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Solvability to some strongly degenerate parabolic problems

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A B S T R A C T

Nonlinear parabolic equations of "divergence form," $u_t = (\varphi(u)\,\psi(u_x))_x$, are considered under the assumption that the "material flux," $\varphi(u)\,\psi(v)$, is bounded for all values of arguments, u and v. In literature such equations have been referred to as "strongly degenerate" equations. This is due to the fact that the coefficient, $\varphi(u)\,\psi'(u_x)$, of the second derivative, u_{xx}, can be arbitrarily small for large value of the gradient, u_x. The "hyperbolic phenomena" (unbounded growth of space derivatives within a finite time) have been established in literature for solutions to Cauchy problem for the above-mentioned equations. Accordingly one can expect a correct statement of the initial-boundary value problem for such equations only under additional assumptions on the problem data. In this paper we describe several restrictions, under which the initial-boundary value problems for strongly degenerate parabolic equations are well-posed.

0. Introduction

The following Cauchy problem was studied in [5]:

$$u_t = \frac{\partial}{\partial x}\big(\varphi(u)\psi(u_x)\big), \tag{0.1}$$
$$u(x,0) = u_0(x),$$

where $\varphi(\xi)$ is a smooth and strictly positive function, and $\psi(s)$ is a smooth and odd function satisfying $\psi'(s) > 0$ for $s \in \mathbf{R}$ and

$$\lim_{s\to\infty} \psi(s) = \psi_\infty < \infty. \tag{0.2}$$

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It is easy to see that under the above assumptions the function $\psi'(s)$ is not uniformly bounded away from zero, or equation (0.1) is degenerate. Following [5], we refer to equation (0.1) as "strongly degenerate." Indeed, in some applications with governing equations of type (0.1) the term under the divergence operator in the right hand side, $\varphi(u)\,\psi(u_x)$, represents a material flux (see [2,3] for the reason to study such strongly degenerate equations). In many cases such a flux is supposed to increase unboundedly as $|u_x| \to \infty$, even in the case of degeneracy for large gradient. Therefore, so far as the boundedness of the flux, $|\varphi(u)\psi(u_x)| < C$, is established, it implies that the gradient is bounded as well, $|u_x| < C_1$. Classical solvability to the problem under consideration follows from an "a priori" estimate: $\sup(|u| + |u_x|) < C_2$ (see [9,10], for example). On the contrary, the flux $\varphi(u)\psi(u_x)$ in equation (0.1) is restricted by the condition (0.2) if the solution itself is bounded, $|u(x,t)| \leq C_0$. In a number of cases the estimate of solution in the uniform norm can be obtained through the maximum principle [12]. It was proved in [5] that the spatial derivative, $u_x(x,t)$, of the solution to Cauchy problem for (0.1) may grow to infinity within a finite time, whose fact gives rise, in particular, to the discontinuity of solution itself (cf. the shock waves in hyperbolic problems, or just the *hyperbolic phenomena* which mean that the space derivatives of the solution grow unboundedly within a finite time). Note that in [5] increasing initial profiles, $u_0(x)$, were considered.

Initial-boundary value problem for equation (0.1) in the simple case of $\varphi(u) \equiv 1$ with Dirichlet boundary conditions was studied in [13], and shown that it is well-posed for any smooth initial profile. In the case of Neumann type boundary conditions similar result was proved in [15] under certain additional assumptions. The so-called "Lyapunov Functionals" technique has been used in the proofs (see [16]). It was based on inequality

$$\sup_{x,t} |\psi(u_x)| \leq \sup_x |\psi(u_0')|, \tag{0.3}$$

which gives an estimate of the derivative, u_x, in case

$$\psi_{-\infty} := \lim_{s \to -\infty} \psi(s) = -\psi_\infty. \tag{0.4}$$

The fact

$$\psi(0) = 0 \tag{0.5}$$

was also essentially used. Note that re-scaling the problem by $\hat{\psi} = \psi + c$, we can easily achieve either symmetric property of the limits in (0.4) or vanishing property in (0.5), but in general only one of them.

In both cases of Dirichlet and Neumann type boundary conditions gradient of the solution could be evaluated by applying maximum principle to the derivative, $v(x,t) := u_x(x,t)$. "Lyapunov Functionals" technique, developed in [15] for the case $\varphi \equiv 1$ (see [4,16] for more details concerning such functionals), was then used in [14] to study more general type of function $\varphi(u)$. Namely, the estimate

$$\sup_{x,t} |\varphi(u)\psi(u_x)| \leq \sup_x |\varphi(u_0)\psi(u_0')| \tag{0.6}$$

was established for the solution to the problem with Dirichlet boundary conditions. Such an inequality, similar to estimate (0.3), gives the desired bound on the gradient, u_x, in a special case of "small variation of $\varphi(u)$."

Note that the cases $\varphi'(s) \equiv 0$ and $\varphi'(s) \neq 0$ *qualitatively* differ from each other. Let us discuss them in detail. Really, the estimate of the gradient, $|u_x(x,t)|$, plays a key role in studying the solvability for equations of type (0.1). In case the gradient is bounded, $|u_x| \leq M_1$ (note that the fact $|u| \leq M$ follows from the maximum principle in the case of Dirichlet boundary conditions for equation (0.1)), the equation may be regarded as a regular, *non degenerate* parabolic equation. Classical solvability of different boundary value problems in such cases has been well studied upon the proper smoothness conditions (see, e.g., [9–12,16]).

Estimate for the gradient, $|u_x| \le M_1$ (under the assumption that the solution itself is bounded as well, $|u| \le M$), was proved for a solution to a more general quasilinear equation

$$u_t = a(x, u, u_x)\, u_{xx} + b(x, u, u_x), \tag{0.7}$$

when the non-linear terms satisfy the famous "quadratic growth condition." For the case of one spatial dimension under study, the above-mentioned growth condition takes the form (see [9,10,12,16])

$$\left|\frac{b(x, u, u_x)}{a(x, u, u_x)}\right| \le C(u)\, f(u_x) \qquad \text{with} \qquad \int\limits_1^\infty \frac{s}{f(s)}\, \mathrm{d}s = \infty. \tag{0.8}$$

Remark 0.1. Roughly speaking, the function $f(s)$ in (0.8) has not more than quadratic growth at infinity. Obviously, for $f(s) = 1+s^{2+\alpha}$ the integral in (0.8) converges for any $\alpha > 0$. However, growth like $f(s) \sim s^2 \ln s$ is acceptable as well.

Violation of condition (0.8) may cause to give a "derivative blow-up" solution, *i.e., hyperbolic phenomenon* (see, e.g., [1,7,8,12,15]). Precisely, it has been proved that the bounded classical solution can not possess a finite derivative even within a certain time interval. In other words, the classical solution does not exist after a finite time.

Suppose that $\varphi(s) \equiv$ *constant* in (0.1), and compare it with its general form in (0.7). We observe that the function $b \equiv 0$, and therefore we can put $f(s) \equiv 1$ and the growth condition (0.8) is fulfilled. Alternatively, for the more general case $\varphi' \ne 0$ in (0.1) we have

$$\frac{b(x, u, u_x)}{a(x, u, u_x)} = \frac{\varphi'(u)\psi(u_x)u_x}{\varphi(u)\psi'(u_x)},$$

and the integral in (0.8) is finite in view of (0.2):

$$\int\limits_1^\infty \frac{s\psi'(s)}{s\psi(s)}\, \mathrm{d}s = \int\limits_1^\infty \mathrm{d}\,(\ln \psi(s)) < \infty.$$

Thus, the growth condition in (0.8) fails.

The present paper is organized as follows. In Section 1 we consider the case that the multiplier $\varphi(x, u, u_x)$ is not the subject of differentiation; besides, the maximum principle is not applicable in estimating the derivative u_x. Global-in-time solvability theorem is established under certain conditions. In Section 2 we give a precise statement of the problems for equation (0.1) to study in the following sections. In Section 3 we prove the "global-in-time" existence theorem for a sufficiently small initial profile. In Section 4 solvability to the problem under study is established for arbitrary initial datum under certain restrictions on the nonlinear term $\psi(u_x)$ in (0.1). Finally, in Section 5 a condition for the existence theorem is formulated in terms of the so-called generalized distance between the initial profile and a set of singular solutions to the corresponding ordinary differential equation.

1. Solvability of degenerate equation in non-divergence form

In this section in a rectangle $Q_T = \{(x, t) \in (a_0, a_1) \times (0, T)\}$ we consider the following problem

$$u_t = \varphi(x, u, u_x)\, \frac{\partial}{\partial x}\left[\psi(u_x) + \varepsilon f(x)\right], \tag{1.1}$$

$$\alpha_i u_x - h_i(u)|_{x=a_i} = 0 \quad (i = 0, 1), \tag{1.2}$$

$$u(x, 0) = u_0(x). \tag{1.3}$$

All of the functions in (1.1)–(1.3) are supposed to be smooth enough, say f, φ, ψ, h_i, and $u_0 \in C^4$; $\varepsilon > 0$ is a constant. Constants α_i $(i = 0, 1)$ in (1.2) are supposed to take values 0 or 1, $\alpha_i \in \{0, 1\}$. Moreover, the boundary conditions in (1.2) are supposed to satisfy relations $\alpha_i^2 + [h_i'(s)]^2 \neq 0$ (to have "non trivial" boundary conditions). In case $\alpha_i = 0$ we suppose that the corresponding boundary condition is of Dirichlet type, $u|_{x=a_i} = constant$ (this assumption is done in order to exclude the "absurd cases," like $\exp(u)|_{x=a_i} = 0$, e.g.). Note that under the above assumptions, all of the classical boundary conditions, namely of Dirichlet, Neumann, and the "flux" type, are admissible. In the following we also assume that the compatibility conditions (see [12] for details) among the initial and boundary conditions are fulfilled.

Nonlinearities (for both smooth functions) are supposed to be as:
$\varphi(x, u, u_x) \geq \delta = constant > 0$, and $\psi(s)$ is a monotone increasing

$$\psi'(s) > 0 \tag{1.4}$$

and bounded function with "symmetric limits,"

$$-\lim_{s\to-\infty} \psi(s) = -\psi_{-\infty} = \psi_\infty = \lim_{s\to\infty} \psi(s) \qquad (\psi_\infty \neq 0). \tag{1.5}$$

For any function $\psi(s)$ having the above properties there exists a unique zero, λ,

$$\psi(\lambda) = 0. \tag{1.6}$$

Let us introduce the following notation:

$$g_i(u) = \begin{cases} \lambda & \text{for} \quad \alpha_i = 0, \\ h_i(u) & \text{for} \quad \alpha_i = 1; \end{cases} \tag{1.7}$$

$$G_i(u) := \psi(g_i(u)) + \varepsilon f(a_i), \tag{1.8}$$

and the sequence of functionals

$$J_n(u) := \frac{1}{m} \int_{a_0}^{a_1} \int_0^u \left(G_0^{2n+1}(\xi) - G_1^{2n+1}(\xi)\right) \mathrm{d}\xi \, \mathrm{d}x$$

$$+ \int_{a_0}^{a_1} \left[\int_\lambda^{u_x} (\psi(\eta) + \varepsilon f(x))^{2n+1} \mathrm{d}\eta + \frac{u_x}{m} \left((x - a_1) G_0^{2n+1}(u) - (x - a_0) G_1^{2n+1}(u)\right) \right] \mathrm{d}x, \tag{1.9}$$

where $n \in \mathbf{N}$ and m is the length of spatial interval,

$$m := a_1 - a_0. \tag{1.10}$$

Obviously, for any smooth function $u(x, t)$ we have (see (1.8))

$$\frac{\mathrm{d}\, J_n}{\mathrm{d}t} = \int_{a_0}^{a_1} (\psi(u_x) + \varepsilon f(x))^{2n+1} u_{xt} \, \mathrm{d}x + \int_{a_0}^{a_1} \left(\frac{x - a_1}{m} G_0^{2n+1}(u) - \frac{x - a_0}{m} G_1^{2n+1}(u) \right) u_{xt} \, \mathrm{d}x$$

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$$+\frac{2n+1}{m}\int_{a_0}^{a_1}\left((x-a_1)G_0^{2n}(u)\psi'(g_0)g_0'(u)-(x-a_0)G_1^{2n}(u)\psi'(g_1)g_1'(u)\right)u_xu_t\,\mathrm{d}x$$

$$+\frac{1}{m}\int_{a_0}^{a_1}\left(G_0^{2n+1}(u)-G_1^{2n+1}(u)\right)u_t\,\mathrm{d}x. \tag{1.11}$$

Integrations by parts give the following identities:

$$\int_{a_0}^{a_1}(\psi(u_x)+\varepsilon f(x))^{2n+1}u_{xt}\,\mathrm{d}x=(\psi(u_x)+\varepsilon f(x))^{2n+1}u_t\Big|_{x=a_0}^{a_1}$$

$$-(2n+1)\int_{a_0}^{a_1}(\psi(u_x)+\varepsilon f(x))^{2n}\left(\psi'(u_x)\,u_{xx}+\varepsilon f'(x)\right)u_t\,\mathrm{d}x, \tag{1.12}$$

$$\frac{1}{m}\int_{a_0}^{a_1}\left((x-a_1)G_0^{2n+1}(u)-(x-a_0)G_1^{2n+1}(u)\right)u_{xt}\,\mathrm{d}x$$

$$=\left(\frac{x-a_1}{a_1-a_0}G_0^{2n+1}(u)-\frac{x-a_0}{a_1-a_0}G_1^{2n+1}(u)\right)u_t\Big|_{x=a_0}^{a_1}-\frac{1}{m}\int_{a_0}^{a_1}\left(G_0^{2n+1}(u)-G_1^{2n+1}(u)\right)u_t\,\mathrm{d}x$$

$$-\frac{2n+1}{m}\int_{a_0}^{a_1}\left((x-a_1)G_0^{2n}(u)\psi'(g_0)g_0'(u)-(x-a_0)G_1^{2n}(u)\psi'(g_1)g_1'(u)\right)u_xu_t\,\mathrm{d}x. \tag{1.13}$$

Note that the terms out of integrals in relations (1.12) and (1.13) possess the same values with opposite signs for any smooth function, $u(x,t)$, satisfying the boundary conditions in (1.2) (see (1.8)). Therefore, from (1.11)–(1.13) we obtain that, for a smooth solution, $u(x,t)$, to problem (1.1)–(1.3) all the functions $J_n(u)(t)$ do not increase with time:

$$\frac{\mathrm{d}\,J_n(u)}{\mathrm{d}t}=-(2n+1)\int_{a_0}^{a_1}(\psi(u_x)+\varepsilon f(x))^{2n}\frac{u_t^2}{\varphi(x,u,u_x)}\,\mathrm{d}x\le 0,$$

or, the inequalities

$$J_n(u(x,t))\le J_n(u_0(x)) \tag{1.14}$$

hold for all $n\in\mathbf{N}$.

Below we describe the basic idea how relation (1.14) leads to the estimate of the “flux,” $|\psi(u_x)+\varepsilon f(x)|<C$.

First, note that the functionals $J_n(u)$ in (1.9) can be represented in the form

$$J_n=\int_{a_0}^{a_1}\int_{\lambda}^{u_x}(\psi(\eta)+\varepsilon f(x))^{2n+1}\,\mathrm{d}\eta\,\mathrm{d}x+\int_0^u G_0^{2n+1}(\xi)\,\mathrm{d}\xi\Big|_{x=a_0}-\int_0^u G_1^{2n+1}(\xi)\,\mathrm{d}\xi\Big|_{x=a_1}. \tag{1.15}$$

Now we are going to evaluate from below the integral (with respect to x) term in (1.15).

For a given function $u(x,t)$ and any $t>0$ consider the following decomposition of the spatial interval $[a_0,a_1]=E^0\cup E^+\cup E^-$, where

$$\begin{cases} E^+(t) := \Big\{x \in [a_0, a_1] \;\Big|\; u_x > \lambda,\ \psi(u_x) + \varepsilon f(x) \geq 0\Big\}, \\ E^-(t) := \Big\{x \in [a_0, a_1] \;\Big|\; u_x < \lambda,\ \psi(u_x) + \varepsilon f(x) \leq 0\Big\}. \end{cases} \tag{1.16}$$

Remark 1.1. By the definition in (1.6) we have $\psi(u_x) > 0$ for $x \in E^+(t)$ and $\psi(u_x) < 0$ for $x \in E^-(t)$, so that either $\psi(u_x) = 0$ or $\operatorname{sgn}\psi(u_x)\operatorname{sgn} f(x) < 0$ for $x \in E^0(t)$. Therefore, the estimate

$$|\psi(u_x) + \varepsilon f(x)| \leq \varepsilon\, |f(x)| \tag{1.17}$$

holds for $x \in E^0(t)$.

By virtue of (1.16) and (1.6) it is easy to see that

$$\begin{aligned} &\int\limits_{E^+\cup E^-} \int\limits_{\lambda}^{u_x} (\psi(\eta) + \varepsilon f(x))^{2n+1} \frac{\psi'(\eta)}{\psi'(\eta)}\, \mathrm{d}\eta\, \mathrm{d}x \\ \geq\ & \frac{(\max \psi'(s))^{-1}}{2n+2} \left(\int\limits_{E^+\cup E^-} (\psi(u_x) + \varepsilon f(x))^{2n+2}\, \mathrm{d}x - \int\limits_{E^+\cup E^-} (\varepsilon f(x))^{2n+2}\, \mathrm{d}x \right). \end{aligned} \tag{1.18}$$

Similarly, in view of (1.17), we have

$$\left| \int\limits_{E^0} \int\limits_{\lambda}^{u_x} (\psi(\eta) + \varepsilon f(x))^{2n+1}\, \mathrm{d}\eta\, \mathrm{d}x \right| \leq C \int\limits_{E^0} (\varepsilon f(x))^{2n+2}\, \mathrm{d}x \tag{1.19}$$

with some constant $C > 0$. Both (1.18) and (1.19) yield that

$$\int\limits_{a_0}^{a_1} \int\limits_{\lambda}^{u_x} (\psi(\eta) + \varepsilon f(x))^{2n+1}\, \mathrm{d}\eta\, \mathrm{d}x \geq \frac{(\max \psi'(s))^{-1}}{2n+2} \int\limits_{a_0}^{a_1} (\psi(u_x) + \varepsilon f(x))^{2n+2}\, \mathrm{d}x - C \int\limits_{a_0}^{a_1} (\varepsilon f(x))^{2n+2}\, \mathrm{d}x \tag{1.20}$$

with some constant $C > 0$.

By similar reasons to the above we obtain that the estimate

$$\left| \int\limits_{a_0}^{a_1} \int\limits_{\lambda}^{u_0'} (\psi(\eta) + \varepsilon f(x))^{2n+1}\, \mathrm{d}\eta\, \mathrm{d}x \right| \leq \frac{(\min \psi'(s))^{-1}}{2n+2} \int\limits_{a_0}^{a_1} (\psi(u_0') + \varepsilon f(x))^{2n+2}\, \mathrm{d}x + C \int\limits_{a_0}^{a_1} (\varepsilon f(x))^{2n+2}\, \mathrm{d}x \tag{1.21}$$

holds, where the minimum is calculated for $|s| \leq \sup_x |u_0'(x)|$.

Using inequalities (1.20), (1.21), (1.14) and representation (1.15), we get

$$\begin{aligned} \int\limits_{a_0}^{a_1} (\psi(u_x) + \varepsilon f(x))^{2n+2}\, \mathrm{d}x \leq (2n+2)\, C \Bigg(& \int\limits_{a_0}^{a_1} (\psi(u_0') + \varepsilon f(x))^{2n+2}\, \mathrm{d}x \\ & + \int\limits_{a_0}^{a_1} (\varepsilon f(x))^{2n+2}\, \mathrm{d}x + \int\limits_{u}^{u_0} G_0^{2n+1}(\xi)\, \mathrm{d}\xi\, \Big|_{x=a_0} - \int\limits_{u}^{u_0} G_1^{2n+1}(\xi)\, \mathrm{d}\xi\, \Big|_{x=a_1} \Bigg), \end{aligned} \tag{1.22}$$

where the constant C depends on the data, but, which is important, does not depend on n. Extracting $2n+2$ root from both sides of (1.22), and then passing to the limit as $n \to \infty$, we obtain

$$\sup_{x,t} |\psi(u_x) + \varepsilon f(x)| \le \max\left\{ \sup_x |\psi(u_0') + \varepsilon f(x)|,\ \sup_{i,t} |G_i(u(a_i,t))|,\ \sup_x |\varepsilon f(x)| \right\}. \tag{1.23}$$

If the traces of solution $u|_{x=a_i}$ are bounded, then estimate (1.23) yields that under a certain restriction on the data the gradient, $u_x(x,t)$, is bounded too.

Theorem 1.1. *Assume that the initial profile $u_0(x)$ in (1.3) and the traces at the boundary of solution to problem (1.1)–(1.3) satisfy inequalities*

$$\sup_x |\psi(u_0'(x))| + 2\varepsilon \sup_x |f(x)| < \psi_\infty, \tag{1.24}$$

$$\sup_{i,t} |\psi(g_i(u(a_i,t))) + \varepsilon f(a_i)| + \varepsilon \sup_x |f(x)| < \psi_\infty. \tag{1.25}$$

Then the solution $u(x,t)$ to problem (1.1)–(1.3) satisfies the estimate

$$\sup_{x,t} |u_x(x,t)| \le M_1. \tag{1.26}$$

Remark 1.2. In special cases, depending on the boundary conditions in (1.2) and some properties of the solution, Theorem 1.1 can be formulated in a better way by evaluating the different terms in (1.22) or (1.23). Below we give some of such observations.

1) In case of Dirichlet boundary conditions, $\alpha_i = 0$, i.e., $u(a_i,t) = constant$ $(i = 0,1)$, it follows from (1.6)–(1.8) that

$$G_i(u) = \varepsilon f(a_i);$$

2) For the Neumann type conditions in (1.2), $\alpha_i = 1$, and $h_i(u) = H_i$, function $G_i(u)$ in (1.8) is nothing but

$$G_i(u) = \psi(H_i) + \varepsilon f(a_i);$$

3) Suppose that $u_x(x,t) \ge \lambda$ and $f(x) \ge 0$. Then, it follows from (1.6) that for all $t \ge 0$

$$|\psi(u_x) + \varepsilon f(x)| = \psi(u_x) + \varepsilon f(x);$$

4) Suppose that $u \ge 0$, $(-1)^i(h_i(u) - \lambda) \ge 0$, and $(-1)^i f(a_i) > 0$. Then we have

$$(-1)^i \int_0^u G_i^{2n+1}(\xi)\,\mathrm{d}\xi \ge 0.$$

Remark 1.3. Note that the estimate for the gradient in (1.26) in a number of cases can be obtained through the maximum principle. Indeed, let $\varepsilon = 0$ and $\varphi = \varphi(x,u)$ in (1.1). After formal differentiation of equation (1.1) we obtain that the gradient, $v(x,t) := u_x(x,t)$, solves the following equation:

$$v_t = \varphi(x,u)\psi'(v)v_{xx} + \varphi\psi''(v)v_x^2 + (\varphi_x + \varphi_u\, v)\,\psi'(v)v_x. \tag{1.27}$$

Any constant, $v = C$, satisfies equation (1.27). Therefore, the solution to Dirichlet problem for equation (1.27) remains bounded, which corresponds to the bounded spatial gradient of solution to Neumann problem for equation (1.1). Similar observation is true in the case of $\alpha_i = 1$ in (1.2) along with the “maximum principle

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inequalities," $(-1)^i h_i(u) \geq 0$ $(i = 0, 1)$. In this case the solution to equation (1.1) remains bounded, and therefore the gradient, $v = u_x$, is bounded as well through the maximum principle for equation (1.27).

Remark 1.4. It is to be noted that in the general case of equation (1.1) with $\varepsilon \neq 0$ and $\varphi = \varphi(x, u, u_x)$, maximum principle is not applicable to evaluate the gradient, $v = u_x$. Such a principle is effective (see [12]) only under the additional assumption that for all x and u the inequality

$$\varepsilon \left(f''\varphi + f'\varphi_x + f'\varphi_u u_x\right) u_x \geq -b_1 u_x^2 - b_2$$

holds with some non-negative constants b_1 and b_2.

Remark 1.5. Note that in case $\alpha_i = 1$ and $h_i'(u) \neq 0$ $(i = 0, 1)$ in (1.2) (the "flux type" boundary conditions) it is necessary to add a certain restriction on $h_i(u)$ for the global-in-time classical solvability. As is well known, "strong" enough "flux" through the boundary may cause blow-up solutions (see [6] and the references therein).

Remark 1.6. By the maximum principle in [12], the solution to problem (1.1)–(1.3) remains bounded, $\sup_{x,t} |u(x, t)| < M$, for suitable boundary conditions if

$$\varepsilon\varphi(x, u, 0) f'(x) u \geq -|u|\Phi(|u|) - b_2, \tag{1.28}$$

where $b_2 \geq 0$, and $\Phi(\tau)$ is a non-decreasing positive function of $\tau \geq 0$ satisfying

$$\int_1^\infty \frac{\mathrm{d}\tau}{\Phi(\tau)} = \infty.$$

Remark 1.7. As have been noted in [9,10], the estimate

$$\sup_{x,t} \left(|u| + |u_x|\right) \leq M_2 \tag{1.29}$$

for the solution to problem of type (1.1)–(1.3) leads to the existence theorem under the proper smoothness conditions on the data.

Therefore, in view of Remarks 1.6 and 1.7, Theorem 1.1 leads to solvability theorem in a number of cases. Note also that once the estimate of the gradient, $u_x(x, t)$, is established, the equation under study could be regarded *non-degenerate*. Hence, the unique classical solvability follows. Below we formulate such a result.

Theorem 1.2. *There exists a global-in-time unique classical solution to problem* (1.1)–(1.3) *if one of the following conditions is fulfilled:*

I: $\alpha_0 = \alpha_1 = 1$ *and* $h_0 = h_1 = 0$ *in* (1.2), $\lambda = 0$ *in* (1.6), $\varepsilon f'(x) \geq 0$, $u_0'(x) \geq 0$, *inequality* (1.28) *holds, and*

$$\sup_x \psi(u_0'(x)) + \varepsilon \left(\sup_x f(x) - \inf_x f(x)\right) < \psi_\infty.$$

II: *The initial profile* $u_0(x)$ *in* (1.3) *satisfies inequality* (1.24) *and one of the options below is fulfilled:*

1) $\alpha_0 = \alpha_1 = 0$ *in* (1.2);

2) *$\alpha_0 = \alpha_1 = 1$ and $h_i = H_i$ in (1.2), inequality (1.28) holds, and*

$$\sup_{i,t} |\psi(g_i(H_i)) + \varepsilon f(a_i)| + \varepsilon \sup_x |f(x)| < \psi_\infty; \tag{1.30}$$

3) *inequalities $\varepsilon f'(x)\,\varphi_u < 0$, $(-1)^i(h_i(u) - \lambda) \geq 0$, $(-1)^i f(a_i) \geq 0$, and $u_0 \geq 0$ hold along with (1.28), and*

$$\sup_i |\psi(g_i(u_0(a_i))) + \varepsilon f(a_i)| + \varepsilon \sup_x |f(x)| < \psi_\infty; \tag{1.31}$$

4) *inequalities (1.28) and (1.25) hold.*

Remark 1.8. Note that case **II**, 1) in Theorem 1.2 gives the unique classical solvability without any reference to the maximum principle. Boundedness of the solution, $\sup_{x,t} |u| < M$, follows from the estimate of the gradient, $\sup_{x,t} |u_x| < M_1$, by integration.

As was already mentioned, it is sufficient to establish estimate (1.29) to prove any statement of Theorem 1.2 (see [9,10], e.g.). Thus, we just give brief comments. In the case of **I** comparison theorem gives $u_x \geq 0$, so that from 3) of Remark 1.2 we rewrite (1.23) as

$$\sup_{x,t} \psi(u_x) - \varepsilon \inf_x f(x) \leq \sup_x \psi(u_0') + \varepsilon \sup_x f(x),$$

which gives (1.26), and, in view of Remark 1.6, (1.29) as well.

Similarly, in order to get (1.26) we make use of the following arguments. In case **II**, 1) we use (1.25), improved by 1) in Remark 1.2, to get (1.26) with the help of (1.23). Estimate $\sup_{x,t} |u| < M$ follows from (1.26) by integration due to the boundary conditions. In case **II**, 2) inequality (1.25) in Theorem 1.1 is replaced by (1.30), while for **II**, 3) we have $u(x,t) \geq 0$ through the comparison principle. Then, we use 4) in Remark 1.2 to evaluate the last two terms of the right hand side in (1.22). Therefore, we replace (1.25) in Theorem 1.1 with (1.31).

2. Equation in divergence form: statement of the problem

In a rectangle $Q_T = \{(x,t) \in (a_0, a_1) \times (0,T)\}$, for equation

$$u_t = \frac{\partial}{\partial x}\big(\varphi(u)\psi(u_x)\big) = \varphi(u)\,\psi'(u_x)\,u_{xx} + \varphi'(u)\,\psi(u_x)u_x \tag{2.1}$$

we consider one of the following boundary conditions:

$$u\,|_{x=a_0} = u\,|_{x=a_1} = A; \tag{2.2}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=a_0} = \left.\frac{\partial u}{\partial x}\right|_{x=a_1} = 0; \tag{2.3}$$

$$(u, u_x)\,|_{x=a_0} = (u, u_x)\,|_{x=a_1}; \tag{2.4}$$

and the initial condition

$$u(x,0) = u_0(x). \tag{2.5}$$

Remark 2.1. In the sequel we shall refer to problem (2.1), (2.2), (2.5) as Problem 2 (P2), while the case that boundary condition (2.3) or (2.4) replaces (2.2) will be referred to as Problem 3 or 4 (P3 or P4) in turn.

All the data are supposed to be sufficiently smooth, *i.e.*, $u_0 \in C^{3+\alpha}([a_0, a_1])$ with $\alpha \in (0, 1)$ and $\varphi,\ \psi \in C^4(\mathbf{R})$. Moreover, the initial profile u_0 is assumed to satisfy the compatibility conditions of zero-th and first orders with the boundary data for all cases under study (see [12], for instance).

Furthermore, we assume that the function $\varphi(s)$ is strictly positive

$$\varphi(s) \geq \delta > 0, \tag{2.6}$$

and, in the case of Dirichlet boundary condition (2.2), satisfies an additional restriction

$$\varphi'(A) = 0; \tag{2.7}$$

the function $\psi(s)$ is monotonically increasing and bounded

$$\psi'(s) > 0, \qquad |\psi(s)| < C, \tag{2.8}$$

and satisfies $\lim_{s\to\infty} \psi'(s) = 0$ and $s^2\psi'(s),\ s^2\psi''(s)^2/\psi'(s) \leq C$ for $s \in \mathbf{R}$.

Therefore, equation (2.1) is degenerate. Moreover, for the bounded function $\psi(s)$ this equation is referred to as the *strongly degenerate* parabolic equation (see [5], for instance).

Inequality

$$s\,\psi(s) \geq 0 \quad \text{for} \quad s \in \mathbf{R} \tag{2.9}$$

is also required, which necessarily means that $\psi(0) = 0$.

For example, the function

$$\psi(s) = \arctan s \tag{2.10}$$

satisfies all the conditions mentioned above.

As has been well known (see [12], e.g.), all the problems above have unique classical solutions, local-in-time, since all the functions are assumed to be smooth and the compatibility conditions are fulfilled. However, as was proved in [5], a global (in time) classical solution to Cauchy problem for equation (2.1) with a monotone initial profile upon some additional assumptions does not exist.

Note that solutions $u(x, t)$ to all of the above problems satisfy the "strong maximum principle":

$$\min_{x\in[a_0,a_1]} u_0(x) \leq u(x, t) \leq \max_{x\in[a_0,a_1]} u_0(x). \tag{2.11}$$

In the sequel we shall also use the fact that for any type of boundary conditions in (2.2)–(2.4), and for any time t (assume that the classical solution to the corresponding problem exists) there is at least one point $x^*(t) \in [a_0, a_1]$ such that the spatial gradient vanishes:

$$u_x(x^*(t), t) = 0.$$

Therefore, at any time while the solution exists, inequality

$$u_x^2(x, t) \leq m \int\limits_{a_0}^{a_1} u_{xx}^2(x, t)\,\mathrm{d}x, \quad m = a_1 - a_0 \tag{2.12}$$

holds.

3. Solvability for small initial data

As we have already mentioned, solvability for the strongly degenerate equations necessarily requires additional conditions. These could be a sort of "smallness" (solvability for small time interval, solvability for small initial data, and so on), as is often used in the theory of parabolic equations. Here we provide results of such type. In addition, we establish some relations among the problem data (non-linear terms, initial profile), which compose sufficient conditions for the unique solvability of the problems under study.

In this section we establish some solvability, global-in-time, result for the problems under study with small initial functions, $u_0(x)$. In order to get a priori estimates necessary for that we shall multiply equation (2.1) by certain selected expressions, integrate with respect to a spatial variable, and integrate by parts. Note that all terms at the boundaries disappear for all of the considered types of boundary conditions in (2.2)–(2.4). We shall not indicate these separately throughout this section.

It is easy to see that the solutions to all problems above, P2, P3, and P4, satisfy identity

$$\int_{a_0}^{a_1} u_t\, u_x^{2n} u_{xx}\,\mathrm{d}x = \frac{1}{2n+1}\int_{a_0}^{a_1} u_t \frac{\partial}{\partial x} u_x^{2n+1}\,\mathrm{d}x = -\frac{1}{2n+1}\int_{a_0}^{a_1} u_x^{2n+1} u_{xt}\,\mathrm{d}x$$
$$= -\frac{1}{(2n+1)(2n+2)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{a_0}^{a_1} u_x^{2n+2}\,\mathrm{d}x.$$

Therefore, multiplying equation (2.1) by $u_x^{2n} u_{xx}$ and integrating with respect to x, we obtain

$$\frac{1}{(2n+1)(2n+2)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{a_0}^{a_1} u_x^{2n+2}\,\mathrm{d}x + \int_{a_0}^{a_1} \varphi(u)\psi'(u_x)u_x^{2n}u_{xx}^2\,\mathrm{d}x = -\int_{a_0}^{a_1} \varphi'(u)\psi(u_x)u_x^{2n+1}u_{xx}\,\mathrm{d}x. \tag{3.1}$$

Integration by parts leads to

$$\begin{aligned}
&-(2n+2)\int_{a_0}^{a_1} \varphi'(u)\psi(u_x)u_x^{2n+1}u_{xx}\,\mathrm{d}x\\
&= \int_{a_0}^{a_1} \varphi'(u)\psi'(u_x)u_x^{2n+2}u_{xx}\,\mathrm{d}x + \int_{a_0}^{a_1} \varphi''(u)\psi(u_x)u_x^{2n+3}\,\mathrm{d}x\\
&= -\frac{1}{2n+3}\left(\int_{a_0}^{a_1} \varphi''(u)\psi'(u_x)u_x^{2n+4}\,\mathrm{d}x + \int_{a_0}^{a_1} \varphi'(u)\psi''(u_x)u_x^{2n+3}u_{xx}\,\mathrm{d}x\right) + \int_{a_0}^{a_1} \varphi''(u)\psi(u_x)u_x^{2n+3}\,\mathrm{d}x\\
&=: I_1 + I_2.
\end{aligned} \tag{3.2}$$

Now we consider the case

$$\varphi''(s) \le 0 \qquad \text{for} \quad s \in [\min_x u_0(x), \max_x u_0(x)]. \tag{3.3}$$

In view of (2.11) it means that $\varphi''(u(x,t)) \le 0$, and therefore, together with (2.9), the first summand in the right side of (3.2) is not positive,

$$I_1 \le 0. \tag{3.4}$$

Second term I_2 can be evaluated as follows:

$$(2n+3)|I_2| \le \int_{a_0}^{a_1} \varphi(u)\psi'(u_x)u_x^{2n}u_{xx}^2\,\mathrm{d}x + C\int_{a_0}^{a_1} u_x^{2n+2}\,\mathrm{d}x, \tag{3.5}$$

where

$$C = \sup\left|\varphi''(u)\psi'(u_x)u_x^2\right| + \sup\left|\frac{\varphi'(u)^2\psi''(u_x)^2}{4\varphi(u)\psi'(u_x)}\,u_x^4\right|.$$

Using (3.2), (3.4) and (3.5), we derive from (3.1) that

$$\int_{a_0}^{a_1} u_x^{2n+2}\,\mathrm{d}x \le \int_{a_0}^{a_1} (u_0')^{2n+2}\,\mathrm{d}x + C\int_0^t\int_{a_0}^{a_1} u_x^{2n+2}\,\mathrm{d}x\,\mathrm{d}t,$$

and therefore, by Gronwall's lemma, inequality

$$\int_{a_0}^{a_1} u_x^{2n+2}\,\mathrm{d}x \le \left(1+\mathrm{e}^{CT}\right)\int_{a_0}^{a_1} \left(u_0'\right)^{2n+2}\,\mathrm{d}x \tag{3.6}$$

holds. Since the constant C in (3.6) does not depend on n (cf. (3.5)), we get

$$\sup_{(x,t)\in Q_T} |u_x(x,t)| \le \sup_{x\in[a_0,a_1]} |u_0'(x)|\,. \tag{3.7}$$

As has been already mentioned, both estimates (2.11) and (3.7) yield

Theorem 3.1. *In addition to the conditions of Section* 2*, under assumption* (3.3) *there exist global-in-time unique classical solutions to all the problems* P2, P3*, and* P4 *under study.*

Now we consider the case $\psi(s) = \arctan s$ in (2.1) (see (2.10)) and suppose that

$$\varphi''(s) \ge 0 \qquad \text{for} \quad s \in [\min_x u_0(x), \max_x u_0(x)] \tag{3.8}$$

(cf. (3.3)).

In order to use identities (3.1) with $n = 0$ and $n = 1$, let $\Psi_j(s)$ be the primitive function of $\psi(s)s^{2j+1}$ $(j = 0, 1)$. Obviously,

$$\begin{cases} \Psi_0(s) = \dfrac{1}{2}\left((s^2+1)\arctan s - s\right), \\[2ex] \Psi_1(s) = \dfrac{1}{4}\left((s^4-1)\arctan s - \dfrac{1}{3}s^3 + s\right). \end{cases} \tag{3.9}$$

Adding (3.1) with $n = 0$ multiplied by 1/2 to (3.1) with $n = 1$, we obtain

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_{a_0}^{a_1}\left(u_x^2 + \frac{1}{3}u_x^4\right)\mathrm{d}x + \int_{a_0}^{a_1}\varphi(u)\frac{\frac{1}{2}+u_x^2}{1+u_x^2}\,u_{xx}^2\,\mathrm{d}x = -\int_{a_0}^{a_1}\varphi'(u)\left(\frac{1}{2}\frac{\partial}{\partial x}\Psi_0(u_x) + \frac{\partial}{\partial x}\Psi_1(u_x)\right)\mathrm{d}x. \tag{3.10}$$

Using exact formulae in (3.9), after integration by parts we see that

$$\int_{a_0}^{a_1} \varphi'(u) \left(\frac{1}{2} \frac{\partial}{\partial x} \Psi_0(u_x) + \frac{\partial}{\partial x} \Psi_1(u_x) \right) \mathrm{d}x = -\frac{1}{4} \int_{a_0}^{a_1} \varphi''(u) \left[\left(u_x^5 + u_x^3 \right) \arctan u_x - \frac{1}{3} u_x^4 \right] \mathrm{d}x.$$

Therefore, by taking into account the trivial observation that

$$f(\xi) := \frac{\alpha + \xi}{1 + \xi} \geq \alpha \quad \text{for } \alpha \in (0,1) \text{ and } \xi \geq 0,$$

it follows from (3.10) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{a_0}^{a_1} \left(u_x^2 + \frac{1}{3} u_x^4 \right) \mathrm{d}x + 2 \int_{a_0}^{a_1} \varphi(u) u_{xx}^2 \,\mathrm{d}x + \frac{1}{3} \int_{a_0}^{a_1} \varphi''(u) u_x^4 \,\mathrm{d}x \leq \int_{a_0}^{a_1} \varphi''(u) \left(|u_x|^5 + |u_x|^3 \right) \mathrm{d}x. \tag{3.11}$$

Using Young's inequality

$$|u_x|^5 + |u_x|^3 \leq \lambda\, u_x^4 + \frac{1}{2\lambda} \left(u_x^6 + u_x^2 \right) \tag{3.12}$$

with $\lambda = 1/3$, and estimates in (2.12) and (3.8), from (3.11) we obtain that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{a_0}^{a_1} \left(u_x^2 + \frac{1}{3} u_x^4 \right) \mathrm{d}x + 2 \min_x \varphi(u) \int_{a_0}^{a_1} u_{xx}^2 \,\mathrm{d}x \\ &\leq \frac{3}{2} m^2 \max_x \varphi''(u) \left[m^2 \left(\int_{a_0}^{a_1} u_{xx}^2 \,\mathrm{d}x \right)^2 + 1 \right] \int_{a_0}^{a_1} u_{xx}^2 \,\mathrm{d}x. \end{aligned} \tag{3.13}$$

Inequality (3.13) leads to

$$\int_{a_0}^{a_1} \left(u_x^2 + \frac{1}{3} u_x^4 \right) \mathrm{d}x \leq \int_{a_0}^{a_1} \left((u_0')^2 + \frac{1}{3} (u_0')^4 \right) \mathrm{d}x,$$

if the relation

$$2 \min_x \varphi(u) \geq \frac{3}{2} m^2 \max_x \varphi''(u) \, (m^2 X^2 + 1), \qquad X := \int_{a_0}^{a_1} u_{xx}^2 \,\mathrm{d}x \tag{3.14}$$

holds.

Suppose now that the relation

$$2 \min_x \varphi(u_0) > \frac{3}{2} m^2 \max_x \varphi''(u_0) \tag{3.15}$$

holds. Then there exists a constant $X_0 > 0$ such that

$$2 \min_x \varphi(u_0) > \frac{3}{2} m^2 \max_x \varphi''(u_0) \, \left(m^2 X_0^2 + 1 \right). \tag{3.16}$$

Moreover, let the initial profile $u_0(x)$ in (2.5) satisfy

$$\int_{a_0}^{a_1} \left((u_0')^2 + \frac{1}{3} (u_0')^4 \right) \mathrm{d}x < \varepsilon \tag{3.17}$$

for a suitably small $\varepsilon > 0$ determined later. It is easy to see that inequality (3.17) yields

$$\max_x u_0 - \min_x u_0 \leq \sqrt{m\varepsilon},$$

$$|\min_x \varphi(u) - \min_x \varphi(u_0)|, \ |\max_x \varphi''(u) - \max_x \varphi''(u_0)| \leq K\sqrt{m\varepsilon}$$

for some positive constant K. Then the left hand side of (3.14) is bounded from below by $2\min_x \varphi(u_0) - 2K\sqrt{m\varepsilon}$, and the right hand side is bounded from above by $3m^2 \left(\max_x \varphi''(u_0) + K\sqrt{m\varepsilon}\right)(m^2X^2+1)/2$, so that by using (3.16) and by choosing $\varepsilon > 0$ in (3.17) such as

$$2\left(4+3m^2\right) K\sqrt{m\varepsilon} \leq 3m^4 \max_x \varphi''(u_0) X_0^2, \tag{3.18}$$

the inequality (3.14) holds for any $X = X(t) (> 0)$ satisfying

$$X \leq X_0 \left(\frac{4+3m^2}{2(4+3m^2)+3m^4X_0^2} \right)^{1/2}.$$

Thus, we have established that under smallness hypothesis (3.17) with ε being chosen according to (3.18) inequality

$$\int_{a_0}^{a_1} u_{xx}^2 \, \mathrm{d}x \leq C \tag{3.19}$$

is fulfilled.

Therefore, by (2.11), (2.12) we have (see Remark 1.7)

Theorem 3.2. *Consider equation (2.1) with $\psi(s) = \arctan s$. If the initial profile $u_0(x)$ in (2.5) satisfies relations (3.15) and (3.17) with $\varepsilon > 0$ being chosen according to (3.18), then there exist global-in-time unique classical solutions to all the problems* P2, P3, *and* P4.

4. Estimates of higher order derivatives

In this section we establish some solvability results for the problems under study which do not necessarily require smallness assumption on the initial function, $u_0(x)$. Besides, we introduce alternative hypotheses that some quantities are small. Criteria stated below can be achieved either by small length of the spatial interval m, or through the small values of derivatives $\varphi'(s)$, $\varphi''(s)$, and $\varphi'''(s)$ in comparison with $\varphi(s)$ itself, along with some other problem parameters.

In the rest of this section we assume that in the case of Neumann type boundary conditions (2.3), the function $\psi(s)$, in addition to all restrictions mentioned above, satisfies the identities

$$\psi(0) = \psi''(0) = 0. \tag{4.1}$$

In the special case of (2.10) both relations (4.1) are satisfied. In the case of Dirichlet type boundary conditions (2.2), identity (2.7) is also assumed. Note that the solution to problem (2.1), (2.2), (2.5) is considered to be smooth enough, and hence such a solution satisfies additional boundary conditions

$$u_{xx}|_{x=a_0} = u_{xx}|_{x=a_1} = 0, \tag{4.2}$$

which directly follow from equation (2.1) under the proper smoothness of $u(x,t)$.

Again, the additional requirements (4.1) and (4.2) imply that all the terms at the boundaries in the calculations below disappear. We shall not mention this fact separately.

Formal differentiation of equation (2.1) with respect to x gives

$$u_{xt} = \varphi\psi'\, u_{xxx} + \varphi\psi''\, u_{xx}^2 + 2\varphi'\psi'\, u_x u_{xx} + \varphi'\psi\, u_{xx} + \varphi''\psi\, u_x^2. \tag{4.3}$$

Multiply equation (4.3) by u_{xxx}, and integrate with respect to x. Noting the auxiliary calculations

$$\int_{a_0}^{a_1} u_{xxx}u_{xt}\,\mathrm{d}x = -\int_{a_0}^{a_1} u_{xx}u_{xxt}\,\mathrm{d}x = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{a_0}^{a_1} u_{xx}^2\,\mathrm{d}x,$$

$$-\int_{a_0}^{a_1} \varphi\psi''\, u_{xx}^2 u_{xxx}\,\mathrm{d}x = \frac{1}{3}\left(\int_{a_0}^{a_1} \varphi\psi'''\, u_{xx}^4\,\mathrm{d}x + \int_{a_0}^{a_1} \varphi'\psi''\, u_x u_{xx}^3\,\mathrm{d}x\right),$$

$$-2\int_{a_0}^{a_1} \varphi'\psi'\, u_x u_{xx} u_{xxx}\,\mathrm{d}x = \int_{a_0}^{a_1} \varphi'\psi''\, u_x u_{xx}^3\,\mathrm{d}x + \int_{a_0}^{a_1} \varphi'\psi'\, u_{xx}^3\,\mathrm{d}x + \int_{a_0}^{a_1} \varphi''\psi'\, u_x^2 u_{xx}^2\,\mathrm{d}x,$$

$$-\int_{a_0}^{a_1} \varphi'\psi\, u_{xx}u_{xxx}\,\mathrm{d}x = \frac{1}{2}\left(\int_{a_0}^{a_1} \varphi'\psi'\, u_{xx}^3\,\mathrm{d}x + \int_{a_0}^{a_1} \varphi''\psi\, u_x u_{xx}^2\,\mathrm{d}x\right),$$

and

$$-\int_{a_0}^{a_1} \varphi''\psi\, u_x^2 u_{xxx}\,\mathrm{d}x = \int_{a_0}^{a_1} \varphi''\psi'\, u_x^2 u_{xx}^2\,\mathrm{d}x + 2\int_{a_0}^{a_1} \varphi''\psi\, u_x u_{xx}^2\,\mathrm{d}x + \int_{a_0}^{a_1} \varphi'''\psi\, u_x^3 u_{xx}\,\mathrm{d}x,$$

we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{a_0}^{a_1} u_{xx}^2\,\mathrm{d}x + \int_{a_0}^{a_1} \varphi\psi'\, u_{xxx}^2\,\mathrm{d}x \\
&= \frac{1}{3}\int_{a_0}^{a_1} \varphi\psi'''\, u_{xx}^4\,\mathrm{d}x + \int_{a_0}^{a_1} \left(\frac{4}{3}\varphi'\psi''\, u_x + \frac{3}{2}\varphi'\psi'\right) u_{xx}^3\,\mathrm{d}x \\
&+ \int_{a_0}^{a_1} \left(2\varphi''\psi'\, u_x^2 + \frac{5}{2}\varphi''\psi\, u_x\right) u_{xx}^2\,\mathrm{d}x + \int_{a_0}^{a_1} \varphi'''\psi\, u_x^3 u_{xx}\,\mathrm{d}x.
\end{aligned} \tag{4.4}$$

Suppose that there exists an $\varepsilon > 0$ such that

$$\psi'''(u_x(x,t)) \le -\varepsilon. \tag{4.5}$$

In the case of particular function $\psi(s)$ of the form (2.10), (4.5) simply means that $|u_x(x,t)| < 1/\sqrt{3}$. In a general case, this assumption on function ψ implies that there exists an interval (s_1, s_2), on which $\psi'''(s) < 0$. Therefore, inequality (4.5) may be formulated in terms of some bounds on the gradient of the considered solution, $u_x(x,t)$.

Remark 4.1. The fact $\psi'''(s) < 0$ for $s \in (s_1, s_2)$ and (4.1) imply $s_1 < 0 < s_2$ and $\psi'''(s_1) = \psi'''(s_2) = 0$. In other words, $\psi''(s)$ is monotonically decreasing on $[s_1, s_2]$, and $s_1 < 0$ and $s_2 > 0$ are the first maximum and minimum points of $\psi''(s)$, respectively.

Now, using the Hölder inequality, we evaluate each term in the right side of (4.4) which does not contain u_{xx}^4. In some of the relations below we shall also use estimate (2.12). Then we have

$$\left| \int_{a_0}^{a_1} \left(\frac{4}{3}\varphi'\psi'' u_x + \frac{3}{2}\varphi'\psi' \right) u_{xx}^3 \,\mathrm{d}x \right| \le C_1 m^{1/4} \left(\int_{a_0}^{a_1} u_{xx}^4 \,\mathrm{d}x \right)^{3/4} \tag{4.6}$$

with

$$C_1 = \sup_x \left| \frac{4}{3}\psi'' u_x + \frac{3}{2}\psi' \right| \sup_x |\varphi'| \tag{4.7}$$

and

$$2 \left| \int_{a_0}^{a_1} \varphi''\psi' u_x^2 u_{xx}^2 \,\mathrm{d}x \right| \le C_2 \sqrt{m} \left(\int_{a_0}^{a_1} u_{xx}^4 \,\mathrm{d}x \right)^{1/2} \tag{4.8}$$

with

$$C_2 = 2 \sup_x \left| \psi' \, u_x^2 \right| \sup_x |\varphi''| \,. \tag{4.9}$$

Remark 4.2. For more general function ψ satisfying $\psi'(s) = O(s^{-2})$ ($|s| \to \infty$) constants C_1 and C_2 are bounded.

Similarly, in view of (2.12), it is easy to see that

$$\frac{5}{2} \left| \int_{a_0}^{a_1} \varphi''\psi u_x u_{xx}^2 \,\mathrm{d}x \right| \le C_3 \sqrt{m} \left(\int_{a_0}^{a_1} u_{xx}^2 \,\mathrm{d}x \right)^{3/2} \le C_3 \, m^{5/4} \left(\int_{a_0}^{a_1} u_{xx}^4 \,\mathrm{d}x \right)^{3/4} \tag{4.10}$$

with

$$C_3 = 3 \sup_x |\psi| \, \sup_x |\varphi''|, \tag{4.11}$$

and

$$\left| \int_{a_0}^{a_1} \varphi'''\psi u_x^3 u_{xx} \,\mathrm{d}x \right| \le C_4 \, m^2 \left(\int_{a_0}^{a_1} u_{xx}^2 \,\mathrm{d}x \right)^2 \le C_4 \, m^3 \left(\int_{a_0}^{a_1} u_{xx}^4 \,\mathrm{d}x \right) \tag{4.12}$$

with

$$C_4 = \sup_x |\psi| \, \sup_x |\varphi'''|. \tag{4.13}$$

It follows from (4.6)–(4.13), (2.12), (2.6) and (4.5) that the right hand side of (4.4) is bounded from above by

$$\left[\left(-\frac{\varepsilon\delta}{3}+C_4 m^3\right)\left(\int\limits_{a_0}^{a_1} u_{xx}^4\,\mathrm{d}x\right)^{1/2} + m^{1/4}\left(C_1+C_3 m\right)\left(\int\limits_{a_0}^{a_1} u_{xx}^4\,\mathrm{d}x\right)^{1/4} + C_2\sqrt{m}\right]\left(\int\limits_{a_0}^{a_1} u_{xx}^4\,\mathrm{d}x\right)^{1/2}. \quad (4.14)$$

In the case of $\int_{a_0}^{a_1} u_{xx}^4\,\mathrm{d}x < 1$, $\|u_{xx}\|_{L^2(a_0,a_1)} \le m^{1/4}$ holds; otherwise (4.14) is non-positive if

$$C_1\, m^{1/4} + C_2\, m^{1/2} + C_3\, m^{5/4} + C_4\, m^3 \le \frac{\varepsilon\delta}{3} \quad (4.15)$$

holds. In the latter case it immediately follows from (4.4) that $\|u_{xx}\|_{L^2(a_0,a_1)} \le \|u_0''\|_{L^2(a_0,a_1)}$. Therefore, we conclude by using (2.12) that

$$\sup_x |u_x(x,t)| \le \sqrt{m}\,\|u_{xx}\|_{L^2(a_0,a_1)} \le \sqrt{m}\,\max\left\{m^{1/4}, \|u_0''(x)\|_{L^2(a_0,a_1)}\right\}. \quad (4.16)$$

In case the inequality (4.5) holds for the values of gradient, specified in (4.16), we obtain classical solvability of the problem under study. The key requirement, (4.15), can be fulfilled either through the small length of the spatial interval m, or in case the derivatives $\varphi'(u_0)$, $\varphi''(u_0)$, and $\varphi'''(u_0)$ are small enough in comparison with the product $\varepsilon\,\varphi(u_0)$ (see (4.5)).

Under an additional assumption

$$\varphi''''(u) \ge 0, \quad (4.17)$$

which is valuable in view of applied reasons, condition (4.15) can be slightly improved. Indeed, since

$$\int\limits_{a_0}^{a_1} \varphi'''(u)\psi(u_x)\,u_x^3 u_{xx}\,\mathrm{d}x = -\frac{1}{4}\int\limits_{a_0}^{a_1} \varphi'''(u)\psi'(u_x)\,u_x^4 u_{xx}\,\mathrm{d}x - \frac{1}{4}\int\limits_{a_0}^{a_1} \varphi''''(u)\psi(u_x)\,u_x^5\,\mathrm{d}x, \quad (4.18)$$

the value of one of the summands in (4.15) may be changed.

Remark 4.3. According to (4.18), (4.15) is replaced by

$$C_1\, m^{1/4} + C_2\, m^{1/2} + C_3\, m^{5/4} + C_5\, m^{9/4} \le \frac{\varepsilon\delta}{3} \quad (4.19)$$

with

$$C_5 = \frac{1}{4}\sup\left|\psi' u_x^2\right|\,\sup|\varphi'''| = \frac{1}{4}\,\frac{\sup\left|\psi' u_x^2\right|}{\sup|\psi|}\,C_4 =: \gamma_0\, C_4.$$

When the problems under study are discussed on the spatial interval (a_0,a_1) with its length $m > \gamma_0^{4/3}$, inequality (4.19) gives a better information than inequality (4.15) does.

Again both estimates (2.11) and (4.16) yield

Theorem 4.1. *In addition to the conditions of Section* 2*, suppose that* (4.1) *and* (4.5) *hold. All the problems* P2, P3*, and* P4 *with additional boundary conditions* (4.2) *have global-in-time unique solutions on* (a_0,a_1) *whose length* m *is determined by* (4.15).

Furthermore, under an additional assumption (4.17) *the global-in-time solutions mentioned above exist on* (a_0,a_1) *whose length* m *is determined by* (4.19).

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5. Solvability through the distance with singular solutions to ODE

As have been already mentioned, equation (2.1) does not satisfy the well known "quadratic growth condition" in (0.8). Therefore, an ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\varphi(y)\psi(y')\Big) = 0, \qquad y = y(x), \tag{5.1}$$

has bounded singular solutions (see [13,16], e.g.). It means that in extending the solutions to certain Cauchy problems for equation (5.1) their gradients $y'(x)$ may grow to infinity, while the solutions themselves, $y(x)$, remain bounded.

Let us introduce two definitions (see [13]).

Definition 5.1. If a solution $z(x)$ to a certain Cauchy problem for equation (5.1) remains bounded, but is not regularly extensible onto the whole interval $[a_0, a_1]$, such a solution is called a singular solution. Set of all the singular solutions to equation (5.1) will be denoted by $\mathcal{Z}$.

Definition 5.2. A smooth function $f(x)$ is said to possess a positive generalized distance with the set of singular solutions $\mathcal{Z}$ defined in Definition 5.1, if

$$\rho(f(x)) := \inf_{z\in\mathcal{Z}} \inf_{x\in[a_0,a_1]} \inf_{\eta\in I(x)} (|z(x) - f(x)| + |z'(x) - \eta|) > 0, \tag{5.2}$$

where $I(x)$ denotes the segment with endpoints 0 and $f'(x)$.

It is clear that the generalized distance introduced above is positive in the following cases:

1. the function $f(x)$ does not cross any of singular solutions $z \in \mathcal{Z}$ to ODE (5.1);
2. at the point of intersection of the graph of $f(x)$ with any of singular solutions $z \in \mathcal{Z}$ the derivatives $f'(x)$ and $z'(x)$ have different signs,
$$\operatorname{sgn} f'(x) \operatorname{sgn} z'(x) < 0;$$
3. the absolute value of $|z'(x)|$ is strictly greater than $|f'(x)|$ at the point of intersection, if the derivatives have the same sign, $\operatorname{sgn} f'(x) \operatorname{sgn} z'(x) > 0$.

If the initial profile, $u_0(x)$, possesses a positive "generalized distance" with the set of all bounded singular solutions to ODE (5.1), problem P2 was proved to be well-posed in [13].

Obviously, equation (5.1) has the first integral

$$\varphi(y)\,\psi(y') = \varphi(y_0)\,\psi(y_1), \tag{5.3}$$

where y_0 and y_1 are the Cauchy data $y(x_0) = y_0$, $y'(x_0) = y_1$.

It is easy to see that in case

$$\frac{\max \varphi(u_0)}{\min \varphi(u_0)}\,\psi(\sup |u_0'|) < \psi_\infty, \tag{5.4}$$

the initial profile, $u_0(x)$, possesses a positive generalized distance with the set of singular solutions to ODE (5.1).

Consequently, we have

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Theorem 5.1. *In addition to the conditions of Section* 2*, suppose that the initial profile* $u_0(x)$ *in* (2.5) *satisfies* (5.4)*. Then each problem of* P2, P3*, and* P4 *admits a global-in-time unique classical solution.*

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