(u,v) = 0, \end{cases}$$

which minimizes E in a set of the form $K_{rR} := (K_1)_{r_1R_1} \times (K_2)_{r_2R_2}$, where $r=(r_1,r_2)$, $R=(R_1,R_2)$ and

$$(K_i)_{r_iR_i} = \{w \in K_i : \ r_i \le l_i(w) \quad \text{and} \quad \|w\|_i \le R_i\}.$$

Applying Ekeland's principle to the functional E and to the closed subset K_{rR} of $X_1 \times X_2$ we easily obtain the vector versions of Lemma 2.1 and Theorem 2.2.

Lemma 3.1. *Let the following conditions be satisfied:*

$$m := \inf_{(u,v)\in K_{rR}} E(u,v) > -\infty;$$

$$\textit{there is} \ \ \varepsilon > 0 \ \ \textit{such that} \ \ E(u,v) \ge m+\varepsilon \ \ \textit{if}$$

$$l_1(u) = r_1 \ \textit{and} \ \|u\|_1 = R_1, \quad \textit{or} \ \ l_2(v) = r_2 \ \textit{and} \ \|v\|_2 = R_2;$$

$$l_1(N_1(u,v)) \ge r_1 \quad \textit{and} \quad l_2(N_2(u,v)) \ge r_2 \quad \textit{for every} \ \ (u,v) \in K_{rR}.$$

Then there exists a minimizing sequence $(u_n,v_n) \subset K_{rR}$, i.e., $E(u_n,v_n) \to m$ as $n \to \infty$, such that

$$E(u_n,v_n) \le m + \frac{1}{n},$$

$$\|E'_u(u_n,v_n) + \lambda_n u_n\|_1 \le \frac{1}{n} \quad \textit{and} \quad \|E'_v(u_n,v_n) + \mu_n v_n\|_2 \le \frac{1}{n},$$

where

$$\lambda_n = \begin{cases} -\frac{\langle E'_u(u_n,v_n),u_n\rangle_1}{R_1^2} & \textit{if} \ \|u_n\|_1 = R_1 \ \textit{and} \ \langle E'_u(u_n,v_n), u_n\rangle_1 < 0 \\ 0 & \textit{otherwise}, \end{cases}$$

$$\mu_n = \begin{cases} -\frac{\langle E'_v(u_n,v_n),v_n\rangle_1}{R_2^2} & \textit{if} \ \|v_n\|_2 = R_2 \ \textit{and} \ \langle E'_v(u_n,v_n), v_n\rangle_2 < 0 \\ 0 & \textit{otherwise}. \end{cases}$$

Theorem 3.2. *Assume that the assumptions of Lemma 3.1 are satisfied. In addition assume that there is a number ν such that*

$$\begin{aligned} \left\langle E_u'(u,v), u\right\rangle_1 &\geq \nu \ \text{ for every } (u,v)\in K_{rR} \text{ with } \|u\|_1 = R_1\ , \\ \left\langle E_v'(u,v), v\right\rangle_2 &\geq \nu \ \text{ for every } (u,v)\in K_{rR} \text{ with } \|v\|_2 = R_2\ , \end{aligned} \tag{3.3}$$

$$\begin{aligned} E_u'(u,v)+\lambda u \neq 0 \ \text{ for all } (u,v)\in K_{rR} \text{ with } \|u\|_1 = R_1 \text{ and } \lambda>0, \\ E_v'(u,v)+\lambda v \neq 0 \ \text{ for all } (u,v)\in K_{rR} \text{ with } \|v\|_2 = R_2 \text{ and } \lambda>0, \end{aligned} \tag{3.4}$$

and the Palais–Smale type condition holds, i.e., any sequence as in the conclusion of Lemma 3.1 has a convergent subsequence. Then there exists $(u,v)\in K_{rR}$ such that

$$E(u,v)=m \ \text{ and } \ E'(u,v)=0.$$

Theorem 3.2 allows to obtain multiple solutions of variational systems, with disjoint localizations of only one or both components.

4. Localization of Nash-type equilibria of nonvariational systems

Now we deal with system (3.2) without assuming the existence of a functional E with property (3.1). Instead, we assume that each equation of the system has a variational structure, i.e., there are two C^1 functionals $E_i : X := X_1\times X_2 \to \mathbb{R}$, such that

$$N_1(u,v) = u - E_{11}(u,v), \quad N_2(u,v) = v - E_{22}(u,v),$$

where by E_{11}, E_{22} we mean the partial derivatives of E_1, E_2 with respect to u and v, respectively. We look for a point (u,v) in a set of the form $K_{rR} := (K_1)_{r_1R_1}\times (K_2)_{r_2R_2}$, which solves (3.2) and is a *Nash-type equilibrium in* K_{rR} of the pair of functionals (E_1,E_2), more exactly

$$\begin{aligned} E_1(u,v) &= \min_{w\in (K_1)_{r_1R_1}} E_1(w,v), \\ E_2(u,v) &= \min_{w\in (K_2)_{r_2R_2}} E_2(u,w). \end{aligned}$$

We say that the operator $N: X\to X$, $N(u,v) = (N_1(u,v), N_2(u,v))$ is a *Perov contraction on* K_{rR} if there exists a matrix $M=[m_{ij}]\in\mathcal{M}_{2,2}(\mathbb{R}_+)$ such that M^n tends to the zero matrix as $n\to\infty$, and the following matricial Lipschitz condition is satisfied

$$\begin{bmatrix} \|N_1(u,v)-N_1(\overline{u},\overline{v})\|_1 \\ \|N_2(u,v)-N_2(\overline{u},\overline{v})\|_2 \end{bmatrix} \leq M \begin{bmatrix} \|u-\overline{u}\|_1 \\ \|v-\overline{v}\|_2 \end{bmatrix} \tag{4.1}$$

for every $u,\overline{u}\in (K_1)_{r_1R_1}$ and $v,\overline{v}\in (K_2)_{r_2R_2}$.

Notice that for a square matrix of nonnegative elements, the property $M^n\to 0$ is equivalent to $\rho(M)<1$, where $\rho(M)$ is the spectral radius of matrix M, and also to the fact that $I-M$ is nonsingular and all the elements of the matrix $(I-M)^{-1}$ are nonnegative (see [14]). In case of square matrices M of order 2, the above property is characterized by the inequality

$$\operatorname{tr}(M) < \min\{2,\ 1+\det(M)\}.$$

Our hypotheses are as follow:

(H1) For each $v \in (K_2)_{r_2R_2}$, the functional $E_1(.,v)$ is bounded from bellow on $(K_1)_{r_1R_1}$; for each $u \in (K_1)_{r_1R_1}$, the functional $E_2(u,.)$ is bounded from bellow on $(K_2)_{r_2R_2}$.

(H2) $l_1(N_1(u,v)) \geq r_1$ for every $(u,v) \in K_{rR}$; $N_1(u,v) \neq (1+\lambda)u$ for all $(u,v) \in K_{rR}$ with $\|u\|_1 = R_1$ and $\lambda > 0$;
$l_2(N_2(u,v)) \geq r_2$ for every $(u,v) \in K_{rR}$; $N_2(u,v) \neq (1+\lambda)v$ for all $(u,v) \in K_{rR}$ with $\|v\|_2 = R_2$ and $\lambda > 0$.

(H3) For each $v \in (K_2)_{r_2R_2}$, there is $\varepsilon > 0$ such that $E_1(u,v) \geq \inf_{(K_1)_{r_1R_1}} E_1(.,v) + \varepsilon$ whenever u simultaneously satisfies $l_1(u) = r_1$ and $\|u\|_1 = R_1$;
for each $u \in (K_1)_{r_1R_1}$, there is $\varepsilon > 0$ such that $E_2(u,v) \geq \inf_{(K_2)_{r_2R_2}} E_2(u,.) + \varepsilon$ whenever v simultaneously satisfies $l_2(v) = r_2$ and $\|v\|_2 = R_2$.

(H4) N is a Perov contraction on K_{rR}.

Let us underline the local character of the contraction condition (H4). This is essential for multiple Nash-type equilibria when (H4) is required to hold on disjoint bounded sets of the type K_{rR} but not on the entire K. Thus the 'slope' of N has to be 'small' on the sets K_{rR} but can be unlimited large between these sets, which makes possible to fulfill the boundary conditions (H2).

Theorem 4.1. *Assume that conditions* (H1)–(H4) *hold. Then there exists a solution* $(u,v) \in K_{rR}$ *of system* (3.2) *which is a Nash-type equilibrium on* K_{rR} *of the pair of functionals* (E_1, E_2).

Proof. The proof follows the same steps as for Theorem 3.1 in [18]. However, for the convenience of readers we give it in details. We shall construct recursively two sequences $(u_n), (v_n)$, based on Lemma 2.1. Let v_0 be any element of $(K_2)_{r_2R_2}$. At any step n $(n \geq 1)$ we may find an $u_n \in (K_1)_{r_1R_1}$ and an $v_n \in (K_2)_{r_2R_2}$ such that

$$E_1(u_n, v_{n-1}) \leq \inf_{(K_1)_{r_1R_1}} E_1(., v_{n-1}) + \frac{1}{n}, \quad \|E_{11}(u_n, v_{n-1}) + \lambda_n u_n\|_1 \leq \frac{1}{n} \tag{4.2}$$

and

$$E_2(u_n, v_n) \leq \inf_{(K_2)_{r_2R_2}} E_2(u_n, .) + \frac{1}{n}, \quad \|E_{22}(u_n, v_n) + \mu_n v_n\|_2 \leq \frac{1}{n}, \tag{4.3}$$

where

$$\lambda_n = \begin{cases} -\frac{\langle E_{11}(u_n, v_{n-1}), u_n\rangle_1}{R_1^2} & \text{if } \|u_n\|_1 = R_1 \text{ and } \langle E_{11}(u_n, v_{n-1}), u_n\rangle_1 < 0 \\ 0 & \text{otherwise,} \end{cases}$$

and the expression of μ_n is analogous.

Condition (H4) guarantees that the operators N_i are bounded, so the boundedness of the sequences of real numbers (λ_n) and (μ_n). Therefore, passing to subsequences, we may assume that the sequences (λ_n) and (μ_n) are convergent.

Let

$$\alpha_n := E_{11}(u_n, v_{n-1}) + \lambda_n u_n \quad \text{and} \quad \beta_n := E_{22}(u_n, v_n) + \mu_n v_n.$$

Clearly $\alpha_n, \beta_n \to 0$. Also

$$\begin{aligned}(1+\lambda_n)\,u_n - N_1(u_n, v_{n-1}) &= \alpha_n \\ (1+\mu_n)\,v_n - N_2(u_n, v_n) &= \beta_n.\end{aligned} \tag{4.4}$$

Since $\lambda_n > 0$, the first equality if (4.4) written for n and $n+p$ yields

$$\begin{aligned}&\|u_{n+p} - u_n\|_1 \\ &\le (1+\lambda_n)\,\|u_{n+p} - u_n\|_1 \\ &= \|(1+\lambda_n)\,u_{n+p} - (1+\lambda_n)\,u_n\|_1 \\ &= \|(1+\lambda_{n+p})\,u_{n+p} - (1+\lambda_n)\,u_n - (\lambda_{n+p} - \lambda_n)\,u_{n+p}\|_1 \\ &\le \|N_1(u_{n+p}, v_{n+p-1}) - N_1(u_n, v_{n-1})\|_1 + \|\alpha_{n+p} - \alpha_n\|_1 + |\lambda_{n+p} - \lambda_n|\,\|u_{n+p}\|_1\,.\end{aligned}$$

Furthermore, using $\|u_{n+p}\|_1 \le R_1$ and (4.1) we deduce

$$\begin{aligned}&\|u_{n+p} - u_n\|_1 \\ &\le m_{11}\,\|u_{n+p} - u_n\|_1 + m_{12}\,\|v_{n+p-1} - v_{n-1}\|_2 + \|\alpha_{n+p} - \alpha_n\|_1 + R_1\,|\lambda_{n+p} - \lambda_n| \\ &= m_{11}\,\|u_{n+p} - u_n\|_1 + m_{12}\,\|v_{n+p} - v_n\|_2 + \|\alpha_{n+p} - \alpha_n\|_1 + R_1\,|\lambda_{n+p} - \lambda_n| \\ &\quad + m_{12}\left(\|v_{n+p-1} - v_{n-1}\|_2 - \|v_{n+p} - v_n\|_2\right).\end{aligned}$$

Denote

$$\begin{aligned}&a_{n,p} = \|u_{n+p} - u_n\|_1\,, \quad b_{n,p} = \|v_{n+p} - v_n\|_2\,, \\ &c_{n,p} = \|\alpha_{n+p} - \alpha_n\|_1 + R_1\,|\lambda_{n+p} - \lambda_n|\,, \quad d_{n,p} = \|\beta_{n+p} - \beta_n\|_2 + R_2\,|\mu_{n+p} - \mu_n|\,.\end{aligned}$$

Clearly, $c_{n,p} \to 0$ and $d_{n,p} \to 0$ uniformly with respect to p. With this notations,

$$a_{n,p} \le m_{11}a_{n,p} + m_{12}b_{n,p} + c_{n,p} + m_{12}\,(b_{n-1,p} - b_{n,p})\,. \tag{4.5}$$

Similarly, from the second equality in (4.4), we find

$$b_{n,p} \le m_{21}a_{n,p} + m_{22}b_{n,p} + d_{n,p}.$$

Hence

$$\begin{bmatrix} a_{n,p} \\ b_{n,p} \end{bmatrix} \le M \begin{bmatrix} a_{n,p} \\ b_{n,p} \end{bmatrix} + \begin{bmatrix} c_{n,p} + m_{12}\,(b_{n-1,p} - b_{n,p}) \\ d_{n,p} \end{bmatrix}.$$

Consequently, since $I - M$ is invertible and its inverse contains only nonnegative elements, we may write

$$\begin{bmatrix} a_{n,p} \\ b_{n,p} \end{bmatrix} \le (I - M)^{-1} \begin{bmatrix} c_{n,p} + m_{12}\,(b_{n-1,p} - b_{n,p}) \\ d_{n,p} \end{bmatrix}.$$

Let $(I - M)^{-1} = [\gamma_{ij}]$. Then

$$\begin{aligned}a_{n,p} &\le \gamma_{11}\,(c_{n,p} + m_{12}\,(b_{n-1,p} - b_{n,p})) + \gamma_{12}d_{n,p} \\ b_{n,p} &\le \gamma_{21}\,(c_{n,p} + m_{12}\,(b_{n-1,p} - b_{n,p})) + \gamma_{22}d_{n,p}.\end{aligned} \tag{4.6}$$

From the second inequality, one has

$$b_{n,p} \leq \frac{\gamma_{21} m_{12}}{1+\gamma_{21} m_{12}} b_{n-1,p} + \frac{\gamma_{21} c_{n,p} + \gamma_{22} d_{n,p}}{1+\gamma_{21} m_{12}}.$$

Clearly $(b_{n,p})$ is bounded uniformly with respect to p. Now we use the following lemma proved in [18].

Lemma 4.2. *Let $(x_{n,p}), (y_{n,p})$ be two sequences of real numbers depending on a parameter p, such that*

$$(x_{n,p}) \text{ is bounded uniformly with respect to } p,$$

and

$$0 \leq x_{n,p} \leq \lambda x_{n-1,p} + y_{n,p} \quad \text{for all } n, p \text{ and some } \lambda \in [0,1).$$

If $y_{n,p} \to 0$ uniformly with respect to p, then $x_{n,p} \to 0$ uniformly with respect to p too.

According to this result, $b_{n,p} \to 0$ uniformly for $p \in \mathbb{N}$, and hence (v_n) is a Cauchy sequence. Next, the first inequality in (4.6) implies that (u_n) is also a Cauchy sequence. Let u^*, v^* be the limits of the sequences $(u_n), (v_n)$, respectively. The conclusion of Theorem 4.1 now follows if we pass to the limit in (4.2), (4.3) and we use (H2). □

5. Applications to periodic problems

5.1. The case of a single equation

We apply the results from Section 2 to the periodic problem

$$\begin{aligned} -u''(t) + a^2 u(t) &= f(u(t)) \quad \text{on } (0,T) \\ u(0) - u(T) &= u'(0) - u'(T) = 0 \end{aligned} \tag{5.1}$$

where $a \neq 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $f(\mathbb{R}_+) \subset \mathbb{R}_+$.

Let $X := H_p^1(0,T)$ be the space of functions of the form

$$u(t) = \int_0^t v(s)\,ds + C,$$

with $u(0) = u(T)$, $C \in \mathbb{R}$ and $v \in L^2(0,T)$, endowed with the inner product

$$\langle u, v\rangle = \int_0^T \left(u'v' + a^2 uv\right) dt$$

and the corresponding norm

$$\|u\| = \left(\int_0^T \left(u'^2 + a^2 u^2\right) dt\right)^{\frac{1}{2}}.$$

Let K be the positive cone of X, i.e., $K = \{u \in H_p^1(0,T) : \; u \geq 0 \text{ on } [0,T]\}$, and let $l : K \to \mathbb{R}_+$ be given by

$$l(u) = \min_{t\in[0,T]} u(t).$$

The energy functional associated to the problem is $E : H_p^1(0,T) \to \mathbb{R}$,

$$E(u) = \frac{1}{2}\|u\|^2 - \int_0^T F(u(t))\,dt,$$

where

$$F(\tau) = \int_0^\tau f(s)\,ds.$$

The identification of the dual $\left(H_p^1(0,T)\right)'$ to the space $H_p^1(0,T)$ via the mapping $J : \left(H_p^1(0,T)\right)' \to H_p^1(0,T)$, $J(v) = w$, where w is the weak solution of the problem

$$\begin{gathered} -w'' + a^2 w = v \quad \text{on} \;\; (0,T), \\ w(0) - w(T) = w'(0) - w'(T) = 0 \end{gathered}$$

yields to the representation

$$E'(u) = u - N(u)$$

where

$$N(u) = J(f(u(\cdot))).$$

Note that the condition $f(\mathbb{R}_+) \subset \mathbb{R}_+$ guarantees that $N(K) \subset K$.

Let $c > 0$ be the embedding constant of the inclusion $H_p^1(0,T) \subset C[0,T]$, that is, $\|u\|_{C[0,T]} \leq c\|u\|$ for all $u \in H_p^1(0,T)$.

Note that for $u \equiv 1$, the above inequality gives $1 \leq ac\sqrt{T}$, whence $a^2 \geq 1/\left(c^2 T\right)$. Also, if r and R are positive numbers and $a\sqrt{T}r \leq R$, then the set K_{rR} is nonempty. Indeed, any constant $\lambda \in \left[r, R/\left(a\sqrt{T}\right)\right]$ belongs to K_{rR}, since $l(\lambda) = \lambda \geq r$ and $\|\lambda\| = \left(\int_0^T a^2\lambda^2 ds\right)^{1/2} = a\lambda\sqrt{T} \leq R$.

Theorem 5.1. *Let r, R be positive constants such that $a\sqrt{T}r \leq R$. Assume that f is nondecreasing on the interval $[r, cR]$ and that the following conditions hold:*

$$E(r) < \frac{R^2}{2} - TF(cR), \tag{5.2}$$

and

$$f(r) \geq a^2 r, \quad f(cR) \leq \frac{R}{cT}. \tag{5.3}$$

Then problem (5.1) has a positive solution u with $r \leq u(t) \leq cR$ for all $t \in [0,T]$, which minimizes E in the set K_{rR}.

Proof. 1. Check of condition (2.1). Let $u \in K_{rR}$. One has $r \leq u(t) \leq cR$ for all $t \in [0,T]$. Then, since F is nondecreasing on $\mathbb{R}_+$,

$$E(u) \geq -\int_0^T F(u(s))\,ds \geq -TF(cR) > -\infty.$$

2. Check of condition (2.2). Take any u with $l(u) = r$ and $\|u\| = R$. Then

$$E(u) = \frac{R^2}{2} - \int_0^T F(u(s))\,ds \geq \frac{R^2}{2} - TF(cR).$$

Thus our claim holds in view of the strict inequality (5.2) and the obvious inequality $m \leq E(r)$ (note that the constant function r belongs to K_{rR}).

3. Check of condition (2.3). Let $u \in K_{rR}$. Then

$$\begin{aligned} l(N(u)) &= l(J(f(u))) \geq l(J(f(r))) = f(r)\,l(J(1)) \\ &= \frac{f(r)}{a^2} \geq r, \end{aligned}$$

in virtue of the first inequality in (5.3).

4. Check of condition (2.10). Assume that $E'(u) + \lambda u = 0$ for some $u \in K_{rR}$ with $\|u\| = R$ and $\lambda > 0$. Then

$$(1+\lambda)\left(-u'' + a^2 u\right) = f(u),$$

whence

$$R^2 < (1+\lambda) R^2 = \langle f(u), u\rangle_{L^2} \leq Tf(cR)\,cR,$$

that is

$$\frac{R}{cT} < f(cR),$$

which contradicts the second inequality in (5.3).

5. Condition (2.9) being immediate and the required Palais–Smale type condition being a consequence of the compact embedding of $H_p^1(0,T)$ into $C[0,T]$, Theorem 2.2 yields the conclusion. □

Example 1. For each $\lambda > 0$, the equation $-u'' + a^2 u = \lambda\sqrt{u}$ has a T-periodic solution satisfying $u(t) \geq \lambda^2/a^4$ for all $t \in [0,T]$.

Indeed, if we take $r = \lambda^2/a^4$, then the first condition from (5.3) is satisfied with equality. Next chose R large enough that (5.2) and the second inequality (5.3) hold, that is,

$$E(r) < \frac{R^2}{2} - \lambda\frac{2}{3}T(cR)^{\frac{3}{2}} \quad \text{and} \quad \lambda\sqrt{cR} \leq \frac{R}{cT}.$$

5.2. The case of a variational system

We now consider the periodic problem for the system

$$\begin{aligned} -u''(t) + a_1^2 u(t) &= f_1(u(t), v(t)) \quad \text{on } (0,T) \\ -v''(t) + a_2^2 v(t) &= f_2(u(t), v(t)) \quad \text{on } (0,T) \end{aligned} \tag{5.4}$$

in the case when f_1, f_2 are the partial derivatives of a function $F : \mathbb{R}^2 \to \mathbb{R}$ with respect to the first and the second variable, respectively. We assume that $a_i \neq 0$ and $f_i(\mathbb{R}_+ \times \mathbb{R}_+) \subset \mathbb{R}_+$, for $i = 1, 2$.

We apply the results in Section 3, where X_1 is the space $H_p^1(0,T)$ endowed with the scalar product $\langle u, v\rangle_1 = \int_0^T \left(u'v' + a_1^2 uv\right) ds$ and the induced norm $\|.\|_1$, while X_2 is the same space endowed with the analogue scalar product and norm $\langle ., .\rangle_2$, $\|.\|_2$. Also $K_1 = K_2$ is the cone of nonnegative functions in $H_p^1(0,T)$, and $l_1(w) = l_2(w) = \min_{t\in[0,T]} w(t)$ for $w \in H_p^1(0,T)$, $w \geq 0$.

The system has a variational structure since its T-periodic solutions (u, v) are the critical points of the energy functional on $H_p^1(0,T) \times H_p^1(0,T)$,

$$E(u,v) = \frac{1}{2}\left(\|u\|_1^2 + \|v\|_2^2\right) - \int_0^T F(u(s), v(s))\, ds.$$

For $i = 1, 2$, let $c_i > 0$ be the embedding constant of the inclusion $X_i \subset C[0,T]$, that is, $\|w\|_{C[0,T]} \leq c_i \|w\|_i$ for all $w \in H_p^1(0,T)$.

Theorem 5.2. *Let r_i, R_i be positive constants such that $a_i\sqrt{T}r_i \leq R_i$ $(i = 1, 2)$. Assume that for $i = 1, 2$, f_i is nondecreasing in each of the variables on $[r_1, c_1R_1] \times [r_2, c_2R_2]$ and that the following conditions hold:*

$$E(r_1, r_2) < \frac{R_i^2}{2} - TF(c_1R_1, c_2R_2),$$

and

$$f_i(r_1, r_2) \geq a_i^2 r_i, \quad f_i(c_1R_1, c_2R_2) \leq \frac{R_i}{c_iT}.$$

Then system (5.4) has a T-solution (u, v) with $r_1 \leq u(t) \leq c_1R_1$ and $r_2 \leq v(t) \leq c_2R_2$ for all $t \in [0,T]$, which minimizes E in the set $K_{rR} := (K_1)_{r_1R_1} \times (K_2)_{r_2R_2}$.

Example 2. The potential of the system

$$\begin{aligned} -u'' + a_1^2 u &= \alpha_1\sqrt{u} + \gamma v \\ -v'' + a_2^2 v &= \alpha_2\sqrt{v} + \gamma u \end{aligned}$$

is

$$F(u,v) = \frac{2}{3}\left(\alpha_1 u^{\frac{3}{2}} + \alpha_2 v^{\frac{3}{2}}\right) + \gamma uv.$$

As for Example 1, we have the following result: For every numbers $\alpha_i > 0$, $i = 1, 2$, $T > 0$ and $0 \leq \gamma < \min\left\{1/\left(2c_i^2T\right) : \ i = 1, 2\right\}$, the system has a T-periodic solution with $u(t) \geq \alpha_1^2/a_1^4$ and $v(t) \geq \alpha_2^2/a_2^4$ and all $t \in [0,T]$. For the proof, take $r_i = \alpha_i^2/a_i^4$ $(i = 1, 2)$ and a sufficiently large $R := R_1 = R_2$.

5.3. The case of a nonvariational system

We now consider the system (5.4) for two arbitrary continuous functions f_1, f_2 and use the notations from the previous section. The energy functionals associated to the equations of the system are $E_i : H_p^1(0,T) \times H_p^1(0,T) \to \mathbb{R}$,

$$E_1(u,v) = \frac{1}{2}\|u\|_1^2 - \int_0^T F_1(u(t), v(t))\,dt,$$

$$E_2(u,v) = \frac{1}{2}\|v\|_2^2 - \int_0^T F_2(u(t), v(t))\,dt,$$

where

$$F_1(\tau_1,\tau_2) = \int_0^{\tau_1} f_1(s,\tau_2)\,ds, \quad F_2(\tau_1,\tau_2) = \int_0^{\tau_2} f_2(\tau_1,s)\,ds.$$

The identification of the dual $\left(H_p^1(0,T)\right)'$ to the space $H_p^1(0,T)$ via the mapping $J_i : \left(H_p^1(0,T)\right)' \to H_p^1(0,T)$, $J_i(v) = w$, where w is the weak solution of the problem

$$-w'' + a_i^2 w = v \quad \text{on} \quad (0,T),$$
$$w(0) - w(T) = w'(0) - w'(T) = 0$$

yields to the representations

$$E_{11}(u,v) = u - N_1(u,v), \quad E_{22}(u,v) = v - N_2(u,v),$$

where E_{11}, E_{22} stand for the partial derivatives of E_1, E_2 with respect to u and v, respectively, and

$$N_i(u,v) = J_i(f_i(u(\cdot), v(\cdot))).$$

Let $r = (r_1, r_2)$ and $R = (R_1, R_2)$ be such that

$$0 < a_i\sqrt{T} r_i \le R_i, \quad i = 1, 2.$$

Check of condition (H1): For every $(u,v) \in K_{rR} = (K_1)_{r_1R_1} \times (K_2)_{r_2R_2}$ and $t \in [0,T]$, one has

$$r_1 \le u(t) \le \|u\|_{C[0,T]} \le c_1\|u\|_1 \le c_1R_1,$$

and similarly $r_2 \le v(t) \le c_2R_2$. It follows that

$$|f_i(\tau_1,\tau_2)| \le \rho_i$$

for every $\tau_1 \in [r_1, c_1R_1]$, $\tau_2 \in [r_2, c_2R_2]$ and some $\rho_i \in \mathbb{R}_+$ $(i = 1, 2)$. Then

$$E_1(u,v) \ge -\int_0^T \int_0^{u(t)} |f_1(s, v(t))|\,dsdt \ge -\int_0^T \int_0^{c_1R_1} |f_1(s, v(t))|\,dsdt$$
$$\ge -c_1R_1T\rho_1 > -\infty,$$

and similarly $E_2(u,v) \ge -c_2R_2T\rho_2 > -\infty$. Hence condition (H1) holds.

Next we assume in addition that for $i \in \{1, 2\}$,

$$f_i(\tau_1, \tau_2) \quad \text{is nonnegative and nondecreasing} \tag{5.5}$$
$$\text{in both variables } \tau_1 \text{ and } \tau_2 \text{ in } [r_1, c_1R_1] \times [r_2, c_2R_2],$$
$$f_i(r_1, r_2) \geq a_i^2 r_i, \tag{5.6}$$
$$f_i(c_1R_1, c_2R_2) \leq R_i/(Tc_i), \tag{5.7}$$

and

$$F_i(c_1R_1, c_2R_2) - F_i(r_1, r_2) < \frac{1}{2T}\left(R_i^2 - a_i^2 T r_i^2\right). \tag{5.8}$$

Check of condition (H2): Let $(u, v) \in K_{rR}$. Then from $u(t) \geq r_1$, $v(t) \geq r_2$ and the monotonicity of f_1, we have

$$f_i(u(t), v(t)) \geq f_i(r_1, r_2).$$

This together with (5.6) implies

$$l_i(N_i(u, v)) \geq l_i(J_i(f_i(r_1, r_2))) = \frac{f_i(r_1, r_2)}{a_i^2} \geq r_i.$$

Thus the first part of (H2) is verified. For the second part, assume that there exists $(u, v) \in K_{rR}$ with $\|u\|_1 = R_1$ and $\lambda > 0$ such that

$$N_1(u, v) = (1 + \lambda) u.$$

Then

$$(1 + \lambda)\left(-u'' + a_1^2 u\right) = f_1(u, v),$$

which gives

$$\begin{aligned} R_1^2 &< (1 + \lambda) R_1^2 = (1 + \lambda) \|u\|_1^2 = \langle f_1(u, v), u\rangle_{L^2} \\ &\leq T f_1(c_1R_1, c_2R_2) c_1R_1, \end{aligned}$$

whence

$$f_1(c_1R_1, c_2R_2) > R_1/(Tc_1),$$

which contradicts (5.7). An analogue reasoning applies if $N_2(u, v) = (1 + \lambda) v$ for some $(u, v) \in K_{rR}$ with $\|v\|_2 = R_2$ and $\lambda > 0$. Therefore (H2) holds.

Check of condition (H3): The constant function r_1 belongs to $(K_1)_{r_1R_1}$ and for any $v \in (K_2)_{r_2R_2}$, one has

$$\begin{aligned} E_1(r_1, v) &= \frac{1}{2} a_1^2 T r_1^2 - \int_0^T F_1(r_1, v(t))\, dt \\ &\leq \frac{1}{2} a_1^2 T r_1^2 - T F_1(r_1, r_2). \end{aligned}$$

Also, for any $(u,v) \in K_{rR}$ with $l_1(u) = r_1$ and $\|u\|_1 = R_1$, one has

$$E_1(u,v) = \frac{1}{2}R_1^2 - \int_0^T F_1(u(t), v(t))\,dt \geq \frac{1}{2}R_1^2 - TF_1(c_1R_1, c_2R_2).$$

Therefore the first part of (H3) holds with

$$\varepsilon = \frac{1}{2}R_1^2 - TF_1(c_1R_1, c_2R_2) - \left(\frac{1}{2}a_1^2Tr_1^2 - TF_1(r_1, r_2)\right)$$

which is positive in view of assumption (5.8). The second part of (H3) can be checked similarly.

Finally, to guarantee (H4) we need some Lipschitz conditions on f_1 and f_2. We assume the existence of nonnegative constants σ_{ij}, $i,j = 1,2$, such that

$$|f_i(\tau_1,\tau_2) - f_i(\overline{\tau}_1,\overline{\tau}_2)| \leq \sigma_{i1}|\tau_1 - \overline{\tau}_1| + \sigma_{i2}|\tau_2 - \overline{\tau}_2|, \quad i = 1,2, \qquad (5.9)$$
$$\text{for } \ \tau_1, \overline{\tau}_1 \in [r_1, c_1R_1] \ \text{ and } \ \tau_2, \overline{\tau}_2 \in [r_2, c_2R_2],$$

and for the matrix $M = [\sigma_{ij}/(a_ia_j)]_{1\leq i,j\leq 2}$ one has

$$M^n \to 0 \ \text{ as } n \to \infty. \qquad (5.10)$$

Check of condition (H4): Notice that for $w \in L^2(0,T)$, from

$$\|J_i(w)\|_i^2 = \langle w, J_i(w)\rangle_{L^2} \leq \|w\|_{L^2}\|J_i(w)\|_{L^2} \leq \frac{1}{a_i}\|w\|_{L^2}\|J_i(w)\|_i,$$

one has

$$\|J_i(w)\|_i \leq \frac{1}{a_i}\|w\|_{L^2}, \quad w \in L^2(0,T). \qquad (5.11)$$

Then using (5.11) and (5.9) we obtain

$$\begin{aligned}\|N_1(u,v) - N_1(\overline{u},\overline{v})\|_1 &= \|J_1(f_1(u,v) - f_1(\overline{u},\overline{v}))\|_1 \\ &\leq \frac{1}{a_1}\|f_1(u,v) - f_1(\overline{u},\overline{v})\|_{L^2} \\ &\leq \frac{\sigma_{11}}{a_1}\|u-\overline{u}\|_{L^2} + \frac{\sigma_{12}}{a_1}\|v-\overline{v}\|_{L^2} \\ &\leq \frac{\sigma_{11}}{a_1^2}\|u-\overline{u}\|_1 + \frac{\sigma_{12}}{a_1a_2}\|v-\overline{v}\|_2.\end{aligned}$$

Similarly,

$$\|N_2(u,v) - N_2(\overline{u},\overline{v})\|_2 \leq \frac{\sigma_{21}}{a_2a_1}\|u-\overline{u}\|_1 + \frac{\sigma_{22}}{a_2^2}\|v-\overline{v}\|_2.$$

Hence (4.1) holds with $m_{ij} = \sigma_{ij}/a_ia_j$.

Therefore we have the following result.

Theorem 5.3. *Under assumptions (5.5)–(5.8), (5.9) and (5.10), there exists a T-periodic solution $(u,v) \in K_{rR}$ of system (5.4) which is a Nash-type equilibrium on K_{rR} of the pair of energy functionals (E_1, E_2).*

Let us underline the fact that all the assumptions on f_1 and f_2 in the above theorem are given with respect to the bounded region $[r_1, c_1R_1] \times [r_2, c_2R_2]$. This makes possible to apply Theorem 5.3 to several disjoint such regions obtaining this way multiple solutions of Nash-type.

Example 3. Consider the problem of positive T-periodic solutions for the system

$$\begin{aligned} -u'' + a_1^2 u &= \alpha_1 \sqrt{u} + \gamma_1 v \\ -v'' + a_2^2 v &= \alpha_2 \sqrt{v} + \gamma_2 u \end{aligned} \tag{5.12}$$

where α_i, γ_i are nonnegative coefficients with $\gamma_i < a_i^2$ $(i = 1, 2)$.

We try to localize a positive solution (u, v) with $r \leq u(t)$ and $r \leq v(t)$ for all $t \in [0, T]$. We apply the previous result with $r_1 = r_2 =: r$ and $R_1 = R_2 =: R$.

(a) The positivity and monotonicity of f_1 and f_2 on $\mathbb{R}_+ \times \mathbb{R}_+$ required by (5.5) are obvious.

(b) Condition (5.6): We have

$$f_1(r, r) = \alpha_1 \sqrt{r} + \gamma_1 r.$$

Thus we need

$$\alpha_1 \sqrt{r} + \gamma_1 r \geq a_1^2 r.$$

Under the assumption $\gamma_1 < a_1^2$ this gives

$$r \leq \left(\frac{\alpha_1}{a_1^2 - \gamma_1} \right)^2.$$

Similarly, for f_2,

$$r \leq \left(\frac{\alpha_2}{a_2^2 - \gamma_2} \right)^2.$$

(c) Condition (5.7): We have

$$f_1(c_1R, c_2R) = \alpha_1 \sqrt{c_1R} + \gamma_1 c_2 R.$$

Hence we need

$$\alpha_1 \sqrt{c_1R} + \gamma_1 c_2 R \leq \frac{R}{Tc_1}.$$

This implies $\gamma_1 < 1/(Tc_1c_2)$ and

$$R \geq \frac{\alpha_1^2 T^2 c_1^3}{(1 - T\gamma_1 c_1 c_2)^2}.$$

Similarly, $\gamma_2 < 1/(Tc_1c_2)$ and

$$R \geq \frac{\alpha_2^2 T^2 c_2^3}{(1 - T\gamma_2 c_1 c_2)^2}.$$

(d) Condition (5.8) for $i = 1$ reads as

$$\frac{2}{3}\alpha_1 (c_1 R)^{\frac{3}{2}} + \gamma_1 c_1 c_2 R^2 - F_1 (r, r) < \frac{1}{2T} \left(R^2 - a_1^2 T r^2\right)$$

and holds for a sufficiently large R provided that

$$\gamma_1 < \frac{1}{2T c_1 c_2}.$$

Similarly,

$$\gamma_2 < \frac{1}{2T c_1 c_2}.$$

(e) Condition (5.9): For $\tau_1 \in [r, c_1 R]$ and $\tau_2 \in [r, c_2 R]$, one has

$$\frac{\partial f_1 (\tau_1, \tau_2)}{\partial \tau_1} = \frac{\alpha_1}{2\sqrt{\tau_1}} \leq \frac{\alpha_1}{2\sqrt{r}}, \quad \frac{\partial f_2 (\tau_1, \tau_2)}{\partial \tau_2} \leq \frac{\alpha_2}{2\sqrt{r}}.$$

In addition

$$\frac{\partial f_1 (\tau_1, \tau_2)}{\partial \tau_2} = \gamma_1, \quad \frac{\partial f_2 (\tau_1, \tau_2)}{\partial \tau_1} = \gamma_2.$$

Hence (5.9) holds with

$$\sigma_{ii} = \frac{\alpha_i}{2\sqrt{r}} \quad \text{and} \quad \sigma_{ij} = \gamma_i \quad \text{for} \quad i \neq j \quad (i, j = 1, 2). \tag{5.13}$$

Consequently we have the following result.

Theorem 5.4. *Assume that*

$$\gamma_i < a_i^2, \quad \gamma_i < \frac{1}{2T c_1 c_2} \quad \textit{for} \quad i = 1, 2,$$

and there exists $r > 0$ with

$$r \leq \min\left\{\left(\frac{\alpha_i}{a_i^2 - \gamma_i}\right)^2 : \ i = 1, 2\right\},$$

such that the matrix $M = [\sigma_{ij}/(a_i a_j)]_{1 \leq i,j \leq 2}$ where σ_{ij} are given by (5.13) satisfies (5.10). Then (5.12) has a unique T-periodic solution (u, v) such that $u(t) \geq r$ and $v(t) \geq r$ for every $t \in [0, T]$, which is a Nash-type equilibrium of the pair of corresponding energy functionals.

Proof. The existence follows from Theorem 5.3 and the uniqueness is a consequence of the Perov contraction property of the operator N. $\square$

In particular, if $a_1 = a_2 =: a$ (when $c_1 = c_2 =: c$) and $\alpha_1 = \alpha_2 =: \alpha$, the assumptions of Theorem 5.4 reduce to the following ones:

$$\gamma_i < \frac{1}{2Tc^2}, \quad r \leq \frac{\alpha^2}{(a^2 - \min\{\gamma_1, \gamma_2\})^2}$$

and

$$4\left(a^4-\gamma_1\gamma_2\right)r-4\alpha a^2\sqrt{r}+\alpha^2>0$$

(the condition for M to satisfy (5.10)). We may choose

$$r=\frac{\alpha^2}{\left(a^2-\min\left\{\gamma_1,\gamma_2\right\}\right)^2}$$

if

$$\min\left\{\gamma_1,\gamma_2\right\}>2\sqrt{\gamma_1\gamma_2}-a^2.$$

Numerical example. Theorem 5.4 applies in particular if $a=T=1$ (when $c=\sqrt{2}$), $\alpha=2$, $\gamma_1=1/5$, $\gamma_2=1/6$ and $r=5.76$.

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