J. Math. Anal. Appl. 474 (2019) 733–747

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Boundedness of solutions to a quasilinear parabolic–parabolic chemotaxis model with nonlinear signal production

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ARTICLE INFO

Article history:
Received 16 October 2018
Available online 1 February 2019
Submitted by P. Yao

Keywords:
Chemotaxis
Nonlinear diffusion
Signal production
Global boundedness

ABSTRACT

This work is concerned with a quasilinear parabolic–parabolic chemotaxis model with nonlinear signal production: $u_t = \nabla \cdot ((1+u)^{-\alpha}\nabla u) - \nabla \cdot (u(1+u)^{\beta-1}\nabla v) + f(u), v_t = \Delta v - v + u^{\gamma}$, with nonnegative initial data under homogeneous Neumann boundary conditions in a smooth bounded domain, where $\alpha, \beta \in \mathbb{R}$ and $\gamma > 0$. The logistic type source term $f(u)$ satisfies that either $f(u) \equiv 0$ or $f(u) = ru - \mu u^k$ with $r \in \mathbb{R}$, $\mu > 0$ and $k > 1$. The global-in-time existence and uniform-in-time boundedness of solutions are established under specific parameters conditions, which improves the known results.

1. Introduction

In this paper, we consider the chemotaxis model for two coupled parabolic equations

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v) + f(u), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + g(u), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with smooth boundary, $\frac{\partial}{\partial \nu}$ denotes differentiation with respect to the outward normal ν on $\partial\Omega$. The initial functions u_0 and v_0 are assumed to be nonnegative.

Systems of this type were initially introduced by Keller and Segel [12,13] to describe the aggregation of cells (bacteria). More specifically, the movement of cells is (partially) directed by the concentration gradient of a signal substance which is produced by cells themselves. This phenomenon, also referred to as chemotaxis, plays an important role in various biological processes such as bacterial aggregation, pattern formation and

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https://doi.org/10.1016/j.jmaa.2019.01.076
0022-247X/

cancer invasion (cf. [8] for a survey providing numerous biological examples). In system (1.1), u denotes the cell density and v is the concentration of the chemical signal. The positive function D represents the diffusivity of the cells, and the nonnegative function S measures the chemotactic sensitivity. The functions f and g are the growth of u and the production of v, respectively. In particular, the importance of both nonlinear diffusion $D(u)$ and nonlinear sensitivity $S(u)$ was initially emphasized in [18], where the authors proposed the presence of a so-called volume-filling effect.

It is an important question whether solutions remain bounded or blow up in finite/infinite time. In the case of linear signal production ($g(u) = u$), when the cell growth is neglected ($f \equiv 0$), Tao and Winkler [19] proved that solutions remain bounded under the condition that $\frac{S(u)}{D(u)} \le cu^\alpha$ with $\alpha < \frac{2}{n}$ and $c > 0$ for all $u > 1$, provided that Ω is a convex domain and $D(u)$ satisfies some other technical conditions. Afterwards, Ishida et al. [11] generalized the result obtained in [19] to non-convex domains. As to the case when logistic source $f(u)$ is emphasized, Zheng [28] considered problem (1.1) under the choices that $D(u) = (1 + u)^{-\alpha}$, $S(u) = u(1 + u)^{\beta-1}$ and $f(u) = r - \mu u^k$, it is shown that if $0 < \alpha + \beta < \max\{k - 1 + \alpha, \frac{2}{n}\}$, or $\beta = k - 1$ and μ is large enough, then all solutions are global and uniformly bounded. In addition, for researches on the corresponding parabolic–elliptic problems, we refer to [5,27] and the references therein.

In comparison to problems with linear signal production, studies on chemotaxis model (1.1) (as well as its parabolic–elliptic variant) with general signal production $g(u)$ are much less complete. When there is no logistic dampening, Liu and Tao [15] analyzed system (1.1) upon the particular choices that $D(u) \equiv 1$, $S(u) = u$ and $g(u) = u^\gamma$, they proved the global boundedness of solutions when $0 < \gamma < \frac{2}{n}$. The same conclusion is true for the parabolic–elliptic variant [24], where the second equation is replaced by $0 = \Delta v + g(u) - \frac{1}{|\Omega|}\int_\Omega g(u)$. Moreover, it is presented in [24] that if Ω is a ball and $\gamma > \frac{2}{n}$, then there exists initial data such that the corresponding radially symmetric solution blows up in finite time, hence $\gamma = \frac{2}{n}$ is critical. When the effect of logistic kinetics is taken into consideration, the parabolic–elliptic version related to problem (1.1) with linear diffusion $D(u) \equiv 1$ are investigated, we refer to Zheng et al. [29], Galakhov et al. [6], Hu and Tao [10], these works established global existence and boundedness of classical solutions. However, only few results concerning the fully parabolic system (1.1) have been found. In space dimension $n = 2$, the authors [30] obtained the global existence of bounded solutions under some technical conditions. In addition, for the studies on the asymptotic behavior of global solutions, we recommend the reader to see the recent papers [4,16,25].

Based on the above observations, the goal of the present work is to investigate the existence of global bounded solutions to the fully parabolic system (1.1). Throughout this paper, we assume that the initial data (u_0, v_0) satisfies

$$\begin{cases} u_0 \in C^0(\bar{\Omega}) & \text{is nonnegative with } \; u_0 \not\equiv 0, \\ v_0 \in C^1(\bar{\Omega}) & \text{is nonnegative.} \end{cases} \tag{1.2}$$

The functions $D, S \in C^2([0,\infty))$ fulfill $S(0) = 0$ and

$$d_0(1+u)^{-\alpha} \le D(u) \le d_1(1+u)^{-\alpha_1}, \quad 0 \le S(u) \le s_1 u(1+u)^{\beta-1} \tag{1.3}$$

for all $u \ge 0$ with some $d_0, d_1, s_1 > 0$ and $\alpha, \alpha_1, \beta \in \mathbb{R}$. Moreover, we assume that $f \in C^0([0,\infty))$ with $f(0) \ge 0$ and $g \in C^1([0,\infty))$ such that

$$f(u) \le ru - \mu u^k, \quad 0 \le g(u) \le g_1 u^\gamma \; \text{ for all } \; u \ge 0, \tag{1.4}$$

where $r \in \mathbb{R}$, $\mu, g_1, \gamma > 0$ and $k > 1$.

Under these hypotheses, our main results read as follows.

Theorem 1.1. *Let $n \geq 2$, $f \equiv 0$ and (u_0, v_0) satisfy (1.2). Suppose that D, S and g fulfill (1.3) and (1.4). If $0 < \gamma \leq 1$ and*

$$\alpha + \beta + \gamma < 1 + \frac{2}{n},$$

then problem (1.1) admits a nonnegative classical solution (u, v) which is globally bounded.

Theorem 1.2. *Let $n \geq 2$ and (u_0, v_0) satisfy (1.2). Suppose that D, S, f and g fulfill (1.3) and (1.4).*

(i) *If $\beta+\gamma < k$, then problem (1.1) admits a nonnegative classical solution (u, v) which is globally bounded.*
(ii) *Assume $\beta+\gamma = k$. Then there exists $\mu_0 > 0$ such that if $\mu \geq \mu_0$, then problem (1.1) admits a nonnegative classical solution (u, v) which is globally bounded.*

Remark 1. When $\gamma = 1$ and Ω is a bounded convex domain, Theorem 1.1 is consistent with Theorem 0.1 in [19]; when $\beta = \gamma = 1$ and $k = 2$, Theorem 1.2 coincide with Theorem 1 in [26].

Under the framework of Theorem 1.2, it is a natural question whether suitably strong logistic dampening can enforce the obtained global bounded solutions to furthermore stabilize toward homogeneous steady states. In fact, some precedent works give affirmative answers for models with $\gamma = 1$ [22,3]. We further underline that the asymptotic behavior of solutions to problem (1.1) remains unknown. According to [22,3,16], for the prototypical choices $f(u) = ru - \mu u^k$ and $g(u) = u^\gamma$, it is our conjecture that the bounded solution guaranteed by Theorem 1.2 has the property:

$$\|u(\cdot, t) - (\frac{r_+}{\mu})^{\frac{1}{k-1}}\|_{L^\infty(\Omega)} \to 0 \quad \text{and} \quad \|v(\cdot, t) - (\frac{r_+}{\mu})^{\frac{\gamma}{k-1}}\|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad t \to \infty,$$

provided that the logistic dampening is strong enough. On the other hand, for the local-in-time classical solution guaranteed by Lemma 2.1 below, it is also meaningful to detect the possibility to exceed carrying capacities to an arbitrary extent when logistic growth is sufficiently weak [23].

This paper is organized as follows. Section 2 gives the local existence of classical solutions to system (1.1) and presents some useful lemmas as preliminaries. In Section 3, without logistic source, we give a proof of Theorem 1.1. In Section 4, we take the effect of logistic source into account and prove the boundedness result exhibited in Theorem 1.2.

2. Preliminaries

To begin with, let us state a basic result on local existence of classical solutions without proof, see the details in [9,21,19,1] and [30].

Lemma 2.1. *Let $n \geq 1$ and (u_0, v_0) satisfy (1.2). Suppose that D, S, f and g fulfill (1.3) and (1.4). Then there exist $T_{max} \in (0, \infty]$ and a pair (u, v) of nonnegative functions from $C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))$ solving problem (1.1) classically. Moreover,*

$$\text{either } T_{max} = \infty, \quad \text{or } \limsup_{t \nearrow T_{max}} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) = \infty.$$

The following lemma shows some fundamental properties of solution (u, v) to problem (1.1) without logistic source.

Lemma 2.2. *Let $n \geq 1$, $f \equiv 0$ and (u_0, v_0) satisfy (1.2). Suppose that D, S and g fulfill (1.3) and (1.4), then the total mass of u is conserved in the sense that*

$$\|u(\cdot,t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad \text{for all } \ t \in (0, T_{max}). \tag{2.1}$$

Moreover, if $0 < \gamma \leq 1$, then for any $s \in [1, \frac{n}{(n\gamma-1)_+})$, there exists $C = C(s,\gamma) > 0$ such that

$$\|v(\cdot,t)\|_{W^{1,s}(\Omega)} \leq C \quad \text{for all } \ t \in (0, T_{max}), \tag{2.2}$$

where $(n\gamma - 1)_+ := \max\{n\gamma - 1, 0\}$.

Proof. The mass conservation property (2.1) can be derived by integrating the first equation in (1.1) over Ω. The assertion (2.2) follows from a method of Neumann semigroup estimates (cf. [14, Lemma 1]). □

The following lemma plays an important role in removing the requirement for convexity of domains, the proof of which can be found in [17, Lemma 4.2].

Lemma 2.3. *Let $n \geq 1$ and $\psi \in C^2(\bar{\Omega})$. If $\frac{\partial \psi}{\partial \nu} = 0$ on $\partial\Omega$, then*

$$\frac{\partial |\nabla \psi|^2}{\partial \nu} \leq 2\kappa_\Omega |\nabla \psi|^2 \quad \text{on } \partial\Omega,$$

where $\kappa_\Omega > 0$ is an upper bound for the curvatures of $\partial\Omega$.

We proceed to give a lemma referred to as a variation of maximal Sobolev regularity, as obtained in [11, Lemma 2.1] and [26, Lemma 2.2].

Lemma 2.4. *Let $n \geq 1$ and m satisfy $n < m < \infty$. Consider the following problem*

$$\begin{cases} z_t = \Delta z - z + w, & (x,t) \in \Omega \times (0,T), \\ \frac{\partial z}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,T), \\ z(x,0) = z_0(x), & x \in \Omega. \end{cases}$$

Then for each $z_0 \in W^{2,m}(\Omega)$ and any $w \in L^m(0,T; L^m(\Omega))$, there exists a unique strong solution

$$z \in W^{1,m}(0,T; L^m(\Omega)) \cap L^m(0,T; W^{2,m}(\Omega)).$$

Moreover, if $t_0 \in [0,T)$ satisfies $z(\cdot,t_0) \in W^{2,m}(\Omega)$ with $\frac{\partial z(\cdot,t_0)}{\partial \nu} = 0$ on $\partial\Omega$, then there exists $C(m) > 0$ such that

$$\begin{aligned} &\int_{t_0}^{T} \int_{\Omega} e^{m\tau} |\Delta z|^m dx d\tau \\ \leq &C(m) \int_{t_0}^{T} \int_{\Omega} e^{m\tau} w^m dx d\tau + C(m) e^{mt_0} (\|z(\cdot,t_0)\|^m_{L^m(\Omega)} + \|\Delta z(\cdot,t_0)\|^m_{L^m(\Omega)}). \end{aligned}$$

3. Global boundedness without logistic source

With $f \equiv 0$, the goal of this section is to establish uniform-in-time bounds for u and ∇v with respect to the norm in $L^p(\Omega)$ in quantitative for arbitrarily large p. To begin with, we adjust some parameters. If $0 < \gamma \le 1$ and $\alpha + \beta + \gamma < 1 + \frac{2}{n}$, then we can fix $s \in [1, \frac{n}{(n\gamma-1)_+})$ such that

$$\gamma - \frac{1}{n} < \frac{1}{s} < 1 + \frac{1}{n} - \alpha - \beta. \tag{3.1}$$

We choose some a, b fulfilling

$$1 < a < \min\{\frac{n}{n-2}, \frac{s}{(s-2)_+}\} \quad \text{and} \quad b > \max\{\frac{n}{2}, \frac{1}{2\gamma}\},$$

then there exist $\bar{p} > \max\{1 + \frac{n\alpha}{2}, 1 + \alpha - \alpha_1\}$ and $\bar{q} > 1 + \frac{s}{2}$ large enough satisfying

$$\begin{cases} \frac{n-2}{n}(1 + \frac{2|\alpha+\beta-1|}{\bar{p}-\alpha}) < \frac{1}{a} < \bar{p} + \alpha + 2\beta - 2, \\ \frac{n-2}{n\bar{q}} < 1 - \frac{1}{a}, \\ \frac{n-2}{n} \cdot \frac{2\gamma}{\bar{p}-\alpha} < \frac{1}{b}, \\ \bar{q} < \frac{\bar{p}-\alpha}{2} s, \end{cases}$$

and hence, it is easy to verify that

$$\frac{n-2}{n} \cdot \frac{p + \alpha + 2\beta - 2}{p - \alpha} < \frac{1}{a} < p + \alpha + 2\beta - 2, \tag{3.2}$$

$$1 - \frac{2}{s} < \frac{1}{a} < 1 - \frac{n-2}{nq}, \tag{3.3}$$

and

$$\frac{n-2}{n} \cdot \frac{2\gamma}{p-\alpha} < \frac{1}{b} < \frac{2}{n} + \frac{1}{q}(1 - \frac{2}{n}) \quad \text{and} \quad \frac{2b(q-1)}{b-1} > s \tag{3.4}$$

for all $p \ge \bar{p}$ and $q \ge \bar{q}$. We are now in the position to establish a key proposition as below.

Proposition 3.1. *Let $n \ge 2$, $f \equiv 0$ and (u_0, v_0) satisfy (1.2). Suppose that D, S and g fulfill (1.3) and (1.4). If $0 < \gamma \le 1$ and*

$$\alpha + \beta + \gamma < 1 + \frac{2}{n},$$

then for all $p \in [1, \infty)$ and each $q \in [1, \infty)$, there exists $C = C(p, q, \alpha, \alpha_1, \beta, \gamma) > 0$ such that

$$\|u(\cdot, t)\|_{L^p(\Omega)} \le C \quad \textit{and} \quad \|\nabla v(\cdot, t)\|_{L^q(\Omega)} \le C \quad \textit{for all } t \in (0, T_{max}).$$

Proof. Let $p \ge \bar{p}$ and $q \ge \bar{q}$. Define

$$\phi(z) := \int_0^z \int_0^\rho \frac{(1+\sigma)^{p-\alpha-2}}{D(\sigma)} d\sigma d\rho \quad \text{for } z \ge 0,$$

here (1.3) implies that ϕ is well-defined and nonnegative. Multiplying the first equation in (1.1) by $\phi'(u)$, integrating by parts over Ω, we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega \phi(u)dx &= -\int_\Omega \phi''(u)D(u)|\nabla u|^2dx + \int_\Omega \phi''(u)S(u)\nabla u\cdot\nabla v dx\\ &= -\int_\Omega (1+u)^{p-\alpha-2}|\nabla u|^2dx + \int_\Omega (1+u)^{p-\alpha-2}\frac{S(u)}{D(u)}\nabla u\cdot\nabla vdx\\ &\le -\int_\Omega (1+u)^{p-\alpha-2}|\nabla u|^2dx + \frac{s_1}{d_0}\int_\Omega (1+u)^{p+\beta-2}|\nabla u||\nabla v|dx\\ &\le -\frac{1}{2}\int_\Omega (1+u)^{p-\alpha-2}|\nabla u|^2dx + \frac{s_1^2}{2d_0^2}\int_\Omega (1+u)^{p+\alpha+2\beta-2}|\nabla v|^2dx, \end{aligned} \tag{3.5}$$

we have used (1.3) and Young's inequality in the last two inequalities. According to (3.5), we write

$$\frac{d}{dt}\int_\Omega \phi(u)dx + \frac{2}{(p-\alpha)^2}\int_\Omega |\nabla(1+u)^{\frac{p-\alpha}{2}}|^2dx \le \frac{s_1^2}{2d_0^2}\int_\Omega (1+u)^{p+\alpha+2\beta-2}|\nabla v|^2dx. \tag{3.6}$$

By a straightforward computation, we get $\Delta|\nabla v|^2 = 2|D^2v|^2 + 2\nabla v\cdot\nabla\Delta v$. Utilizing the second equation in (1.1) and the pointwise estimate $|\Delta v|^2 \le n|D^2v|^2$, we have

$$(|\nabla v|^2)_t + \frac{2}{n}|\Delta v|^2 + 2|\nabla v|^2 \le \Delta|\nabla v|^2 + 2\nabla v\cdot\nabla g(u). \tag{3.7}$$

Testing (3.7) by $|\nabla v|^{2(q-1)}$ and recalling Lemma 2.3, we can find some positive constant $C_1 = C_1(q)$ such that

$$\begin{aligned}&\frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q}dx + \frac{2}{n}\int_\Omega |\nabla v|^{2(q-1)}|\Delta v|^2dx + 2\int_\Omega |\nabla v|^{2q}dx\\ \le &\int_\Omega |\nabla v|^{2(q-1)}\Delta|\nabla v|^2dx + 2\int_\Omega |\nabla v|^{2(q-1)}\nabla v\cdot\nabla g(u)dx\\ = &-(q-1)\int_\Omega |\nabla v|^{2(q-2)}|\nabla|\nabla v|^2|^2dx + \int_{\partial\Omega} |\nabla v|^{2(q-1)}\frac{\partial|\nabla v|^2}{\partial\nu}dS\\ &-2(q-1)\int_\Omega |\nabla v|^{2(q-2)}\nabla|\nabla v|^2\cdot\nabla vg(u)dx - 2\int_\Omega |\nabla v|^{2(q-1)}\Delta vg(u)dx\\ \le &-\frac{q-1}{2}\int_\Omega |\nabla v|^{2(q-2)}|\nabla|\nabla v|^2|^2dx + 2\kappa_\Omega\int_{\partial\Omega} |\nabla v|^{2q}dS\\ &+\frac{2}{n}\int_\Omega |\nabla v|^{2(q-1)}|\Delta v|^2dx + (2(q-1)+\frac{n}{2})\int_\Omega |\nabla v|^{2(q-1)}g^2(u)dx\end{aligned}$$

$$\leq -\frac{2(q-1)}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 dx + 2\kappa_\Omega \int_{\partial\Omega} |\nabla v|^{2q} dS + \frac{2}{n}\int_\Omega |\nabla v|^{2(q-1)}|\Delta v|^2 dx$$
$$+ (2(q-1)+\frac{n}{2})\int_\Omega |\nabla v|^{2(q-1)} g^2(u) dx$$
$$\leq -\frac{q-1}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 dx + C_1\int_\Omega |\nabla v|^{2q} dx + \frac{2}{n}\int_\Omega |\nabla v|^{2(q-1)}|\Delta v|^2 dx$$
$$+ (2(q-1)+\frac{n}{2})\int_\Omega |\nabla v|^{2(q-1)} g^2(u) dx, \tag{3.8}$$

here we have used the trace inequality (cf. [7, Proposition 4.22, Proposition 4.24])

$$2\kappa_\Omega \|w\|^2_{L^2(\partial\Omega)} \leq \frac{q-1}{q^2}\|\nabla w\|^2_{L^2(\Omega)} + C_1\|w\|^2_{L^2(\Omega)}.$$

(3.8) along with (1.4) implies

$$\frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q} dx + \frac{q-1}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 dx$$
$$\leq g_1^2(2(q-1)+\frac{n}{2})\int_\Omega u^{2\gamma}|\nabla v|^{2(q-1)} dx + (C_1-2)\int_\Omega |\nabla v|^{2q} dx. \tag{3.9}$$

We combine (3.6) with (3.9) to see that

$$\frac{d}{dt}\int_\Omega (\phi(u)+\frac{1}{q}|\nabla v|^{2q}) dx + \frac{2}{(p-\alpha)^2}\int_\Omega |\nabla(1+u)^{\frac{p-\alpha}{2}}|^2 dx + \frac{q-1}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 dx$$
$$\leq C_2\int_\Omega (1+u)^{p+\alpha+2\beta-2}|\nabla v|^2 dx + C_2\int_\Omega |\nabla v|^{2q} dx$$
$$+ C_2\int_\Omega (1+u)^{2\gamma}|\nabla v|^{2(q-1)} dx, \tag{3.10}$$

where $C_2 = \max\{\frac{s_1^2}{2d_0^2}, g_1^2(2(q-1)+\frac{n}{2}), C_1-2\} > 0$. Since $a, b > 1$, we can use Hölder's inequality to estimate the integrals on the right hand side of (3.10) according to

$$\int_\Omega (1+u)^{p+\alpha+2\beta-2}|\nabla v|^2 dx \leq (\int_\Omega (1+u)^{(p+\alpha+2\beta-2)a} dx)^{\frac{1}{a}}(\int_\Omega |\nabla v|^{2a'} dx)^{\frac{1}{a'}}, \tag{3.11}$$

and

$$\int_\Omega (1+u)^{2\gamma}|\nabla v|^{2(q-1)} dx \leq (\int_\Omega (1+u)^{2\gamma b} dx)^{\frac{1}{b}}(\int_\Omega |\nabla v|^{2(q-1)b'} dx)^{\frac{1}{b'}}, \tag{3.12}$$

where $a' = \frac{a}{a-1} > 1$ and $b' = \frac{b}{b-1} > 1$. By virtue of (2.1) and (3.2), we employ the Gagliardo–Nirenberg inequality to get

$$
\begin{aligned}
(\int_\Omega (1+u)^{(p+\alpha+2\beta-2)a}dx)^{\frac{1}{a}} =& \|(1+u)^{\frac{p-\alpha}{2}}\|^{\frac{2(p+\alpha+2\beta-2)}{p-\alpha}}_{L^{\frac{2a(p+\alpha+2\beta-2)}{p-\alpha}}(\Omega)} \\
\le& C_3\|\nabla(1+u)^{\frac{p-\alpha}{2}}\|^{\frac{2(p+\alpha+2\beta-2)\theta}{p-\alpha}}_{L^2(\Omega)}\|(1+u)^{\frac{p-\alpha}{2}}\|^{\frac{2(p+\alpha+2\beta-2)(1-\theta)}{p-\alpha}}_{L^{\frac{2}{p-\alpha}}(\Omega)} \\
&+ C_3\|(1+u)^{\frac{p-\alpha}{2}}\|^{\frac{2(p+\alpha+2\beta-2)}{p-\alpha}}_{L^{\frac{2}{p-\alpha}}(\Omega)} \\
\le& C_4(\int_\Omega |\nabla(1+u)^{\frac{p-\alpha}{2}}|^2dx)^{\frac{p+\alpha+2\beta-2}{p-\alpha}\theta} + C_4,
\end{aligned}
\tag{3.13}
$$

where $\theta = \frac{\frac{p-\alpha}{2} - \frac{p-\alpha}{2a(p+\alpha+2\beta-2)}}{\frac{1}{n}-\frac{1}{2}+\frac{p-\alpha}{2}} \in (0,1)$, and C_3, C_4 are some positive constants depending on p, α and β. In view of (3.3), we use (2.2) and the Gagliardo–Nirenberg inequality once again to obtain

$$
\begin{aligned}
(\int_\Omega |\nabla v|^{2a'}dx)^{\frac{1}{a'}} =& \||\nabla v|^q\|^{\frac{2}{q}}_{L^{\frac{2a'}{q}}(\Omega)} \\
\le& C_5\|\nabla|\nabla v|^q\|^{\frac{2\delta}{q}}_{L^2(\Omega)}\||\nabla v|^q\|^{\frac{2(1-\delta)}{q}}_{L^{\frac{s}{q}}(\Omega)} + C_5\||\nabla v|^q\|^{\frac{2}{q}}_{L^{\frac{s}{q}}(\Omega)} \\
\le& C_6(\int_\Omega |\nabla|\nabla v|^q|^2dx)^{\frac{\delta}{q}} + C_6,
\end{aligned}
\tag{3.14}
$$

where $\delta = \frac{\frac{q}{s}+\frac{q}{2a}-\frac{q}{2}}{\frac{1}{n}-\frac{1}{2}+\frac{q}{s}} \in (0,1)$, $C_5 = C_5(q)$ and $C_6 = C_6(q,\gamma)$ are positive constants. Substituting (3.13), (3.14) into (3.11), we can find a positive constant $C_7 = C_7(p,q,\alpha,\beta,\gamma)$ fulfilling

$$
\begin{aligned}
&C_2\int_\Omega (1+u)^{p+\alpha+2\beta-2}|\nabla v|^2dx \\
\le& C_7(\int_\Omega |\nabla(1+u)^{\frac{p-\alpha}{2}}|^2dx)^{\frac{p+\alpha+2\beta-2}{p-\alpha}\theta}(\int_\Omega |\nabla|\nabla v|^q|^2dx)^{\frac{\delta}{q}} + C_7.
\end{aligned}
\tag{3.15}
$$

Similarly, in light of (3.4), Lemma 2.2 along with the Gagliardo–Nirenberg inequality indicates that

$$
\begin{aligned}
(\int_\Omega (1+u)^{2\gamma b}dx)^{\frac{1}{b}} =& \|(1+u)^{\frac{p-\alpha}{2}}\|^{\frac{4\gamma}{p-\alpha}}_{L^{\frac{4\gamma b}{p-\alpha}}(\Omega)} \\
\le& C_8(\int_\Omega |\nabla(1+u)^{\frac{p-\alpha}{2}}|^2dx)^{\frac{2\gamma\bar\theta}{p-\alpha}} + C_8,
\end{aligned}
$$

and

$$
(\int_\Omega |\nabla v|^{2(q-1)b'}dx)^{\frac{1}{b'}} = \||\nabla v|^q\|^{\frac{2(q-1)}{q}}_{L^{\frac{2(q-1)b'}{q}}(\Omega)} \le C_9(\int_\Omega |\nabla|\nabla v|^q|^2dx)^{\frac{(q-1)\bar\delta}{q}} + C_9
$$

with some positive constants $C_8 = C_8(p,\alpha,\gamma)$ and $C_9 = C_9(q,\gamma)$, where

$$
\bar\theta = \frac{\frac{p-\alpha}{2} - \frac{p-\alpha}{4\gamma b}}{\frac{1}{n}-\frac{1}{2}+\frac{p-\alpha}{2}} \in (0,1) \quad \text{and} \quad \bar\delta = \frac{\frac{q}{s}+\frac{q}{2(q-1)b}-\frac{q}{2(q-1)}}{\frac{1}{n}-\frac{1}{2}+\frac{q}{s}} \in (0,1).
$$

Then it follows from (3.12) that

$$
\begin{aligned}
&C_2\int_\Omega (1+u)^{2\gamma}|\nabla v|^{2(q-1)}dx\\
\le& C_{10}\Big(\int_\Omega |\nabla(1+u)^{\frac{p-\alpha}{2}}|^2dx\Big)^{\frac{2\gamma\bar\theta}{p-\alpha}}\Big(\int_\Omega |\nabla|\nabla v|^q|^2dx\Big)^{\frac{(q-1)\bar\delta}{q}}+C_{10},
\end{aligned}
\tag{3.16}
$$

where $C_{10}=C_{10}(p,q,\alpha,\gamma)>0$. Inserting (3.15) and (3.16) into (3.10) results in

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega (\phi(u)+\frac{1}{q}|\nabla v|^{2q})dx+\frac{2}{(p-\alpha)^2}\int_\Omega |\nabla(1+u)^{\frac{p-\alpha}{2}}|^2dx\\
&+\frac{q-1}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2dx\\
\le& C_7\Big(\int_\Omega |\nabla(1+u)^{\frac{p-\alpha}{2}}|^2dx\Big)^{\frac{p+\alpha+2\beta-2}{p-\alpha}\theta}\Big(\int_\Omega |\nabla|\nabla v|^q|^2dx\Big)^{\frac{\delta}{q}}\\
&+C_{10}\Big(\int_\Omega |\nabla(1+u)^{\frac{p-\alpha}{2}}|^2dx\Big)^{\frac{2\gamma\bar\theta}{p-\alpha}}\Big(\int_\Omega |\nabla|\nabla v|^q|^2dx\Big)^{\frac{(q-1)\bar\delta}{q}}\\
&+C_2\int_\Omega |\nabla v|^{2q}dx+C_7+C_{10}.
\end{aligned}
\tag{3.17}
$$

If

$$
\frac{p+\alpha+2\beta-2}{p-\alpha}\theta+\frac{\delta}{q}<1 \quad\text{and}\quad \frac{2\gamma\bar\theta}{p-\alpha}+\frac{(q-1)\bar\delta}{q}<1,
\tag{3.18}
$$

then applying Young's inequality to (3.17), we can find $C_{11}=C_{11}(p,q,\alpha,\beta,\gamma)>0$ such that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega (\phi(u)+\frac{1}{q}|\nabla v|^{2q})dx+\frac{1}{(p-\alpha)^2}\int_\Omega |\nabla(1+u)^{\frac{p-\alpha}{2}}|^2dx\\
&+\frac{q-1}{2q^2}\int_\Omega |\nabla|\nabla v|^q|^2dx\le C_2\int_\Omega |\nabla v|^{2q}dx+C_{11}.
\end{aligned}
\tag{3.19}
$$

In order to obtain (3.18), we define

$$
\begin{aligned}
h(q)&:=\frac{p+\alpha+2\beta-2}{p-\alpha}\theta+\frac{\delta}{q}=\frac{\frac{p+\alpha+2\beta-2}{2}-\frac{1}{2a}}{\frac1n-\frac12+\frac{p-\alpha}{2}}+\frac{\frac1s+\frac{1}{2a}-\frac12}{\frac1n-\frac12+\frac qs},\\
\bar h(q)&:=\frac{2\gamma\bar\theta}{p-\alpha}+\frac{(q-1)\bar\delta}{q}=\frac{\gamma-\frac{1}{2b}}{\frac1n-\frac12+\frac{p-\alpha}{2}}+\frac{\frac{q-1}{s}+\frac{1}{2b}-\frac12}{\frac1n-\frac12+\frac qs}.
\end{aligned}
$$

Thanks to (3.1), we know that

$$
h(q(p))<1 \quad\text{and}\quad \bar h(q(p))<1,
$$

where $q(p):=\frac{p-\alpha}{2}s$. By a continuity argument, for any $p\ge\bar p$, there exists $q\in[\bar q,q(p))$ close to $q(p)$ satisfying

$$
h(q)<1 \quad\text{and}\quad \bar h(q)<1,
$$

which together with the fact that $q(p) \to +\infty$ as $p \to \infty$ guarantees (3.18) for all $p \geq \bar{p}$ and $q \geq \bar{q}$. Recalling the definition of ϕ, we have from (1.3) and the Gagliardo–Nirenberg inequality that

$$\begin{aligned}\int_\Omega \phi(u)dx &\leq \frac{1}{d_0 p(p-1)} \int_\Omega (1+u)^p dx \\ &= \frac{1}{d_0 p(p-1)} \|(1+u)^{\frac{p-\alpha}{2}}\|_{L^{\frac{2p}{p-\alpha}}(\Omega)}^{\frac{2p}{p-\alpha}} \\ &\leq C_{12}(\int_\Omega |\nabla(1+u)^{\frac{p-\alpha}{2}}|^2 dx)^{\frac{p\sigma}{p-\alpha}} + C_{12}, \end{aligned} \tag{3.20}$$

where $\sigma = \frac{\frac{p-\alpha}{2} - \frac{p-\alpha}{2p}}{\frac{1}{n} - \frac{1}{2} + \frac{p-\alpha}{2}} \in (0,1)$ and $C_{12} = C_{12}(p,\alpha) > 0$. By the same argument, there exist some positive constants C_{13}, C_{14} depending on q and γ such that

$$\begin{aligned}(\frac{1}{q} + C_2) \int_\Omega |\nabla v|^{2q} dx &= (\frac{1}{q} + C_2) \||\nabla v|^q\|_{L^2(\Omega)}^2 \\ &\leq C_{13}(\int_\Omega |\nabla|\nabla v|^q|^2 dx)^{\bar{\sigma}} + C_{13} \\ &\leq \frac{q-1}{2q^2} \int_\Omega |\nabla|\nabla v|^q|^2 dx + C_{14}, \end{aligned} \tag{3.21}$$

where $\bar{\sigma} = \frac{\frac{q}{s} - \frac{1}{2}}{\frac{1}{n} - \frac{1}{2} + \frac{q}{s}} \in (0,1)$. It can be observed from (3.19)–(3.21) that

$$\frac{d}{dt} \int_\Omega (\phi(u) + \frac{1}{q}|\nabla v|^{2q})dx + C_{15}(\int_\Omega \phi(u)dx)^{\frac{p-\alpha}{p\sigma}} + \frac{1}{q} \int_\Omega |\nabla v|^{2q} dx \leq C_{16}$$

with some positive constants $C_{15} = C_{15}(p,\alpha)$ and $C_{16} = C_{16}(p,q,\alpha,\beta,\gamma)$, this allows us to have

$$\frac{d}{dt} \int_\Omega (\phi(u) + \frac{1}{q}|\nabla v|^{2q})dx + C_{17} \int_\Omega (\phi(u) + \frac{1}{q}|\nabla v|^{2q})dx \leq C_{18},$$

where $C_{17} = C_{17}(p,\alpha)$ and $C_{18} = C_{18}(p,q,\alpha,\beta,\gamma)$ are some positive constants. By a straightforward ODE comparison argument, we arrive at

$$\int_\Omega (\phi(u) + \frac{1}{q}|\nabla v|^{2q})dx \leq \max\{\int_\Omega (\phi(u_0) + \frac{1}{q}|\nabla v_0|^{2q})dx, \frac{C_{18}}{C_{17}}\} \quad \text{for all } t \in (0, T_{max}).$$

It follows from (1.3) that $(1+u)^{p+\alpha_1-\alpha} \leq C_{19}(\phi(u) + u + 1)$ with $C_{19} = C_{19}(p,\alpha,\alpha_1) > 0$, and hence, for any $p \geq \bar{p}$ and each $q \geq \bar{q}$, we can find $C_{20} = C_{20}(p,q,\alpha,\alpha_1,\beta,\gamma) > 0$ fulfilling

$$\int_\Omega (1+u)^{p+\alpha_1-\alpha} dx \leq C_{20} \quad \text{and} \quad \int_\Omega |\nabla v|^{2q} dx \leq C_{20} \quad \text{for all } t \in (0, T_{max}).$$

These end our proof. □

Proof of Theorem 1.1. Theorem 1.1 is an immediate consequence of Proposition 3.1 and [19, Lemma A.1]. □

4. Global boundedness with logistic source

In this section, we establish the global existence and boundedness of classical solutions to problem (1.1) with logistic source, and the following proposition is of great help to get the main result.

Proposition 4.1. *Let $n \geq 2$ and (u_0, v_0) satisfy (1.2). Suppose that D, S, f and g fulfill (1.3) and (1.4).*

(i) *If $\beta + \gamma < k$, then for any $p \in [1, \infty)$, there exists $C = C(p, \beta, \gamma, r, \mu, k) > 0$ such that*

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq C \quad \text{for all } t \in (0, T_{max}).$$

(ii) *Assume $\beta + \gamma = k$. Then for any $p \in [1, \infty)$, there exists $\mu_p = \mu_p(p, \beta, \gamma, \mu, k) > 0$ with the property that if $\mu \geq \mu_p$, then there exists $\tilde{C} = \tilde{C}(p, \beta, \gamma, r, \mu, k) > 0$ such that*

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq \tilde{C} \quad \text{for all } t \in (0, T_{max}).$$

Proof. We abbreviate $t_0 := \min\{1, \frac{1}{2}T_{max}\}$. Let $p > \max\{1, n(k - \beta) + 1 - k\}$ and $t \in (t_0, T_{max})$. We multiply the first equation in (1.1) by $(1 + u)^{p-1}$, then (1.3) and (1.4) entail that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega (1+u)^p dx \\
=& -(p-1)\int_\Omega (1+u)^{p-2}D(u)|\nabla u|^2 dx + (p-1)\int_\Omega (1+u)^{p-2}S(u)\nabla u \cdot \nabla v dx \\
&+ \int_\Omega (1+u)^{p-1}f(u)dx \\
\leq& (p-1)\int_\Omega (1+u)^{p-2}S(u)\nabla u \cdot \nabla v dx + r\int_\Omega (1+u)^{p-1}u dx - \mu\int_\Omega (1+u)^{p-1}u^k dx \\
\leq& (p-1)\int_\Omega (1+u)^{p-2}S(u)\nabla u \cdot \nabla v dx + r\int_\Omega (1+u)^{p-1}u dx \\
&- \frac{\mu}{2^{k-1}}\int_\Omega (1+u)^{p+k-1}dx + \mu\int_\Omega (1+u)^{p-1}dx \\
\leq& (p-1)\int_\Omega (1+u)^{p-2}S(u)\nabla u \cdot \nabla v dx + 2r_+\int_\Omega (1+u)^p dx \\
&- \frac{\mu}{2^{k-1}}\int_\Omega (1+u)^{p+k-1}dx + C_1,
\end{aligned} \tag{4.1}$$

where $C_1 = C_1(p, r, \mu) > 0$. Let

$$m := \frac{p+k-1}{k-\beta},$$

then $p > n(k - \beta) + 1 - k$ ensures $m > n$. It follows from (4.1) that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega (1+u)^p dx + \frac{m}{p}\int_\Omega (1+u)^p dx \\
\le&(p-1)\int_\Omega (1+u)^{p-2}S(u)\nabla u\cdot\nabla v dx + (\frac{m}{p}+2r_+)\int_\Omega (1+u)^p dx \\
&-\frac{\mu}{2^{k-1}}\int_\Omega (1+u)^{p+k-1}dx + C_1 \\
\le&(p-1)\int_\Omega (1+u)^{p-2}S(u)\nabla u\cdot\nabla v dx - \frac{\mu}{2^k}\int_\Omega (1+u)^{p+k-1}dx + C_2 \qquad (4.2)
\end{aligned}$$

with some positive constant $C_2 = C_2(p, r, \mu, k)$. Define

$$\phi_1(z) := (p-1)\int_0^z (1+\sigma)^{p-2}S(\sigma)d\sigma \quad \text{for } z \ge 0.$$

The inequality $p > n(k-\beta) + 1 - k$ along with $k > \beta$ implies $p + \beta - 1 > 0$, we infer from (1.3) that

$$0 \le \phi_1(z) \le \frac{s_1(p-1)}{p+\beta-1}(1+z)^{p+\beta-1} \quad \text{for } z \ge 0.$$

Integrating by parts, we therefore get that

$$\begin{aligned}
&(p-1)\int_\Omega (1+u)^{p-2}S(u)\nabla u\cdot\nabla v dx \\
=&\int_\Omega \nabla\phi_1(u)\cdot\nabla v dx \\
\le&\frac{s_1(p-1)}{p+\beta-1}\int_\Omega (1+u)^{p+\beta-1}|\Delta v| dx \\
\le&\frac{\mu}{2^{k+1}}\int_\Omega (1+u)^{p+k-1}dx + C_3\int_\Omega |\Delta v|^{\frac{p+k-1}{k-\beta}}dx, \qquad (4.3)
\end{aligned}$$

where $C_3 = C_3(p, \beta, \mu, k) > 0$. Combining (4.2) with (4.3) yields

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega (1+u)^p dx + \frac{m}{p}\int_\Omega (1+u)^p dx \\
\le& -\frac{\mu}{2^{k+1}}\int_\Omega (1+u)^{p+k-1}dx + C_3\int_\Omega |\Delta v|^m dx + C_2,
\end{aligned}$$

this together with the variation-of-constants formula shows that

$$\begin{aligned}
&\frac{1}{p}\int_\Omega (1+u(x,t))^p dx \\
\le& -\frac{\mu}{2^{k+1}}\int_{t_0}^t\int_\Omega e^{-m(t-\tau)}(1+u)^{p+k-1}dxd\tau + C_3\int_{t_0}^t\int_\Omega e^{-m(t-\tau)}|\Delta v|^m dxd\tau
\end{aligned}$$

$$+ C_2 \int_{t_0}^{t} e^{-m(t-\tau)} d\tau + \frac{1}{p} e^{-m(t-t_0)} \int_\Omega (1+u(x,t_0))^p dx$$

$$\le -\frac{\mu}{2^{k+1}} \int_{t_0}^{t} \int_\Omega e^{-m(t-\tau)} (1+u)^{p+k-1} dx d\tau + C_3 \int_{t_0}^{t} \int_\Omega e^{-m(t-\tau)} |\Delta v|^m dx d\tau + C_4, \tag{4.4}$$

where $C_4 = \frac{C_2}{m} + \frac{1}{p}\int_\Omega (1+u(x,t_0))^p dx$. Recalling (1.4) and Lemma 2.4, it can be obtained from (4.4) that

$$\begin{aligned} &\frac{1}{p}\int_\Omega (1+u(x,t))^p dx \\ &\le -\frac{\mu}{2^{k+1}} \int_{t_0}^{t} \int_\Omega e^{-m(t-\tau)} (1+u)^{p+k-1} dx d\tau + C_5 \int_{t_0}^{t} \int_\Omega e^{-m(t-\tau)} g^m(u) dx d\tau \\ &\quad + C_5 e^{-m(t-t_0)} (\|v(\cdot,t_0)\|^m_{L^m(\Omega)} + \|\Delta v(\cdot,t_0)\|^m_{L^m(\Omega)}) + C_4 \\ &\le -\frac{\mu}{2^{k+1}} \int_{t_0}^{t} \int_\Omega e^{-m(t-\tau)} (1+u)^{p+k-1} dx d\tau \\ &\quad + C_5 g_1^m \int_{t_0}^{t} \int_\Omega e^{-m(t-\tau)} u^{m\gamma} dx d\tau + C_6, \end{aligned} \tag{4.5}$$

where $C_5 = C_5(p,\beta,\mu,k) > 0$ and $C_6 = C_5(\|v(\cdot,t_0)\|^m_{L^m(\Omega)} + \|\Delta v(\cdot,t_0)\|^m_{L^m(\Omega)}) + C_4$.

In the case of $\beta+\gamma < k$, we have $m\gamma < p+k-1$, then we can find a positive constant $C_7 = C_7(p,\beta,\gamma,\mu,k)$ fulfilling

$$C_5 g_1^m \int_{t_0}^{t} \int_\Omega e^{-m(t-\tau)} u^{m\gamma} dx d\tau \le \frac{\mu}{2^{k+1}} \int_{t_0}^{t} \int_\Omega e^{-m(t-\tau)} (1+u)^{p+k-1} dx d\tau + C_7.$$

In conjunction with (4.5), this results in

$$\frac{1}{p}\int_\Omega (1+u(x,t))^p dx \le C_6 + C_7 \quad \text{for all} \quad t \in (t_0, T_{max}).$$

In the case of $\beta+\gamma = k$, it can be easily verified that $m\gamma = p+k-1$. Define

$$\mu_p := 2^{k+1} C_5 g_1^m.$$

If $\mu \ge \mu_p$, then (4.5) enables us to conclude that

$$\frac{1}{p}\int_\Omega (1+u(x,t))^p dx \le C_6 \quad \text{for all} \quad t \in (t_0, T_{max}).$$

Hence we complete the proof. □

Proof of Theorem 1.2. We take $p_0 > \max\{1, n(k-\beta)+1-k\}$ large enough fulfilling (A.8)–(A.10) in [19]. Thanks to Proposition 4.1, if

$$\beta + \gamma < k,$$

or

$$\beta + \gamma = k \ \text{ and } \ \mu \geq \mu_0 := \mu_{p_0},$$

then we can find a positive constant $C_7 = C_7(\beta, \gamma, r, \mu, k)$ satisfying

$$\|u(\cdot, t)\|_{L^{p_0}(\Omega)} \leq C_7 \ \text{ for all } \ t \in (0, T_{max}),$$

this together with the $L^p - L^q$ estimates for the Neumann semigroup (cf. [20, Lemma 1.3], [2, Lemma 2.1]) warrants the existence of $q_1 > n + 2$ and $q_2 > \frac{n+2}{2}$ fulfilling

$$S(u(\cdot, t))\nabla v(\cdot, t) \in L^{q_1}(\Omega) \quad \text{and} \quad f(u(\cdot, t)) \in L^{q_2}(\Omega) \quad \text{for all } \ t \in (0, T_{max}).$$

By means of a Moser-type iteration [19, Lemma A.1], we arrive at

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_8 \quad \text{ for all } t \in (0, T_{max}),$$

where $C_8 = C_8(\beta, \gamma, r, \mu, k) > 0$. Now we fix $q_0 > n$ and apply a standard semigroup technique once again to see that

$$\|v(\cdot, t)\|_{W^{1,q_0}(\Omega)} \leq C_9 \quad \text{ for all } t \in (0, T_{max}),$$

where $C_9 = C_9(\beta, \gamma, r, \mu, k) > 0$. The boundedness of $\|v(\cdot, t)\|_{L^\infty(\Omega)}$ follows readily from the embedding theorem. These prove Theorem 1.2. $\square$

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Grant No. 11571020 and No. 11671021). The authors wish to thank the anonymous reviewers for many valuable comments and suggestions to improve the expressions.

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