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Global boundedness of classical solutions to a two species cancer invasion haptotaxis model with tissue remodeling [☆]

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ABSTRACT

In this paper, we deal with a haptotaxis cancer invasion model describing the migration and proliferation of two families of cancer cells, the epithelial-mesenchymal transition between two families cancer cells, the dynamics of matrix degrading enzymes, and the evolution and re-modeling of the extracellular matrix. Under appropriate regularity assumptions on initial data, by making some a priori estimates and applying iterative techniques, we establish the global existence and uniform boundedness of the unique classical solution in two-dimensional spatial domain for arbitrary cancer cells proliferation rates and in three-dimensional spatial setting for large cancer cells proliferation rates. These results improve and extend previously known ones.

1. Introduction

In this paper, we consider the following two species cancer invasion haptotaxis model

$$\begin{cases} c_t^D = \Delta c^D - \chi_D \nabla\cdot(c^D\nabla v) - \mu_{EMT}c^D + \mu_D c^D(1-c^D-c^S-v), & x\in\Omega, t>0,\\ c_t^S = \Delta c^S - \chi_S \nabla\cdot(c^S\nabla v) + \mu_{EMT}c^D + \mu_S c^S(1-c^D-c^S-v), & x\in\Omega, t>0,\\ \tau m_t = \Delta m + c^D + c^S - m, & x\in\Omega, t>0,\\ v_t = -mv + \mu_v v(1-c^D-c^S-v), & x\in\Omega, t>0,\\ \frac{\partial c^D}{\partial\nu} - \chi_D c^D\frac{\partial v}{\partial \nu} = \frac{\partial c^S}{\partial\nu} - \chi_S c^S\frac{\partial v}{\partial \nu} = \frac{\partial m}{\partial\nu} = 0, & x\in\partial\Omega, t>0,\\ c^D(x,0)=c_0^D(x),\ c^S(x,0)=c_0^S(x),\ \tau m(x,0)=\tau m_0(x),\ v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

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in a bounded domain $\Omega \subseteq \mathbb{R}^n$ $(n = 2, 3)$ with smooth boundary $\partial\Omega$, where ∂_ν denotes the outward normal derivative on $\partial\Omega$, the unknown functions $c^D(x,t)$, $c^S(x,t)$, $m(x,t)$, $v(x,t)$ represent the density of *differentiated cancer cells* (DCCs), the density of *cancer stem cells* (CSCs), the concentration of *the matrix metalloproteinases* (MMPs) and the concentration of *extracellular matrix* (ECM), respectively, $\chi_D, \chi_S > 0$ are the haptotactic coefficients of DCCs and CSCs correspondingly. The terms $-\mu_{EMT}c^D$ and $+\mu_{EMT}c^D$ stand for *the epithelial-mesenchymal transition* (EMT) from DCCs to CSCs, the terms $\mu_D(1 - c^D - c^S - v)$, $\mu_S(1 - c^D - c^S - v)$ with coefficients $\mu_D, \mu_S > 0$ describe the proliferation of DCCs and CSCs according to a logistic law which is influenced by the local density of the total biomass and includes competition for free space as well as resources with the ECM, the term $+c^D + c^S$ indicates the spontaneous production of the matrix degrading enzyme MMPs by DCCs and CSCs, the term $-m$ shows the decay of MMPs, the term $-mv$ illustrates the degradation of ECM by MMPs upon contact, the term $\mu_v v(1 - c^D - c^S - v)$ with coefficient $\mu_v > 0$ describes that the ECM is able to be self-remodeled in a typical logistic manner in the absence of cancer cells (DCCs and CSCs) and assumed to compete for free space and resources with cancer cells.

This model describes the process of two families of cancer cells invasion of surrounding healthy tissue, which involves many biological mechanisms, for instance, the migration of cancer cells arising from random diffusion and haptotaxis (the movement of cancer cells is biased towards a gradient of the non-diffusible ECM [5,27]), the epithelial-mesenchymal transition from DCCs to CSCs [22], the proliferation of cancer cells and their competition for space with ECM, the production and decay of MMPs, the degradation and self-construction of ECM. The parameter $\tau \in \{0, 1\}$. When $\tau = 0$, it indicates the diffusion rate of MMPs is much faster than that of cancer cells [7]. When $\tau = 1$, it was recently proposed by Hellmann et al. [13] and Sfakianakis et al. [28] as a modified tumor invasion model with haptotaxis effect of Anderson et al. type [2]. A novel point or a key feature of this model is that it includes not only two families of cancer cells with haptotactic movement but the epithelial-mesenchymal transition, as for more biological background and explanations of it, we refer to [13,28] and cited references therein.

Mathematicians have been extensively attracted by the haptotaxis cancer invasion model to develop a detailed analysis of the global existence and asymptotic behavior of solution. Walker and Webb [35] considered the haptotaxis model of Chaplain and Anderson [6] which includes one cancer cell species, the matrix degrading enzyme, ECM and oxygen, and they proved the existence of unique global classical solution. An Perumpanani and Byrne's haptotaxis model [27] consisting of tumor cell, tumor cell-derived protease and the collagen gel was investigated by Tao and Zhu [34], and the existence and uniqueness of global classical solution was proved by a priori estimates, together with the parabolic L^p estimates and Schauder estimates. Subsequently, Liţcanu and Morales-Rodrigo [20] analytically studied the asymptotic behavior of solutions to the Perumpanani and Byrne's model. The global existence of weak solutions to the simplified haptotaxis model of [2] was discussed by Marciniak-Crzochra and Ptashnyk [23] and the uniform boundedness of solutions was showed by applying the method of bounded invariant rectangles. What is worth mentioning is that Tao [30] is the first attempt to investigate the global existence of classical solution to the haptotaxis model with tissue remodeling proposed in [6]. More related works are referred to [3,8,21,37] and abundant references cited therein.

It is important to remark that there is merely one cancer cell species in above mentioned works on haptotaxis cancer invasion models, because those haptotaxis cancer invasion models with two or more cancer cell species may be difficult to be analyzed as a result of the complex structure and strong coupling between various cancer cell species and ECM. Based on the go-or-grow hypothesis assuming cancer cells can either move or proliferate, a strongly coupled PDE-ODE-ODE two species cancer invasion haptotaxis model was proposed by Stinner et al. [29], and the global existence of weak solutions was obtained in arbitrary dimensions. More recently, we particularly mentioned that, for the case of $\tau = 1$ in two dimensional space of haptotaxis model (1.1), Giesselmann et al. [10] investigated the global existence and uniqueness of the classical solutions for large μ_D and μ_S. A natural question followed by the later contribution [10] is whether or not the global solvability of haptotaxis model (1.1) remains valid for small $\mu_D, \mu_S > 0$ or arbitrary $\mu_D, \mu_S > 0$ in two dimensions, to the best of our knowledge, it remains open. In addition, the global solvability of haptotaxis model (1.1) in three dimensions has never been touched. No matter biological relevance or mathematical meaning, we find it is worth addressing above question.

In order to answer above question, we discuss the global solvability and boundedness of (1.1) in dimensions 2 and 3. As opposed to [10], we investigate the global solvability of (1.1) with $\tau \in \{0,1\}$ in two-dimensional case for arbitrary $\mu_D, \mu_S > 0$ and in three-dimensional setting for large $\mu_D, \mu_S > 0$, however, only the case of $n = 2$ and $\tau = 1$ for large $\mu_D, \mu_S > 0$ was discussed in [10]. In detail, **for the case $\boldsymbol{n = 2}$ and $\boldsymbol{\tau = 1}$**, we only assume that $\mu_D, \mu_S > 0$ other than $\mu_D \ge \chi_D \mu_v,\ \mu_S \ge \chi_S \mu_v$ (see (1.6) in [10]), the latter only need to estimate the terms $\int_\Omega \left(a^D\right)^p, \int_\Omega \left(a^S\right)^p$, but in the present paper, we need to estimate $\int_\Omega \left(a^D\right)^{p+1}, \int_\Omega \left(a^S\right)^{p+1}$, this leads to more obstacles. To overcome these obstacles, we make some a priori estimates. Firstly, we develop a certain dissipative property of the functionals $\int_\Omega e^{\chi_D v}\left(a^D\right)^2, \int_\Omega e^{\chi_S v}\left(a^S\right)^2$ which will serve as a starting point to establish an iteration step resulting in the $L^\infty(\Omega)$ boundedness of a^D and a^S. Then, by building a bridge between $\|\nabla v(\cdot,t)\|^q_{L^q(\Omega)}$ and $\int_0^t \left(\left\|\nabla a^D(\cdot,s)\right\|^q_{L^q(\Omega)} + \left\|\nabla a^S(\cdot,s)\right\|^q_{L^q(\Omega)}\right) ds$, we show the estimates of $\|\nabla v\|_{L^5(\Omega)}$, thereby complete the proof. **In the case of $\boldsymbol{n = 2}$ and $\boldsymbol{\tau = 0}$**, one can establish the estimate of $\|m(\cdot,t)\|_{L^3(\Omega)}$ by the L^1 estimate on semi-linear elliptic equations, then one can prove the main result by proceeding in like manner as the case of $n = 2$ and $\tau = 1$. **In the condition $\boldsymbol{n = 3}$ and $\boldsymbol{\tau = 1}$**, we derive an adapted iteration criterion to raise successfully the regularities of a^D, a^S from $L^1(\Omega)$ to $L^p(\Omega)$ for any $p > 1$, then complete the proof by applying the similar way as the case of $n = 2$. **Under the circumstance $\boldsymbol{n = 3}$ and $\boldsymbol{\tau = 0}$**, the estimates of $\|m(\cdot,t)\|_{L^3(\Omega)}$ can be turned into a priori estimates of $\left\|a^D(\cdot,t)\right\|_{L^3(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^3(\Omega)}$ with the help of the standard elliptic L^p theory, which can guarantee the iteration criterion used in the case of $n = 3$ and $\tau = 1$ also can be applied here.

Let us give the following hypotheses for the need of analysis.

(H_1) Suppose that in what follows that the prescribed initial data satisfy

$$\begin{cases} \left(c_0^D, c_0^S, m_0, v_0\right) \in C^{2+\alpha}(\bar{\Omega}) \text{ for some } \alpha \in (0,1), \\ c_0^D, c_0^S, m_0 > 0,\ 0 < v_0 \le 1 \quad \text{in } \Omega, \\ \frac{\partial c_0^D}{\partial \nu} - \chi_D c_0^D \frac{\partial v_0}{\partial \nu} = \frac{\partial c_0^S}{\partial \nu} - \chi_S c_0^S \frac{\partial v_0}{\partial \nu} = \frac{\partial m_0}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

(H_2) Assume that the EMT rate function $\mu_{EMT}\left(c^D, c^S, m, v\right) : \mathbb{R}^4 \to \mathbb{R}$ satisfies $0 \le \mu_{EMT} \le \mu_M$ for some constant $\mu_M > 0$. Moreover, $\mu_{EMT}\left(c^D, c^S, m, v\right)$ is Lipschitz continuous, that is, for some finite constant $L > 0$, there holds

$$\begin{aligned} &\left\|\mu_{EMT}\left(c_1^D, c_1^S, m_1, v_1\right) - \mu_{EMT}\left(c_2^D, c_2^S, m_2, v_2\right)\right\|_{C(\bar{Q}_T)} \\ &\le L\left(\left\|c_1^D - c_2^D\right\|_{C(\bar{Q}_T)} + \left\|c_1^S - c_2^S\right\|_{C(\bar{Q}_T)} + \|m_1 - m_2\|_{C(\bar{Q}_T)} + \|v_1 - v_2\|_{C(\bar{Q}_T)}\right). \end{aligned}$$

Based on above hypotheses, the main results of the present paper read as follows.

Theorem 1.1. *(Global existence in 2 dimensions) Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega \in C^{2+\alpha}$ for some $\alpha \in (0,1)$ and $\tau \in \{0,1\}$. Suppose that $\chi_D, \chi_S, \mu_D, \mu_S > 0$ and $\mu_v > 0$, and the hypotheses $(H_1) - (H_2)$ hold. Then the problem (1.1) admits a unique classical solution with $c^D, c^S, m > 0$ and $0 < v \le 1$, where c^D, c^S, m, v are bounded uniformly in the following sense*

$$\sup_{t \in (0,\infty)} \left\|c^D(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|c^S(\cdot,t)\right\|_{L^\infty(\Omega)} + \|m(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \le C \tag{1.2}$$

for some $C > 0$ independent of time.

Theorem 1.2. *(Global existence in 3 dimensions) Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega \in C^{2+\alpha}$ for some $\alpha \in (0,1)$ and $\tau \in \{0,1\}$. Suppose that $\chi_D, \chi_S, \mu_D, \mu_S > 0$ and $\mu_v > 0$, and the hypotheses $(H_1) - (H_2)$ hold. In addition, assume that $\mu_D \geq \chi_D\mu_v$, $\mu_S \geq \chi_S\mu_v$. Then the problem (1.1) admits a unique classical solution with $c^D, c^S, m > 0$ and $0 < v \leq 1$, where c^D, c^S, m, v are bounded uniformly in the following sense*

$$\sup_{t\in(0,\infty)} \left\|c^D(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|c^S(\cdot,t)\right\|_{L^\infty(\Omega)} + \|m(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \leq C \tag{1.3}$$

for some $C > 0$ independent of time.

The remaining part of this article is organized as follows. In Section 2, some necessary notations of this problem are introduced and some preliminary results which are useful for our investigation are given. Section 3 is devoted to derive the local existence of unique classical solution for (1.1) on account of the Banach's fixed point theorem and provide a weakened extensibility criterion of local solutions with the help of the L^p theory and Schauder estimates of parabolic and elliptic equations. In Section 4, we prove Theorem 1.1 by making some a priori estimates. Section 5 focus on giving the proof of Theorem 1.2. Finally, we conclude this paper in Section 6.

2. Notations and preliminaries

In this section, we shall introduce some notations and give some preliminary results which will be often used in sequel and indispensable for dealing with our problem.

For consistency, in what follows, we denote $Q_T = \Omega \times (0,T)$ for any fixed $T \in (0,\infty)$, $Q_t = \Omega \times (0,t)$ for any $t \in (0,T]$, $\Sigma_T = \partial\Omega \times (0,T)$ and $|\Omega|$ represent the measure of Ω. For simplicity, we abbreviate $\int_\Omega y(x)dx$ as $\int_\Omega y$, the variables x will not be omitted in this integral if we emphasize the spatial dependence of y. In addition, for the convenience of notation, throughout this section, we denote various positive constants by A_0 that may be different in different places.

Let us first recall the following derivate of Poincaré's inequality [14].

Lemma 2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth enough boundary. Then, for any $u \in W^{1,p}(\Omega)$, there exists a positive constant A_0 such that*

$$\|u\|_{W^{1,p}(\Omega)} \leq A_0 \left(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^q(\Omega)}\right) \tag{2.1}$$

with arbitrary $p > 1$ and $q > 0$.

Next, we shall need the following well-known Gagliardo-Nirenberg interpolation inequality in several places [9,32].

Lemma 2.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth enough boundary. Let l, k be any integers satisfying $0 \leq l < k$ and $p > 0$. Then, for any function $u \in W^{k,q}(\Omega)\bigcap L^r(\Omega)$, there exists a positive constant C_{GN} depending only on Ω, q, k, r, n such that the following inequality holds:*

$$\|D^l u\|_{L^p(\Omega)} = C_{GN}\left(\|D^k u\|^{\lambda}_{L^q(\Omega)}\|u\|^{1-\lambda}_{L^r(\Omega)} + \|u\|_{L^r(\Omega)}\right), \tag{2.2}$$

where

$$\frac{l}{k} \leq \lambda \leq 1, \quad 1 \leq q, r \leq \infty, \quad \frac{1}{p} - \frac{l}{n} = \lambda\left(\frac{1}{q} - \frac{k}{n}\right) + \frac{1-\lambda}{r}$$

when $k-l-\frac{n}{q}$ is not a nonnegative integer;

$$\frac{l}{k} \leq \lambda < 1, \quad 1 < q < \infty, \quad r > 1, \quad \frac{1}{p} - \frac{l}{n} = \lambda\left(\frac{1}{q} - \frac{k}{n}\right) + \frac{1-\lambda}{r}$$

when $k-l-\frac{n}{q}$ is a nonnegative integer.

In addition, the following result will play an essential role in the proof of Lemma 4.2 bellow, we would like refer the reader to [25, Lemma 2.3] for its proof.

Lemma 2.3. *Let $T > 0$, $\theta \in (0,T)$ and assume that y is a nonnegative absolutely continuous function satisfying*

$$y'(t) + a(t)y(t) \leq b(t)y(t) + c(t) \qquad \text{for } a.e.\ t \in (0,T) \tag{2.3}$$

with some functions $a(t) > 0$, $b(t) \geq 0$, $c(t) \geq 0$ and $a, b, c \in L^1_{loc}(0,T)$ for which there exist $b_1, c_1 > 0$ and $\gamma > 0$ such that

$$\sup_{0\leq t\leq T-\theta} \int_t^{t+\theta} b(s)ds \leq b_1, \qquad \sup_{0\leq t\leq T-\theta} \int_t^{t+\theta} c(s)ds \leq c_1$$

and

$$\sup_{0\leq t\leq T-\theta} \int_t^{t+\theta} a(s)ds - \sup_{0\leq t\leq T-\theta} \int_t^{t+\theta} b(s)ds \geq \gamma \quad \text{for any } t \in (0, T-\theta).$$

Then for all $t \in (0,T)$, we have

$$y(t) \leq y(0)e^{b_1} + \frac{c_1 e^{2b_1}}{1-e^{-\gamma}} + c_1 e^{b_1}. \tag{2.4}$$

For the estimates of m when $\tau = 1$ in (1.1), we need following Lemma 2.4. We omit its proof for simplicity and refer the reader to [18, Lemma 1 and 32, Lemmata 3.2-3.3].

Lemma 2.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth enough boundary, $T > 0$ and $u_0 \in W^{1,\infty}(\Omega)$. Suppose that $\|f(\cdot,t)\|_{L^p(\Omega)} \leq A_0$ for all $t \in (0,T)$ and (u,f) satisfies the following inhomogeneous linear heat equation*

$$\begin{cases} u_t = \Delta u - u + f, & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0, & \text{in } \Omega. \end{cases} \tag{2.5}$$

Then, for all $t \in (0,T)$ and for every $1 \leq p < n$, we have

$$\|u(\cdot,t)\|_{W^{1,q}(\Omega)} \leq A_0, \tag{2.6}$$

where

$$q < \frac{np}{n-p}. \tag{2.7}$$

If $p = n$, then (2.6) holds with every $q < \infty$; if $p > n$, then (2.6) holds with $q = \infty$. In addition, there holds

$$\|u(\cdot,t)\|_{L^s(\Omega)} \le A_0 \tag{2.8}$$

for all $t \in (0,T)$ and any $s > p$ satisfying

$$\frac{1}{s} + \frac{2}{n} > \frac{1}{p}. \tag{2.9}$$

3. Local existence and extensibility criterion

It is worthwhile to point out that, if (c^D, c^S, m, v) is a local classical solution of (1.1), we require $v \in C^{2,1}(Q_T)$ at least from the two terms $-\chi_D c^D \Delta v$ and $-\chi_S c^S \Delta v$ of (1.1). However, above mentioned regularity of v is difficult to obtain due to the v-equation of (1.1) is only an ODE. Hence, for the convenience of subsequent analysis, we make the following variable transformations followed [17,31,33]:

$$\begin{cases} a^D = c^D e^{-\chi_D v}, \\ a^S = c^S e^{-\chi_S v}. \end{cases}$$

Consequently, the original system (1.1) can be transformed as

$$\begin{cases} a_t^D = e^{-\chi_D v}\nabla\cdot\left(e^{\chi_D v}\nabla a^D\right) + \chi_D a^D m v - \mu_{EMT} a^D & \\ \qquad + \left(\mu_D - \chi_D \mu_v v\right) a^D \left(1 - e^{\chi_D v} a^D - e^{\chi_S v} a^S - v\right), & x \in \Omega, t > 0, \\ a_t^S = e^{-\chi_S v}\nabla\cdot\left(e^{\chi_S v}\nabla a^S\right) + \chi_S a^S m v + \mu_{EMT} a^D e^{\chi_D v - \chi_S v} & \\ \qquad + \left(\mu_S - \chi_S \mu_v v\right) a^S \left(1 - e^{\chi_D v} a^D - e^{\chi_S v} a^S - v\right), & x \in \Omega, t > 0, \\ \tau m_t = \Delta m + e^{\chi_D v} a^D + e^{\chi_S v} a^S - m, & x \in \Omega, t > 0, \\ v_t = -mv + \mu_v v\left(1 - e^{\chi_D v} a^D - e^{\chi_S v} a^S - v\right), & x \in \Omega, t > 0, \\ \frac{\partial a^D}{\partial \nu} = \frac{\partial a^S}{\partial \nu} = \frac{\partial m}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ a^D(x,0) = a_0^D(x) = c_0^D(x) e^{-\chi_D v_0(x)},\ a^S(x,0) = a_0^S(x) = c_0^S(x) e^{-\chi_S v_0(x)}, & \\ \tau m(x,0) = \tau m_0(x),\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{3.1}$$

It is worth noting that the systems (1.1) and (3.1) are equivalent in the sense of classical solution. Therefore, in what follows we only need discuss the classical solution of (3.1).

In addition, it follows from the assumptions (H_1) that

$$\begin{cases} \left(a_0^D, a_0^S, m_0, v_0\right) \in C^{2+\alpha}(\bar{\Omega}) & \text{for some } \alpha \in (0,1), \\ a_0^D, a_0^S, m_0 > 0,\ 0 < v_0 \le 1 & \text{for } x \in \Omega, \\ \frac{\partial a_0^D}{\partial \nu} = \frac{\partial a_0^S}{\partial \nu} = \frac{\partial m_0}{\partial \nu} = 0. & \end{cases} \tag{3.2}$$

By appropriate adaption of a fixed point arguments and results in [30, Theorems 2.1-2.2 and 34, Lemma 2.1], we have the following two statements on local existence of classical solutions to the problem (3.1). For the convenience of notation, throughout this section, we shall use a universal positive constant B_0 to denote various constants that may vary in different places, and denote them by $B_0(a.b...)$ while we need emphasize this constant depending on some parameters $a, b...$

Lemma 3.1. *(Local existence and extensibility criterion for $\tau = 1$) Let $\Omega \subseteq \mathbb{R}^n$ $(n = 2,3)$ be a bounded domain with smooth enough boundary. Suppose that $\chi_D, \chi_S, \mu_D, \mu_S, \mu_v > 0$, $\tau = 1$, and that the assumptions*

(H_2) and (3.2) hold. Then there exists $T_{max} \in (0, \infty]$ such that the problem (3.1) admits a unique classical solution $\left(a^D, a^S, m, v\right) \in \left(C^{2+\alpha,1+\frac{\alpha}{2}}\left(\bar{\Omega} \times [0, T_{max})\right)\right)^4$ for $\alpha \in (0, 1)$ satisfying

$$a^D > 0, \quad a^S > 0, \quad m > 0, \quad 0 < v \leq 1, \qquad \text{for all } (x, t) \in \Omega \times (0, T_{max}), \tag{3.3}$$

and which are such that

$$\text{either } T_{max} = \infty \text{ or } \limsup_{t \nearrow T_{max}} \left\{ \left\|a^D(\cdot, t)\right\|_{C^{2+\alpha}(\bar{\Omega})} + \left\|a^S(\cdot, t)\right\|_{C^{2+\alpha}(\bar{\Omega})} + \|v(\cdot, t)\|_{C^{2+\alpha}(\bar{\Omega})} \right\} = \infty. \tag{3.4}$$

Lemma 3.2. *(Local existence and extensibility criterion for $\tau = 0$) Let $\Omega \subseteq \mathbb{R}^n$ $(n = 2, 3)$ be a bounded domain with smooth enough boundary. Suppose that $\chi_D, \chi_S, \mu_D, \mu_S, \mu_v > 0$, $\tau = 0$, and that the assumptions (H_2) and (3.2) hold. Then there exists $T_{max} \in (0, \infty]$ such that the problem (3.1) admits a unique classical solution $\left(a^D, a^S, m, v\right) \in \left(C^{2+\alpha,1+\frac{\alpha}{2}}\left(\bar{\Omega} \times [0, T_{max})\right)\right)^2 \times C^{2+\alpha,\frac{\alpha}{2}}\left(\bar{\Omega} \times [0, T_{max})\right) \times C^{2+\alpha,1+\frac{\alpha}{2}}\left(\bar{\Omega} \times [0, T_{max})\right)$ for $\alpha \in (0, 1)$ satisfying*

$$a^D > 0, \quad a^S > 0, \quad m > 0, \quad 0 < v \leq 1, \qquad \text{for all } (x, t) \in \Omega \times (0, T_{max}), \tag{3.5}$$

and which are such that

$$\text{either } T_{max} = \infty \text{ or } \limsup_{t \nearrow T_{max}} \left\{ \left\|a^D(\cdot, t)\right\|_{C^{2+\alpha}(\bar{\Omega})} + \left\|a^S(\cdot, t)\right\|_{C^{2+\alpha}(\bar{\Omega})} + \|v(\cdot, t)\|_{C^{2+\alpha}(\bar{\Omega})} \right\} = \infty. \tag{3.6}$$

Remark 3.1. It is worthwhile to point out that we only assume the nonnegative function $\mu_{EMT}(c^D, c^S, m, v)$ is bounded and Lipschitz continuous other than the Lipschitz continuous of its first derivatives. This assumption on $\mu_{EMT}(c^D, c^S, m, v)$ is weaker than that supposed in [10] (see (1.7a)-(1.7c) of [10]).

According to Lemma 3.1 and Lemma 3.2, in order to obtain the global existence of the unique classical solution of problem (3.1), we only need show the boundedness of $\limsup_{t \nearrow T_{max}} \left\{ \left\|a^D(\cdot, t)\right\|_{C^{2+\alpha}(\bar{\Omega})} + \left\|a^S(\cdot, t)\right\|_{C^{2+\alpha}(\bar{\Omega})} + \|v(\cdot, t)\|_{C^{2+\alpha}(\bar{\Omega})} \right\}$, but it is always difficult to achieve it. Therefore, inspired by [24, Lemma 2.2 and 16, Lemma 3.2], we weaken the extensibility criterions (3.4) and (3.6) as follows.

Lemma 3.3. *(Weakened extensibility criterion) Let $\tau \in \{0, 1\}$. Assume that the assumptions (H_2) and (3.2) hold. Then the solutions $\left(a^D, a^S, m, v\right)$ of (3.1) constructed in Lemma 3.1 and Lemma 3.2 have the property that*

$$\text{either } T_{max} = \infty \text{ or } \limsup_{t \nearrow T_{max}} \left(\left\|a^D(\cdot, t)\right\|_{L^\infty(\Omega)} + \left\|a^S(\cdot, t)\right\|_{L^\infty(\Omega)} + \|\nabla v(\cdot, t)\|_{L^5(\Omega)} \right) = \infty. \tag{3.7}$$

Proof. Let us consider the case $\tau = 1$. We proceed the proof by contradiction. Assume that $T_{max} < \infty$, but

$$\sup_{t \in (0, T_{max})} \left(\left\|a^D(\cdot, t)\right\|_{L^\infty(\Omega)} + \left\|a^S(\cdot, t)\right\|_{L^\infty(\Omega)} + \|\nabla v(\cdot, t)\|_{L^5(\Omega)} \right) \leq B_0. \tag{3.8}$$

The a^D, a^S-equations of (3.1) can be rewritten as the following linear forms

$$a_t^D = \Delta a^D + \chi_D \nabla v \nabla a^D + g_8 a^D \tag{3.9}$$

and

$$a_t^S = \Delta a^S + \chi_S \nabla v \nabla a^S + g_9 a^S + \mu_{EMT} e^{\chi_D v - \chi_S v} a^D, \tag{3.10}$$

where $g_8 = (\chi_D mv - \mu_{EMT}) + (\mu_D - \chi_D \mu_v v)\left(1 - e^{\chi_D v} a^D - e^{\chi_S v} a^S - v\right)$, $g_9 = \chi_S mv + (\mu_S - \chi_S \mu_v v)\,(1 - e^{\chi_D v} a^D - e^{\chi_S v} a^S - v)$.
By using Lemma 2.4, we infer from (3.8) that

$$\|m\|_{W^{1,\infty}(\Omega)} \le B_0 \quad \text{for all } t \in (0, T_{max}). \tag{3.11}$$

This in conjunction with the assumption (H_2) and $0 < v \le 1$, we deduce that there exist some $B_0 > 0$ such that

$$\|g_8\|_{L^\infty(\Omega)} \le B_0 \quad \text{and} \quad \|g_9\|_{L^\infty(\Omega)} \le B_0 \quad \text{for all } t \in (0, T_{max}). \tag{3.12}$$

This together with the maximal parabolic regularity results (see [19, Theorem IV.9.1] and [11, Theorem 2.3]) yields

$$\left\|a^D\right\|_{W_4^{2,1}(Q_{T_{max}})} \le B_0 \quad \text{and} \quad \left\|a^S\right\|_{W_4^{2,1}(Q_{T_{max}})} \le B_0. \tag{3.13}$$

Hence, by the Sobolev embedding theorem (see Lemma II.3.3 of [19]), we have

$$\left\|\nabla a^D\right\|_{L^{20}(Q_{T_{max}})} \le B_0 \quad \text{and} \quad \left\|\nabla a^S\right\|_{L^{20}(Q_{T_{max}})} \le B_0. \tag{3.14}$$

Now, let us deal with v. Applying ∇ to the v-equation of (3.1), we arrive

$$\nabla v_t = g_{10} \nabla v + g_{11}, \tag{3.15}$$

where $g_{10} = -m + \mu_v\left(1 - e^{\chi_D v} a^D - e^{\chi_S v} a^S - v\right) - \mu_v v\left(1 + \chi_D e^{\chi_D v} a^D + \chi_S e^{\chi_S v} a^S\right)$, $g_{11} = -v\nabla m - \mu_v v\left(e^{\chi_D v} \nabla a^D + e^{\chi_S v} \nabla a^S\right)$.
Furthermore, we find that $g_{10} \le \mu_v$ due to (3.3). Multiplying (3.15) by $q|\nabla v|^{q-2}\nabla v$ for $q \ge 2$, in view of (3.3) and (3.11), it follows from Young's inequality that

$$\begin{aligned}
\left(|\nabla v|^q\right)_t &= q g_{10} |\nabla v|^q - qv|\nabla v|^{q-2}\nabla v \cdot \nabla m - q\mu_v v |\nabla v|^{q-2}\nabla v \cdot \left(e^{\chi_D v}\nabla a^D + e^{\chi_S v}\nabla a^S\right) \\
&\le q\mu_v |\nabla v|^q + q|\nabla m||\nabla v|^{q-1} + q\mu_v|\nabla v|^{q-1}\left(e^{\chi_D}|\nabla a^D| + e^{\chi_S}|\nabla a^S|\right) \\
&\le B_0(q)\left(|\nabla v|^q + |\nabla a^D|^q + |\nabla a^S|^q + 1\right).
\end{aligned} \tag{3.16}$$

Integrating (3.16) over Ω and using the Gronwall's inequality, we obtain

$$\int_\Omega |\nabla v|^q \le B_0(q)\left(\int_0^t \int_\Omega |\nabla a^D|^q + \int_0^t \int_\Omega |\nabla a^S|^q + 1\right) \quad \text{for all } t \in (0, T_{max}). \tag{3.17}$$

This in conjunction with (3.14) entails

$$\|\nabla v(\cdot, t)\|_{L^{20}(\Omega)} \le B_0 \quad \text{for all } t \in (0, T_{max}). \tag{3.18}$$

According to (3.18) and repeating above process, we further get

$$\left\|a^D\right\|_{W_{20}^{2,1}(Q_{T_{max}})} \le B_0, \quad \text{and} \quad \left\|a^S\right\|_{W_{20}^{2,1}(Q_{T_{max}})} \le B_0. \tag{3.19}$$

Thanks to the t-anisotropic embedding argument $W_p^{2,1}(Q_T) \hookrightarrow C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$ for any $\alpha \in (0, 2 - \frac{n+2}{p}]$ if $p > \frac{n+2}{2}$ [19], this implies

$$\left\|a^D\right\|_{C^{\frac{7}{4},\frac{7}{8}}(\bar{Q}_{T_{max}})} \leq B_0, \quad \text{and} \quad \left\|a^S\right\|_{C^{\frac{7}{4},\frac{7}{8}}(\bar{Q}_{T_{max}})} \leq B_0. \tag{3.20}$$

For m, we also have

$$\|m\|_{C^{\frac{7}{4},\frac{7}{8}}(\bar{Q}_{T_{max}})} \leq B_0. \tag{3.21}$$

As for v, by the fourth equation of (3.1), we have

$$\|v(\cdot,t)\|_{C^{\frac{3}{4}}(\bar{\Omega})} + \|v_t(\cdot,t)\|_{C^{\frac{3}{4}}(\bar{\Omega})} \leq B_0 \quad \text{for all } t \in (0, T_{max}). \tag{3.22}$$

From (3.15), (3.20) and (3.21), we obtain

$$\|\nabla v(\cdot,t)\|_{C^{\frac{3}{4}}(\bar{\Omega})} + \|\nabla v_t(\cdot,t)\|_{C^{\frac{3}{4}}(\bar{\Omega})} \leq B_0 \quad \text{for all } t \in (0, T_{max}). \tag{3.23}$$

In virtue of (3.2), if we take $\beta = \min\left\{\frac{3}{4}, \alpha\right\}$, thus by using the regularities of a^D, a^S and v and the parabolic Schauder theory [19], we infer from the m-equations of (3.1) that

$$\|m\|_{C^{2+\beta,\frac{2+\beta}{2}}(\bar{Q}_{T_{max}})} \leq B_0. \tag{3.24}$$

Similar to m, it follows from (3.9) and (3.10) that

$$\left\|a^D\right\|_{C^{2+\beta,\frac{2+\beta}{2}}(\bar{Q}_{T_{max}})} + \left\|a^S\right\|_{C^{2+\beta,\frac{2+\beta}{2}}(\bar{Q}_{T_{max}})} \leq B_0. \tag{3.25}$$

Therefore, a combination of (3.24), (3.25) and the v-equations of (3.1) immediately yields

$$\|v\|_{C^{2+\beta}(\bar{\Omega})} + \|v_t\|_{C^{2+\beta}(\bar{\Omega})} \leq B_0 \quad \text{for all } t \in (0, T_{max}). \tag{3.26}$$

Recalling of the m-equations of (3.1) and applying the parabolic Schauder theory, we further have

$$\|m\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{Q}_{T_{max}})} \leq B_0. \tag{3.27}$$

Going back to (3.9), (3.10) and in conjunction with (3.26), one can obtain

$$\left\|a^D\right\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{Q}_{T_{max}})} + \left\|a^S\right\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{Q}_{T_{max}})} + \|v\|_{C^{2+\beta}(\bar{\Omega})} + \|v_t\|_{C^{2+\beta}(\bar{\Omega})} \leq B_0. \tag{3.28}$$

But this contradicts the extensibility criterion (3.4) established in Lemma 3.1. Therefore, we assert that $T_{max} = \infty$.

As for the case $\tau = 0$, it is not hard to get it by proceeding as in the proof of case $\tau = 1$, so we omit it for simplicity.

Thus, the proof of Lemma 3.3 is completed. □

Before finishing this section, let us give the following observations for the local classical solution of (3.1) which will be used frequently below.

Lemma 3.4. *Let $\left(a^D, a^S, m, v\right)$ be the local classical solution of (3.1) constructed in Lemma 3.1 and Lemma 3.2. Suppose that the hypotheses of Theorem 1.1 hold. Then, for any $p > 1$, we have*

$$
\begin{aligned}
&\frac{d}{dt}\int\limits_{\Omega} e^{\chi_D v}\left(a^D\right)^p+\frac{4(p-1)}{p}\int\limits_{\Omega} e^{\chi_D v}\left|\nabla\left(a^D\right)^{\frac{p}{2}}\right|^2+\left(p\mu_D-(p-1)\chi_D\mu_v\right)\int\limits_{\Omega} e^{2\chi_D v}\left(a^D\right)^{p+1}\\
&\le\left(p\mu_D+(p-1)\chi_D\mu_v\right)\int\limits_{\Omega} e^{\chi_D v}\left(a^D\right)^p+(p-1)\chi_D\int\limits_{\Omega} e^{\chi_D v}\left(a^D\right)^p m\\
&\quad+\left((p-1)\chi_D\mu_v-p\mu_D\right)\int\limits_{\Omega} e^{\chi_D v+\chi_S v}\left(a^D\right)^p a^S
\end{aligned}
\tag{3.29}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}\int\limits_{\Omega} e^{\chi_S v}\left(a^S\right)^p+\frac{4(p-1)}{p}\int\limits_{\Omega} e^{\chi_S v}\left|\nabla\left(a^S\right)^{\frac{p}{2}}\right|^2+\left(p\mu_S-(p-1)\chi_S\mu_v\right)\int\limits_{\Omega} e^{2\chi_S v}\left(a^S\right)^{p+1}\\
&\le\left(p\mu_S+(p-1)\chi_S\mu_v\right)\int\limits_{\Omega} e^{\chi_S v}\left(a^S\right)^p+(p-1)\chi_S\int\limits_{\Omega} e^{\chi_S v}\left(a^S\right)^p m\\
&\quad+\left((p-1)\chi_S\mu_v-p\mu_S\right)\int\limits_{\Omega} e^{\chi_D v+\chi_S v}\left(a^S\right)^p a^D+p\mu_M\int\limits_{\Omega} e^{\chi_D v}a^D\left(a^S\right)^{p-1}.
\end{aligned}
\tag{3.30}
$$

Proof. Multiplying the a^D-equation in (3.1) by $p\left(a^D\right)^{p-1}$ with $p>1$ and integrating the resulting equation over Ω by parts, then combining like terms, leaving out some negative terms and noting $0<v\le 1$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int\limits_{\Omega} e^{\chi_D v}\left(a^D\right)^p\\
&=\int\limits_{\Omega}\chi_D e^{\chi_D v}\left(a^D\right)^p v_t+p\int\limits_{\Omega} e^{\chi_D v}\left(a^D\right)^{p-1}a_t^D\\
&=-\chi_D\int\limits_{\Omega} e^{\chi_D v}\left(a^D\right)^p mv+\chi_D\mu_v\int\limits_{\Omega} e^{\chi_D v}\left(a^D\right)^p v\left(1-e^{\chi_D v}a^D-e^{\chi_S v}a^S-v\right)\\
&\quad-\frac{4(p-1)}{p}\int\limits_{\Omega} e^{\chi_D v}\left|\nabla\left(a^D\right)^{\frac{p}{2}}\right|^2+p\chi_D\int\limits_{\Omega} e^{\chi_D v}\left(a^D\right)^p mv-p\int\limits_{\Omega}\mu_{EMT}e^{\chi_D v}\left(a^D\right)^p\\
&\quad+p\int\limits_{\Omega}\left(\mu_D-\chi_D\mu_v v\right)e^{\chi_D v}\left(a^D\right)^p\left(1-e^{\chi_D v}a^D-e^{\chi_S v}a^S-v\right)\\
&\le-\frac{4(p-1)}{p}\int\limits_{\Omega} e^{\chi_D v}\left|\nabla\left(a^D\right)^{\frac{p}{2}}\right|^2+\left((p-1)\chi_D\mu_v-p\mu_D\right)\int\limits_{\Omega} e^{2\chi_D v}\left(a^D\right)^{p+1}\\
&\quad+\left((p-1)\chi_D\mu_v-p\mu_D\right)\int\limits_{\Omega} e^{\chi_D v+\chi_S v}\left(a^D\right)^p a^S+\left(p\mu_D+(p-1)\chi_D\mu_v\right)\int\limits_{\Omega} e^{\chi_D v}\left(a^D\right)^p\\
&\quad+(p-1)\chi_D\int\limits_{\Omega} e^{\chi_D v}\left(a^D\right)^p m.
\end{aligned}
\tag{3.31}
$$

Hence, (3.29) holds. Proceeding in a same way as (3.29), one can obtain (3.30) immediately.

Thus, the proof of Lemma 3.4 is completed. □

Lemma 3.5. *Let $\left(a^D,a^S,m,v\right)$ be the local classical solutions of (3.1) constructed in Lemma 3.1 and Lemma 3.2. Suppose that the hypotheses of Theorem 1.2 hold. Then, for any $p>1$, we have*

$$\frac{d}{dt}\int_\Omega e^{\chi_D v}\left(a^D\right)^p+\frac{4(p-1)}{p}\int_\Omega\left|\nabla\left(a^D\right)^{\frac{p}{2}}\right|^2$$
$$\leq\left(\chi_D\mu_v+p\mu_D\right)e^{\chi_D}\int_\Omega\left(a^D\right)^p+p\chi_De^{\chi_D}\int_\Omega\left(a^D\right)^p m \tag{3.32}$$

and

$$\frac{d}{dt}\int_\Omega e^{\chi_S v}\left(a^S\right)^p+\frac{4(p-1)}{p}\int_\Omega\left|\nabla\left(a^S\right)^{\frac{p}{2}}\right|^2$$
$$\leq\left[\left(\chi_S\mu_v+p\mu_S\right)e^{\chi_S}+(p-1)\mu_Me^{\chi_D}\right]\int_\Omega\left(a^S\right)^p+\mu_Me^{\chi_D}\int_\Omega\left(a^D\right)^p+p\chi_Se^{\chi_S}\int_\Omega\left(a^S\right)^p m. \tag{3.33}$$

Proof. The proof is similar to the proof of Lemma 3.4 but take the additional assumptions $\mu_D\geq\chi_D\mu_v$, $\mu_S\geq\chi_S\mu_v$ into account, so we omit it here. $\square$

4. Proof of Theorem 1.1

The key point in the proof of Theorem 1.1 is to derive a priori estimates of $\left\|a^D\right\|_{L^\infty(\Omega)}$, $\left\|a^S\right\|_{L^\infty(\Omega)}$ and $\|\nabla v\|_{L^5(\Omega)}$ on account of Lemma 3.3. To this end, enlightened by [25], we will develop a certain dissipative property of the functionals $\int_\Omega e^{\chi_D v}\left(a^D\right)^2, \int_\Omega e^{\chi_S v}\left(a^S\right)^2$ which will serve as a starting point to establish an iteration step resulting in the $L^\infty(\Omega)$ boundedness of a^D and a^S. Furthermore, inspired by [33], we will build a bridge between $\|\nabla v(\cdot,t)\|_{L^q(\Omega)}$ and $\int_0^t\left(\left\|\nabla a^D(\cdot,s)\right\|_{L^q(\Omega)}+\left\|\nabla a^S(\cdot,s)\right\|_{L^q(\Omega)}\right)ds$ which is crucial of estimating $\|\nabla v\|_{L^5(\Omega)}$.

For notational convenience, in what follows, we shall use C_i $(i=1,2...)$ to denote the positive constants independent of time, whereas we use $C_i(T)$ $(i=1,2...)$ to denote the positive constants depending on time. These constants C_i and $C_i(T)$ may vary from line to line. In addition, according to the above local existence results Lemma 3.1 and Lemma 3.2, without loss of generality, we can assume that

$$\left\|c_0^D\right\|_{C^2(\bar\Omega)}+\left\|c_0^S\right\|_{C^2(\bar\Omega)}+\|m_0\|_{C^2(\bar\Omega)}+\|v_0\|_{C^2(\bar\Omega)}\leq C_1. \tag{4.1}$$

4.1. The case of $\tau=1$

Firstly, based on the ideas of [15, Lemma 2.1], some basic but important properties of solutions of (1.1) and (3.1) when $\tau=1$ are derived in the following Lemma.

Lemma 4.1. *Let $\left(c^D,c^S,m,v\right)$ and $\left(a^D,a^S,m,v\right)$ be the classical solutions of (1.1) and (3.1) with $\tau=1$, respectively. Then we have*

(i) $\left\|a^D(\cdot,t)\right\|_{L^1(\Omega)}\leq\left\|c^D(\cdot,t)\right\|_{L^1(\Omega)}\leq M_1:=\max\left\{|\Omega|,\left\|c_0^D\right\|_{L^1(\Omega)}\right\}$ *for all $t\in(0,T_{max})$;*

(ii) $\left\|a^S(\cdot,t)\right\|_{L^1(\Omega)}\leq\left\|c^S(\cdot,t)\right\|_{L^1(\Omega)}\leq M_2:=\max\left\{\left\|c_0^S\right\|_{L^1(\Omega)},\frac{|\Omega|}{2}\left(1+\sqrt{1+\frac{4\mu_MM_1}{\mu_S|\Omega|}}\right)\right\}$ *for all $t\in(0,T_{max})$;*

(iii) $\int_t^{t+\theta}\left\|a^D(\cdot,s)\right\|_{L^2(\Omega)}^2ds\leq\int_t^{t+\theta}\left\|c^D(\cdot,t)\right\|_{L^2(\Omega)}^2ds\leq M_3:=|\Omega|+\frac{2M_1}{\mu_D}$ *for any $0<\theta\leq\min\left\{1,\frac{T_{max}}{2}\right\}$ and all $t\in(0,T_{max}-\theta)$;*

(*iv*) $\int_t^{t+\theta} \left\|a^S(\cdot,s)\right\|^2_{L^2(\Omega)} ds \le \int_t^{t+\theta} \left\|c^S(\cdot,t)\right\|^2_{L^2(\Omega)} ds \le M_4 := |\Omega| + \frac{2(\mu_M M_1 + M_2)}{\mu_S}$ *for any* $0 < \theta \le \min\left\{1, \frac{T_{max}}{2}\right\}$ *and all* $t \in (0, T_{max} - \theta)$;

(*v*) $\|m(\cdot,t)\|_{L^1(\Omega)} \le M_5 := \max\left\{\|m_0\|_{L^1(\Omega)}, M_1 + M_2\right\}$ *for all* $t \in (0, T_{max})$;

(*vi*) $\|\nabla m(\cdot,t)\|^2_{L^2(\Omega)} \le M_6 := \frac{(2+\mu_D+\mu_M)M_1+(2+\mu_S)M_2+\mu\|\nabla m_0\|^2_{L^2(\Omega)}}{\mu}$ *for all* $t \in (0, T_{max})$, *where* $\mu := \min\{\mu_D, \mu_S\}$.

Proof. (*i*) Integrating the c^D-equation of (1.1) over Ω immediately yields

$$\frac{d}{dt}\int_\Omega c^D(x,t) \le \mu_D \int_\Omega c^D(x,t) - \mu_D \int_\Omega \left|c^D(x,t)\right|^2 \tag{4.2}$$

by (3.3) and $a^D := e^{-\chi_D v} c^D$. Moreover, by the Cauchy-Schwartz inequality, we have

$$\frac{d}{dt}\int_\Omega c^D(x,t) \le \mu_D \int_\Omega c^D(x,t) - \frac{\mu_D}{|\Omega|}\left(\int_\Omega c^D(x,t)\right)^2, \tag{4.3}$$

which means from the ODE comparison principle that

$$\left\|c^D(\cdot,t)\right\|_{L^1(\Omega)} \le \max\left\{|\Omega|, \left\|c_0^D\right\|_{L^1(\Omega)}\right\} := M_1, \tag{4.4}$$

this together with $a^D := e^{-\chi_D v} c^D$ and $0 < v \le 1$ implies (*i*).

(*ii*) Integrating the c^S-equation of (1.1) over Ω, due to (3.3) and $a^S := e^{-\chi_S v} c^S$ as well as hypothesis (H_2), we get

$$\frac{d}{dt}\int_\Omega c^S(x,t) \le \mu_M \int_\Omega c^D(x,t) + \mu_S \int_\Omega c^S(x,t) - \mu_S \int_\Omega \left|c^S(x,t)\right|^2, \tag{4.5}$$

which, by means of (*i*) and the Cauchy-Schwartz inequality, yields

$$\frac{d}{dt}\int_\Omega c^S(x,t) \le \mu_M M_1 + \mu_S \int_\Omega c^S(x,t) - \frac{\mu_S}{|\Omega|}\left(\int_\Omega c^S(x,t)\right)^2. \tag{4.6}$$

Similarly, it follows from the ODE comparison principle that

$$\left\|c^S(\cdot,t)\right\|_{L^1(\Omega)} \le \max\left\{\left\|c_0^S\right\|_{L^1(\Omega)}, \frac{|\Omega|}{2}\left(1 + \sqrt{1 + \frac{4\mu_M M_1}{\mu_S|\Omega|}}\right)\right\} := M_2, \tag{4.7}$$

which, combined with $a^S := e^{-\chi_S v} c^S$ and $0 < v \le 1$, gives (*ii*).

(*iii*) In view of (4.2), one infer from the Cauchy's inequality that

$$\frac{d}{dt}\int_\Omega c^D(x,t) + \frac{\mu_D}{2}\int_\Omega \left(c^D(x,t)\right)^2 \le \frac{\mu_D}{2}|\Omega|. \tag{4.8}$$

Integrating (4.8) over $(t, t+\theta)$ for any $0 < \theta \le \min\left\{1, \frac{T_{max}}{2}\right\}$, and using (3.3) and (*i*), we have

$$\int_t^{t+\theta}\int_\Omega \left|c^D(x,s)\right|^2 \leq |\Omega|\theta + \frac{2}{\mu_D}\int_\Omega c^D(x,t) \leq |\Omega| + \frac{2M_1}{\mu_D} := M_3. \tag{4.9}$$

Noting $a^D := e^{-\chi_D v} c^D$ and the fact $0 < v \leq 1$, thus (iii) obviously holds.

(iv) Analogous to the proof of (iii), we obtain

$$\frac{d}{dt}\int_\Omega c^S(x,t) + \frac{\mu_S}{2}\int_\Omega \left(c^D(x,t)\right)^2 \leq \mu_M \int_\Omega c^D(x,t) + \frac{\mu_S}{2}|\Omega| \leq \mu_M M_1 + \frac{\mu_S}{2}|\Omega|. \tag{4.10}$$

Then we integrate above inequality over $(t, t+\theta)$ for any $0 < \theta \leq \min\left\{1, \frac{T_{max}}{2}\right\}$ to yield

$$\int_t^{t+\theta}\int_\Omega \left(c^S(x,s)\right)^2 \leq \frac{2\mu_M M_1 \theta}{\mu_S} + |\Omega|\theta + \frac{2}{\mu_S}\int_\Omega c^S(x,t) \leq |\Omega| + \frac{2\left(\mu_M M_1 + M_2\right)}{\mu_S} := M_4. \tag{4.11}$$

Thanks to $a^S := e^{-\chi_S v} c^S$ and $0 < v \leq 1$, it is easy to find that (iv) is valid.

(v) Integrating the m-equation of (1.1) over Ω yields

$$\frac{d}{dt}\int_\Omega m(x,t) + \int_\Omega m(x,t) = \int_\Omega c^D(x,t) + \int_\Omega c^S(x,t) \leq M_1 + M_2, \tag{4.12}$$

which, applied to the Gronwall's inequality, gives (v).

(vi) Testing the m-equation of (1.1) by $-\Delta m$ and integrating the resulting equation over Ω by parts, and using the Cauchy's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla m(x,t)|^2 + \int_\Omega |\Delta m(x,t)|^2 + \int_\Omega |\nabla m(x,t)|^2 \\
&= -\int_\Omega \left(c^D \Delta m\right)(x,t) - \int_\Omega \left(c^S \Delta m\right)(x,t) \\
&\leq \int_\Omega |\Delta m(x,t)|^2 + \frac{1}{2}\int_\Omega \left|c^D(x,t)\right|^2 + \frac{1}{2}\int_\Omega \left|c^S(x,t)\right|^2,
\end{aligned} \tag{4.13}$$

which implies

$$\frac{d}{dt}\int_\Omega |\nabla m(x,t)|^2 + \int_\Omega |\nabla m(x,t)|^2 \leq \int_\Omega \left|c^D(x,t)\right|^2 + \int_\Omega \left|c^S(x,t)\right|^2. \tag{4.14}$$

Let us set $\mu := \min\{\mu_D, \mu_S\}$. (4.14) together with (4.2) and (4.5) directly yields

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(c^D(x,t) + c^S(x,t) + \mu|\nabla m(x,t)|^2\right) + \int_\Omega \left(c^D(x,t) + c^S(x,t) + \mu|\nabla m(x,t)|^2\right) \\
&\leq \left(1 + \mu_D + \mu_M\right)\int_\Omega c^D(x,t) + \left(1 + \mu_S\right)\int_\Omega c^S(x,t) \\
&\leq \left(1 + \mu_D + \mu_M\right) M_1 + \left(1 + \mu_S\right) M_2,
\end{aligned} \tag{4.15}$$

which, applied to the Gronwall's inequality, yields

$$
\begin{aligned}
&\mu \int_\Omega |\nabla m(x,t)|^2 \\
&\quad \leq (1+\mu_D+\mu_M) M_1 + (1+\mu_S) M_2 + \left\|a_0^D\right\|_{L^1(\Omega)} + \left\|a_0^S\right\|_{L^1(\Omega)} + \mu \|\nabla m_0\|^2_{L^2(\Omega)} \\
&\quad \leq (2+\mu_D+\mu_M) M_1 + (2+\mu_S) M_2 + \mu \|\nabla m_0\|^2_{L^2(\Omega)}, \qquad (4.16)
\end{aligned}
$$

that is, (vi) holds.

Thus, the proof of Lemma 4.1 is completed. □

Lemma 4.2. *Let $\left(a^D, a^S, m, v\right)$ be the classical solutions of (3.1) with $\tau = 1$ constructed in Lemma 3.1. Suppose that the assumptions of Theorem 1.1 hold. Then there exists $C_2\left(\min\left\{1, \frac{T_{max}}{4}\right\}\right) > 0$ such that*

$$
\left\|a^D(\cdot,t)\right\|_{L^2(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^2(\Omega)} \leq C_2\left(\min\left\{1, \frac{T_{max}}{4}\right\}\right) \quad \textit{for all } t \in (0, T_{max}). \qquad (4.17)
$$

Proof. Let us take $p = 2$ in (3.29) and (3.30), we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega e^{\chi_D v}\left(a^D\right)^2 + 2\int_\Omega e^{\chi_D v}\left|\nabla a^D\right|^2 + (2\mu_D - \chi_D\mu_v)\int_\Omega e^{2\chi_D v}\left(a^D\right)^3 \\
&\leq (2\mu_D + \chi_D\mu_v)\int_\Omega e^{\chi_D v}\left(a^D\right)^2 + \chi_D\int_\Omega e^{\chi_D v}\left(a^D\right)^2 m \\
&\quad + (\chi_D\mu_v - 2\mu_D)\int_\Omega e^{\chi_D v + \chi_S v}\left(a^D\right)^2 a^S \qquad (4.18)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega e^{\chi_S v}\left(a^S\right)^2 + 2\int_\Omega e^{\chi_S v}\left|\nabla a^S\right|^2 + (2\mu_S - \chi_S\mu_v)\int_\Omega e^{2\chi_S v}\left(a^S\right)^3 \\
&\leq (2\mu_S + \chi_S\mu_v)\int_\Omega e^{\chi_S v}\left(a^S\right)^2 + \chi_S\int_\Omega e^{\chi_S v}\left(a^S\right)^2 m + (\chi_S\mu_v - 2\mu_S)\int_\Omega e^{\chi_D v + \chi_S v}\left(a^S\right)^2 a^D \\
&\quad + 2\mu_M\int_\Omega e^{\chi_D v} a^D a^S, \qquad (4.19)
\end{aligned}
$$

which, applied to the Young's inequality, give

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega e^{\chi_D v}\left(a^D\right)^2 + \int_\Omega e^{\chi_D v}\left|\nabla\left(a^D\right)\right|^2 + \int_\Omega e^{\chi_D v}\left(a^D\right)^2 \\
&\leq C_3\left\|a^D\right\|^3_{L^3(\Omega)} + C_3\left\|a^S\right\|^3_{L^3(\Omega)} + C_3\|m\|^3_{L^3(\Omega)} + C_3 \qquad (4.20)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega e^{\chi_S v}\left(a^S\right)^2 + \int_\Omega e^{\chi_S v}\left|\nabla\left(a^S\right)\right|^2 + \int_\Omega e^{\chi_S v}\left(a^S\right)^2 \\
&\leq C_4\left\|a^S\right\|^3_{L^3(\Omega)} + C_4\left\|a^D\right\|^3_{L^3(\Omega)} + C_4\|m\|^3_{L^3(\Omega)} + C_4. \qquad (4.21)
\end{aligned}
$$

Noting $n=2$, by the Sobolev embedding theorem, Lemma 2.1 and the facts of (v), (vi) in Lemma 4.1, we obtain

$$\|m\|_{L^3(\Omega)} \le C_5\|m\|_{W^{1,2}(\Omega)} \le C_5 A_0 \left(\|\nabla m\|_{L^2(\Omega)} + \|m\|_{L^1(\Omega)}\right) \le C_6. \tag{4.22}$$

Substituting (4.22) into (4.20) and (4.21), one can deduce that

$$\frac{d}{dt}\int_\Omega e^{\chi_D v}\left(a^D\right)^2 + \int_\Omega e^{\chi_D v}\left|\nabla\left(a^D\right)\right|^2 + \int_\Omega e^{\chi_D v}\left(a^D\right)^2 \le C_7\left\|a^D\right\|^3_{L^3(\Omega)} + C_7\left\|a^S\right\|^3_{L^3(\Omega)} + C_7 \tag{4.23}$$

and

$$\frac{d}{dt}\int_\Omega e^{\chi_S v}\left(a^S\right)^2 + \int_\Omega e^{\chi_S v}\left|\nabla\left(a^S\right)\right|^2 + \int_\Omega e^{\chi_S v}\left(a^S\right)^2 \le C_8\left\|a^S\right\|^3_{L^3(\Omega)} + C_8\left\|a^D\right\|^3_{L^3(\Omega)} + C_8. \tag{4.24}$$

Combining (4.23) and (4.24), and using the fact $v>0$ gives

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \left(e^{\chi_D v}\left(a^D\right)^2 + e^{\chi_S v}\left(a^S\right)^2\right) + \int_\Omega \left(\left|\nabla\left(a^D\right)\right|^2 + \left|\nabla\left(a^S\right)\right|^2 + \left(a^D\right)^2 + \left(a^S\right)^2\right)\\ &\le \left(C_7+C_8\right)\left\|a^D\right\|^3_{L^3(\Omega)} + \left(C_7+C_8\right)\left\|a^S\right\|^3_{L^3(\Omega)} + C_7 + C_8.\end{aligned} \tag{4.25}$$

On the other hand, by using the Gagliardo-Nirenberg's inequality [9,14] and the Young's inequality, for any $\varepsilon>0$, we have

$$\|a\|^2_{W^{1,2}(\Omega)} \ge \frac{1}{\varepsilon}\|a\|^3_{L^3(\Omega)} - \frac{C_{GN}^6}{\varepsilon^2}\|a\|^4_{L^2(\Omega)}, \tag{4.26}$$

which after inserting into (4.25) for $a=a^D$ and $a=a^S$ correspondingly, and taking $\varepsilon = \frac{1}{2(C_7+C_8)}$ yields

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \left(e^{\chi_D v}\left(a^D\right)^2 + e^{\chi_S v}\left(a^S\right)^2\right) + \left(C_7+C_8\right)\left\|a^D\right\|^3_{L^3(\Omega)} + \left(C_7+C_8\right)\left\|a^S\right\|^3_{L^3(\Omega)}\\ &\le 4\left(C_7+C_8\right)^2 C_{GN}^6\left\|a^D\right\|^4_{L^2(\Omega)} + 4\left(C_7+C_8\right)^2 C_{GN}^6\left\|a^S\right\|^4_{L^2(\Omega)} + C_7 + C_8.\end{aligned} \tag{4.27}$$

Thanks to $a^3 \ge \frac{1}{\varepsilon}a^2 - \frac{1}{\varepsilon^3}$ for all $a\ge 0$ and any $\varepsilon>0$, we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \left(e^{\chi_D v}\left(a^D\right)^2 + e^{\chi_S v}\left(a^S\right)^2\right) + \frac{1}{\varepsilon}\int_\Omega \left(e^{\chi_D v}(a^D)^2 + e^{\chi_S v}(a^S)^2\right)\\ &\le 4\left(C_7+C_8\right)^2 C_{GN}^6\left(\left\|a^D\right\|^4_{L^2(\Omega)} + \left\|a^S\right\|^4_{L^2(\Omega)}\right) + C_7 + C_8 + \frac{\left(e^{3\chi_D} + e^{3\chi_S}\right)|\Omega|}{\varepsilon^3\left(C_7+C_8\right)^2}\\ &\le 4\left(C_7+C_8\right)^2 C_{GN}^6\left(\left\|a^D\right\|^2_{L^2(\Omega)} + \left\|a^S\right\|^2_{L^2(\Omega)}\right)\left(\int_\Omega \left(e^{\chi_D v}(a^D)^2 + e^{\chi_S v}(a^S)^2\right)\right)\\ &\quad + C_7 + C_8 + \frac{\left(e^{3\chi_D} + e^{3\chi_S}\right)|\Omega|}{\varepsilon^3\left(C_7+C_8\right)^2}.\end{aligned} \tag{4.28}$$

Let $\theta = \min\left\{1, \frac{T_{max}}{4}\right\}$, $\varepsilon = \frac{\theta}{1+4(C_7+C_8)^2 C_{GN}^6 (M_3+M_4)}$, $a(t) := \frac{1}{\varepsilon}$, $b(t) := 4\left(C_7 + C_8\right)^2 C_{GN}^6 \left(\left\|a^D\right\|_{L^2(\Omega)}^2 + \left\|a^S\right\|_{L^2(\Omega)}^2\right)$, $c(t) := C_7 + C_8 + \frac{(e^{3\chi_D}+e^{3\chi_S})|\Omega|}{\varepsilon^3(C_7+C_8)^2}$ and $y(t) := \int_\Omega \left(e^{\chi_D v}(a^D)^2 + e^{\chi_S v}(a^S)^2\right)$. Consequently, (4.28) can be rewritten as the following ODE inequality

$$y'(t) + a(t)y(t) \le b(t)y(t) + c(t). \tag{4.29}$$

Furthermore, by a simple calculation, using Lemma 2.3 to (4.28) with $b_1 = 4\left(C_7 + C_8\right)^2 C_{GN}^6 (M_3 + M_4)$, $c_1 = C_7 + C_8 + \frac{(e^{3\chi_D}+e^{3\chi_S})|\Omega|}{\varepsilon^3(C_7+C_8)^2}$ and $\gamma = 1$ directly yields

$$\int_\Omega \left(e^{\chi_D v}(a^D)^2 + e^{\chi_S v}(a^S)^2\right) \le \left(e^{\chi_D}\left\|a_0^D\right\|_{L^2(\Omega)} + e^{\chi_S}\left\|a_0^S\right\|_{L^2(\Omega)}\right)e^{b_1} + \frac{c_1 e^{2b_1}}{1-e^{-1}} + c_1 e^{b_1}, \tag{4.30}$$

which verifies (4.17).

Thus, the proof of Lemma 4.2 is completed. $\square$

Now, we can raise the regularity of $L^2(\Omega)$ to $L^p(\Omega)$ for any $p > 1$ with respect to the boundedness of a^D and a^S.

Lemma 4.3. *Assume that the hypotheses of Lemma 4.2 remains valid. Then for all $p > 1$, there exists $C_9\left(\min\left\{1, \frac{T_{max}}{4}\right\}\right) > 0$ such that*

$$\left\|a^D(\cdot,t)\right\|_{L^p(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^p(\Omega)} \le C_9\left(\min\left\{1, \frac{T_{max}}{4}\right\}\right) \quad \text{for all } t \in (0, T_{max}). \tag{4.31}$$

Proof. Noting $n = 2$, then it follows from Lemma 4.2 and Lemma 2.4 that

$$\|m\|_{L^\infty(\Omega)} \le C_{10}\left(\min\left\{1, \frac{T_{max}}{4}\right\}\right) \quad \text{for all } t \in (0, T_{max}). \tag{4.32}$$

Applying the Young's inequality to (3.29) and (3.30) for any $p > 1$ and using the fact $0 < v \le 1$, we derive

$$\begin{aligned} &\frac{d}{dt}\int_\Omega e^{\chi_D v}\left(a^D\right)^p + \frac{4(p-1)}{p}\int_\Omega e^{\chi_D v}\left|\nabla\left(a^D\right)^{\frac{p}{2}}\right|^2 + \int_\Omega e^{\chi_D v}\left(a^D\right)^p \\ &\le C_{11}\left(\min\left\{1, \frac{T_{max}}{4}\right\}\right)\left\|a^D\right\|_{L^{p+1}(\Omega)}^{p+1} + C_{12}\left\|a^S\right\|_{L^{p+1}(\Omega)}^{p+1} + C_{12} \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} &\frac{d}{dt}\int_\Omega e^{\chi_S v}\left(a^S\right)^p + \frac{4(p-1)}{p}\int_\Omega e^{\chi_S v}\left|\nabla\left(a^S\right)^{\frac{p}{2}}\right|^2 + \int_\Omega e^{\chi_S v}\left(a^S\right)^p \\ &\le C_{13}\left(\min\left\{1, \frac{T_{max}}{4}\right\}\right)\left\|a^S\right\|_{L^{p+1}(\Omega)}^{p+1} + C_{14}\left\|a^D\right\|_{L^{p+1}(\Omega)}^{p+1} + C_{14}. \end{aligned} \tag{4.34}$$

Now, it follows from Lemma 2.2 and Lemma 4.2 that

$$\begin{aligned} \left\|a^D\right\|_{L^{p+1}(\Omega)}^{p+1} &= \left\|\left(a^D\right)^{\frac{p}{2}}\right\|_{L^{\frac{2(p+1)}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \\ &\le C_{15}\left(\left\|\nabla\left(a^D\right)^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{\lambda\frac{2(p+1)}{p}}\left\|\left(a^D\right)^{\frac{p}{2}}\right\|_{L^{\frac{4}{p}}(\Omega)}^{(1-\lambda)\frac{2(p+1)}{p}} + \left\|\left(a^D\right)^{\frac{p}{2}}\right\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{2(p+1)}{p}}\right) \end{aligned}$$

$$\leq C_{16}\left(\min\left\{1, \frac{T_{max}}{4}\right\}\right)\left(\left\|\nabla\left(a^D\right)^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{\lambda\frac{2(p+1)}{p}} + 1\right), \tag{4.35}$$

where $\lambda = \frac{p-1}{p+1}$, thus $\lambda\frac{2(p+1)}{p} = 2-\frac{2}{p} \in (0,2)$ for any $p > 1$. Hence, by the Young's inequality and Lemma 4.2, we have

$$\begin{aligned}&\left(C_{11}\left(\min\left\{1, \frac{T_{max}}{4}\right\}\right) + C_{14}\right)\|a^D\|_{L^{p+1}(\Omega)}^{p+1}\\ &\quad\leq \frac{4(p-1)}{p}\int_\Omega e^{\chi_D v}\left|\nabla\left(a^D\right)^{\frac{p}{2}}\right|^2 + C_{17}\left(\min\left\{1, \frac{T_{max}}{4}\right\}\right).\end{aligned} \tag{4.36}$$

Analogous to (4.36), it is not hard to deduce that

$$\begin{aligned}&\left(C_{13}\left(\min\left\{1, \frac{T_{max}}{4}\right\}\right) + C_{12}\right)\|a^S\|_{L^{p+1}(\Omega)}^{p+1}\\ &\quad\leq \frac{4(p-1)}{p}\int_\Omega e^{\chi_S v}\left|\nabla\left(a^S\right)^{\frac{p}{2}}\right|^2 + C_{18}\left(\min\left\{1, \frac{T_{max}}{4}\right\}\right).\end{aligned} \tag{4.37}$$

Adding (4.33) to (4.34), and substituting (4.36) and (4.37) into the resulting inequality, this yields

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\left(e^{\chi_D v}\left(a^D\right)^p + e^{\chi_S v}\left(a^S\right)^p\right) + \int_\Omega\left(e^{\chi_D v}\left(a^D\right)^p + e^{\chi_S v}\left(a^S\right)^p\right)\\ &\leq C_{19}\left(\min\left\{1, \frac{T_{max}}{4}\right\}\right),\end{aligned} \tag{4.38}$$

which, applied to the Gronwall's inequality and combined with the fact $0 < v \leq 1$, obviously gives (4.31).

Thus, the proof of Lemma 4.3 is completed. □

Next, we can establish the $L^\infty(\Omega)$-boundedness of a^D and a^S by making a adaptation of the well-known Moser-Alikakos L^p iteration technique [1] (see also [17,33]).

Lemma 4.4. *Under the hypotheses of Lemma 4.2. Then there exists $C_{20}(T_{max}) > 0$ such that*

$$\left\|a^D(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^\infty(\Omega)} \leq C_{20}(T_{max}) \quad \textit{for all } t \in (0, T_{max}). \tag{4.39}$$

Proof. Combining (3.29) and (3.30), noting (4.32), then using the Young's inequality to the resulting inequality, for any $p \geq 3$, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\left(e^{\chi_D v}\left(a^D\right)^p + e^{\chi_S v}\left(a^S\right)^p\right) + \int_\Omega\left(e^{\chi_D v}\left(a^D\right)^p + e^{\chi_S v}\left(a^S\right)^p\right)\\ &\quad + \int_\Omega\left(\left|\nabla\left(a^D\right)^{\frac{p}{2}}\right|^2 + \left|\nabla\left(a^S\right)^{\frac{p}{2}}\right|^2\right)\\ &\leq C_{21}\left(\min\left\{1, \frac{T_{max}}{4}\right\}\right)p\left(\|a^D\|_{L^{p+1}(\Omega)}^{p+1} + \|a^S\|_{L^{p+1}(\Omega)}^{p+1} + 1\right),\end{aligned} \tag{4.40}$$

where $C_{21}\left(\min\left\{1, \frac{T_{max}}{4}\right\}\right)$ is independent of $p \geq 3$. On the other hand, from Lemma 2.2, we have

$$
\begin{aligned}
& C_{21}\left(\min\left\{1,\frac{T_{max}}{4}\right\}\right)p\,\|a\|_{L^{p+1}(\Omega)}^{p+1} \\
& = C_{21}\left(\min\left\{1,\frac{T_{max}}{4}\right\}\right)p\left\|a^{\frac{p}{2}}\right\|_{L^{\frac{2(p+1)}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \\
& \leq C_{22}\left(\min\left\{1,\frac{T_{max}}{4}\right\}\right)p\left(\left\|\nabla a^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{\frac{p+2}{p}}\left\|a^{\frac{p}{2}}\right\|_{L^1(\Omega)}+\left\|a^{\frac{p}{2}}\right\|_{L^1(\Omega)}^{\frac{2(p+1)}{p}}\right) \\
& \leq \left\|\nabla a^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2+C_{23}\left(\min\left\{1,\frac{T_{max}}{4}\right\}\right)p^{\frac{2p}{p-2}}\left(\left\|a^{\frac{p}{2}}\right\|_{L^1(\Omega)}^{\frac{2p}{p-2}}+1\right) \\
& \leq \left\|\nabla a^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2+C_{24}\left(\min\left\{1,\frac{T_{max}}{4}\right\}\right)p^{6}\left(\left\|a^{\frac{p}{2}}\right\|_{L^1(\Omega)}^{\frac{2p}{p-2}}+1\right).
\end{aligned}
\tag{4.41}
$$

Here, we have used the Young's inequality guaranteed by $\frac{p+2}{p}\in(1,\frac{5}{3})$ and $\frac{2(p+1)}{p}<\frac{2p}{p-2}$, we also used the fact $\frac{2p}{p-2}\leq 6$ for $p\geq 3$. Inserting (4.41) with $a=a^D, a=a^S$ into (4.40), one has

$$
\begin{aligned}
& \frac{d}{dt}\int_\Omega\left(e^{\chi_D v}\left(a^D\right)^p+e^{\chi_S v}\left(a^S\right)^p\right)+\int_\Omega\left(e^{\chi_D v}\left(a^D\right)^p+e^{\chi_S v}\left(a^S\right)^p\right) \\
& \leq C_{25}\left(\min\left\{1,\frac{T_{max}}{4}\right\}\right)p^6\left(\left\|\left(a^D\right)^{\frac{p}{2}}\right\|_{L^1(\Omega)}^{\frac{2p}{p-2}}+\left\|\left(a^S\right)^{\frac{p}{2}}\right\|_{L^1(\Omega)}^{\frac{2p}{p-2}}+1\right) \\
& \leq C_{26}\left(\min\left\{1,\frac{T_{max}}{4}\right\}\right)p^6\left(\max\left\{1,\left\|\left(a^D\right)^{\frac{p}{2}}\right\|_{L^1(\Omega)}+\left\|\left(a^D\right)^{\frac{p}{2}}\right\|_{L^1(\Omega)}\right\}\right)^{\frac{2p}{p-2}}.
\end{aligned}
\tag{4.42}
$$

Let $p_k:=3\cdot 2^k$, $q_k:=\frac{2p_k}{p_k-2}$ and $M_k(T_{max}):=\max\left\{1,\sup_{t\in(0,T_{max})}\left(\left\|\left(a^D\right)^{p_k}\right\|_{L^1(\Omega)}+\left\|\left(a^S\right)^{p_k}\right\|_{L^1(\Omega)}\right)\right\}$ for $k=0,1,2...$ Therefore, upon the ODE comparison principle, we infer from (4.42) that there exists $\eta>1$ depending on T_{max} but independent of k such that

$$
M_k(T_{max})\leq\max\left\{\eta^k M_{k-1}^{q_k}(T_{max}),e^{\chi_D}|\Omega|\left\|a_0^D\right\|_{L^\infty(\Omega)}^{p_k}+e^{\chi_S}|\Omega|\left\|a_0^S\right\|_{L^\infty(\Omega)}^{p_k}\right\}\quad\text{for all } k\geq 1. \tag{4.43}
$$

Consequently, if $\eta^k M_{k-1}^{q_k}(T_{max})\leq e^{\chi_D}|\Omega|\left\|a_0^D\right\|_{L^\infty(\Omega)}^{p_k}+e^{\chi_S}|\Omega|\left\|a_0^S\right\|_{L^\infty(\Omega)}^{p_k}$ for infinitely many $k\geq 1$, we have

$$
\begin{aligned}
& \sup_{t\in(0,T_{max})}\left(\int_\Omega\left(a^D\right)^{p_{k-1}}\right)^{\frac{1}{p_{k-1}}}+\sup_{t\in(0,T_{max})}\left(\int_\Omega\left(a^S\right)^{p_{k-1}}\right)^{\frac{1}{p_{k-1}}} \\
& \leq\left(\frac{e^{\chi_D}|\Omega|\left\|a_0^D\right\|_{L^\infty(\Omega)}^{p_k}+e^{\chi_S}|\Omega|\left\|a_0^S\right\|_{L^\infty(\Omega)}^{p_k}}{\eta^k}\right)^{\frac{1}{p_{k-1}q_k}},
\end{aligned}
\tag{4.44}
$$

which implies

$$
\sup_{t\in(0,T_{max})}\left\|a^D(\cdot,t)\right\|_{L^\infty(\Omega)}+\sup_{t\in(0,T_{max})}\left\|a^S(\cdot,t)\right\|_{L^\infty(\Omega)}\leq\left\|a_0^D\right\|_{L^\infty(\Omega)}+\left\|a_0^S\right\|_{L^\infty(\Omega)}. \tag{4.45}
$$

Conversely, if $\eta^k M_{k-1}^{q_k}(T_{max})>e^{\chi_D}|\Omega|\left\|a_0^D\right\|_{L^\infty(\Omega)}^{p_k}+e^{\chi_S}|\Omega|\left\|a_0^S\right\|_{L^\infty(\Omega)}^{p_k}$ for all sufficiently large k, then it follows from (4.43) that

$$
M_k(T_{max})\leq\eta^k M_{k-1}^{q_k}(T_{max})\quad\text{for all sufficiently large } k, \tag{4.46}
$$

therefore, (4.46) still holds for all $k \geq 1$ by enlarging η if necessary, this amounts to say,

$$M_k(T_{max}) \leq \eta^k M_{k-1}^{q_k}(T_{max}) \quad \text{for all } k \geq 1. \tag{4.47}$$

Thus, by a simple induction, we get

$$M_k(T_{max}) \leq \eta^{k+\sum_{j=2}^{k}(j-1)\cdot\prod_{i=j}^{k} q_i} \cdot M_0(T_{max})^{\prod_{i=1}^{k} q_i} \quad \text{for all } k \geq 1. \tag{4.48}$$

Let us set $\zeta_k := \frac{2}{p_k-2}$, then $q_k = \frac{2p_k}{p_k-2} = 2\left(1+\frac{2}{p_k-2}\right) = 2(1+\zeta_k)$ for $k \geq 1$. In addition, thanks to $p_k := 3\cdot 2^k$, this means that $\zeta_k := \frac{2}{p_k-2} = \frac{2}{3\cdot 2^k-2} \leq 2^{-k}$, which, combined with the fact $\ln(1+x) \leq x$ for any $x \geq 0$, yields

$$\begin{aligned}\prod_{i=j}^{k} q_i &= 2^{k+1-j} e^{\Sigma_{i=j}^{k} \ln(1+\zeta_i)} \leq 2^{k+1-j} e^{\Sigma_{i=j}^{k}\zeta_i} \\ &\leq 2^{k+1-j} e^{\Sigma_{i=j}^{k} 2^{-i}} \leq 2^{k+1-j} e \quad \text{for all } k \geq 1 \text{ and } j \in \{1, 2..., k\}.\end{aligned} \tag{4.49}$$

Thus, we have

$$\frac{\sum_{j=2}^{k}(j-1)\cdot\prod_{i=j}^{k} q_i}{3\cdot 2^k} \leq \frac{\sum_{j=2}^{k}(j-1)2^{k+1-j}e}{3\cdot 2^k} \leq \frac{2e}{3}\sum_{j=2}^{k}\frac{j-1}{2^j} \leq \frac{2e}{3}\cdot\frac{3}{4} = \frac{e}{2}. \tag{4.50}$$

Then, we deduce from (4.48) that

$$M_k^{\frac{1}{p_k}}(T_{max}) \leq \eta^{\frac{k}{3\cdot 2^k}+\frac{\sum_{j=2}^{k}(j-1)\cdot\prod_{i=j}^{k} q_i}{3\cdot 2^k}} \cdot M_0^{\frac{\prod_{i=1}^{k} q_i}{3\cdot 2^k}}(T_{max}) \leq \eta^{\frac{k}{3\cdot 2^k}+\frac{e}{2}} \cdot M_0^{\frac{e}{3}}(T_{max}), \tag{4.51}$$

which after passing to $k \to \infty$ immediately entails

$$\sup_{t\in(0,T_{max})} \left\|a^D(\cdot,t)\right\|_{L^\infty(\Omega)} + \sup_{t\in(0,T_{max})} \left\|a^S(\cdot,t)\right\|_{L^\infty(\Omega)} \leq \eta^{\frac{e}{2}} \cdot M_0^{\frac{e}{3}}(T_{max}). \tag{4.52}$$

Letting $C_{20}(T_{max}) := \max\left\{\left\|a_0^D\right\|_{L^\infty(\Omega)} + \left\|a_0^S\right\|_{L^\infty(\Omega)}, \eta^{\frac{e}{2}} \cdot M_0^{\frac{e}{3}}(T_{max})\right\}$, hence a combination of (4.45) and (4.52) directly implies (4.39).

Thus, the proof of Lemma 4.4 is completed. $\square$

Remark 4.1. Recalling the proof of Lemma 4.3 and Lemma 4.4, it is worth noting that the time-dependent boundedness of $\left\|a^D(\cdot,t)\right\|_{L^p(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^p(\Omega)}$ for any $p > 1$ derived in Lemma 4.3 and $\left\|a^D(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^\infty(\Omega)}$ established in Lemma 4.4 are as a result of the time-dependent boundedness of $\left\|a^D(\cdot,t)\right\|_{L^2(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^2(\Omega)}$ shown in Lemma 4.2.

According to the weakened extensibility criterion (3.7) of Lemma 3.3, it remains to derive a priori estimates for $\|\nabla v(\cdot,t)\|_{L^5(\Omega)}$. To this end, the following result bridging $\|\nabla v(\cdot,t)\|_{L^q(\Omega)}^q$ with $\int_0^t \left(\|\nabla a^D(\cdot,s)\|_{L^q(\Omega)}^q + \|\nabla a^S(\cdot,s)\|_{L^q(\Omega)}^q\right) ds$ is crucial.

Lemma 4.5. *Under the assumptions of Lemma 4.2. Then for all $t \in (0, T_{max})$ and $q \geq 2$, there exists a $C_{27}(q) > 0$ independent of time such that*

$$\|\nabla v(\cdot,t)\|^q_{L^q(\Omega)} \le c_{27}e^{c_{27}t}\left(\|\nabla v_0\|^q_{L^q(\Omega)} + 1 + \int_0^t \left(\|\nabla a^D(\cdot,s)\|^q_{L^q(\Omega)} + \|\nabla a^S(\cdot,s)\|^q_{L^q(\Omega)}\right) ds\right). \quad (4.53)$$

Proof. Proceeding in a same way as (3.15)–(3.17) in Lemma 3.3, it is easy to prove (4.53). We note $\|m\|_{W^{1,\infty}(\Omega)} \le C_{28}(T_{max})$ for all $t \in (0, T_{max})$ which can be ensured by combining Lemma 4.4 and Lemma 2.4 for $n = 2$.

Thus, the proof of Lemma 4.5 is completed. □

Furthermore, the following Lemma is also needed.

Lemma 4.6. *Suppose that the assumptions of Lemma 4.2 are valid. Then, for all $t \in (0, T_{max})$, there exists $C_{29}(T_{max}) > 0$ such that*

$$\left\|\nabla a^D(\cdot,t)\right\|^2_{L^2(\Omega)} + \left\|\nabla a^S(\cdot,t)\right\|^2_{L^2(\Omega)} + \int_0^t \left(\left\|\Delta a^D(\cdot,s)\right\|^2_{L^2(\Omega)} + \left\|\Delta a^S(\cdot,s)\right\|^2_{L^2(\Omega)}\right) ds$$
$$\le C_{29}(T_{max}). \quad (4.54)$$

Proof. Firstly, we have the following linear forms with respect to the a^D, a^S-equations of (3.1)

$$a^D_t = \Delta a^D + \chi_D \nabla v \nabla a^D + g_{12} \quad (4.55)$$

and

$$a^S_t = \Delta a^S + \chi_S \nabla v \nabla a^S + g_{13}, \quad (4.56)$$

where $g_{12} = (\chi_D m v - \mu_{EMT})\, a^D + (\mu_D - \chi_D \mu_v v)\, a^D \left(1 - e^{\chi_D v} a^D - e^{\chi_S v} a^S - v\right)$, $g_{13} = \chi_S a^S m v + (\mu_S - \chi_S \mu_v v)\, a^S \left(1 - e^{\chi_D v} a^D - e^{\chi_S v} a^S - v\right) + \mu_{EMT} e^{\chi_D v - \chi_S v} a^D$.
Combining Lemma 4.4 and Lemma 2.4, we obtain

$$\|m(\cdot,t)\|_{L^\infty(\Omega)} \le C_{30}(T_{max}) \quad \text{for all } t \in (0, T_{max}), \quad (4.57)$$

which, combined with the assumption (H_2) and $0 < v \le 1$, yields

$$\|g_{12}\|_{L^\infty(\Omega)} \le C_{31}(T_{max}) \quad \text{and} \quad \|g_{13}\|_{L^\infty(\Omega)} \le C_{31}(T_{max}) \quad \text{for all } t \in (0, T_{max}). \quad (4.58)$$

Multiplying (4.55) by $-\Delta a^D$, integrating the resulting equation over Ω by parts and applying the Young's inequality yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\nabla a^D\right|^2 + \int_\Omega \left|\Delta a^D\right|^2 = -\int_\Omega \chi_D \nabla v \cdot \nabla a^D \Delta a^D - \int_\Omega g_{12}\Delta a^D$$
$$\le \frac{1}{2}\int_\Omega \left|\Delta a^D\right|^2 + \frac{\varepsilon}{2}\int_\Omega \left|\nabla a^D\right|^4 + C_{32}(\varepsilon)\int_\Omega |\nabla v|^4 + C_{33}(T_{max}), \quad (4.59)$$

which implies that

$$\frac{d}{dt}\int_\Omega \left|\nabla a^D\right|^2 + \int_\Omega \left|\Delta a^D\right|^2 \le \varepsilon \int_\Omega \left|\nabla a^D\right|^4 + 2C_{32}(\varepsilon)\int_\Omega |\nabla v|^4 + 2C_{33}(T_{max}). \quad (4.60)$$

By Lemma 2.2 and Lemma 4.4, we have

$$\begin{aligned}\left\|\nabla a^D\right\|_{L^4(\Omega)}^4 &\le C_{GN}^4\left(\left\|\Delta a^D\right\|_{L^2(\Omega)}^2\left\|a^D\right\|_{L^\infty(\Omega)}^2+\left\|a^D\right\|_{L^\infty(\Omega)}^4\right)\\&\le C_{34}(T_{max})\left\|\Delta a^D\right\|_{L^2(\Omega)}^2+C_{34}(T_{max}).\end{aligned}\tag{4.61}$$

Substituting (4.61) into (4.60) and taking $\varepsilon=\frac{1}{2C_{34}(T_{max})}$, we get

$$\frac{d}{dt}\int_\Omega\left|\nabla a^D\right|^2+\frac12\int_\Omega\left|\Delta a^D\right|^2\le C_{35}(T_{max})\int_\Omega|\nabla v|^4+C_{35}(T_{max}).\tag{4.62}$$

Analogously, we obtain

$$\frac{d}{dt}\int_\Omega\left|\nabla a^S\right|^2+\frac12\int_\Omega\left|\Delta a^S\right|^2\le C_{36}(T_{max})\int_\Omega|\nabla v|^4+C_{36}(T_{max}).\tag{4.63}$$

Combining (4.62) and (4.63) gives

$$\frac{d}{dt}\int_\Omega\left(\left|\nabla a^D\right|^2+\left|\nabla a^S\right|^2\right)+\frac12\int_\Omega\left(\left|\Delta a^D\right|^2+\left|\Delta a^S\right|^2\right)\le C_{37}(T_{max})\int_\Omega|\nabla v|^4+C_{37}(T_{max}).\tag{4.64}$$

Inserting (4.53) with $q=4$ into (4.64) and applying (4.61), we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\left(\left|\nabla a^D\right|^2+\left|\nabla a^S\right|^2\right)+\frac12\int_\Omega\left(\left|\Delta a^D\right|^2+\left|\Delta a^S\right|^2\right)\\&\le C_{37}(T_{max})c_{27}e^{c_{27}t}\left(\|\nabla v_0\|_{L^4(\Omega)}^4+1+\int_0^t\left(\|\nabla a^D(\cdot,s)\|_{L^4(\Omega)}^4+\|\nabla a^S(\cdot,s)\|_{L^4(\Omega)}^4\right)ds\right)+C_{37}(T_{max})\\&\le C_{37}(T_{max})c_{27}e^{c_{27}t}\left(\|\nabla v_0\|_{L^4(\Omega)}^4+1+C_{34}(T_{max})\int_0^t\left(\left\|\Delta a^D(\cdot,s)\right\|_{L^2(\Omega)}^2+\left\|\Delta a^S(\cdot,s)\right\|_{L^2(\Omega)}^2\right)ds\right.\\&\quad\left.+2C_{34}(T_{max})t\right)+C_{37}(T_{max}).\end{aligned}\tag{4.65}$$

Integrating (4.65) from 0 to t, we see that

$$\begin{aligned}&\left\|\nabla a^D(\cdot,t)\right\|_{L^2(\Omega)}^2+\left\|\nabla a^S(\cdot,t)\right\|_{L^2(\Omega)}^2+\frac12\int_0^t\left(\left\|\Delta a^D(\cdot,s)\right\|_{L^2(\Omega)}^2+\left\|\Delta a^S(\cdot,s)\right\|_{L^2(\Omega)}^2\right)\\&\le C_{37}(T_{max})c_{27}e^{c_{27}t}\left(t\|\nabla v_0\|_{L^4(\Omega)}^4+t+C_{34}(T_{max})t\int_0^t\left(\left\|\Delta a^D(\cdot,s)\right\|_{L^2(\Omega)}^2+\left\|\Delta a^S(\cdot,s)\right\|_{L^2(\Omega)}^2\right)ds\right.\\&\quad\left.+2C_{34}(T_{max})t^2\right)+C_{37}(T_{max})t+\left\|\nabla a_0^D\right\|_{L^2(\Omega)}^2+\left\|\nabla a_0^S\right\|_{L^2(\Omega)}^2.\end{aligned}\tag{4.66}$$

Let us take $0<t_1<\min\{1,T_{max}\}$ such that $C_{37}(T_{max})C_{34}(T_{max})c_{27}e^{c_{27}t_1}t_1\le\frac14$. Thus, by (4.66), for all $t\in(0,t_1]$, we have

$$\left\|\nabla a^D(\cdot,t)\right\|_{L^2(\Omega)}^2+\left\|\nabla a^S(\cdot,t)\right\|_{L^2(\Omega)}^2+\frac{1}{4}\int\limits_0^t\left(\left\|\Delta a^D(\cdot,s)\right\|_{L^2(\Omega)}^2+\left\|\Delta a^S(\cdot,s)\right\|_{L^2(\Omega)}^2\right)$$
$$\le\left\|\nabla a_0^D\right\|_{L^2(\Omega)}^2+\left\|\nabla a_0^S\right\|_{L^2(\Omega)}^2+C_{38}(T_{max})\|\nabla v_0\|_{L^4(\Omega)}^4+C_{38}(T_{max}). \tag{4.67}$$

In virtue of (4.61), (4.67) and (4.53), we see that

$$\|\nabla v(\cdot,t_1)\|_{L^4(\Omega)}^4\le C_{39}(T_{max})\left(\left\|\nabla a_0^D\right\|_{L^2(\Omega)}^2,\left\|\nabla a_0^S\right\|_{L^2(\Omega)}^2,\|\nabla v_0\|_{L^4(\Omega)}^4\right), \tag{4.68}$$

which guarantee we can repeat the above procedure by taking t_1 as the initial time. Consequently, it is not hard for us to extend the estimate (4.67) to the whole time interval $(0,T_{max})$ after finitely many steps. Thus, (4.54) holds.

Thus, the proof of Lemma 4.6 is completed. □

Now, we are ready to prove Theorem 1.1 with $\tau=1$.

The proof of Theorem 1.1 in the case of $\tau=1$

Proof. Suppose to the contrary that the maximal existence time T_{max} is finite. Thanks to Lemma 4.4, there exists a positive constant C_{40} such that

$$\left\|a^D(\cdot,t)\right\|_{L^\infty(\Omega)}+\left\|a^S(\cdot,t)\right\|_{L^\infty(\Omega)}\le C_{40}. \tag{4.69}$$

On the other hand, a combination of Lemma 4.6 and (4.61) (we note that (4.61) remains valid when a^D replacing a^D with a^S) directly yields

$$\int\limits_0^t\left(\left\|\nabla a^D(\cdot,s)\right\|_{L^4(\Omega)}^4+\left\|\nabla a^S(\cdot,s)\right\|_{L^4(\Omega)}^4\right)ds\le C_{41}, \tag{4.70}$$

which, applied to Lemma 4.5 with $q=4$, entails

$$\|\nabla v(\cdot,t)\|_{L^4(\Omega)}^4\le C_{42}\quad\text{for all } t\in(0,T_{max}). \tag{4.71}$$

Recalling (4.58), and applying the parabolic L^p theory (see [19, Theorem IV.9.1] to (4.55) and (4.56) and the Sobolev embedding theorem (see [19, Lemma II.3.3]) yields

$$\left\|\nabla a^D\right\|_{L^5(Q_{T_{max}})}+\left\|\nabla a^S\right\|_{L^5(Q_{T_{max}})}\le C_{43}\left\|a^D\right\|_{W_3^{2,1}(Q_{T_{max}})}+\left\|a^S\right\|_{W_3^{2,1}(Q_{T_{max}})}\le C_{44}, \tag{4.72}$$

which, applied to Lemma 4.5 with $q=5$, gives

$$\|\nabla v(\cdot,t)\|_{L^5(\Omega)}^5\le C_{45}\quad\text{for all } t\in(0,T_{max}). \tag{4.73}$$

Combining (4.69) with (4.73), which contradicts the weakened extensibility criterion (3.7) established in Lemma 3.3 and thereby proves that $T_{max}=\infty$. As for the uniform boundedness of a^D and a^S with respect to $t\in(0,\infty)$ is a straightforward consequence of Remark 4.1. Indeed, thanks to $T_{max}=\infty$, the positive constants $C_2\left(\min\left\{1,\frac{T_{max}}{4}\right\}\right)$ and $C_9\left(\min\left\{1,\frac{T_{max}}{4}\right\}\right)$ in (4.17) and (4.31) are independent of T_{max}, thus the bounds in Lemmata 4.2-4.4 are time-independent. From the equivalent of (1.1) and (3.1), we have the uniform boundedness of c^D and c^S. Furthermore, the uniform boundedness of m in the sense of $W^{1,\infty}(\Omega)$

with regard to $t \in (0, \infty)$ is immediately obtained by Lemma 2.4 and the uniform boundedness of c^D and c^S. Indeed, let us consider the m-equations of (1.1) with $\tau = 1$

$$\begin{cases} m_t = \Delta m + c^D + c^S - m, & x \in \Omega, t > 0, \\ \frac{\partial m}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ m(x, 0) = m_0(x), & x \in \Omega. \end{cases} \tag{4.74}$$

By Duhamel's principle, the solution of (4.74) can be expressed as follows

$$m(t) = e^{-t} e^{t\Delta} m_0 + \int_0^t e^{-(t-s)} e^{(t-s)\Delta} \left(c^D + c^S\right)(s) ds, \tag{4.75}$$

where $\left\{e^{t\Delta}\right\}_{t \geq 0}$ is the Neumann heat semigroup in Ω. By applying the well-known $L^q - L^p$ estimates of the heat semigroup (see [38, Lemma 2.3] and [36, Lemma 1.3] or [26, Lemma 2.1]), for some $\gamma > 0$ and $\alpha \in (0, 1)$, we have

$$\begin{aligned} &\|m(\cdot, t)\|_{W^{1,\infty}(\Omega)} \\ &\leq e^{-t} \left\|e^{t\Delta} m_0\right\|_{W^{1,\infty}(\Omega)} + \int_0^t \left\|e^{-(t-s)(-\Delta+1)} \left(c^D + c^S\right)(s)\right\|_{W^{1,\infty}(\Omega)} ds \\ &\leq C_{46} \|m_0\|_{W^{1,\infty}(\Omega)} + C_{46} \int_0^t \left\|(-\Delta + 1)^\alpha e^{-(t-s)(-\Delta+1)} \left(c^D + c^S\right)(s)\right\|_{L^q(\Omega)} ds \\ &\leq C_{46} \|m_0\|_{W^{1,\infty}(\Omega)} + C_{47} \left(\left\|c^D(\cdot, t\right\|_{L^\infty(\Omega)} + \left\|c^S(\cdot, t)\right\|_{L^\infty(\Omega)}\right) \int_0^t (t-s)^{-\alpha} e^{-\gamma(t-s)} ds. \end{aligned} \tag{4.76}$$

Here, for estimating the first term of the second inequality in (4.76), we used the maximal principle of parabolic equations, the well-known results [36, (1.5) and (1.13)]. As to the second term of it, we used the result of [38, Lemma 2.3]. The second inequality of (4.76) holds if and only if $\alpha > \frac{1}{2} + \frac{1}{q}$. Thus, one can take $\alpha \in \left(\frac{1}{2} + \frac{1}{q}, 1\right)$ provided $q > 2$ such that above integral is finite. Consequently, $\|m(\cdot, t)\|_{W^{1,\infty}(\Omega)}$ is uniformly bounded with respect to $t \in (0, \infty)$. In conclusion, (1.2) holds.

4.2. The case of $\tau = 0$

Analogous to Lemma 4.1, we have some essential results of solutions to the problems (1.1) and (3.1) with $\tau = 0$ as follows.

Lemma 4.7. *Let $\left(c^D, c^S, m, v\right)$ and $\left(a^D, a^S, m, v\right)$ be the classical solutions of (1.1) and (3.1) with $\tau = 0$, respectively. Then we get*

(i) $\left\|a^D(\cdot, t)\right\|_{L^1(\Omega)} \leq \left\|c^D(\cdot, t)\right\|_{L^1(\Omega)} \leq M_1 := \max\left\{|\Omega|, \left\|c_0^D\right\|_{L^1(\Omega)}\right\}$ *for all* $t \in (0, T_{max})$*;*

(ii) $\left\|a^S(\cdot, t)\right\|_{L^1(\Omega)} \leq \left\|c^S(\cdot, t)\right\|_{L^1(\Omega)} \leq M_2 := \max\left\{\left\|c_0^S\right\|_{L^1(\Omega)}, \frac{|\Omega|}{2}\left(1 + \sqrt{1 + \frac{4\mu_M M_1}{\mu_S |\Omega|}}\right)\right\}$ *for all* $t \in (0, T_{max})$*;*

(iii) $\displaystyle\int_t^{t+\theta} \left\|a^D(\cdot, s)\right\|_{L^2(\Omega)}^2 ds \leq \int_t^{t+\theta} \left\|c^D(\cdot, t)\right\|_{L^2(\Omega)}^2 ds \leq M_3 := |\Omega| + \frac{2M_1}{\mu_D}$ *for any* $0 < \theta \leq \min\left\{1, \frac{T_{max}}{2}\right\}$ *and all* $t \in (0, T_{max} - \theta)$*;*

(*iv*) $\int\limits_t^{t+\theta} \left\|a^S(\cdot,s)\right\|^2_{L^2(\Omega)} ds \le \int\limits_t^{t+\theta} \left\|c^S(\cdot,t)\right\|^2_{L^2(\Omega)} ds \le M_4 := |\Omega| + \frac{2(\mu_M M_1 + M_2)}{\mu_S}$ *for any* $0 < \theta \le \min\left\{1, \frac{T_{max}}{2}\right\}$ *and all* $t \in (0, T_{max} - \theta)$*;*

(*v*) $\|m(\cdot,t)\|_{L^1(\Omega)} \le M_1 + M_2$ *for all* $t \in (0, T_{max})$*.*

Proof. Thanks to the results of $(i) - (iv)$ are the same as Lemma 4.1, we only need to prove (v) here. Integrating the m-equation of (1.1) over Ω and applying (i) and (ii), we have

$$\int\limits_\Omega m(\cdot,t) = \int\limits_\Omega c^D(\cdot,t) + \int\limits_\Omega c^S(\cdot,t) \le M_1 + M_2. \tag{4.77}$$

Thus, the proof of Lemma 4.7 is completed. □

Retracing the proof of Lemma 4.2, the boundedness of $\|m(\cdot,t)\|_{L^3(\Omega)}$ plays an essential in the proof of the boundedness of $\left\|a^D(\cdot,t)\right\|_{L^2(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^2(\Omega)}$. Inspired by this point, we have following result which can guarantee the boundedness of $\|m(\cdot,t)\|_{L^3(\Omega)}$.

Lemma 4.8. *Let* $\left(c^D, c^S, m, v\right)$ *be the solutions of (1.1) with* $\tau = 0$*. Assume that the hypotheses of Theorem 1.1 are true. Then, for any* $q \in [1,2)$*, there exists* $C_{49} > 0$ *independent of time such that*

$$\|m(\cdot,t)\|_{W^{1,q}(\Omega)} \le C_{49} \quad \text{for all } t \in (0, T_{max}). \tag{4.78}$$

Proof. Noting $n = 2$, let us consider the m-equation of (1.1), in view of (i) and (ii) in Lemma 4.7, (4.78) is a straightforward consequence of the well-known regularity result on semi-linear second-order elliptic equations with L^1 right hand term (see [4, Lemma 23]).

Thus, the proof of Lemma 4.8 is completed. □

Now, we can derive the following result of $\left\|a^D(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^\infty(\Omega)}$ by proceeding as in the proof Lemma 4.2-Lemma 4.4.

Lemma 4.9. *Let* $\left(a^D, a^S, m, v\right)$ *be the classical solutions of (3.1) with* $\tau = 0$ *constructed in Lemma 3.2. Suppose that the assumptions of Theorem 1.1 are valid. Then there exists* $C_{50}(T_{max}) > 0$ *such that*

$$\left\|a^D(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^\infty(\Omega)} \le C_{50}(T_{max}) \quad \textit{for all } t \in (0, T_{max}). \tag{4.79}$$

Proof. Since this proof is very similar to that proof of Theorem 1.1 with $\tau = 1$, we give the outline of it here. Firstly, on account of Lemma 4.8, by using the Sobolev embedding theorem $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $q \in [1, \frac{np}{n-p}]$ if $p < n$ (see [12], pp. 171), we have

$$\|m(\cdot,t)\|_{L^3(\Omega)} \le C_{51} \quad \text{for all } t \in (0, T_{max}). \tag{4.80}$$

In reality, we only need choose $p \in [\frac{6}{5}, 2)$ to ensure (4.80). Secondly, it is not hard to prove that

$$\left\|a^D(\cdot,t)\right\|_{L^2(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^2(\Omega)} \le C_{52}(T_{max}) \quad \text{for all } t \in (0, T_{max}) \tag{4.81}$$

by applying the same method as Lemma 4.2. Next, in view of (4.81), using the elliptic L^p theory to the m-equation of (3.1) with $\tau = 0$, from the Sobolev embedding theorem, we can find that

$$\|m(\cdot,t)\|_{L^\infty(\Omega)} \leq C_{53}\|m(\cdot,t)\|_{W^{2,2}(\Omega)}$$
$$\leq C_{54}\left(\left\|a^D(\cdot,t)+a^S(\cdot,t)\right\|_{L^2(\Omega)}\right) \leq C_{55}(T_{max}) \quad \text{for all } t\in(0,T_{max}), \tag{4.82}$$

which, applied to the similar way as the proof of Lemma 4.3, implies

$$\left\|a^D(\cdot,t)\right\|_{L^p(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^p(\Omega)} \leq C_{56}(T_{max}) \quad \text{for all } t\in(0,T_{max}) \tag{4.83}$$

for any $p>1$. Finally, we can derive (4.79) by making a adaptation of the well-known Moser-Alikakos L^p iteration technique (see the proof of Lemma 4.4).

Thus, the proof of Lemma 4.9 is completed. □

Remark 4.2. Analogous to Remark 4.1, we also remark that the time-dependent boundedness of $\left\|a^D(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^\infty(\Omega)}$ established in Lemma 4.9 is due to the time-dependent boundedness of $\left\|a^D(\cdot,t)\right\|_{L^2(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^2(\Omega)}$, while the latter is as a result of the time-dependent of the choice of θ (see the proof of Lemma 4.2). In addition, in the present paper, thanks to the absence of chemotaxis term, we do not need to deal with the $\int_\Omega a^p|\nabla v|^2$ (see (3.21) of [33]), which is different from [33]. Therefore, we only need the boundedness of $\|m\|_{L^3(\Omega)}$ other than $\|\nabla v\|_{L^2(\Omega)}$ when establishing the a priori estimates of $\|a(\cdot,t)\|_{L^p(\Omega)}$ for all $p>2$ (see [33, Lemma 3.11]).

Furthermore, based on (4.83), by the standard elliptic L^p theory and the Sobolev embedding theorem, it follows from the third equation of (3.1) with $\tau=0$ that $\|m(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C_{57}\|m(\cdot,t)\|_{W^{2,p}(\Omega)} \leq C_{58}(T_{max})$ for all $t\in(0,T_{max})$, which, combined with almost exactly the same arguments as that in the proof of Lemma 4.5 and Lemma 4.6, yields the following results.

Lemma 4.10. *Under the hypotheses of Lemma 4.9. Then, for all $t\in(0,T_{max})$ and $q\geq 2$, there exists $C_{59}(q)>0$ independent of time such that*

$$\|\nabla v(\cdot,t)\|^q_{L^q(\Omega)} \leq c_{59}e^{c_{59}t}\left(\|\nabla v_0\|^q_{L^q(\Omega)} + 1 + \int_0^t \left(\|\nabla a^D(\cdot,s)\|^q_{L^q(\Omega)} + \|\nabla a^S(\cdot,s)\|^q_{L^q(\Omega)}\right)ds\right). \tag{4.84}$$

Lemma 4.11. *Suppose that the assumptions of Lemma 4.9 are valid. Then, for all $t\in(0,T_{max})$, there exists $C_{60}(T_{max})>0$ such that*

$$\left\|\nabla a^D(\cdot,t)\right\|^2_{L^2(\Omega)} + \left\|\nabla a^S(\cdot,t)\right\|^2_{L^2(\Omega)} + \int_0^t \left(\left\|\Delta a^D(\cdot,s)\right\|^2_{L^2(\Omega)} + \left\|\Delta a^S(\cdot,s)\right\|^2_{L^2(\Omega)}\right)ds$$
$$\leq C_{60}(T_{max}). \tag{4.85}$$

The proof of Theorem 1.1 in the case of $\tau=0$

Proof. From the weakened existence criterion (3.32) in Lemma 3.3, the global existence of the unique classical solution to the system (1.1) with $\tau=0$ is a straightforward consequence of combining Lemma 4.9 with Lemma 4.10 and Lemma 4.11. Analogous to the proof of Theorem 1.1 in the case of $\tau=1$, for the uniform boundedness of a^D, a^S, from Remark 4.2, we can achieve it by taking $\theta=\min\left\{1,\frac{T_{max}}{4}\right\}=1$ when $T_{max}=\infty$. As for the uniform boundedness of m, it is not hard to obtain by the standard elliptic L^p theory and the Sobolev embedding theorem. Consequently, (1.2) is true.

5. Proof of Theorem 1.2

Analogous to the proof of Theorem 1.1, to extend the local solution established in Lemma 3.1 and Lemma 3.2, assuming on the contrary that $T_{max} < \infty$, we need establish the boundedness of $\left\|a^D\right\|_{L^\infty(\Omega)}$, $\left\|a^S\right\|_{L^\infty(\Omega)}$ and $\|\nabla v\|_{L^5(\Omega)}$ by Lemma 3.3. In contrast to the proof of Theorem 1.1, inspired by [30], we will derive an adapted iteration criterion (see Lemma 5.1 and Lemma 5.2 below) to raise successfully the regularities of a^D, a^S from $L^1(\Omega)$ to $L^p(\Omega)$ for any $p > 1$, then use the iterative technique of Alikakos [1] or [8,24] to obtain the boundedness of $\left\|a^D\right\|_{L^\infty(\Omega)}$ and $\left\|a^S\right\|_{L^\infty(\Omega)}$.

5.1. The case of $\tau = 1$

Lemma 5.1. *Let $\left(a^D, a^S, m, v\right)$ be the classical solutions of (3.1) with $\tau = 1$ constructed in Lemma 3.1. Suppose that the assumptions of Theorem 1.2 are valid. Then, for $p = \frac{7}{6}q$, there exist some $C_{61}, C_{62} > 0$ independent of time such that*

$$\left\|a^D(\cdot,t)\right\|_{L^p(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^p(\Omega)} \le C_{62} \quad \text{for all } t \in (0, T_{max}) \tag{5.1}$$

provided $\left\|a^D(\cdot,t)\right\|_{L^q(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^q(\Omega)} \le C_{61}$ for all $t \in (0, T_{max})$ and any $q \ge 1$.

Proof. Noting $n = 3$, $p = \frac{7}{6}q$ for any $q \ge 1$, then $\frac{np}{np+2q} < 1 + \frac{2}{n} - \frac{1}{q}$ holds obviously. Therefore, one can find $r > 1$ such that

$$\frac{np}{np+2q} < \frac{1}{r} < 1 + \frac{2}{n} - \frac{1}{q}. \tag{5.2}$$

From $\frac{np}{np+2q} < \frac{1}{r}$, we have $nr - n < \frac{2q}{p}$. Thus, by the Gagliardo-Nirenberg's inequality [9,14], we obtain

$$\|u\|_{L^{2r}(\Omega)}^{2r} \le C_{63}\|u\|_{W^{1,2}(\Omega)}^2\|u\|_{L^{nr-n}(\Omega)}^{2(r-1)} \le C_{64}\|u\|_{W^{1,2}(\Omega)}^2\|u\|_{L^{\frac{2q}{p}}(\Omega)}^{2(r-1)} \quad \text{for } u \in W^{1,2}(\Omega). \tag{5.3}$$

On the other hand, from $\frac{1}{r} < 1 + \frac{2}{n} - \frac{1}{q}$, we have $\frac{1}{r'} > \frac{1}{q} - \frac{2}{n}$ for $\frac{1}{r} + \frac{1}{r'} = 1$. Thus, by $\left\|a^D(\cdot,t)\right\|_{L^q(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^q(\Omega)} \le C_{61}$ and Lemma 2.4, it follows from the m-equation of (3.1) that $\|m\|_{L^{r'}(\Omega)} \le C_{65}$. Therefore, we infer from the Young's inequality and (5.3) that

$$\begin{aligned}
\int_\Omega \left(a^D\right)^p m &\le \varepsilon \int_\Omega \left(a^D\right)^{pr} + C_{66}(\varepsilon)\int_\Omega m^{r'} \\
&\le \varepsilon \left\|\left(a^D\right)^{\frac{p}{2}}\right\|_{L^{2r}(\Omega)}^{2r} + C_{67}(\varepsilon) \\
&\le \varepsilon C_{64}\left\|\left(a^D\right)^{\frac{p}{2}}\right\|_{W^{1,2}(\Omega)}^2 \left\|\left(a^D\right)^{\frac{p}{2}}\right\|_{L^{\frac{2q}{p}}(\Omega)}^{2(r-1)} + C_{67}(\varepsilon) \\
&= \varepsilon C_{64}\left\|\left(a^D\right)^{\frac{p}{2}}\right\|_{W^{1,2}(\Omega)}^2 \left\|\left(a^D\right)\right\|_{L^q(\Omega)}^{p(r-1)} + C_{67}(\varepsilon) \\
&\le \varepsilon C_{68}\int_\Omega \left(a^D\right)^p + \varepsilon C_{68}\int_\Omega \left|\nabla\left(a^D\right)^{\frac{p}{2}}\right|^2 + C_{67}(\varepsilon).
\end{aligned} \tag{5.4}$$

Similarly, we have

$$\int_\Omega \left(a^S\right)^p m \le \varepsilon C_{68}\int_\Omega \left(a^S\right)^p + \varepsilon C_{68}\int_\Omega \left|\nabla\left(a^S\right)^{\frac{p}{2}}\right|^2 + C_{67}(\varepsilon). \tag{5.5}$$

Substituting (5.3) and (5.4) into (3.32) and (3.33), then choosing $\varepsilon > 0$ sufficiently small such that $\varepsilon C_{68} p \max\{\chi_D e^{\chi_D}, \chi_S e^{\chi_S}\} \le \frac{2(p-1)}{p}$, we obtain

$$\frac{d}{dt}\int_\Omega e^{\chi_D v}\left(a^D\right)^p + \frac{2(p-1)}{p}\int_\Omega \left|\nabla\left(a^D\right)^{\frac{p}{2}}\right|^2 \le C_{69}\int_\Omega \left(a^D\right)^p + C_{69} \tag{5.6}$$

and

$$\frac{d}{dt}\int_\Omega e^{\chi_S v}\left(a^S\right)^p + \frac{2(p-1)}{p}\int_\Omega \left|\nabla\left(a^S\right)^{\frac{p}{2}}\right|^2 \le C_{70}\int_\Omega \left(a^S\right)^p + C_{70}\int_\Omega \left(a^D\right)^p + C_{70}. \tag{5.7}$$

Combining (5.6) with (5.7) and adding $\int_\Omega \left(\left(a^D\right)^p + \left(a^S\right)^p\right)$ in both sides of the resulting inequality, we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(e^{\chi_D v}\left(a^D\right)^p + e^{\chi_S v}\left(a^S\right)^p\right) + \frac{2(p-1)}{p}\int_\Omega \left(\left|\nabla\left(a^D\right)^{\frac{p}{2}}\right|^2 + \left|\nabla\left(a^S\right)^{\frac{p}{2}}\right|^2\right)\\
&\quad + \int_\Omega \left(\left(a^D\right)^p + \left(a^S\right)^p\right)\\
&\quad \le C_{71}\int_\Omega \left(\left(a^D\right)^p + \left(a^S\right)^p\right) + C_{71}.
\end{aligned} \tag{5.8}$$

It follows from the Gagliardo-Nirenberg's inequality [9,14] and the Young's inequality that

$$\begin{aligned}
\int_\Omega \left(a^D\right)^p &= \left\|\left(a^D\right)^{\frac{p}{2}}\right\|^2_{L^2(\Omega)}\\
&= \left\|\left(a^D\right)^{\frac{p}{2}}\right\|^{2\cdot\frac{1}{5}}_{W^{1,2}(\Omega)} \left\|\left(a^D\right)^{\frac{p}{2}}\right\|^{2\cdot\frac{4}{5}}_{L^{\frac{12}{7}}(\Omega)}\\
&\le \varepsilon\left\|\left(a^D\right)^{\frac{p}{2}}\right\|^2_{W^{1,2}(\Omega)} + C_{72}(\varepsilon)\left\|a^D\right\|^p_{L^{\frac{6p}{7}}(\Omega)}\\
&\le \varepsilon\left\|\left(a^D\right)^{\frac{p}{2}}\right\|^2_{W^{1,2}(\Omega)} + C_{72}(\varepsilon)C_{61}.
\end{aligned} \tag{5.9}$$

Analogously, we have

$$\int_\Omega \left(a^S\right)^p \le \varepsilon\left\|\left(a^S\right)^{\frac{p}{2}}\right\|^2_{W^{1,2}(\Omega)} + C_{72}(\varepsilon)C_{61}. \tag{5.10}$$

Substituting (5.9), (5.10) into (5.8), and taking sufficiently small $\varepsilon > 0$ such that $C_{71}\varepsilon \le \frac{2}{7}$ (we note that $\frac{2(p-1)}{p} \in \left[\frac{2}{7}, 2\right)$ for $p = \frac{7}{6}q$ and $q \ge 1$). Then, from (5.8), we obtain

$$\frac{d}{dt}\int_\Omega \left(e^{\chi_D v}\left(a^D\right)^p + e^{\chi_S v}\left(a^S\right)^p\right) + \frac{5}{7}\int_\Omega \left(\left(a^D\right)^p + \left(a^S\right)^p\right) \le C_{73}. \tag{5.11}$$

We derive from $0 < v \le 1$ that

$$\frac{d}{dt}\int_\Omega \left(e^{\chi_D v}\left(a^D\right)^p + e^{\chi_S v}\left(a^S\right)^p\right) + \frac{5}{7e^{\max\{\chi_D,\chi_S\}}}\int_\Omega \left(e^{\chi_D v}\left(a^D\right)^p + e^{\chi_S v}\left(a^S\right)^p\right) \le C_{73}. \tag{5.12}$$

Consequently, by the Gronwall's inequality, we get

$$\int_\Omega \left(e^{\chi_D v}\left(a^D\right)^p + e^{\chi_S v}\left(a^S\right)^p\right) \le C_{74}, \tag{5.13}$$

which implies (5.1).

Thus, the proof of Lemma 5.1 is completed. □

Lemma 5.2. *Suppose that the assumptions of Lemma 5.1 remain valid. Then, for all $p > 1$, there exists $C_{75} > 0$ independent of time such that*

$$\left\|a^D(\cdot,t)\right\|_{L^p(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^p(\Omega)} \le C_{75} \quad \textit{for all } t \in (0, T_{max}). \tag{5.14}$$

Proof. From Lemma 4.1 (i) and (ii), we have $\left\|a^D(\cdot,t)\right\|_{L^1(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^1(\Omega)} \le M_1 + M_2$ for all $t \in (0, T_{max})$. Therefore, we infer from Lemma 5.1 that

$$\left\|a^D(\cdot,t)\right\|_{L^{\frac{7}{6}}(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^{\frac{7}{6}}(\Omega)} \le C_{76} \quad \text{for all } t \in (0, T_{max}). \tag{5.15}$$

Using a bootstrap argument to raise the regularity estimate of $\left\|a^D(\cdot,t)\right\|_{L^p(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^p(\Omega)}$ as above, which proves (5.14).

Thus, the proof of Lemma 5.2 is completed. □

Now, let us make a priori estimates of $\left\|a^D\right\|_{L^\infty(\Omega)}$ and $\left\|a^S\right\|_{L^\infty(\Omega)}$ by the iterative technique of Alikakos.

Lemma 5.3. *Under the assumptions of Lemma 5.1. Then there exists $C_{77} > 0$ independent of time such that*

$$\left\|a^D(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^\infty(\Omega)} \le C_{77} \quad \textit{for all } t \in (0, T_{max}). \tag{5.16}$$

Proof. From Lemma 5.2, we can obtain that there exists a $p > \frac{3}{2}$ such that $\left\|a^D(\cdot,t)\right\|_{L^p(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^p(\Omega)} \le C_{78}$. Hence, by using Lemma 2.4, we get

$$\|m(\cdot,t)\|_{L^\infty(\Omega)} \le C_{79} \quad \text{for all } t \in (0, T_{max}). \tag{5.17}$$

Combining (3.32) with (3.33), inserting (5.17) into the resulting equation, and adding $\int_\Omega \left(e^{\chi_D v}\left(a^D\right)^p + e^{\chi_S v}\left(a^S\right)^p\right)$ to both sides of the final result yields

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(e^{\chi_D v}\left(a^D\right)^p + e^{\chi_S v}\left(a^S\right)^p\right) + \int_\Omega \left(\left|\nabla\left(a^D\right)^{\frac{p}{2}}\right|^2 + \left|\nabla\left(a^S\right)^{\frac{p}{2}}\right|^2\right) \\
&\quad + \int_\Omega \left(e^{\chi_D v}\left(a^D\right)^p + e^{\chi_S v}\left(a^S\right)^p\right) \\
&\le C_{80}p\left(\left\|a^D\right\|^p_{L^p(\Omega)} + \left\|a^S\right\|^p_{L^p(\Omega)}\right)
\end{aligned} \tag{5.18}$$

for any $p \ge 2$, where C_{80} is independent of p. On the other hand, by Lemma 2.2 and the Young's inequality, we obtain

$$\begin{aligned} C_{80}p\left\|a\right\|_{L^p(\Omega)}^p &= C_{80}p\left\|a^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 \\ &\le C_{81}p\left(\left\|\nabla a^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{\frac{6}{5}}\left\|a^{\frac{p}{2}}\right\|_{L^1(\Omega)}^{\frac{4}{5}}+\left\|a^{\frac{p}{2}}\right\|_{L^1(\Omega)}^2\right) \\ &\le \left\|\nabla a^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2+C_{82}p^{\frac{5}{2}}\left\|a^{\frac{p}{2}}\right\|_{L^1(\Omega)}^2. \end{aligned} \tag{5.19}$$

Inserting (5.19) with $a = a^D, a = a^S$ into (5.18), we can conclude that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \left(e^{\chi_D v}\left(a^D\right)^p+e^{\chi_S v}\left(a^S\right)^p\right)+\int_\Omega\left(e^{\chi_D v}\left(a^D\right)^p+e^{\chi_S v}\left(a^S\right)^p\right) \\ &\le C_{82}p^{\frac{5}{2}}\left(\left\|\left(a^D\right)^{\frac{p}{2}}\right\|_{L^1(\Omega)}^2+\left\|\left(a^S\right)^{\frac{p}{2}}\right\|_{L^1(\Omega)}^2\right) \\ &\le C_{83}p^{\frac{5}{2}}\left(\max\left\{1,\left\|\left(a^D\right)^{\frac{p}{2}}\right\|_{L^1(\Omega)}+\left\|\left(a^D\right)^{\frac{p}{2}}\right\|_{L^1(\Omega)}\right\}\right)^2. \end{aligned} \tag{5.20}$$

Let $p_k := 2^k$ and $M_k := \max\left\{1, \sup_{t\in(0,T_{max})}\left(\left\|\left(a^D\right)^{p_k}\right\|_{L^1(\Omega)}+\left\|\left(a^S\right)^{p_k}\right\|_{L^1(\Omega)}\right)\right\}$ for $k = 1, 2...$ Therefore, upon the ODE comparison principle, we infer from (5.20) that there exists $\eta > 1$ independent of k such that

$$M_k \le \max\left\{\eta^k M_{k-1}^2, e^{\chi_D}|\Omega|\left\|a_0^D\right\|_{L^\infty(\Omega)}^{p_k}+e^{\chi_S}|\Omega|\left\|a_0^S\right\|_{L^\infty(\Omega)}^{p_k}\right\} \quad \text{for all } k \ge 1. \tag{5.21}$$

Consequently, if $\eta^k M_{k-1}^2 \le e^{\chi_D}|\Omega|\left\|a_0^D\right\|_{L^\infty(\Omega)}^{p_k}+e^{\chi_S}|\Omega|\left\|a_0^S\right\|_{L^\infty(\Omega)}^{p_k}$ for infinitely many $k \ge 1$, it is not hard to obtain

$$\begin{aligned} &\sup_{t\in(0,T_{max})}\left(\int_\Omega\left(a^D\right)^{p_{k-1}}\right)^{\frac{1}{p_{k-1}}}+\sup_{t\in(0,T_{max})}\left(\int_\Omega\left(a^S\right)^{p_{k-1}}\right)^{\frac{1}{p_{k-1}}} \\ &\le\left(\frac{e^{\chi_D}|\Omega|\left\|a_0^D\right\|_{L^\infty(\Omega)}^{p_k}+e^{\chi_S}|\Omega|\left\|a_0^S\right\|_{L^\infty(\Omega)}^{p_k}}{\eta^k}\right)^{\frac{1}{2p_{k-1}}}, \end{aligned} \tag{5.22}$$

which implies that

$$\sup_{t\in(0,T_{max})}\left\|a^D(\cdot,t)\right\|_{L^\infty(\Omega)}+\sup_{t\in(0,T_{max})}\left\|a^S(\cdot,t)\right\|_{L^\infty(\Omega)} \le \left\|a_0^D\right\|_{L^\infty(\Omega)}+\left\|a_0^S\right\|_{L^\infty(\Omega)}. \tag{5.23}$$

Conversely, if $\eta^k M_{k-1}^2 > e^{\chi_D}|\Omega|\left\|a_0^D\right\|_{L^\infty(\Omega)}^{p_k}+e^{\chi_S}|\Omega|\left\|a_0^S\right\|_{L^\infty(\Omega)}^{p_k}$ for all sufficiently large k, then it follows from (5.21) that

$$M_k \le \eta^k M_{k-1}^2 \quad \text{for all sufficiently large } k, \tag{5.24}$$

therefore, (5.24) still holds for all $k \ge 1$ by enlarging η if necessary. By a straightforward induction, we get

$$M_k \le \eta^{k+2+2^{k+1}} M_0^{2^k} \quad \text{for all } k \ge 1, \tag{5.25}$$

which implies that

$$M_k^{\frac{1}{p_k}} \le \eta^{\frac{k+2}{2^k}+2} M_0 \quad \text{for all } k \ge 1, \tag{5.26}$$

which after passing to $k \to \infty$ immediately entails that

$$\sup_{t\in(0,T_{max})} \left\|a^D(\cdot,t)\right\|_{L^\infty(\Omega)} + \sup_{t\in(0,T_{max})} \left\|a^S(\cdot,t)\right\|_{L^\infty(\Omega)} \leq \eta^2 \cdot M_0. \tag{5.27}$$

Letting $C_{77} := \max\left\{\left\|a_0^D\right\|_{L^\infty(\Omega)} + \left\|a_0^S\right\|_{L^\infty(\Omega)}, \eta^2 \cdot M_0\right\}$, hence a combination of (5.23) and (5.27) and by Lemma 4.1 (i) and (ii) immediately implies (5.16).

Thus, the proof of Lemma 5.3 is completed. □

Now, we are ready to prove Theorem 1.2 in the case of $\tau = 1$.

The proof of Theorem 1.2 in the case of $\tau = 1$

Proof. Since this proof is very similar to the proof of Theorem 1.1 in the case of $\tau = 1$, we omit it here (see Lemma 4.5 and Lemma 4.6 and the proof of Theorem 1.1 in the case of $\tau = 1$, we note that Lemma 4.5 holds for any dimension if only $\|m\|_{W^{1,\infty}(\Omega)} \leq C_{84}$, in addition, Lemma 4.6 holds if only $\|m\|_{L^\infty(\Omega)} \leq C_{85}$).

5.2. The case of $\tau = 0$

As opposed to the proof of Theorem 1.1 in the case of $\tau = 0$, we can not use the well-known regularity result on semi-linear second-order elliptic equations with L^1 right-hand term [4] (see Lemma 4.8 for details). Indeed, for the case of $n = 3$, on account of [4, Lemma 23] and the facts of (i) and (ii) in Lemma 4.7, we can only obtain the estimates of $\|m(\cdot,t)\|_{W^{1,q}(\Omega)}$ for all $p \in [1, \frac{3}{2})$, but by the Sobolev embedding theorem, the boundedness of $\|m(\cdot,t)\|_{L^3(\Omega)}$ is guaranteed by the boundedness of $\|m(\cdot,t)\|_{W^{1,q}(\Omega)}$ for $q \geq \frac{3}{2}$. However, by the standard elliptic L^p theory, we transform the estimate of $\|m(\cdot,t)\|_{L^3(\Omega)}$ into the estimate of $\left\|a^D(\cdot,t)\right\|_{L^3(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^3(\Omega)}$ which can be controlled by some terms successfully (see the proof of Lemma 5.4 below).

Lemma 5.4. *Let $\left(a^D, a^S, m, v\right)$ be the classical solutions of (3.1) with $\tau = 0$ constructed in Lemma 3.2. Suppose that the assumptions of Theorem 1.2 hold. Then there exists $C_{86} > 0$ independent of time such that*

$$\left\|a^D(\cdot,t)\right\|_{L^2(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^2(\Omega)} \leq C_{86} \quad \text{for all } t \in (0, T_{max}). \tag{5.28}$$

Proof. Multiplying the first equation of (3.1) by $2a^D$ and the second equation of (3.1) by $2a^S$, adding $\int_\Omega e^{\chi_D v}\left(a^D\right)^2$ and $\int_\Omega e^{\chi_S v}\left(a^S\right)^2$ to both sides of the resulting equalities, then integrating the results over Ω by parts and using the Young's inequality yields correspondingly

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega e^{\chi_D v}\left(a^D\right)^2 + 2\int_\Omega \left|\nabla a^D\right|^2 + \int_\Omega e^{\chi_D v}\left(a^D\right)^2 \\
&\leq \left(\chi_D \mu_v + 2\mu_D + 1\right) e^{\chi_D} \int_\Omega \left(a^D\right)^2 + \chi_D e^{\chi_D} \int_\Omega \left(a^D\right)^2 m - \mu_D \int_\Omega \left(a^D\right)^3 \\
&\leq \varepsilon_1 \int_\Omega \left(a^D\right)^3 + C_{87}(\varepsilon_1) + \varepsilon_2 \int_\Omega \left(a^D\right)^3 + C_{88}(\varepsilon_2) \int_\Omega m^3 - \mu_D \int_\Omega \left(a^D\right)^3
\end{aligned} \tag{5.29}$$

and

$$\frac{d}{dt}\int_\Omega e^{\chi_S v}\left(a^S\right)^2 + 2\int_\Omega \left|\nabla a^S\right|^2 + \int_\Omega e^{\chi_S v}\left(a^S\right)^2$$

$$
\begin{aligned}
&\leq \left(\left(\chi_S\mu_v + 2\mu_S + 1\right)e^{\chi_S} + \mu_M e^{\chi_D}\right)\int_\Omega \left(a^S\right)^2 + \chi_S e^{\chi_S}\int_\Omega \left(a^S\right)^2 m \\
&\quad + \mu_M e^{\chi_D}\int_\Omega \left(a^D\right)^2 - \mu_S\int_\Omega \left(a^S\right)^3 \\
&\leq \varepsilon_3\int_\Omega \left(a^S\right)^3 + C_{89}(\varepsilon_3) + \varepsilon_4\int_\Omega \left(a^S\right)^3 + C_{90}(\varepsilon_4)\int_\Omega m^3 \\
&\quad + \varepsilon_5\int_\Omega \left(a^D\right)^3 + C_{91}(\varepsilon_5) - \mu_S\int_\Omega \left(a^S\right)^3.
\end{aligned} \tag{5.30}
$$

Combining (5.29) with (5.30) entails

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(e^{\chi_D v}\left(a^D\right)^2 + e^{\chi_S v}\left(a^S\right)^2\right) + 2\int_\Omega \left(\left|\nabla a^D\right|^2 + \left|\nabla a^S\right|^2\right) + \int_\Omega \left(e^{\chi_D v}\left(a^D\right)^2 + e^{\chi_S v}\left(a^S\right)^2\right) \\
&\leq \left(\varepsilon_1 + \varepsilon_2 + \varepsilon_5 - \mu_D\right)\int_\Omega \left(a^D\right)^3 + \left(\varepsilon_3 + \varepsilon_4 - \mu_S\right)\int_\Omega \left(a^S\right)^3 + \left(C_{88}(\varepsilon_2) + C_{90}(\varepsilon_4)\right)\int_\Omega m^3 \\
&\quad + C_{87}(\varepsilon_1) + C_{89}(\varepsilon_3) + C_{91}(\varepsilon_5).
\end{aligned} \tag{5.31}
$$

It follows from the Gagliardo-Nirenberg's inequality, the standard elliptic L^p theory and the fact of (v) in Lemma 4.7 as well as the Young's inequality that

$$
\begin{aligned}
&\left(C_{88}(\varepsilon_2) + C_{90}(\varepsilon_4)\right)\int_\Omega m^3 \\
&= \left(C_{88}(\varepsilon_2) + C_{90}(\varepsilon_4)\right)\|m\|^3_{L^3(\Omega)} \\
&\leq C_{92}(\varepsilon_2,\varepsilon_4)\|m\|^2_{W^{1,3}(\Omega)}\|m\|_{L^1(\Omega)} \leq C_{92}(\varepsilon_2,\varepsilon_4)(M_1+M_2)\|m\|^2_{W^{1,3}(\Omega)} \\
&\leq C_{93}(\varepsilon_2,\varepsilon_4)\left(\left\|a^D\right\|^2_{L^3(\Omega)} + \left\|a^S\right\|^2_{L^3(\Omega)}\right) \leq \varepsilon_6\left(\left\|a^D\right\|^3_{L^3(\Omega)} + \left\|a^S\right\|^3_{L^3(\Omega)}\right) + C_{94}(\varepsilon_6).
\end{aligned} \tag{5.32}
$$

Substituting (5.32) into (5.31) gives

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(e^{\chi_D v}\left(a^D\right)^2 + e^{\chi_S v}\left(a^S\right)^2\right) + \int_\Omega \left(e^{\chi_D v}\left(a^D\right)^2 + e^{\chi_S v}\left(a^S\right)^2\right) \\
&\leq \left(\varepsilon_1 + \varepsilon_2 + \varepsilon_5 + \varepsilon_6 - \mu_D\right)\int_\Omega \left(a^D\right)^3 + \left(\varepsilon_3 + \varepsilon_4 + \varepsilon_6 - \mu_S\right)\int_\Omega \left(a^S\right)^3 + C_{95}(\varepsilon_1,\varepsilon_3,\varepsilon_5,\varepsilon_6).
\end{aligned} \tag{5.33}
$$

Taking sufficiently small $\varepsilon_i > 0,\ (i = 1, 2, ..., 6)$ such that $\varepsilon_1+\varepsilon_2+\varepsilon_5+\varepsilon_6-\mu_D < 0$ and $\varepsilon_3+\varepsilon_4+\varepsilon_6-\mu_S < 0$, then by the Gronwall's inequality and the fact $0 < v \leq 1$, we conclude from (5.33) that (5.28) is valid.

Thus, the proof of Lemma 5.4 is completed. □

Based on the estimate of Lemma 5.4, by applying the standard elliptic L^p theory to the m-equation of (3.1) and the Sobolev embedding theorem, it is not hard to show the boundedness of $\|m(\cdot,t)\|_{L^\infty(\Omega)} \leq C_{96}\|m(\cdot,t)\|_{W^{2,2}(\Omega)} \leq C_{97}$ for all $t \in (0, T_{max})$. Consequently, using the same arguments that for the case $\tau = 1$ (we note (5.4) obviously holds on account of $\|m(\cdot,t)\|_{L^\infty(\Omega)} \leq C_{97}$), we can obtain the estimate of $\left\|a^D(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^\infty(\Omega)}$ as follows.

Lemma 5.5. *Under the assumptions of Lemma 5.4. Then there exists a constant $C_{98} > 0$ independent of time such that*

$$\left\|a^D(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|a^S(\cdot,t)\right\|_{L^\infty(\Omega)} \le C_{98} \quad \textit{for all } t \in (0, T_{max}). \tag{5.34}$$

Now, one can prove the main result.

The proof of Theorem 1.2 in the case of $\tau = 0$

Proof. Proceeding in a same way as the proof of Theorem 1.1 in the case $\tau = 0$, one can complete this proof, we omit it here for simplicity. □

6. Conclusions

This paper is concerned with the global solvability of the two species cancer invasion haptotaxis-only system in two and three dimensional spatial settings. We obtained the global existence and boundedness of unique classical solution for arbitrary $\mu_D, \mu_S > 0$ in dimension 2, this result improves the existing result in [10]. Moreover, the global existence and boundedness of unique classical solution for large μ_D, μ_S in dimension 3 are investigated, which extend the previous result [10]. Unfortunately, we can not get rid of the technical assumption $\mu_D \ge \chi_D \mu_v$, $\mu_S \ge \chi_S \mu_v$ for the case of dimension 3. It is worth noting that some recent works are dedicated to investigate the global boundedness of solutions in 3-dimensional setting for the chemotaxis-haptotaxis model with tissue remodeling. Undoubtedly, since the chemotaxis-haptotaxis model with tissue remodeling is more complicated than the chemotaxis-only model, the researches of the global bounded solutions in dimension 3 are more challenging. For instance, the paper [16] studied the global solvability under some strong restriction on generalized logistic damping of cell for $q > \frac{8}{7}$ in dimension 3 (see [16, Theorem 1.1] for its details), but the cases $q \le \frac{8}{7}$ or $q = 1$ are open. In [25], the authors used a sufficient small birth-rate parameter of cell r and suitable small initial data to control the quantity $\int_\Omega a^2(t) + \int_\Omega |\nabla v(t)|^4$ such that it satisfies an autonomous ordinary differential inequality, then established the global solvability for $n = 3$ (see [25, Theorem 1.2] for its details), but the case $r = 1$ is open. Inspired by [16,25], we expect that the similar results for our haptotaxis-only model (1.1) can be obtained. Furthermore, we hope that we can solve the open questions in [16,25] mentioned above for our haptotaxis-only model. Therefore, some of further explorations are as follows:

(1) Introducing the generalized logistic damping as [16] into two cancer cell equations of model (1.1), for the case of arbitrary $\mu_D, \mu_S > 0$, discussing the global boundedness of solution in the n-dimensional setting ($n \ge 1$), especially in the physically most relevant case $n = 3$.

(2) Taking the logistic type as [25] into account for model (1.1), studying the global boundedness of solution in the n-dimensional setting ($n \ge 3$).

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