J. Math. Anal. Appl. 480 (2019) 123448

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Minimal and weakly trapped submanifolds in standard static spacetimes

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ARTICLE INFO

Article history:
Received 14 May 2019
Available online 28 August 2019
Submitted by P. Yao

Keywords:
Minimal submanifold
Weakly trapped submanifold
Mean curvature
Standard static spacetime

ABSTRACT

In this article we obtain some rigidity results for spacelike submanifolds of arbitrary codimension in standard static spacetimes. In particular, under appropriate assumptions we prove that they must be contained in a spacelike slice, which enables us to characterize minimal closed submanifolds of codimension two. We also prove new rigidity results for complete non-compact weakly trapped submanifolds of arbitrary codimension in these ambient spacetimes.

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1. Introduction

The study of spacelike submanifolds of codimension greater than one began when Penrose introduced the notion of trapped surfaces in four dimensional spacetimes in order to study the singularities of a spacetime [30]. Namely, the existence of a trapped surface is a sign of the presence of a black hole. Thus, spacelike submanifolds of codimension two have played a crucial role in the development of some of the most important singularity theorems (see [15], [16]), the analysis of gravitational collapse [6] as well as in the study of the comsmic censorship hypothesis [12] and the related Penrose inequality [19].

The usual definition of trapped surfaces in terms of null expansions is related to the causal orientation of the mean curvature vector, which provides a better and more powerful characterization of these surfaces and allows the extension of the concepts of trapped and marginally trapped submanifolds to spacelike submanifolds of arbitrary codimension [22]. In particular, let us consider a spacetime $\overline{M}$ and let M be an immersed spacelike submanifold of arbitrary codimension k. That is, a connected manifold admitting a smooth immersion $\psi : M^n \longrightarrow \overline{M}^{n+k}$ such that the induced metric on M is Riemannian. If we denote by $\mathbf{H}$ the mean curvature vector field of the submanifold, following the standard terminology in General Relativity (see [22] and [25]), $\psi(M)$ is said to be:

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https://doi.org/10.1016/j.jmaa.2019.123448
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- *Future trapped* if $\mathbf{H}$ is timelike and future-pointing everywhere on M. Similarly for *past trapped*.
- *Marginally future trapped* if $\mathbf{H}$ is lightlike and future-pointing everywhere on M. Similarly for *marginally past trapped*.
- *Weakly future trapped* if $\mathbf{H}$ is causal and future-pointing everywhere on M. Similarly for *weakly past trapped*.
- *Minimal* if $\mathbf{H} = 0$ on M.

Due to the aforementioned reasons, there has recently been an increasing interest in obtaining new rigidity results for trapped submanifolds and in general, spacelike submanifolds of arbitrary codimension, in a wide variety of spacetimes like Lorentzian space forms (see [5] and [24]) and Generalized Robertson-Walker spacetimes [4].

In this article we will study spacelike submanifolds of arbitrary codimension in the class of spacetimes known as standard static spacetimes, given by the product manifold $\overline{M} = B \times \mathbb{R}$, where (B, g_B) is a (connected) Riemannian manifold, endowed with the Lorentzian metric

$$\overline{g} = \pi_B^*(g_B) - h(\pi_B)^2 \pi_{\mathbb{R}}^*(dt^2), \tag{1}$$

where π_B and $\pi_{\mathbb{R}}$ denote, respectively, the projections on B and $\mathbb{R}$ and h is a smooth positive function on B. A standard static spacetime $(\overline{M} = B \times \mathbb{R}, \overline{g})$ is a warped product in the sense of [27, Ch. 7] with base (B, g_B), fiber $(\mathbb{R}, -dt^2)$ and warping function h endowed with the time orientation given by $\partial_t = \partial/\partial t$. These spacetimes present an infinitesimal causal symmetry given by the timelike Killing vector field ∂_t. Moreover, any static spacetime (i.e., a spacetime admitting an irrotational timelike Killing vector field) is locally isometric to a standard static one [27]. Indeed, under certain conditions every static spacetime admits a global standard static structure (see [1] and [32]).

The family of standard static spacetimes includes some classical spacetime models like Lorentz-Minkowski spacetime, Einstein static universe as well as exterior Schwarzschild and Reissner-Nördstrom spacetimes [31]. These last two models describe a universe where there is only a spherically symmetric non-rotating mass, like a star or a black hole. In the first model the mass has no electric charge, whereas in the second it is uniformly charged (indeed, Reissner-Nördstrom spacetime can be regarded as an extension of Schwarchild's model [10]). Therefore, since the existence of a trapped surface signals the onset of gravitational collapse under certain energy conditions, the study of trapped submanifolds in standard static spacetimes is crucial to understand the geometry of black holes.

This article is organized as follows. Section 2 is devoted to define our ambient spacetimes as well as to present the main geometric objects that we will use along the paper. In Section 3 we will focus on the particular case of codimension two submanifolds that are contained in a spacelike slice, obtaining some results that will help us characterizing codimension two spacelike submanifolds in some more general cases. Finally, in Section 4 we obtain our main rigidity results for general spacelike submanifolds of arbitrary codimension which state that, under certain conditions, they are contained in a spacelike slice. In particular, we obtain a half-space theorem (Theorem 4) that yields as a corollary the scarcity of closed spacelike submanifolds (i.e., compact without boundary) in these spacetimes. Taking this into account, for the relevant case of closed minimal spacelike submanifolds we obtain our main rigidity result in Theorem 6, which enables us to characterize them in the codimension two case. Finally, Theorems 9, 10 and 12 extend these rigidity results to the case of complete non-compact weakly trapped or minimal submanifolds of arbitrary codimension.

2. Preliminaries

Consider a standard static spacetime $\overline{M} = B \times_h \mathbb{R}$ as defined in the previous section. As we have mentioned, the timelike vector field ∂_t is Killing. Thus, the integral curves of the unitary timelike vector

field $\frac{1}{h}\partial_t$ define a distinguished family of observers that measure a spatial universe that always looks the same. This is due to the fact that the local flows of ∂_t are isometries in $\overline{M}$ that preserve the restpaces of the observers in $\frac{1}{h}\partial_t$.

Moreover, the restspaces of the observers in $\frac{1}{h}\partial_t$ are given by the spacelike slices $B \times \{t_0\}$, $t_0 \in \mathbb{R}$, which define a foliation of the whole spacetime. It can be easily seen that a spacelike submanifold in $\overline{M}$ is contained in a spacelike slice if and only if the restriction of the projection $\pi_{\mathbb{R}}$ along the immersion ψ, that we will denote by $\tau := \pi_{\mathbb{R}} \circ \psi$, is constant on M. Furthermore, the shape operator of any spacelike slice vanishes identically, i.e., they are totally geodesic.

Let $\psi : M \longrightarrow \overline{M} = B \times_h \mathbb{R}$ be an immersed n-dimensional spacelike submanifold of an $(n+k)$-dimensional standard static spacetime. Let us denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections of $\overline{M}$ and M, respectively. The Gauss formula of M in $\overline{M}$ is given by

$$\overline{\nabla}_X Y = \nabla_X Y - \sigma(X, Y), \tag{2}$$

for every $X, Y \in \mathfrak{X}(M)$, where σ denotes the second fundamental form of M. Note that we are following the usual convention in General Relativity to define σ (which is opposite to the commonly used one in Differential Geometry). On the other hand, Weingarten formula is

$$\overline{\nabla}_X N = A_N X + \nabla^{\perp}_X N \tag{3}$$

for $X \in \mathfrak{X}(M)$ and $N \in \mathfrak{X}^{\perp}(M)$, where A_N denotes the shape operator associated to N and $\nabla^{\perp}$ is the normal connection of the submanifold. Moreover, we can define the mean curvature vector $\mathbf{H}$ of $\psi(M)$ by

$$\mathbf{H} = \frac{1}{n}\mathrm{trace}(\sigma). \tag{4}$$

Moreover, we can decompose the vector field ∂_t as

$$\partial_t = \partial_t^T + \partial_t^N, \tag{5}$$

where $\partial_t^T \in \mathfrak{X}(M)$ and $\partial_t^N \in \mathfrak{X}^{\perp}(M)$ denote, respectively, the tangent and normal components of ∂_t. Taking this into account, since the gradient of τ on $\overline{M}$ is $\overline{\nabla}\tau = -\frac{1}{h^2}\partial_t$, it is easy to check that the gradient of τ on M is

$$\nabla\tau = -\frac{1}{h^2}\partial_t^T, \tag{6}$$

where h is evaluated on M. Now, using (6) and taking a local orthonormal reference frame $\{E_1, \ldots, E_n\}$ on (M, g) we can compute the Laplacian of τ on M, obtaining

$$\Delta\tau = -\sum_{i=1}^{n} g\left(\nabla_{E_i}\left(\frac{1}{h^2}\partial_t^T\right), E_i\right) = -\sum_{i=1}^{n} E_i\left(\frac{1}{h^2}\right)\, g(\partial_t^T, E_i) - \frac{1}{h^2}\sum_{i=1}^{n} g(\nabla_{E_i}\partial_t^T, E_i). \tag{7}$$

From (5) and (6), (7) leads to

$$\Delta\tau = -\frac{2}{h}\sum_{i=1}^{n} g(E_i, \nabla h)\, g(\nabla\tau, E_i) - \frac{1}{h^2}\sum_{i=1}^{n} g(\nabla_{E_i}\partial_t, E_i) + \frac{1}{h^2}\sum_{i=1}^{n} g(\nabla_{E_i}\partial_t^N, E_i). \tag{8}$$

Since ∂_t is a Killing vector field on $\overline{M}$, the second addend vanishes. Furthermore, using Weingarten formula (3) we can write (8) as

$$\Delta\tau = -\frac{2}{h} g(\nabla h, \nabla\tau) + \frac{1}{h^2} \sum_{i=1}^{n} g(A_{\partial_t^N} E_i, E_i). \tag{9}$$

From the relation between the shape operator and the second fundamental form ($\overline{g}(\sigma(X,Y),N) = g(A_N X, Y)$), we get

$$\Delta\tau = -\frac{2}{h} g(\nabla h, \nabla\tau) + \frac{1}{h^2} \sum_{i=1}^{n} \overline{g}\left((\sigma(E_i, E_i), \partial_t^N\right). \tag{10}$$

To conclude, we can write the Laplacian of τ on M in terms of the mean curvature vector defined in (4) as

$$\Delta\tau = -\frac{2}{h}\ g(\nabla h, \nabla\tau) + \frac{n}{h^2}\ \overline{g}(\mathbf{H}, \partial_t). \tag{11}$$

2.1. Parabolic Riemannian manifolds

A complete (non-compact) Riemannian manifold is called parabolic if the only superharmonic functions bounded from below that it admits are the constants (see [21] for instance). The study of parabolicity has been approached from different standpoints.

From a physical perspective, parabolicity is closely related with the behaviour of the Brownian motion on a Riemannian manifold. Namely, the recurrence of the Brownian motion is equivalent to the parabolicity of the Riemannian manifold [14].

From a mathematical viewpoint, parabolicity of Riemannian surfaces is closely related to their Gaussian curvature. Indeed, complete spacelike surfaces with non-negative Gaussian curvature are parabolic [18]. In dimension greater than two, parabolicity of Riemannian manifolds has no clear relation with the sectional curvature. For instance, note that the Euclidean space $\mathbb{R}^n$ is parabolic if and only if $n \leq 2$. Nevertheless, there are sufficient conditions for parabolicity of a Riemannian manifold of arbitrary dimension based on the volume growth of its geodesic balls (see [3], [14] and references therein). For instance, if the spacelike submanifold M is complete and for a fixed origin $o \in M$ the geodesic ball centered at o with radius R, B_R, satisfies

$$\frac{R}{\mathrm{vol}(B_R)} \notin L^1(+\infty),$$

or

$$\frac{1}{\mathrm{vol}(\partial B_R)} \notin L^1(+\infty),$$

then M is parabolic.

A key fact about parabolicity is that it is invariant under quasi-isometries [21], [14, Cor. 5.3]. We recall that given two Riemannian manifolds (M_1, g_1) and (M_2, g_2), a global diffeomorphism φ from M_1 onto M_2 is called a quasi-isometry if there exists a constant $c \geq 1$ such that

$$c^{-1}\ g_1(v,v) \leq g_2(d\varphi(v), d\varphi(v)) \leq c\ g_1(v,v)$$

for all $v \in T_pM_1$, $p \in M_1$. In particular, in this article we will need the following result concerning the invariance of parabolicity under certain conformal changes of metric.

Lemma 1. *Let (M,g) be a parabolic Riemannian manifold and consider the conformal metric given by $\widehat{g} = \rho g$. If the conformal function ρ is bounded, then $(M,\widehat{g})$ is also parabolic.*

Proof. To prove this lemma, we need to realize that the conformal metric satisfies

$$\left(\inf_M \rho\right) g \leq \widehat{g} \leq \left(\sup_M \rho\right) g.$$

Thus, we see that if ρ is bounded (M, g) and $(M, \widehat{g})$ are quasi-isometric and therefore, both are parabolic. □

3. Codimension two submanifolds contained in spacelike slices

Let $\phi : M^n \longrightarrow B^{n+1}$ be an immersed hypersurface into the Riemannian manifold (B, g_B) and denote by g the Riemannian metric induced on M by ϕ, i.e., $g = \phi^*(g_B)$. If we consider for a fixed $t_0 \in \mathbb{R}$ the map $\phi_{t_0} : M \longrightarrow \overline{M} = B \times_h \mathbb{R}$ into the standard static spacetime $(\overline{M} = B \times_h \mathbb{R}, \overline{g})$ given by

$$\phi_{t_0}(p) = (\phi(p), t_0), \text{ for every } p \in M,$$

it is not hard to see that ϕ_{t_0} is a spacelike immersion of M into $\overline{M}$ which is contained in the slice $B \times \{t_0\}$. Moreover, the induced metric on M via ϕ_{t_0} is

$$g_{t_0} = \phi_{t_0}^*(\overline{g}) = \phi^*(g_B) = g.$$

Conversely, given an immersion $\psi : M^n \longrightarrow \overline{M}^{n+2} = B^{n+1} \times_h \mathbb{R}$ which is contained in a spacelike slice $B \times \{t_0\}$, the projection $\phi = \pi_B \circ \psi : M^n \longrightarrow B^{n+1}$ defines an immersed hypersurface for which $\psi(p) = (\phi(p), t_0) = \phi_{t_0}(p)$. Indeed, $\psi : M \longrightarrow \overline{M} = B \times_h \mathbb{R}$ is an embedding if and only if $\phi : M \longrightarrow B$ is an embedding.

Therefore, we can relate the extrinsic geometry of the codimension two submanifold contained in a spacelike slice $\psi : M \longrightarrow \overline{M} = B \times_h \mathbb{R}$ with the extrinsic geometry of the hypersurface $\phi : M \longrightarrow B$. In order to do so, given a unitary normal vector field N to the hypersurface $\phi(M)$ in B, we have that

$$\left\{ N(p), \frac{\partial_t}{h}\Big|_{(\phi(p),t_0)} \right\}, \ p \in M$$

defines a local orthonormal reference frame along $\psi(M)$ with

$$\overline{g}(N, N) = 1, \ \ \overline{g}\left(N, \frac{\partial_t}{h}\right) = 0 \ \text{ and } \ \overline{g}\left(\frac{\partial_t}{h}, \frac{\partial_t}{h}\right) = -1.$$

Taking this into account, we can write the second fundamental form σ_ψ of the immersion ψ as

$$\begin{aligned} \sigma_\psi(X, Y) &= \overline{g}(\sigma_\psi(X, Y), N)N + \overline{g}\left(\sigma_\psi(X, Y), \frac{\partial_t}{h}\right)\frac{\partial_t}{h} \\ &= \overline{g}(A_N X, Y)N + \overline{g}\left(A_{\frac{\partial_t}{h}} X, Y\right)\frac{\partial_t}{h}, \end{aligned} \tag{12}$$

for every $X, Y \in \mathfrak{X}(M)$. Moreover, since

$$\overline{\nabla}_X N = \nabla_X N,$$

for every $X \in \mathfrak{X}(M)$, where ∇ denotes the Levi-Civita connection of (M, g), we have that the shape operator satisfies

$$A_N X = AX, \tag{13}$$

where $A : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ is the shape operator of the hypersurface $\phi : M \longrightarrow B$ associated to N. On the other hand, for every $X \in \mathfrak{X}(M)$,

$$\overline{\nabla}_X \frac{\partial_t}{h} = -\frac{1}{h^2} X(h)\partial_t + \frac{1}{h}\overline{\nabla}_X \partial_t.$$

Thus, from Weingarten formula (3) we obtain

$$A_{\frac{\partial_t}{h}} X = \frac{1}{h}\nabla_X \partial_t. \tag{14}$$

Hence, using (13) and (14) we can rewrite (12) as

$$\sigma_\psi(X,Y) = \overline{g}(AX,Y)N - \frac{1}{h^2}\,\overline{g}(\nabla_X \partial_t, Y)\partial_t, \tag{15}$$

for every $X, Y \in \mathfrak{X}(M)$. If we compute the trace of (15) using that ∂_t is a timelike Killing vector field in $\overline{M}$ we obtain that the mean curvature vector field of ψ is

$$\mathbf{H} = \frac{1}{n}\mathrm{trace}(\sigma_\psi) = HN, \tag{16}$$

where $H = \frac{1}{n}\mathrm{trace}(A)$ is the mean curvature function of the hypersurface $\phi : M \longrightarrow B$. As a consequence, we obtain that a codimension two submanifold contained in a spacelike slice of a standard static spacetime is minimal if and only if $\phi : M \longrightarrow B$ is a minimal hypersurface.

Furthermore, from (16) we also obtain

$$\overline{g}(\mathbf{H}, \mathbf{H}) = H^2, \tag{17}$$

which yields to the next result.

Proposition 2. *In a standard static spacetime $\overline{M} = B \times_h \mathbb{R}$ there are no weakly trapped, marginally trapped nor trapped codimension two submanifolds contained in a spacelike slice. Moreover, a minimal codimension two submanifold in $\overline{M} = B \times_h \mathbb{R}$ is contained in a spacelike slice if and only if it is a minimal hypersurface in B.*

This proposition provides a method to construct minimal submanifolds of codimension two in standard static spacetimes. As a consequence, we have the next existence result for minimal surfaces in four dimensional standard static spacetimes.

Corollary 3. *There exist embedded minimal surfaces with arbitrary genus in standard static spacetimes $\overline{M} = B \times_h \mathbb{R}$ with base either $B = \mathbb{R}^3$, $B = \mathbb{H}^3$ or $B = \mathbb{S}^3$.*

Proof. From Proposition 2 and our previous comments, any embedded minimal surface in B defines a minimal surface in the spacelike slice $B \times \{t_0\}$, for any $t_0 \in \mathbb{R}$. If the base is $B = \mathbb{R}^3$, following the ideas in [20] and [33] we can construct a minimal surface with prescribed geometry of arbitrary topological type using the Weierstrass representation for minimal surfaces in $\mathbb{R}^3$. Moreover, [17, Thm. 1] guarantees the existence of complete, non-flat, properly embedded minimal surfaces in $\mathbb{R}^3$ of finite arbitrary genus with one end that are asymptotic to a helicoid at infinity.

On the other hand, in [26, Thm. A] the authors showed that any open, connected, orientable surface can be properly embedded in $\mathbb{H}^3$ as an area-minimizing surface. Finally, it was proved in [23, Thm. 2] that there also exist embedded minimal surfaces with arbitrary genus in $\mathbb{S}^3$, which concludes the proof. □

4. Main results

Now, we will focus on the general case of spacelike submanifolds of arbitrary codimension which are not necessarily contained in a spacelike slice of a standard static spacetime. As a first result, we are able to prove the next half-space theorem.

Theorem 4. *Let $\psi : M \longrightarrow \overline{M} = B \times_h \mathbb{R}$ be a spacelike submanifold of arbitrary codimension in a standard static spacetime.*

(i) If $\psi(M)$ is weakly past trapped, then the height function τ has no local maximum. In particular, $\psi(M)$ is not contained in any "lower half-space" of the form $B \times (-\infty, t_0]$, $t_0 \in \mathbb{R}$.
(ii) If $\psi(M)$ is weakly future trapped, then τ has no local minimum. In particular, $\psi(M)$ is not contained in any "upper half-space" of the form $B \times [t_0, +\infty)$, $t_0 \in \mathbb{R}$.

Proof. Let us suppose that $\mathbf{H}$ is causal and past-pointing, i.e., $\overline{g}(\mathbf{H}, \partial_t) > 0$, and that there exists a point $p \in M$ where τ attains a local maximum. Thus, $\nabla\tau(p) = 0$ and $\Delta\tau(p) \leq 0$. However, since $\overline{g}(\mathbf{H}, \partial_t) > 0$, from (11) we obtain

$$\Delta\tau(p) = \frac{n}{h^2}\, \overline{g}(\mathbf{H}, \partial_t)(p) > 0,$$

reaching a contradiction. The second statement is proved analogously. □

As a consequence of this theorem we have the following corollary, which is a particular case of the result obtained in [25, Thm. 1] and shows the scarcity of closed spacelike submanifolds (i.e., compact without boundary) in standard static spacetimes.

Corollary 5. *In a standard static spacetime there are no weakly trapped, marginally trapped nor trapped closed submanifolds.*

In fact, we can go one step further and characterize the closed minimal submanifolds in these ambiences by means of the next result.

Theorem 6. *Let $\psi : M \longrightarrow \overline{M} = B \times_h \mathbb{R}$ be a closed spacelike submanifold of arbitrary codimension in a standard static spacetime. If $\psi(M)$ is minimal, then it is contained in a spacelike slice $B \times \{t_0\}$, $t_0 \in \mathbb{R}$. Moreover, if $\psi(M)$ has codimension two, it must be a minimal hypersurface in $B \times \{t_0\}$.*

Proof. If $\mathbf{H} = 0$, we have from (11)

$$\Delta\tau + \frac{2}{h}\, g(\nabla h, \nabla\tau) = 0. \tag{18}$$

Now, we recall the f-Laplacian operator, which naturally appears in the study of weighted Riemannian manifolds $(M, g, e^{-f}d\mu)$, being $d\mu$ the Riemannian volume density, and for a function u is defined by

$$\Delta_f u = \Delta u - g(\nabla f, \nabla u).$$

Therefore, choosing as weight function $f = -2\log h$, (18) yields to

$$\Delta_{-2\log h}\tau = 0. \tag{19}$$

Thus, τ defines a f-harmonic function on a weighted compact Riemannian manifold without boundary and therefore it must be constant. The last statement of the theorem follows from Proposition 2. □

This result extends [28, Cor. 2] to the parametric case in arbitrary codimension. In fact, as a consequence of Theorem 6, we can obtain the following result recalling that there are no closed embedded minimal hypersurfaces in $\mathbb{R}^{n+1}$ nor in $\mathbb{H}^{n+1}$ (see [2] and [9]).

Corollary 7. *There are no closed embedded codimension two minimal submanifolds in a standard static spacetime* $\overline{M} = B \times_h \mathbb{R}$ *with base either* $B = \mathbb{R}^{n+1}$ *or* $B = \mathbb{H}^{n+1}$.

Moreover, when the base is $\mathbb{S}^3$, since the only minimal topological 2-spheres in $\mathbb{S}^3$ are the totally geodesic ones (see [11]), Theorem 6 implies

Corollary 8. *The only minimal topological 2-spheres in a standard static spacetime with base* $\mathbb{S}^3$ *are the totally geodesic ones in* $\mathbb{S}^3$.

Furthermore, this technique of considering the f-Laplacian enables us to use Liouville-type theorems for weighted Riemannian manifolds to extend our results to the non-compact case.

Theorem 9. *Let* $\psi : M \longrightarrow \overline{M} = B \times_h \mathbb{R}$ *be a complete minimal submanifold of arbitrary codimension in a standard static spacetime such that* $\mathrm{Ric} \geq 0$ *and* $\mathrm{Hess}(\log h) \leq 0$, *where* Ric *and* Hess *denote the Ricci tensor and Hessian operator on* (M, g), *respectively. If* τ *is bounded, then* $\psi(M)$ *is contained in a spacelike slice.*

Proof. Proceeding as in the proof of Theorem 6, we see that τ is a bounded f-harmonic function on the complete weighted Riemannian manifold $(M, g, \frac{1}{h^2}d\mu)$. Moreover, under our assumptions, the Bakry-Emery Ricci tensor satisfies

$$\mathrm{Ric}_{-2\log h} = \mathrm{Ric} - 2\ \mathrm{Hess}(\log h) \geq 0.$$

Thus, we can apply [8, Cor. 2] to conclude that τ is constant. □

Moreover, we can obtain new rigidity results for non-compact weakly trapped and minimal submanifolds by means of parabolicity. We recall that parabolicity is a weaker assumption than compactness that has been previously used in the study of spacelike submanifolds.

Theorem 10. *Let* $\psi : M \longrightarrow \overline{M} = B \times_h \mathbb{R}$ *be a parabolic spacelike submanifold of arbitrary codimension in a standard static spacetime such that* $\inf_M h > 0$ *and* $\sup_M h < +\infty$.

(i) If $\overline{g}(\mathbf{H}, \partial_t) \leq 0$ *and* $\inf_M \tau > -\infty$, *then* $\psi(M)$ *is contained in a spacelike slice.*
(ii) If $\overline{g}(\mathbf{H}, \partial_t) \geq 0$ *and* $\sup_M \tau < +\infty$, *then* $\psi(M)$ *is contained in a spacelike slice.*

Proof. The key idea to prove this result is to endow the parabolic spacelike submanifold (M, g) with the conformal metric given by

$$\widehat{g} = h^{\frac{4}{n-2}}\ g. \tag{20}$$

Since $\inf_M h > 0$ and $\sup_M h < +\infty$, Lemma 1 guarantees the parabolicity of $(M, \widehat{g})$. Moreover, taking into account how differential operators behave under conformal changes of metric (see [7]) we can get from (11) that the Laplacian of τ on $(M, \widehat{g})$ is

$$\widehat{\Delta}\tau = n\, h^{-\frac{2n}{n-2}}\, \overline{g}(\mathbf{H}, \partial_t). \tag{21}$$

Hence, if $\overline{g}(\mathbf{H}, \partial_t) \leq 0$, the function τ is $\widehat{\Delta}$-superharmonic on M. Now, since $\tau_* = \inf_M \tau > -\infty$ we get that the function $\widetilde{\tau} = \tau + |\tau_*| + 1$ is a positive $\widehat{\Delta}$-superharmonic function on the parabolic Riemannian manifold $(M, \widehat{g})$ and thus, it must be constant. We can proceed analogously to prove the second statement. □

Remark 11. Although the conformal change of metric (20) does not work in the 2-dimensional case, we can use the same idea as in [29] to use it. Namely, given a spacelike surface $\psi : M \to B \times_h \mathbb{R}$, consider a new spacelike submanifold in the extended spacetime $\psi' : \mathbb{S}^2 \times M \to \mathbb{S}^2 \times B \times_h \mathbb{R}$ given by $\psi'(s, p) = (s, \psi(p))$, with $s \in \mathbb{S}^2$ and $p \in M$. Note that the mean curvature vector field of ψ' is the same as the one of ψ lifted to the extended spacetime and that the Riemannian product of a compact Riemannian manifold and a parabolic one is also parabolic [21]. In this new ambient we can use the conformal change of metric to prove Theorem 10.

To conclude, we can extend these results by means of certain assumptions on the warping function.

Theorem 12. *Let $\psi : M \longrightarrow \overline{M} = B \times_h \mathbb{R}$ be a complete, non-compact, spacelike submanifold of arbitrary codimension in a standard static spacetime such that*

$$\int_1^{+\infty} \left(\int_{\partial B_r(o)} h^2 \right)^{-1} \mathrm{d}r = +\infty, \tag{22}$$

for some reference point $o \in M$, where $\partial B_r(o)$ denotes the boundary of the geodesic ball in M centered at o with radius r. If $\overline{g}(\mathbf{H}, \partial_t) \geq 0$ and τ is bounded from above, then $\psi(M)$ is contained in a spacelike slice.

Proof. Since $\overline{g}(\mathbf{H}, \partial_t) \geq 0$, we obtain from (11) that the weighted Laplacian $\Delta_{-2\log h}\tau$ satisfies

$$\Delta_{-2\log h}\tau = \frac{n}{h^2}\, \overline{g}(\mathbf{H}, \partial_t) \geq 0.$$

Completeness of M and (22) imply that the operator $\Delta_{-2\log h}$ is parabolic (see [13, Prop. 4]). Since τ is bounded from above, it is constant on M. Therefore, $\psi(M)$ is contained in a spacelike slice. □

Acknowledgments

The author is supported by Spanish MINECO and ERDF project MTM2016-78807-C2-1-P. The author would also like to thank the referee for his valuable comments.

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