



# Exact controllability of wave equations with locally distributed control in non-cylindrical domain



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## ABSTRACT

Exact controllability of a one-dimensional wave equation with locally distributed control in non-cylindrical domain is considered. This equation characterizes the motion of a string with a fixed endpoint and the other moving one. If the adjoint system is observable, this establishes exact controllability of the original system. The adjoint system is observable by multiplier method. Therefore we obtain exact controllability of this equation, when the speed of the moving endpoint is less than wave speed.

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## 1. Introduction

Given  $T > 0$ . For any  $0 < k < 1$ , set

$$\alpha_k(t) = 1 + kt \quad \text{for } t \in [0, T]. \quad (1.1)$$

Let us consider the subsets  $\Omega_t$  of  $\mathbb{R}$  given by

$$\Omega_t = \{x \in \mathbb{R}; 0 < x < \alpha_k(t)\}, \quad 0 < t < T$$

and  $\hat{Q}_T^k$  the non-cylindrical domain of  $\mathbb{R}^2$ ,

$$\hat{Q}_T^k = \bigcup_{0 < t < T} \Omega_t \times \{t\}.$$

For any  $0 < m < \frac{n}{k+1} < n < 1$ ,

$$\omega_t = \{x \in \mathbb{R}; m\alpha_k(t) < x < n\alpha_k(t)\} \subset \Omega_t$$

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and

$$\widehat{Q}_1 = \bigcup_{0 < t < T} \omega_t \times \{t\}.$$

In this paper, we consider the following control problem associated with a one-dimensional wave equation:

$$\begin{cases} u_{tt}(x, t) - u_{xx}(x, t) = B(x, t)v(x, t) & \text{in } \widehat{Q}_T^k, \\ u(0, t) = u(\alpha_k(t), t) = 0 & \text{on } (0, T), \\ u(x, 0) = u^0, \quad u_t(x, 0) = u^1 & \text{in } (0, 1), \end{cases} \quad (1.2)$$

where  $u$  is the state variable,  $(u^0, u^1) \in L^2(0, 1) \times H^{-1}(0, 1)$  is any given initial value,  $v$  is the control variable and the domain where the control is applied is  $\widehat{Q}_1$  lying in  $\widehat{Q}_T^k$ . The constant  $k$  is called speed of the moving endpoint and the operator  $B$  is defined as follows:

$$B(x, t) = \begin{cases} 0, & (x, t) \in \widehat{Q}_T^k \setminus \widehat{Q}_1, \\ 1, & (x, t) \in \widehat{Q}_1. \end{cases} \quad (1.3)$$

In this paper, we consider irregular controls in the classes

$$v \in L^2([0, T]; (H^1(\omega_t))^*),$$

where

$$H^1(\omega_t) = \left\{ f, \frac{\partial f}{\partial x} \in L^2(\omega_t) \right\}$$

and we denote the dual space to  $H^1(\omega_t)$  by  $(H^1(\omega_t))^*$ .

By [10], it is easy to check that (1.2) has a unique weak solution

$$u \in C([0, T]; L^2(\Omega_t)) \bigcap C^1([0, T]; H^{-1}(\Omega_t)).$$

The problem of exact controllability for (1.2) is formulated as follows:

**Definition 1.1.** (1.2) is called to be exactly controllable at the time  $T$ , if for any initial value  $(u^0, u^1) \in L^2(0, 1) \times H^{-1}(0, 1)$  and any target  $(u_d^0, u_d^1) \in L^2(\Omega_T) \times H^{-1}(\Omega_T)$ , one can always find a control  $v \in L^2([0, T]; (H^1(\omega_t))^*)$  such that corresponding weak solution  $u$  of (1.2) satisfies

$$u(T) = u_d^0, \quad u_t(T) = u_d^1. \quad (1.4)$$

The main goal of this article is to obtain the exact controllability of (1.2). As we all know, there exist numerous literatures on the controllability problems of wave equations in a cylindrical domain, see e.g. [8, 9, 11, 13, 15–17]. However, there are only a few works on the exact controllability for wave equations defined in non-cylindrical domains. We refer to [1–7, 12, 14] for some known results in this respect. In [3–6, 12, 14], boundary controllability for wave equations with moving boundary has been obtained. In [1], distributed controllability of a wave equation with constant coefficients in a non-cylindrical domain was established, when a control entered the system through the whole non-cylindrical domain. While in [2], locally distributed control of a one-dimensional wave equation in a non-cylindrical domain was obtained, when  $k \in (0, \tilde{k})$ ,  $0 < \tilde{k} < 1$ . Motivated by [9], [13] and [1, 2, 7, 14], we extend the result in [2], and locally distributed control is

obtained when  $k \in (0, 1)$  and the control class is narrowed. The key point is to define directly the energy function of a wave equation in the non-cylindrical domain and choose an appropriate multiplier to overcome these difficulties.

Our paper is divided into four sections. In Section 2, we state the principal results. In Section 3, we choose a multiplier with discontinuous satisfying the first-order differential equation. In Section 4, using the multiplier method, we consider the homogeneous wave equation and establish observability inequality.

## 2. Preliminaries and main results

Set  $T_k^* > 0$  for the controllability time. The main result of this paper is stated as follows.

**Theorem 2.1.** *Let  $0 < k < 1$  and  $T > T_k^*$ , (1.2) is exactly controllable at time  $T$  in the sense of Definition 1.1.*

**Remark 2.1.**  $T_k^*$  will be defined during the course of the later proof.

**Remark 2.2.** We can obtain the same result as that of in this paper for a more general function  $\alpha_k(t)$ , as long as it meets the condition  $0 < \alpha_k'(t) < 1$ .

**Remark 2.3.** We expect to obtain the same result in the forthcoming paper, when the control variable  $v \in L^2([0, T]; \omega(t))$ ,  $\omega(t) \subseteq (0, \alpha_k(t))$ .

To prove this, let  $u = \xi + \eta$ , where  $\xi$  and  $\eta$  satisfy the following systems:

$$\begin{cases} \xi_{tt}(x, t) - \xi_{xx}(x, t) = 0 & (x, t) \in \widehat{Q}_T^k, \\ \xi(0, t) = \xi(\alpha_k(t), t) = 0 & t \in (0, T), \\ \xi(x, 0) = u^0, \quad \xi_t(x, 0) = u^1 & x \in (0, 1), \end{cases} \quad (2.1)$$

$$\begin{cases} \eta_{tt}(x, t) - \eta_{xx}(x, t) = B(x, t)v(x, t) & (x, t) \in \widehat{Q}_T^k, \\ \eta(0, t) = \eta(\alpha_k(t), t) = 0 & t \in (0, T), \\ \eta(x, 0) = \eta_t(x, 0) = 0 & x \in (0, 1). \end{cases} \quad (2.2)$$

Therefore, we only need to obtain internal controllability of (2.2).

**Theorem 2.2.** *Let  $T > T_k^*$ . Then for any target  $(u_d^0, u_d^1) \in L^2(\Omega_T) \times H^{-1}(\Omega_T)$ , there exists a control  $v \in L^2([0, T]; (H^1(\omega_t))^*)$  such that corresponding weak solution  $\eta$  of (2.2) satisfies*

$$\eta(T) = u_d^0, \quad \eta_t(T) = u_d^1.$$

**Remark 2.4.** If Theorem 2.2 holds, then Theorem 2.1 holds. In fact, for any  $(u^0, u^1) \in L^2(0, 1) \times H^{-1}(0, 1)$  and any target  $(u_d^0, u_d^1) \in L^2(\Omega_T) \times H^{-1}(\Omega_T)$ ,  $(u_d^0 - \xi(T), u_d^1 - \xi_t(T))$  belongs to  $L^2(\Omega_T) \times H^{-1}(\Omega_T)$ , where  $\xi$  is the solution of (2.1) associated to  $(u^0, u^1)$ . By Theorem 2.2, we choose  $v \in L^2([0, T]; (H^1(\omega_t))^*)$ , then  $\eta$  satisfies

$$\eta(T) = u_d^0 - \xi(T), \quad \eta_t(T) = u_d^1 - \xi_t(T).$$

This implies that  $u = \xi + \eta$  satisfies (1.2) and (1.4).

In the following, we prove Theorem 2.2. Let us introduce some notations. Write  $U = L^2([0, T]; (H^1(\omega_t))^*)$ ,  $F = L^2(\Omega_T) \times H^{-1}(\Omega_T)$ .  $U^* = L^2([0, T]; H^1(\omega_t))$  and  $F^* = H_0^1(\Omega_T) \times L^2(\Omega_T)$  denote their dual spaces respectively. Then we define the scalar products between  $F$  and  $F^*$

$$\begin{aligned} & \langle (w(x, T), w_t(x, T)), (z(x, T), z_t(x, T)) \rangle_{F, F^*} \\ &= \int_0^{\alpha_k(T)} w_t(x, T) z(x, T) dx - \int_0^{\alpha_k(T)} w(x, T) z_t(x, T) dx, \end{aligned}$$

where for any  $(w(x, T), w_t(x, T)) \in F$  and any  $(z(x, T), z_t(x, T)) \in F^*$ .

Define a linear operator  $A$ :

$$\begin{aligned} A : U &\rightarrow F, \\ Av &= (\eta(x, T), \eta_t(x, T)) \quad \forall v \in U, \end{aligned}$$

where we use  $\eta$  to denote the solution of (2.2) associated to  $v$ . Then,  $A$  is surjective if and only if internal controllability of the wave equation (2.2) is obtained. While  $A$  is surjective, by the method in [9] and [13], we only need to prove

$$\|A^*(z^0, z^1)\|_{U^*}^2 \geq C(T, k) \|(z^0, z^1)\|_{F^*}^2, \quad \forall (z^0, z^1) \in F^* \quad (2.3)$$

for  $T > T_k^*$ , where  $A^* : F^* \rightarrow U^*$  denotes the dual operator to  $A$ ,  $C(T, k)$  denotes a positive constant depending only  $T$  and  $k$ .

The dual operator  $A^*$  is associated with the following homogeneous wave equation:

$$\begin{cases} z_{tt}(x, t) - z_{xx}(x, t) = 0 & (x, t) \in \widehat{Q}_T^k, \\ z(0, t) = z(\alpha_k(t), t) = 0 & t \in (0, T), \\ z(x, T) = z^0, z_t(x, T) = z^1 & x \in (0, \alpha_k(T)), \end{cases} \quad (2.4)$$

where  $(z^0, z^1) \in F^*$  is any given initial value. (2.4) has a unique weak solution

$$z \in C([0, T]; H_0^1(\Omega_t)) \bigcap C^1([0, T]; L^2(\Omega_t)).$$

Multiplying the first equation of (2.2) by  $z$  and integrating on  $\widehat{Q}_T^k$ , we have

$$\begin{aligned} \langle Bv, z \rangle &= \int_0^{\alpha_k(T)} \eta_t(x, T) z^0 dx - \int_0^{\alpha_k(T)} \eta(x, T) z^1 dx \\ &= \langle (\eta(x, T), \eta_t(x, T)), (z^0, z^1) \rangle \\ &= \langle Av, (z^0, z^1) \rangle. \end{aligned}$$

Let  $B^*$  be the adjoint operator of  $B$  in (1.3) and if  $v \in L^2([0, T]; (H^1(\omega_t))^*)$ , then

$$B^* : L^2([0, T]; H_0^1(\Omega_t)) \rightarrow L^2([0, T]; H^1(\omega_t)).$$

From which, we have

$$\langle v, B^* z \rangle = \langle v, A^*(z^0, z^1) \rangle.$$

Hence  $A^*$  is defined as followed:

$$A^*(z^0, z^1) = B^*z(x, t) = z(x, t), \quad (x, t) \in \widehat{Q_1}, \quad \forall (z^0, z^1) \in F^*,$$

where  $z$  is the solution of (2.4).

Therefore (2.3) is equivalent to the following inequality:

$$|z|_{U^*} \geq C(T, k)|(z^0, z^1)|_{F^*} \quad \forall (z^0, z^1) \in F^*. \quad (2.5)$$

In the sequel, we denote by  $C$  a positive constant depending only on  $T$  and  $k$ , which may be different from one place to another.

### 3. The choice of the multiplier

In this section, motivated by [13], we choose a multiplier satisfying the following lemma.

**Lemma 3.1.** *There exists an  $\varepsilon_0 > 0$ , such that for any  $\varepsilon \in (0, \varepsilon_0)$ , one always can find a constant  $\lambda_\varepsilon \in (0, 1)$  and  $x_0(t) \in \omega_t$  for any  $t \in (0, T)$ , so that the function  $p(\cdot, t) \in L^1(\Omega_t)$  defined as*

$$p(x, t) = \begin{cases} (\lambda_\varepsilon - 1)x & (x, t) \in (0, m\alpha_k(t)) \times (0, T), \\ \lambda_\varepsilon[x - x_0(t)] & (x, t) \in \omega_t \times (0, T), \\ (\lambda_\varepsilon - 1)[x - \alpha_k(t)] & (x, t) \in (n\alpha_k(t), \alpha_k(t)) \times (0, T), \end{cases} \quad (3.1)$$

satisfies the following conditions:

$$p(0, t) = p(\alpha_k(t), t) = p(x_0(t), t) = 0, \quad \forall t \in (0, T); \quad (3.2)$$

$$p(m\alpha_k(t) + 0, t) - p(m\alpha_k(t) - 0, t) = -\varepsilon\alpha_k(t), \quad \forall t \in (0, T); \quad (3.3)$$

$$p(n\alpha_k(t) + 0, t) - p(n\alpha_k(t) - 0, t) = -\varepsilon\alpha_k(t), \quad \forall t \in (0, T); \quad (3.4)$$

and for any  $0 < t < T$ ,

$$\frac{dp(x, t)}{\partial x} = \begin{cases} \lambda_\varepsilon - 1 & x \in \Omega_t \setminus \overline{\omega_t}, \\ \lambda_\varepsilon & x \in \omega_t, \end{cases} \quad (3.5)$$

where  $p(m\alpha_k(t) + 0, t) = \lim_{x \rightarrow [m\alpha_k(t)]^+} p(x, t)$  and  $p(m\alpha_k(t) - 0, t) = \lim_{x \rightarrow [m\alpha_k(t)]^-} p(x, t)$ .

**Proof.** It is easy to check that  $p$  satisfies (3.2) and (3.5). To make  $p$  satisfy (3.3), we set

$$\lambda_\varepsilon \triangleq \lambda_{\varepsilon-} = \frac{(m + \varepsilon)\alpha_k(t)}{x_0(t)}, \quad x \in (0, x_0(t)). \quad (3.6)$$

Similarly, from (3.4)

$$\lambda_\varepsilon \triangleq \lambda_{\varepsilon+} = \frac{[(1 - n) + \varepsilon]\alpha_k(t)}{\alpha_k(t) - x_0(t)}, \quad x \in (x_0(t), \alpha_k(t)). \quad (3.7)$$

From (3.6) and (3.7), both  $\lambda_{\varepsilon-}$  and  $\lambda_{\varepsilon+}$  continuously depend on  $x_0(t) \in \omega_t$  and  $\varepsilon > 0$ . Moreover, it holds that  $\lambda_{\varepsilon-}$  and  $\lambda_{\varepsilon+}$  are monotone decreasing and increasing with respect to  $x_0(t)$  and for sufficiently small  $\varepsilon > 0$

$$\lambda_{\varepsilon-}, \lambda_{\varepsilon+} \in (0, 1); \lambda_{\varepsilon-}(m\alpha_k(t)) > \lambda_{\varepsilon+}(m\alpha_k(t)); \lambda_{\varepsilon-}(n\alpha_k(t)) < \lambda_{\varepsilon+}(n\alpha_k(t)).$$

Therefore for such values of  $\varepsilon$ , there exists a unique  $x_0(t) \in \omega_t$  of the equation  $\lambda_{\varepsilon-}(x_0(t)) = \lambda_{\varepsilon+}(x_0(t))$ , the corresponding value

$$\lambda_{\varepsilon} = \lambda_{\varepsilon-} = \lambda_{\varepsilon+} \in (0, 1).$$

In addition, after calculating, we have

$$x_0(t) = \frac{(m + \varepsilon)\alpha_k(t)}{1 - n + m + 2\varepsilon} \quad (3.8)$$

and

$$\lambda_{\varepsilon} = 1 - n + m + 2\varepsilon. \quad \square \quad (3.9)$$

**Remark 3.1.** It is easy to verify that

$$M_1 \triangleq \max_{(x,t) \in \hat{Q}_T^k} |p(x,t)| = \max\{|p(m\alpha_k(t) + 0)|, |p(n\alpha_k(t) - 0)|\}.$$

#### 4. Observability inequality in $L^2([0, T]; H^1(\omega_t))$

In the following, we shall give proof of (2.5) by multiplier method. The energy function of system of (2.4) is defined as follows:

$$E(t) = \frac{1}{2} \int_0^{\alpha_k(t)} [|z_t(x,t)|^2 + |z_x(x,t)|^2] dx \quad \text{for } t \geq 0,$$

where  $z$  is the solution of (2.4). In particular,

$$E_T = \frac{1}{2} \int_0^{\alpha_k(T)} [|z^1(x)|^2 + |z_x^0(x)|^2] dx.$$

By similar method, we obtain the following lemma about a growth estimate of the energy function (see the detailed proof in [14]).

**Lemma 4.1.** For any  $(z^0, z^1) \in H_0^1(\Omega_T) \times L^2(\Omega_T)$  and  $t \in [0, T]$ , the corresponding solution  $z$  of (2.4) follows

$$\frac{(1-k)\alpha_k(T)}{(1+k)\alpha_k(t)} E_T \leq E(t) \leq \frac{(1+k)\alpha_k(T)}{(1-k)\alpha_k(t)} E_T. \quad (4.1)$$

**Remark 4.1.** From Lemma 3.1, we obtain that

$$\frac{(1-k)}{(1+k)\alpha_k(t)} E_T \leq E(t) \leq \frac{(1+k)}{(1-k)} \alpha_k(T) E_T$$

and

$$\frac{(1-k)\alpha_k(T)}{(1+k)}E_T \leq E(0) \leq \frac{(1+k)\alpha_k(T)}{(1-k)}E_T.$$

In the following, we prove (2.5) by the multiplier (3.1).

Multiplying the first equation of (2.4) by  $p z_x$  and integrating on  $\widehat{Q}_T^k$ , we obtain

$$0 = \int_0^T \int_0^{\alpha_k(t)} \{z_{tt} p z_x - z_{xx} p z_x\} dx dt \triangleq J_1 + J_2.$$

We calculate the above two integrals  $J_i$  ( $i = 1, 2$ ). By (3.2), (3.3) and (3.4), we find the expression

$$\begin{aligned} J_1 &= \int_0^T \int_0^{\alpha_k(t)} z_{tt} p z_x dx dt \\ &= \int_0^T \int_0^{\alpha_k(t)} \left\{ \frac{\partial}{\partial t} [p z_x z_t] - p_t z_x z_t - p \frac{\partial}{\partial x} \left( \frac{1}{2} |z_t|^2 \right) \right\} dx dt \\ &= \int_0^{\alpha_k(t)} [p z_x z_t] dx \Big|_0^T - \int_0^T \int_0^{\alpha_k(t)} p_t z_x z_t dx dt \\ &\quad - \left[ \int_0^T p \frac{1}{2} |z_t|^2 \Big|_0^{m\alpha_k(t)} dt + \int_0^T p \frac{1}{2} |z_t|^2 \Big|_{m\alpha_k(t)}^{n\alpha_k(t)} dt \right. \\ &\quad \left. + \int_0^T p \frac{1}{2} |z_t|^2 \Big|_{n\alpha_k(t)}^{\alpha_k(t)} dt \right] + \int_0^T \int_0^{\alpha_k(t)} p_x \frac{1}{2} |z_t|^2 dx dt \\ &= \int_0^{\alpha_k(t)} [p z_x z_t] dx \Big|_0^T - \int_0^T \int_0^{\alpha_k(t)} p_t z_x z_t dx dt \\ &\quad - \frac{\varepsilon}{2} \int_0^T [ |z_t|^2 \Big|_{x=m\alpha_k(t)} + |z_t|^2 \Big|_{x=n\alpha_k(t)} ] \alpha_k(t) dt \\ &\quad + \int_0^T \int_0^{\alpha_k(t)} p_x \frac{1}{2} |z_t|^2 dx dt. \end{aligned} \tag{4.2}$$

Further, we have

$$\begin{aligned}
J_2 &= - \int_0^T \int_0^{\alpha_k(t)} z_{xx} p z_x dx dt \\
&= - \int_0^T \int_0^{\alpha_k(t)} \left\{ p \frac{\partial}{\partial x} \left[ \frac{1}{2} |z_x|^2 \right] \right\} dx dt \\
&= - \left[ \int_0^T p \frac{1}{2} |z_x|^2 \Big|_0^{m\alpha_k(t)} dt + \int_0^T p \frac{1}{2} |z_x|^2 \Big|_{n\alpha_k(t)}^{m\alpha_k(t)} dt \right. \\
&\quad \left. + \int_0^T p \frac{1}{2} |z_x|^2 \Big|_{\alpha_k(t)}^{n\alpha_k(t)} dt \right] + \int_0^T \int_0^{\alpha_k(t)} \frac{1}{2} p_x |z_x|^2 dx dt \\
&= - \frac{\varepsilon}{2} \int_0^T [ |z_x|^2 |_{x=m\alpha_k(t)} + |z_x|^2 |_{x=n\alpha_k(t)} ] \alpha_k(t) dt + \int_0^T \int_0^{\alpha_k(t)} \frac{1}{2} p_x |z_x|^2 dx dt.
\end{aligned} \tag{4.3}$$

Therefore, by (4.2) and (4.3), it follows that

$$\begin{aligned}
&\int_0^T \int_0^{\alpha_k(t)} \frac{1}{2} p_x (|z_t|^2 + |z_x|^2) dx dt \\
&= \int_0^T \int_0^{\alpha_k(t)} p_t z_x z_t dx dt - \left[ \int_0^{\alpha_k(t)} p z_x z_t dx \right] \Big|_0^T \\
&\quad + \frac{\varepsilon}{2} \int_0^T [ (|z_t|^2 + |z_x|^2) |_{x=m\alpha_k(t)} + (|z_t|^2 + |z_x|^2) |_{x=n\alpha_k(t)} ] \alpha_k(t) dt.
\end{aligned} \tag{4.4}$$

By the expression of  $p$ , we deduce

$$\begin{aligned}
&\lambda_\varepsilon \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} [|z_t|^2 + |z_x|^2] dx dt \\
&\quad + (\lambda_\varepsilon - 1) \int_0^T \int_{(0, \alpha_k(t)) \setminus (m\alpha_k(t), n\alpha_k(t))} [|z_t|^2 + |z_x|^2] dx dt \\
&= \int_0^T \int_0^{\alpha_k(t)} p_t z_x z_t dx dt - \left[ \int_0^{\alpha_k(t)} p z_x z_t dx \right] \Big|_0^T \\
&\quad + \frac{\varepsilon}{2} \int_0^T [ (|z_t|^2 + |z_x|^2) |_{x=m\alpha_k(t)} + (|z_t|^2 + |z_x|^2) |_{x=n\alpha_k(t)} ] \alpha_k(t) dt.
\end{aligned}$$



From which, we obtain that

$$\begin{aligned}
 & \frac{1}{2} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} [|z_t|^2 + |z_x|^2] dx dt \\
 &= \frac{1-\lambda_\varepsilon}{2} \int_0^T \int_0^{\alpha_k(t)} [|z_t|^2 + |z_x|^2] dx dt \\
 & \quad + \int_0^T \int_0^{\alpha_k(t)} p_t z_x z_t dx dt - \left[ \int_0^{\alpha_k(t)} p z_x z_t dx \right] \Big|_0^T \\
 & \quad + \frac{\varepsilon}{2} \int_0^T [ (|z_t|^2 + |z_x|^2)|_{x=m\alpha_k(t)} + (|z_t|^2 + |z_x|^2)|_{x=n\alpha_k(t)} ] \alpha_k(t) dt.
 \end{aligned} \tag{4.5}$$

Next, we estimate every term in the right side of (4.5).

By (4.5) and (3.9), we have that

$$\begin{aligned}
 & \frac{1-\lambda_\varepsilon}{2} \int_0^T \int_0^{\alpha_k(t)} [|z_t(x,t)|^2 + |z_x(x,t)|^2] dx dt \\
 &= (1-\lambda_\varepsilon) \int_0^T E(t) dt \\
 &= (n-m-2\varepsilon) \int_0^T E(t) dt.
 \end{aligned} \tag{4.6}$$

By (3.1), (3.8) and (3.9), it follows that

$$\begin{aligned}
 M_2 &\triangleq \max_{(x,t) \in \hat{Q}_T^k} |p_t(x,t)| = \max\{(m+\varepsilon)k, (n-m-2\varepsilon)k\}. \\
 & \left| \int_0^T \int_0^{\alpha_k(t)} p_t z_x z_t dx dt \right| \\
 & \leq M_2 \int_0^T E(t) dt.
 \end{aligned} \tag{4.7}$$

And for  $0 < \varepsilon < \frac{n-m-mk}{k+2}$ , there exists

$$n - m - 2\varepsilon > M_2.$$

Note that for any  $t$ , by (4.1)

$$\begin{aligned} & \left| \int_0^{\alpha_k(t)} pz_x z_t dx \right| \\ & \leq M_1 E(t) \\ & \leq M_1 \frac{(1+k)\alpha_k(T)}{(1-k)\alpha_k(t)} E_T \\ & \leq \frac{M_1(1+k)\alpha_k(T)}{1-k} E_T. \end{aligned}$$

From which, we deduce that

$$\left| \int_0^{\alpha_k(t)} [pz_x z_t dx] \right|_0^T \leq \frac{2M_1(1+k)\alpha_k(T)}{1-k} E_T. \quad (4.8)$$

By (4.1), (4.5), (4.6), (4.7) and (4.8), we derive that,

$$\begin{aligned} & \frac{1}{2} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} [|z_t|^2 + |z_x|^2] dx dt \\ & \geq (n-m-2\varepsilon-M_2) \int_0^T E(t) dt - \frac{2M_1(1+k)\alpha_k(T)}{1-k} E_T \\ & \quad + \frac{\varepsilon}{2} \int_0^T [(z_t^2 + z_x^2)|_{x=m\alpha_k(t)} + (z_t^2 + z_x^2)|_{x=n\alpha_k(t)}] \alpha_k(t) dt \\ & \geq \left[ \frac{(n-m-2\varepsilon-M_2)(1-k)}{k(1+k)} \ln(1+kT) - \frac{2M_1(1+k)}{1-k} \right] \alpha_k(T) E_T \\ & \quad + \frac{\varepsilon}{2} \int_0^T [(z_t^2 + z_x^2)|_{x=m\alpha_k(t)} + (z_t^2 + z_x^2)|_{x=n\alpha_k(t)}] \alpha_k(t) dt \\ & \geq \left[ \frac{(n-m-2\varepsilon-M_2)(1-k)}{k(1+k)} \ln(1+kT) - \frac{2M_1(1+k)}{1-k} \right] \alpha_k(T) E_T \\ & \quad + \frac{\varepsilon}{2} \int_0^T [(z_t^2 + z_x^2)|_{x=m\alpha_k(t)} + (z_t^2 + z_x^2)|_{x=n\alpha_k(t)}] dt. \end{aligned} \quad (4.9)$$

To eliminate the integral of the derivative  $p_t$  from the left-hand side of inequality (4.9), we need to make the following estimates.

Multiplying the first equation of (2.4) by  $\frac{z}{2}$  and integrating on  $\omega_t \times (0, T)$ , we have

$$0 = \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} \left\{ z_{tt} \frac{z}{2} - z_{xx} \frac{z}{2} \right\} dx dt \triangleq L_1 + L_2.$$

We calculate the above two integrals  $L_i$  ( $i = 1, 2$ ).

$$\begin{aligned}
 L_1 &= \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} z_{tt} \frac{z}{2} dx dt \\
 &= \frac{1}{2} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} \left\{ \frac{\partial}{\partial t} [zz_t] - |z_t|^2 \right\} dx dt \\
 &= \frac{1}{2} \int_0^T \frac{\partial}{\partial t} \int_{m\alpha_k(t)}^{n\alpha_k(t)} zz_t dx dt - \frac{1}{2} \int_0^T (zz_t) \Big|_{x=n\alpha_k(t)} \cdot nk dt \\
 &\quad + \frac{1}{2} \int_0^T (zz_t) \Big|_{x=m\alpha_k(t)} \cdot mk dt - \frac{1}{2} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} |z_t|^2 dx dt \\
 &= \frac{1}{2} \int_{m\alpha_k(t)}^{n\alpha_k(t)} zz_t dx \Big|_0^T - \frac{1}{2} \int_0^T (zz_t) \Big|_{x=n\alpha_k(t)} \cdot nk dt \\
 &\quad + \frac{1}{2} \int_0^T (zz_t) \Big|_{x=m\alpha_k(t)} \cdot mk dt - \frac{1}{2} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} |z_t|^2 dx dt.
 \end{aligned} \tag{4.10}$$

Further, we have

$$\begin{aligned}
 L_2 &= -\frac{1}{2} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} z_{xx} z dx dt \\
 &= \frac{1}{2} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} \left\{ |z_x|^2 - \frac{\partial}{\partial x} [zz_x] \right\} dx dt \\
 &= \frac{1}{2} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} |z_x|^2 dx dt - \frac{1}{2} \int_0^T (zz_x) \Big|_{x=n\alpha_k(t)} dt \\
 &\quad + \frac{1}{2} \int_0^T (zz_x) \Big|_{x=m\alpha_k(t)} dt.
 \end{aligned} \tag{4.11}$$

By (4.10) and (4.11), we have

$$\begin{aligned}
& \frac{1}{2} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} |z_t|^2 dx dt \\
&= \frac{1}{2} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} |z_x|^2 dx dt + \frac{1}{2} \int_{m\alpha_k(t)}^{n\alpha_k(t)} z z_t dx \Bigg|_0^T \\
&\quad - \frac{1}{2} \int_0^T [(z z_t) \cdot nk + (z z_x)]|_{x=n\alpha_k(t)} dt \\
&\quad + \frac{1}{2} \int_0^T [(z z_t) \cdot mk + (z z_x)]|_{x=m\alpha_k(t)} dt.
\end{aligned} \tag{4.12}$$

Now, we estimate every term in the right side of (4.12).

$$\begin{aligned}
& \left| \frac{1}{2} \int_{m\alpha_k(t)}^{n\alpha_k(t)} z z_t dx \right| \\
&\leq \frac{\delta}{4} \int_{m\alpha_k(t)}^{n\alpha_k(t)} |z_t|^2 dx + \frac{1}{4\delta} \int_{m\alpha_k(t)}^{n\alpha_k(t)} z^2 dx, \quad \delta > 0.
\end{aligned}$$

A function  $\phi(t) \in H^1(0, T)$  of one variable admits the estimate

$$\max_{0 \leq t \leq T} |\phi(t)|^2 \leq \left(\frac{1}{T} + \frac{1}{\gamma}\right) \int_0^T \phi^2(s) ds + \gamma \int_0^T |\phi'(s)|^2 ds, \quad \gamma > 0,$$

which permits us to have the following estimate:

$$\begin{aligned}
& \left| \frac{1}{2} \int_{m\alpha_k(t)}^{n\alpha_k(t)} z z_t dx \right| \\
&\leq \frac{\delta}{2} E(t) + \left(\frac{1}{T} + \frac{1}{\gamma}\right) \frac{1}{4\delta} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} z^2 dx dt \\
&\quad + \frac{\gamma}{4\delta} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} |z_t|^2 dx dt.
\end{aligned} \tag{4.13}$$

Therefore by (4.13), it follows that

$$\begin{aligned}
& \left| \frac{1}{2} \int_{m\alpha_k(t)}^{n\alpha_k(t)} z z_t dx \right|_0^T \\
& \leq \delta E(t) + \left( \frac{1}{T} + \frac{1}{\gamma} \right) \frac{1}{2\delta} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} z^2 dx dt \\
& \quad + \frac{\gamma}{2\delta} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} |z_t|^2 dx dt.
\end{aligned} \tag{4.14}$$

In addition, we easily deduce that

$$\begin{aligned}
& \left| \frac{1}{2} \int_0^T [(z z_t) \cdot nk + (z z_x)]|_{x=n\alpha_k(t)} dt \right| \\
& \leq \frac{\delta}{4} \int_0^T (|z_t|^2 + |z_x|^2)|_{x=n\alpha_k(t)} dt \\
& \quad + \frac{1+n^2k^2}{4\delta} \int_0^T z^2|_{x=n\alpha_k(t)} dt
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
& \left| \frac{1}{2} \int_0^T [(z z_t) \cdot mk + (z z_x)]|_{x=m\alpha_k(t)} dt \right| \\
& \leq \frac{\delta}{4} \int_0^T (|z_t|^2 + |z_x|^2)|_{x=m\alpha_k(t)} dt \\
& \quad + \frac{1+m^2k^2}{4\delta} \int_0^T z^2|_{x=m\alpha_k(t)} dt.
\end{aligned} \tag{4.16}$$

Therefore by (4.12), (4.14), (4.15) and (4.16), there exists that

$$\begin{aligned}
& \frac{1}{2} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} |z_t|^2 dx dt \\
& \leq \delta E(t) + \left( \frac{1}{T} + \frac{1}{\gamma} \right) \frac{1}{2\delta} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} z^2 dx dt \\
& \quad + \frac{\gamma}{2\delta} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} |z_t|^2 dx dt + \frac{1}{2} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} |z_x|^2 dx dt
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
& + \frac{\delta}{4} \int_0^T (|z_t|^2 + |z_x|^2) \big|_{x=n\alpha_k(t)} dt + \frac{\delta}{4} \int_0^T (|z_t|^2 + |z_x|^2) \big|_{x=m\alpha_k(t)} dt \\
& + \frac{1+n^2k^2}{4\delta} \int_0^T z^2 \big|_{x=n\alpha_k(t)} dt + \frac{1+m^2k^2}{4\delta} \int_0^T z^2 \big|_{x=m\alpha_k(t)} dt.
\end{aligned}$$

By setting  $\gamma = \delta^2, \delta < 1$ , we obtain that

$$\begin{aligned}
& \frac{1}{2} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} |z_t|^2 dx dt \\
& \leq \frac{\delta}{1-\delta} E(t) + \frac{1}{1-\delta} \left( \frac{1}{T} + \frac{1}{\delta^2} \right) \frac{1}{2\delta} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} z^2 dx dt \\
& \quad + \frac{1}{2(1-\delta)} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} |z_x|^2 dx dt \\
& \quad + \frac{1}{1-\delta} \left\{ \frac{\delta}{4} \int_0^T (|z_t|^2 + |z_x|^2) \big|_{x=n\alpha_k(t)} dt + \frac{\delta}{4} \int_0^T (|z_t|^2 + |z_x|^2) \big|_{x=m\alpha_k(t)} dt \right. \\
& \quad \left. + \frac{1+n^2k^2}{4\delta} \int_0^T z^2 \big|_{x=n\alpha_k(t)} dt + \frac{1+m^2k^2}{4\delta} \int_0^T z^2 \big|_{x=m\alpha_k(t)} dt \right\}.
\end{aligned} \tag{4.18}$$

By (4.9) and (4.18), we have that

$$\begin{aligned}
& \frac{1}{2} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} |z_x|^2 dx dt + \frac{1}{2(1-\delta)} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} |z_x|^2 dx dt \\
& \quad + \frac{\delta}{1-\delta} E(t) + \frac{1}{1-\delta} \left( \frac{1}{T} + \frac{1}{\delta^2} \right) \frac{1}{2\delta} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} z^2 dx dt \\
& \quad + \frac{1}{1-\delta} \left\{ \frac{\delta}{4} \int_0^T (|z_t|^2 + |z_x|^2) \big|_{x=n\alpha_k(t)} dt + \frac{\delta}{4} \int_0^T (|z_t|^2 + |z_x|^2) \big|_{x=m\alpha_k(t)} dt \right. \\
& \quad \left. + \frac{1+n^2k^2}{4\delta} \int_0^T z^2 \big|_{x=n\alpha_k(t)} dt + \frac{1+m^2k^2}{4\delta} \int_0^T z^2 \big|_{x=m\alpha_k(t)} dt \right\} \\
& \geq \left[ \frac{(n-m-2\varepsilon-M_2)(1-k)}{k(1+k)} \ln(1+kT) - \frac{2M_1(1+k)}{1-k} \right] \alpha_k(T) E_T \\
& \quad + \frac{\varepsilon}{2} \int_0^T [(|z_t|^2 + |z_x|^2) \big|_{x=m\alpha_k(t)} + (|z_t|^2 + |z_x|^2) \big|_{x=n\alpha_k(t)}] dt.
\end{aligned} \tag{4.19}$$

Take  $\delta = \frac{2\varepsilon}{2\varepsilon+1}$ , then it is easy to check that

$$0 < \delta < 1, \\ \frac{1}{1-\delta} \cdot \frac{\delta}{4} = \frac{\varepsilon}{2}.$$

From (4.1) and (4.19), it follows that

$$\begin{aligned} & (\varepsilon + 1) \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} |z_x|^2 dx dt \\ & + \left( \frac{1}{T} + \left( \frac{1+2\varepsilon}{2\varepsilon} \right)^2 \right) \frac{(1+2\varepsilon)^2}{4\varepsilon} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} z^2 dx dt \\ & + \frac{(1+n^2k^2)(1+2\varepsilon)}{8\varepsilon} \int_0^T z^2|_{x=n\alpha_k(t)} dt + \frac{(1+m^2k^2)(1+2\varepsilon)}{8\varepsilon} \int_0^T z^2|_{x=m\alpha_k(t)} dt \} \\ & \geq \left[ \frac{(n-m-2\varepsilon-M_2)(1-k)}{k(1+k)} \ln(1+kT) - \frac{2M_1(1+k)}{1-k} \right] \alpha_k(T) E_T - 2\varepsilon E(t) \\ & \geq \left[ \frac{(n-m-2\varepsilon-M_2)(1-k)}{k(1+k)} \ln(1+kT) - \frac{2(M_1+\varepsilon)(1+k)}{1-k} \right] \alpha_k(T) E_T. \end{aligned} \quad (4.20)$$

If

$$T > T_k^* = \frac{\exp^{\frac{2k(M_1+\varepsilon)(1+k)^2}{(1-k)^2(n-m-2\varepsilon-M_2)}} - 1}{k},$$

it holds that

$$\left[ \frac{(n-m-2\varepsilon-M_2)(1-k)}{k(1+k)} \ln(1+kT) - \frac{2(M_1+\varepsilon)(1+k)}{1-k} \right] > 0.$$

Also,

$$\begin{aligned} & (\varepsilon + 1) \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} |z_x|^2 dx dt \\ & + \left( \frac{1}{T} + \left( \frac{1+2\varepsilon}{2\varepsilon} \right)^2 \right) \frac{(1+2\varepsilon)^2}{4\varepsilon} \int_0^T \int_{m\alpha_k(t)}^{n\alpha_k(t)} z^2 dx dt \\ & + \frac{(1+n^2k^2)(1+2\varepsilon)}{8\varepsilon} \int_0^T z^2|_{x=n\alpha_k(t)} dt + \frac{(1+m^2k^2)(1+2\varepsilon)}{8\varepsilon} \int_0^T z^2|_{x=m\alpha_k(t)} dt \} \\ & \geq \left[ \frac{(n-m-2\varepsilon-M_2)(1-k)}{k(1+k)} \ln(1+kT) - \frac{2(M_1+\varepsilon)(1+k)}{1-k} \right] \alpha_k(T) E_T \\ & \geq C \left[ \frac{(n-m-2\varepsilon-M_2)(1-k)}{k(1+k)} \ln(1+kT) - \frac{2(M_1+\varepsilon)(1+k)}{1-k} \right] \alpha_k(T) (|z^0|_{H_0^1(\Omega_T)}^2 + |z^1|_{L^2(\Omega_T)}^2). \end{aligned} \quad (4.21)$$

Note that, for given  $\varepsilon > 0$ , the left side of inequality (4.21) contains one of the equivalent norms in the function space  $H^1(\omega_t)$ :

$$\begin{aligned} \|\phi\|_{H^1(\omega_t)}^2 &= (\varepsilon + 1) \int_{m\alpha_k(t)}^{n\alpha_k(t)} |\phi_x|^2 dx \\ &+ \left(\frac{1}{T} + \left(\frac{1+2\varepsilon}{2\varepsilon}\right)^2\right) \frac{(1+2\varepsilon)^2}{4\varepsilon} \int_{m\alpha_k(t)}^{n\alpha_k(t)} \phi^2 dx \\ &+ \frac{(1+n^2k^2)(1+2\varepsilon)}{8\varepsilon} \phi^2|_{x=n\alpha_k(t)} + \frac{(1+m^2k^2)(1+2\varepsilon)}{8\varepsilon} \phi^2|_{x=m\alpha_k(t)}. \end{aligned}$$

By the choice of that norm, (2.5) follows from (4.21).

**Remark 4.2.** It is easy check that

$$T^0 \triangleq \lim_{k \rightarrow 0} T_k^* = 2 \max\{m, 1 - n\}.$$

It is well known that (1.2) in the cylindrical domain is internally controllable at any time  $T > T^0$ . However,  $T_k^*$  is not sharp.

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