Dampening effect of logistic source in a two-dimensional haptotaxis system with nonlinear zero-order interaction

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A R T I C L E   I N F O

A B S T R A C T

This paper deals with the oncolytic virotherapy model

$$
\begin{cases}
u_t = \Delta u - \nabla \cdot (u \nabla v) - uz + \mu u(1-u), \\
v_t = -(u+w)v, \\
w_t = D_w \Delta w - w + uz, \\
z_t = D_z \Delta z - z - uz + \beta w,
\end{cases} \qquad (\star)
$$

in a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary, where μ, D_w, D_z and β are prescribed positive parameters. For any given suitably regular initial data, the global existence of classical solution to the corresponding homogeneous Neumann initial-boundary problem for a more general model allowing $\mu = 0$ was previously verified in Tao and Winkler (2020) [15]. This work further shows that whenever $\mu > 0$, the above-mentioned global classical solution to $(\star)$ is uniformly bounded; and moreover, if $\beta < 1$, then the solution (u,v,w,z) stabilizes to the constant equilibrium $(1,0,0,0)$ in the topology $L^p(\Omega) \times (L^\infty(\Omega))^3$ with any $p > 1$ in a large time limit.

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1. Introduction

Aim at bypassing the obstacle of drug transfer in traditional chemotherapy ([10], [6]), oncolytic virotherapy becomes an alternative treatment for cancer and it has been undergoing clinical trials (cf. [2] and [9], for instance). Very recently, in order to explore efficiency of this novel therapy, Alzahrani et al. in [1] proposed the following haptotaxis model

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$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) - uz + \mu u(1-u), & x \in \Omega, \ t > 0, \\ v_t = -(u+w)v, & x \in \Omega, \ t > 0, \\ w_t = D_w \Delta w - w + uz, & x \in \Omega, \ t > 0, \\ z_t = D_z \Delta z - z - uz + \beta w, & x \in \Omega, \ t > 0, \\ (\nabla u - u \nabla v) \cdot \nu = \frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), \quad z(x,0) = z_0(x), & x \in \Omega, \end{cases} \qquad (1.1)$$

with positive parameters μ, β, D_w and D_z. Throughout this work we shall assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary. Here u, w, z and v represent the densities of uninfected tumor cells, infected tumor cells, virus particles and normal tissue, respectively. Besides random motion, the uninfected tumor cells direct their movement toward the higher densities of tissue, they are assumed to follow a logistic growth and their number could be reduced due to infection by virus, while the infected cells randomly diffuse and have a natural death; the virus is released by infected cells and it experiences diffusion and loss due to decay and infection; and the tissue is degraded upon contact with cancer cells. The readers may refer to [1] for more detailed biological backgrounds behind (1.1).

Different from standard reaction-diffusion equations, the cross-diffusion term in chemotactic or haptotactic systems like (1.1) may exhibit a destabilizing feature ([5], [22]). Previous qualitative studies on such types of cross-diffusion systems mainly concentrate on global existence ([20], [11], [26], [25], [8]) and only a few address the large time behavior of solutions ([7], [13], [24]).

In contrast to preceding haptotaxis systems, (1.1) contains a nonlinear zero-order interaction term uz in the third equation that gives rise to a new challenge in analysis of this system. Although (1.1) is indeed globally well-posed ([15]), a novel critical parameter phenomenon for infinite-time blow-up was recently detected for (1.1) without any growth or sink term ([17]) and a critical virus production rate for efficiency of this treatment was also analytically verified for (1.1) without any proliferation source ([19]). Moreover, the global existence and large time behavior of classical solutions was also discussed for (1.1) provided that $\beta < 1$ and $\mu = 0$ ([16]). Very recently, an asymptotic stability property of the homogeneous distribution $(u, v, w, z) = (\gamma, 0, 0, 0)$ with $\gamma \in (0, \frac{1}{(\beta-1)_+})$ for any $\beta > 0$, particularly for $\beta \geq 1$, was asserted in ([18]).

The purpose of this work is to identify the stabilization effect of the logistic dampening term in (1.1). To this end, we assume that

$$\begin{cases} u_0, v_0 \text{ and } w_0 \text{ are nonnegative functions from } C^{2+\vartheta}(\overline{\Omega}) \text{ for some } \vartheta > 0, \\ \text{with } u_0 > 0, \ w_0 \not\equiv 0, z_0 \not\equiv 0, \ \sqrt{v_0} \in W^{1,2}(\Omega) \text{ and } \frac{\partial u_0}{\partial \nu} = \frac{\partial v_0}{\partial \nu} = \frac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \qquad (1.2)$$

We shall prove that whenever $\mu > 0$, for any $\beta > 0$ the classical solution of (1.1) is uniformly bounded; furthermore, it is claimed that if $\beta < 1$, then the solution (u, v, w, z) stabilizes to the constant stationary solution $(1, 0, 0, 0)$ in L^∞ topology. More precisely, we have the following:

Theorem 1.1. *Let $\mu > 0$, and let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain. Then for any given (u_0, v_0, w_0, z_0) satisfying (1.2), the solution (u, v, w, z) of (1.1) is bounded in the sense that*

$$\sup_{t>0} \left\{ \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} + \|z(\cdot, t)\|_{L^\infty(\Omega)} \right\} < \infty. \qquad (1.3)$$

If

$$\beta \in (0, 1),$$

then furthermore

$$u(\cdot, t) \to 1 \quad in \ L^p(\Omega) \qquad for \ all \ p \geq 1$$

and

$$v(\cdot, t) \to 0 \qquad in \ L^\infty(\Omega)$$

as well as

$$w(\cdot, t) \to 0 \qquad in \ L^\infty(\Omega)$$

and

$$z(\cdot, t) \to 0 \qquad in \ L^\infty(\Omega)$$

as $t \to \infty$.

In contrast to previous studies on the asymptotic behavior of solutions to chemotaxis-haptotaxis or pure chemotaxis systems in which certain requirements of the minimal size of the logistic dampening coefficient μ are indispensably demanded (cf. [23], [12] and [13], for instance), no any restrictions on the size of μ seem necessary for the stabilization result obtained in Theorem 1.1.

2. L^p bounds for w

Let us begin with the global smooth solvability of (1.1) previously asserted in [15], together with one basic solution property.

Lemma 2.1. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with smooth boundary, and assume that* (u_0, v_0, w_0, z_0) *fulfills (1.2). Then the problem (1.1) admits a uniquely determined classical solution* $(u, v, w, z) \in (C^{2,1}(\overline{\Omega} \times [0, \infty)))^4$ *for which* v *is nonnegative, and for which* u, w *and* z *are positive in* $\overline{\Omega} \times (0, \infty)$. *Moreover,*

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq \|v_0(\cdot)\|_{L^\infty(\Omega)} \qquad for \ all \ t > 0. \tag{2.1}$$

The presence of logistic dampening in the first equation in (1.1) will be decisive for us to further verify the uniform boundedness of the global solution constructed in Lemma 2.1, and correspondingly the assumption that $\mu > 0$ will play a key role in deriving the following L^1 bounds for u, w and z and a space-time L^2 bound for u, which serve as a starting point of our subsequently reasoning.

Lemma 2.2. *There exists* $C > 0$ *such that*

$$\int_\Omega u(\cdot, t) \leq C, \quad \int_\Omega w(\cdot, t) \leq C \quad and \quad \int_\Omega z(\cdot, t) \leq C \tag{2.2}$$

as well as

$$\int_t^{t+1} \int_\Omega u^2 \leq C \tag{2.3}$$

for all $t > 0$.

Proof. We firstly assume that $\beta > 0$ and use (1.1) to compute

$$
\frac{d}{dt}\left\{ 2\beta \int_\Omega u + 2\beta \int_\Omega w + \int_\Omega z \right\} = \left\{ -2\beta \int_\Omega uz + 2\beta\mu \int_\Omega u - 2\beta\mu \int_\Omega u^2 \right\}
$$

$$
+ \left\{ -2\beta \int_\Omega w + 2\beta \int_\Omega uz \right\}
$$

$$
+ \left\{ -\int_\Omega z - \int_\Omega uz + \beta \int_\Omega w \right\}
$$

$$
= 2\beta\mu \int_\Omega u - \beta \int_\Omega w - \int_\Omega z - \int_\Omega uz - 2\beta\mu \int_\Omega u^2
$$

for all $t > 0$, so that by Young's inequality,

$$
\frac{d}{dt}\left\{ 2\beta \int_\Omega u + 2\beta \int_\Omega w + \int_\Omega z \right\} + \frac{1}{2} \cdot \left\{ 2\beta \int_\Omega u + 2\beta \int_\Omega w + \int_\Omega z \right\} + \beta\mu \int_\Omega u^2
$$

$$
= \beta(1 + 2\mu) \int_\Omega u - \beta\mu \int_\Omega u^2 - \frac{1}{2} \int_\Omega z - \int_\Omega uz
$$

$$
\leq \beta(1 + 2\mu) \int_\Omega u - \beta\mu \int_\Omega u^2
$$

$$
\leq \frac{\beta(1 + 2\mu)^2}{4\mu} \cdot |\Omega| \qquad \text{for all } t > 0.
$$

Upon an ODE comparison, this readily implies (2.2) due to $\beta > 0$ and thereby a further integration over time yields (2.3) thanks to the fact that $\mu > 0$. If $\beta = 0$, by estimating $\frac{d}{dt}\left\{ \int_\Omega w + \int_\Omega z \right\}$ and $\frac{d}{dt}\int_\Omega u$ we still can arrive at (2.2) and (2.3) in a same manner. $\square$

With the help of parabolic smooth properties in the spatial two-dimensional setting, the L^1 bound of w actually implies L^p integrability of z for arbitrarily large finite p.

Lemma 2.3. Let $p \in (1, \infty)$. Then there exists $C(p) > 0$ such that

$$
\|z(\cdot, t)\|_{L^p(\Omega)} \leq C(p) \qquad \text{for all } t > 0. \tag{2.4}
$$

Proof. By nonnegativity of uz and z, in light of the order preserving property of $e^{\sigma\Delta}$ for $\sigma \geq 0$ we firstly find that

$$
z(\cdot, t) = e^{t(D_z\Delta - 1)}z_0 - \int_0^t e^{(t-s)(D_z\Delta - 1)}u(\cdot, s)z(\cdot, s)ds + \beta \int_0^t e^{(t-s)(D_z\Delta - 1)}w(\cdot, s)ds
$$

$$
\leq e^{tD_z\Delta}z_0 + \beta \int_0^t e^{(t-s)(D_z\Delta - 1)}w(\cdot, s)ds \quad \text{in } \Omega \qquad \text{for all } t > 0,
$$

and hence, by well-known smoothing properties of the Neumann heat semigroup $(e^{\sigma\Delta})_{\sigma \geq 0}$ ([21]) and Lemma 2.2, secondly see that

$$\|z(\cdot,t)\|_{L^p(\Omega)} \leq \|e^{tD_z\Delta}z_0\|_{L^p(\Omega)} + c_1\beta \int_0^t \left(1 + (t-s)^{-1+\frac{1}{p}}\right)e^{-(t-s)}\|w(\cdot,s)\|_{L^1(\Omega)}ds$$

$$\leq \|z_0\|_{L^p(\Omega)} + c_1 c_2\beta \int_0^t \left(1 + (t-s)^{-1+\frac{1}{p}}\right)e^{-(t-s)}ds \qquad \text{for all } t > 0$$

with some $c_1 > 0$ and $c_2 > 0$. This entails (2.4), because

$$\int_0^t \left(1 + (t-s)^{-1+\frac{1}{p}}\right)e^{-(t-s)}ds \leq \int_0^\infty (1 + \sigma^{-1+\frac{1}{p}})e^{-\sigma}d\sigma < 2 + p \qquad \text{for all } t > 0. \quad \square$$

In light of the above L^p integrability information of z along with a space-time L^2 bound for u in (2.3) and according to parabolic smooth estimates in the two-dimensional case once again, we also obtain L^p bounds of w for arbitrary $p > 2$.

Lemma 2.4. *Let* $p \in (2,\infty)$. *Then there exists* $C(p) > 0$ *with the property*

$$\|w(\cdot,t)\|_{L^p(\Omega)} \leq C(p) \qquad \text{for all } t > 0. \tag{2.5}$$

Proof. Given $p > 2$, we choose $q := \frac{2p}{p+1} \in (1,2)$. We then recall known smoothing estimates for the Neumann heat semigroup on Ω ([21]) to obtain positive constants $c_1 = c_1(p)$ and $c_2 = c_2(p)$ satisfying

$$\|e^{D_w\Delta}\varphi\|_{L^p(\Omega)} \leq c_1\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi \in C^0(\overline{\Omega}) \tag{2.6}$$

and

$$\|e^{\sigma D_w\Delta}\varphi\|_{L^p(\Omega)} \leq c_2\sigma^{-\frac{p-1}{2p}}\|\varphi\|_{L^q(\Omega)} \qquad \text{for all } \sigma \in (0,1) \text{ and each } \varphi \in C^0(\overline{\Omega}). \tag{2.7}$$

Now in light of a variation-of-constants representation of w solving the third equation in (1.1), we can invoke (2.6) and (2.7) to estimate

$$\|w(\cdot,t)\|_{L^p(\Omega)} = \left\|e^{D_w\Delta-1}w(\cdot,t-1) + \int_{t-1}^t e^{(t-s)(D_w\Delta-1)}u(\cdot,s)z(\cdot,s)ds\right\|_{L^p(\Omega)}$$

$$\leq e^{-1} \cdot c_1\|w(\cdot,t-1)\|_{L^1(\Omega)} + c_2 \int_{t-1}^t (t-s)^{-\frac{p-1}{2p}}\|u(\cdot,s)z(\cdot,s)\|_{L^q(\Omega)}ds \tag{2.8}$$

for all $t > 1$, where according to the Hölder inequality and Lemma 2.3, we find $c_3 = c_3(p) > 0$ such that

$$c_2 \int_{t-1}^t (t-s)^{-\frac{p-1}{2p}}\|u(\cdot,s)z(\cdot,s)\|_{L^q(\Omega)}ds \leq c_2 \int_{t-1}^t (t-s)^{-\frac{p-1}{2p}}\|u(\cdot,s)\|_{L^2(\Omega)}\|z(\cdot,s)\|_{L^{\frac{2q}{2-q}}(\Omega)}ds$$

$$\leq c_3 \int_{t-1}^t (t-s)^{-\frac{p-1}{2p}}\|u(\cdot,s)\|_{L^2(\Omega)}ds$$

$$\leq c_3 \cdot \left\{\int_{t-1}^t (t-s)^{-\frac{p-1}{p}}ds\right\}^{\frac{1}{2}} \cdot \left\{\int_{t-1}^t \|u(\cdot,s)\|_{L^2(\Omega)}^2 ds\right\}^{\frac{1}{2}}$$

$$= pc_3 \cdot \left\{ \int_{t-1}^{t} \|u(\cdot,s)\|_{L^2(\Omega)}^2 ds \right\}^{\frac{1}{2}} \qquad \text{for all } t > 1.$$

Since

$$\sup_{t>1} \|w(\cdot, t-1)\|_{L^1(\Omega)} < \infty \qquad \text{and} \qquad \int_{t-1}^{t} \|u(\cdot,s)\|_{L^2(\Omega)}^2 ds = \int_{t-1}^{t} \int_{\Omega} u^2 < \infty \qquad \text{for all } t > 1$$

by Lemma 2.2, (2.8) yields (2.5), because w is bounded in $\overline{\Omega} \times [0,1]$ by Lemma 2.1. □

3. L^∞ estimates on u

In this section we shall establish the boundedness of u in L^∞. For this purpose, as performed in [3], [4], [20] and [11] which dealt with haptotaxis-related systems, we introduce the variable change

$$a := ue^{-v} \tag{3.1}$$

and then in view of (1.1),

$$\begin{cases} a_t = e^{-v}\nabla \cdot (e^v \nabla a) + a(ae^v + w)v - az + \mu a(1 - ae^v), & x \in \Omega, \ t > 0, \\ v_t = -(ae^v + w)v, & x \in \Omega, \ t > 0, \\ \frac{\partial a}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ a(x,0) = u_0(x)e^{-v_0(x)} =: a_0(x), \quad v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{3.2}$$

A direct testing procedure leads to the following primary inequality which will be invoked in Lemma 3.2 and Lemma 3.4 below.

Lemma 3.1. *Let $p > 1$. Then*

$$\frac{d}{dt} \int_\Omega e^v a^p \leq -p(p-1) \int_\Omega e^v a^{p-2} |\nabla a|^2 + (p-1) \int_\Omega e^v a^p (ae^v + w)v + \mu p \int_\Omega e^v a^p \qquad \text{for all } t > 0. \tag{3.3}$$

Proof. From (3.2) we infer that $(ae^v + w)v = -v_t$ in $\Omega \times (0,\infty)$ and that

$$\frac{d}{dt} \int_\Omega e^v a^p = p \int_\Omega e^v a^{p-1} \cdot \left\{ e^{-v} \nabla \cdot (e^v \nabla a) - av_t - az + \mu a - \mu a^2 e^v \right\} + \int_\Omega e^v a^p v_t$$

$$= -p(p-1) \int_\Omega e^v a^{p-2} |\nabla a|^2 - (p-1) \int_\Omega e^v a^p v_t + \mu p \int_\Omega e^v a^p$$

$$- \rho p \int_\Omega e^v a^p z - \mu p \int_\Omega e^{2v} a^{p+1}$$

for all $t > 0$, this leads to (3.3) upon abandoning the rightmost two nonpositive summands. □

Using a bootstrap L^p-estimate technique developed in [14, Lemma 3.8] and relying on (2.3), Lemma 2.4 and Lemma 2.1, an application of Lemma 3.1 to $p = 2$ can yield a bound for a in L^2.

Lemma 3.2. *There exists $C > 0$ fulfilling*

$$\|a(\cdot, t)\|_{L^2(\Omega)} \le C \qquad \text{for all } t > 0. \tag{3.4}$$

Proof. Lemma 2.1 guarantees the existence of $c_1 > 0$ such that

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \le c_1 \qquad \text{for all } t > 0, \tag{3.5}$$

whereas Lemma 2.4 provides $c_2 > 0$ satisfying

$$\int_\Omega w^3(\cdot, t) \le c_2 \qquad \text{for all } t > 0. \tag{3.6}$$

Now from Lemma 3.1 we infer the inequality

$$\frac{d}{dt} \int_\Omega e^v a^2 + 2 \int_\Omega e^v |\nabla a|^2 \le \int_\Omega e^{2v} a^3 v + \int_\Omega e^v a^2 vw + 2\mu \int_\Omega e^v a^2 \qquad \text{for all } t > 0, \tag{3.7}$$

where by Young's inequality, (3.5) and (3.6),

$$\int_\Omega e^{2v} a^3 v + \int_\Omega e^v a^2 vw + 2\mu \int_\Omega e^v a^2 \le c_1 e^{2c_1} \int_\Omega a^3 + c_1 e^{c_1} \int_\Omega a^2 w + 2\mu e^{c_1} \int_\Omega a^2$$

$$\le \left\{ c_1 e^{2c_1} + c_1 e^{c_1} + 2\mu e^{c_1} \right\} \cdot \int_\Omega a^3 + c_1 e^{c_1} \int_\Omega w^3 + 2\mu e^{c_1} \cdot |\Omega|$$

$$\le c_3 \int_\Omega a^3 + c_4 \qquad \text{for all } t > 0$$

with $c_3 := c_1 e^{2c_1} + c_1 e^{c_1} + 2\mu e^{c_1}$ and $c_4 := c_1 e^{c_1} c_2 + 2\mu e^{c_1} \cdot |\Omega|$. As the Gagliardo-Nirenberg inequality along with Young's inequality yields that

$$c_3 \int_\Omega a^3 = c_3 \|a\|_{L^3(\Omega)}^3 \le c_5 \|\nabla a\|_{L^2(\Omega)} \|a\|_{L^2(\Omega)}^2 + c_5 \|a\|_{L^2(\Omega)}^3$$

$$\le \left\{ 2\|\nabla a\|_{L^2(\Omega)}^2 + c_5^2 \|a\|_{L^2(\Omega)}^4 \right\} + \left\{ \|a\|_{L^2(\Omega)}^4 + c_5^4 \right\}$$

$$\le 2 \int_\Omega e^v |\nabla a|^2 + c_6 \|a\|_{L^2(\Omega)}^4 + c_6 \qquad \text{for all } t > 0$$

with some $c_5 > 0$ and $c_6 := \max\{c_5^2 + 1, c_5^4\}$, by $\int_\Omega a^2 \le \int_\Omega e^v a^2$ we thus infer from (3.7) that

$$\frac{d}{dt} \int_\Omega e^v a^2 \le \left\{ c_6 \int_\Omega a^2(\cdot, t) \right\} \cdot \int_\Omega e^v a^2 + c_6 \qquad \text{for all } t > 0. \tag{3.8}$$

Letting

$$h(t) := c_6 \int_\Omega a^2(\cdot, t) \qquad \text{for all } t > 0,$$

we may apply Lemma 2.2 along with (2.1) to find $c_7 > 0$ fulfilling

$$\int_t^{t+1} h(s)ds \le c_7 \qquad \text{for all } t > 0, \tag{3.9}$$

and thereby for any given $t > 1$, it is possible to fix $t_0 = t_0(t) \in (t-1,t)$ such that $t_0 > 0$ and

$$\int_\Omega a^2(\cdot, t_0) \le c_8 := \max\left\{ \int_\Omega a_0^2, \, c_7 \right\},$$

so that integrating (3.8) shows that

$$\int_\Omega e^{v(\cdot,t)}a^2(\cdot,t) \le \left(\int_\Omega e^{v(\cdot,t_0)}a^2(\cdot,t_0) \right) \cdot e^{\int_{t_0}^t h(s)ds} + c_6 \int_{t_0}^t e^{\int_s^t h(\sigma)d\sigma} ds$$

$$\le e^{c_1}c_8 \cdot e^{c_7} + c_6 e^{c_7} \qquad \text{for all } t > 1$$

due to (3.5), (3.9) and the fact that $t - t_0 \le 1$. Since $e^v \ge 1$ and a is bounded in $\overline{\Omega} \times [0,1]$ by Lemma 2.1, this proves (3.4). $\square$

Using this L^2 bound of a together with the smoothing properties of the Neumann heat semigroup on Ω once more, we can further improve the regularity of w and z in the following sense.

Lemma 3.3. *There exists $C > 0$ with the property that*

$$\|w(\cdot,t)\|_{W^{1,3}(\Omega)} \le C \qquad \text{for all } t > 0 \tag{3.10}$$

and

$$\|z(\cdot,t)\|_{W^{1,3}(\Omega)} \le C \qquad \text{for all } t > 0. \tag{3.11}$$

Proof. Combining Lemma 3.2, Lemma 2.4 and Lemma 2.3 with (3.1) and (2.1) we can fix positive constants c_1, c_2 and c_3 satisfying

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le c_1, \quad \|w(\cdot,t)\|_{L^2(\Omega)} \le c_2 \quad \text{and} \quad \|z(\cdot,t)\|_{L^4(\Omega)} \le c_3 \qquad \text{for all } t > 0.$$

Since well-known smoothing properties of the Neumann heat semigroup on Ω ([21]) ensure the existence of $c_4 > 0$ and $c_5 > 0$ such that

$$\|e^{D_w\Delta}\varphi\|_{W^{1,3}(\Omega)} \le c_4\|\varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi \in C^0(\overline{\Omega})$$

and

$$\|e^{\sigma D_w\Delta}\varphi\|_{W^{1,3}(\Omega)} \le c_5\sigma^{-\frac{11}{12}}\|\varphi\|_{L^{\frac{4}{3}}(\Omega)} \qquad \text{for all } \sigma \in (0,1) \text{ and any } \varphi \in C^0(\overline{\Omega}),$$

we thus see that by the Hölder inequality,

$$\|w(\cdot,t)\|_{W^{1,3}(\Omega)} = \left\| e^{D_w \Delta - 1} w(\cdot, t-1) + \int_{t-1}^{t} e^{(t-s)(D_w \Delta - 1)} u(\cdot, s) z(\cdot, s) ds \right\|_{W^{1,3}(\Omega)}$$

$$\leq c_4 e^{-1} \|w(\cdot, t-1)\|_{L^2(\Omega)} + c_5 \int_{t-1}^{t} (t-s)^{-\frac{11}{12}} \|u(\cdot, s) z(\cdot, s)\|_{L^{\frac{4}{3}}(\Omega)} ds$$

$$\leq c_4 e^{-1} \|w(\cdot, t-1)\|_{L^2(\Omega)} + c_5 \int_{t-1}^{t} (t-s)^{-\frac{11}{12}} \|u(\cdot, s)\|_{L^2(\Omega)} \|z(\cdot, s)\|_{L^4(\Omega)} ds$$

$$\leq c_2 c_4 e^{-1} + 12 c_1 c_3 c_5 \qquad \text{for all } t > 1,$$

which entails (3.10), while (3.11) can be proved in a quite similar manner. $\quad\square$

Lemma 3.3 actually implies the boundedness of w in L^∞ due to the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ for $p > n$, with n denoting the space dimension. Depending on this and Lemma 3.2, we can achieve the boundedness of a in L^∞ via a Moser iteration technique.

Lemma 3.4. *There exists $C > 0$ such that*

$$\|a(\cdot, t)\|_{L^\infty(\Omega)} \leq C \qquad \text{for all } t > 0. \tag{3.12}$$

Proof. According to Lemma 2.1 and Lemma 3.3 along with the Sobolev embedding $W^{1,3}(\Omega) \hookrightarrow C^0(\overline{\Omega})$, we can pick $c_1 > 0$ and $c_2 > 0$ such that $v \leq c_1$ and $w \leq c_2$ on $\Omega \times (0, \infty)$. Combining these with the fact that $\frac{4(p-1)}{p} e^v \geq 3$ when $p \geq 4$, from Lemma 3.1 together with Young's inequality we obtain that for all $p \geq 4$,

$$\frac{d}{dt} \int_\Omega e^v a^p + 3 \int_\Omega |\nabla a^{\frac{p}{2}}|^2 + \int_\Omega e^v a^p \leq (p-1) \int_\Omega e^v a^p (ae^v + w) v + (\mu p + 1) \int_\Omega e^v a^p$$

$$\leq c_1 e^{2c_1} p \int_\Omega a^{p+1} + \left\{ (c_1 c_2 e^{c_1} + \mu e^{c_1}) p + e^{c_1} \right\} \int_\Omega a^p$$

$$\leq c_3 p \int_\Omega a^{p+1} + c_3(p+1) \int_\Omega a^p$$

$$\leq c_3 p \int_\Omega a^{p+1} + c_3(p+1) \left\{ \frac{p}{p+1} \int_\Omega a^{p+1} + \frac{|\Omega|}{p+1} \right\}$$

$$= 2 c_3 p \int_\Omega a^{p+1} + c_4 \qquad \text{for all } t > 0, \tag{3.13}$$

where $c_3 := \max\{c_1 e^{2c_1}, c_1 c_2 e^{c_1} + \mu e^{c_1}, e^{c_1}\} > 0$ and $c_4 := c_3 |\Omega| > 0$ are independent of p. Since Lemma 3.2 provides $c_5 > 0$ such that

$$\int_\Omega a^2(\cdot, t) \leq c_5 \qquad \text{for all } t > 0,$$

we thus can invoke a Moser-type iteration method to derive (3.12) (cf. [16, Lemma 6.4] for details). $\quad\square$

4. Stabilization when $\beta < 1$

Throughout this section we shall suppose that $\beta < 1$, and we shall investigate the asymptotic behavior of the global classical solution constructed in Lemma 2.1. In fact, the assumption that $\beta < 1$ readily results in the exponential decay property of w and z.

Lemma 4.1. *Let $\beta < 1$. Then there exist $\gamma \in (0, 1)$ and $C > 0$ such that*

$$\|w(\cdot, t)\|_{L^\infty(\Omega)} \le Ce^{-\gamma t} \qquad \text{for all } t > 0 \tag{4.1}$$

and

$$\|z(\cdot, t)\|_{L^\infty(\Omega)} \le Ce^{-\gamma t} \qquad \text{for all } t > 0 \tag{4.2}$$

Proof. We use the third and fourth equations in (1.1) to compute

$$\frac{d}{dt}\left\{ \int_\Omega w + \int_\Omega z \right\} = \left\{ -\int_\Omega w + \int_\Omega uz \right\} + \left\{ -\int_\Omega z - \int_\Omega uz + \beta \int_\Omega w \right\}$$

$$= -(1-\beta)\int_\Omega w - \int_\Omega z$$

$$\le -(1-\beta) \cdot \left\{ \int_\Omega w + \int_\Omega z \right\} \qquad \text{for all } t > 0$$

which implies that

$$\int_\Omega w + \int_\Omega z \le \left\{ \int_\Omega w_0 + \int_\Omega z_0 \right\} \cdot e^{-(1-\beta)t} \qquad \text{for all } t > 0.$$

This in conjunction with Lemma the Gagliardo-Nirenberg inequality and Lemma 3.3 yields $c_1 > 0$ and $c_2 > 0$ such that

$$\|w(\cdot, t)\|_{L^\infty(\Omega)} \le c_1 \|w(\cdot, t)\|_{W^{1,3}(\Omega)}^{\frac{6}{7}} \|w(\cdot, t)\|_{L^1(\Omega)}^{\frac{1}{7}}$$

$$\le c_2 e^{-\frac{1-\beta}{7}t} \qquad \text{for all } t > 0,$$

and thus the proof is complete with $\gamma := \frac{1-\beta}{7}$ and $C := c_2$.   □

Since z eventually vanishes in L^∞ due to (4.2), from (3.2) and the parabolic comparison principle we can infer that a has a positive lower bound, and thus so does u in view of (3.1) and (2.1).

Lemma 4.2. *If $\beta < 1$, then one can find $\delta > 0$ fulfilling*

$$u(x, t) \ge \delta \qquad \text{for all } (x, t) \in \Omega \times (0, \infty). \tag{4.3}$$

Proof. According to Lemma 4.1 and Lemma 2.1, it is possible for us to pick $t_0 > 0$ and $c_1 > 0$ such that

$$z(x, t) \le c_1 e^{-\gamma t} \le \frac{\mu}{2} \qquad \text{for all } x \in \Omega \text{ and } t > t_0.$$

From the first equation in (3.2) along with this and Lemma 2.1 once again we infer that a satisfies

$$a_t \geq e^{-v} \nabla \cdot (e^v \nabla a) + \mu a \left(\frac{1}{2} - e^{\|v_0\|_{L^\infty(\Omega)}} a \right) \qquad \text{for all } x \in \Omega \text{ and } t > t_0$$

with $\frac{\partial a}{\partial \nu} = 0$ throughout $\partial \Omega \times (t_0, \infty)$. On the other hand, for

$$\underline{a}(x,t) := y(t), \qquad x \in \overline{\Omega}, \ t \geq t_0,$$

with $y \in C^1([t_0, \infty))$ denoting the solution of

$$\begin{cases} y'(t) = \mu y \left(\frac{1}{2} - c_2 y \right), & t > t_0, \\ y(t_0) := \inf_{x \in \Omega} a(x, t_0) > 0, \end{cases}$$

where $c_2 := e^{\|v_0\|_{L^\infty(\Omega)}}$. Then we have

$$\underline{a}_t - e^{-v(x,t)} \nabla \cdot (e^{v(x,t)} \nabla \underline{a}) - \mu \underline{a} \left(\frac{1}{2} - c_2 \underline{a} \right)$$

$$= y'(t) - \mu y \left(\frac{1}{2} - c_2 y \right)$$

$$= 0 \qquad \text{for all } x \in \Omega \text{ and } t > t_0,$$

and since clearly $\frac{\partial \underline{a}}{\partial \nu} = 0$ on $\partial \Omega \times (t_0, \infty)$ and

$$\underline{a}(x, t_0) = y(t_0) = \inf_{x \in \Omega} a(x, t_0) \leq a(x, t_0) \qquad \text{for all } x \in \Omega,$$

an application of the comparison principle readily shows that $a \geq \underline{a}$ in $\Omega \times (t_0, \infty)$ and hence

$$u(x,t) = a(x,t) e^{v(x,t)} \geq a(x,t) \geq y(t) \qquad \text{for all } x \in \Omega \text{ and } t > t_0. \tag{4.4}$$

Since we have $y(t) \to \frac{1}{2c_2}$ as $t \to \infty$ due to $y(t_0) > 0$, we can conclude that there exists $t_1 \geq t_0$ such that

$$y(t) \to \frac{1}{4c_2} \qquad \text{for all } t > t_1. \tag{4.5}$$

Finally, the positivity and continuity of $u(x,t)$ in $\Omega \times (0, \infty)$ by Lemma 2.1 guarantees the existence of $c_3 > 0$ fulfilling

$$c_3 := \min_{\overline{\Omega} \times [0, t_1]} u(x,t) > 0. \tag{4.6}$$

So that (4.3) results from (4.4)-(4.6) with $\delta := \min \left\{ \frac{1}{4c_2}, c_3 \right\} > 0$. $\quad \square$

According to the second equation in (1.1), Lemma 4.2 along with Lemma 2.1 immediately leads to the following.

Corollary 4.3. *Let $\beta < 1$. Then we have*

$$v(x,t) \leq \|v_0(\cdot)\|_{L^\infty(\Omega)} \cdot e^{-\delta t} \qquad \text{for all } x \in \Omega \text{ and } t > 0, \tag{4.7}$$

where $\delta > 0$ is defined by Lemma 4.2.

Proof. From (1.1), Lemma 4.2 and the nonnegativity of v and w we obtain that

$$v_t = -(u+w)v \leq -\delta v \qquad \text{for all } x \in \Omega \text{ and } t > 0,$$

from which (4.7) readily follows. $\square$

With the outcomes of Lemma 3.4, Lemma 4.1 and Corollary 4.3, we firstly obtain the following weak convergence information of a, which is based on a specific testing procedure.

Lemma 4.4. *If $\beta < 1$, then there holds*

$$\int\limits_0^\infty \int\limits_\Omega (ae^v - 1)^2 < \infty. \tag{4.8}$$

Proof. Clearly, Lemma 3.4, Lemma 4.1 and Corollary 4.3 ensure the existence of $c_1 > 0$, $c_2 > 0$, $c_3 > 0$, $c_4 > 0$ and $c_5 > 0$ such that

$$a \leq c_1, \quad v \leq c_2, \quad \text{and} \quad w \leq c_3 \qquad \text{in } \Omega \times (0, \infty) \tag{4.9}$$

as well as

$$\int\limits_0^\infty \int\limits_\Omega v \leq c_4 \quad \text{and} \quad \int\limits_0^\infty \int\limits_\Omega z \leq c_5. \tag{4.10}$$

According to the basic inequality: $s - 1 - \ln s \geq 0$ for all $s > 0$ and in view of the fact that $v_t \leq 0$, we use (3.2) to estimate

$$\frac{d}{dt} \int\limits_\Omega e^v (a - 1 - \ln a) = \int\limits_\Omega e^v \left(1 - \frac{1}{a}\right) a_t + \int\limits_\Omega e^v (a - 1 - \ln a) v_t$$

$$\leq \int\limits_\Omega e^v \left(1 - \frac{1}{a}\right) a_t$$

$$= \int\limits_\Omega e^v \left(1 - \frac{1}{a}\right) \cdot \left\{e^{-v} \nabla \cdot (e^v \nabla a) + a(ae^v + w)v - az + \mu a(1 - ae^v)\right\}$$

$$= -\int\limits_\Omega e^v \frac{|\nabla a|^2}{a^2} + \int\limits_\Omega e^v (a - 1)(ae^v + w)v - \int\limits_\Omega e^v (a - 1)z$$

$$+ \mu \int\limits_\Omega (ae^v - e^v)(1 - ae^v)$$

$$\leq \int\limits_\Omega e^v a(ae^v + w)v + \int\limits_\Omega e^v z + \mu \int\limits_\Omega (1 - e^v)(1 - ae^v)$$

$$- \mu \int\limits_\Omega (ae^v - 1)^2 \qquad \text{for all } t > 0$$

due to the nonnegativity of a, v, w and z. Here by the elementary inequality: $1 - e^{-s} \leq s$ for all $s \geq 0$, we have $(e^v - 1) = e^v(1 - e^{-v}) \leq e^v v$, and thus in light of this and (4.9) we see that

$$\int_\Omega e^v a(ae^v + w)v + \int_\Omega e^v z + \mu \int_\Omega (1 - e^v)(1 - ae^v)$$

$$\leq \int_\Omega e^v a(ae^v + w)v + \int_\Omega e^v z + \mu \int_\Omega e^v(ae^v - 1)v$$

$$\leq \int_\Omega \left(a^2 e^{2v} + awe^v + \mu ae^{2v} \right)v + \int_\Omega e^v z$$

$$\leq \left(c_1^2 e^{2c_2} + c_1 c_3 e^{c_2} + \mu c_1 e^{2c_2} \right) \int_\Omega v + e^{c_2} \int_\Omega z \qquad \text{for all } t > 0.$$

Therefore, we arrive at

$$\frac{d}{dt} \int_\Omega e^v (a - 1 - \ln a) + \mu \int_\Omega (ae^v - 1)^2 \leq c_6 \int_\Omega v + c_7 \int_\Omega z \qquad \text{for all } t > 0$$

with $c_6 := c_1^2 e^{2c_2} + c_1 c_3 e^{c_2} + \mu c_1 e^{2c_2}$ and $c_7 := e^{c_2}$. Integrating this in time entails that

$$\int_\Omega e^{v(\cdot,t)} \{a(\cdot,t) - 1 - \ln a(\cdot,t)\} + \mu \int_0^t \int_\Omega (ae^v - 1)^2$$

$$\leq \int_\Omega e^{v_0}(a_0 - 1 - \ln a_0) + c_6 \int_0^t \int_\Omega v + c_7 \int_0^t \int_\Omega z$$

$$\leq \int_\Omega e^{v_0}(a_0 - 1 - \ln a_0) + c_6 \int_0^\infty \int_\Omega v + c_7 \int_0^\infty \int_\Omega z$$

$$= \int_\Omega e^{v_0}(a_0 - 1 - \ln a_0) + c_4 c_6 + c_5 c_7 \qquad \text{for all } t > 0.$$

Since $s - 1 - \ln s \geq 0$ for all $s > 0$ as mentioned before and since $a(x,t) > 0$ in $\Omega \times (0, \infty)$ by Lemma 2.1, this implies that

$$\int_0^\infty \int_\Omega (ae^v - 1)^2 \leq c_8 := \frac{1}{\mu} \cdot \left\{ \int_\Omega e^{v_0}(a_0 - 1 - \ln a_0) + c_4 c_6 + c_5 c_7 \right\} < \infty$$

thanks to $\mu > 0$ and the fact that $a_0 := u_0 e^{-v_0} > 0$ by (1.2), and thereby completes the proof. $\square$

In order to improve the above weak stabilization information, we need further regularity properties of a, which can be obtained through another testing procedure.

Lemma 4.5. *Let $\beta < 1$, then*

$$\int_0^\infty \int_\Omega a_t^2 < \infty \tag{4.11}$$

and

$$\sup_{t>0} \int_{\Omega} |\nabla a(\cdot, t)|^2 < \infty. \tag{4.12}$$

Proof. Lemma 3.4, Lemma 4.1 and Corollary 4.3 warrant the boundedness of a, v, w and z as well as the exponential decay of v and z, which enables us to find $c_i > 0, i \in \{1, \cdots, 5\}$, fulfilling

$$2e^v a^2(ae^v + w^2) \le c_1, \quad 2e^v a^2 \le c_2 \quad \text{and} \quad e^v a^2 \le c_3 \qquad \text{in } \Omega \times (0, \infty) \tag{4.13}$$

and moreover

$$\int_0^\infty \int_\Omega v^2 \le c_4 \quad \text{and} \quad \int_0^\infty \int_\Omega z^2 \le c_5. \tag{4.14}$$

Lemma 4.4 also provides $c_6 > 0$ satisfying

$$\int_0^\infty \int_\Omega (ae^v - 1)^2 < c_5. \tag{4.15}$$

Next, multiplying the first equation in (3.2) by $e^v a_t$ and integrating by parts, we have

$$\int_\Omega e^v a_t^2 = \int_\Omega a_t \nabla \cdot (e^v \nabla a) + \int_\Omega e^v a_t a(ae^v + w)v - \int_\Omega e^v a_t az + \mu \int_\Omega e^v a_t a(1 - ae^v)$$

$$= -\int_\Omega e^v \nabla a \cdot \nabla a_t + \int_\Omega e^v a_t a(ae^v + w)v - \int_\Omega e^v a_t az + \mu \int_\Omega e^v a_t a(1 - ae^v) \tag{4.16}$$

for all $t > 0$, where by nonpositivity of v_t,

$$-\int_\Omega e^v \nabla a \cdot \nabla a_t = -\frac{1}{2} \int_\Omega e^v \partial_t |\nabla a|^2$$

$$= -\frac{1}{2} \frac{d}{dt} \int_\Omega e^v |\nabla a|^2 + \frac{1}{2} \int_\Omega e^v |\nabla a|^2 v_t$$

$$\le -\frac{1}{2} \frac{d}{dt} \int_\Omega e^v |\nabla a|^2 \qquad \text{for all } t > 0. \tag{4.17}$$

Moreover, in view of Young's inequality and (4.13),

$$\int_\Omega e^v a_t a(ae^v + w)v \le \frac{1}{8} \int_\Omega e^v a_t^2 + 2 \int_\Omega e^v a^2(ae^v + w)^2 v^2$$

$$\le \frac{1}{8} \int_\Omega e^v a_t^2 + c_1 \int_\Omega v^2 \qquad \text{for all } t > 0 \tag{4.18}$$

and similarly,

$$-\int_\Omega e^v a_t a z \le \frac{1}{8}\int_\Omega e^v a_t^2 + 2\int_\Omega e^v a^2 z^2$$

$$\le \frac{1}{8}\int_\Omega e^v a_t^2 + c_2 \int_\Omega z^2 \qquad \text{for all } t > 0 \tag{4.19}$$

as well as

$$\mu \int_\Omega e^v a_t a(1 - ae^v) \le \frac{1}{4}\int_\Omega e^v a_t^2 + \mu^2 \int_\Omega e^v a^2 (1 - ae^v)^2$$

$$\le \frac{1}{4}\int_\Omega e^v a_t^2 + \mu^2 c_3 \int_\Omega (1 - ae^v)^2 \qquad \text{for all } t > 0. \tag{4.20}$$

Collecting (4.16)-(4.20) and integrating in time we obtain

$$\frac{1}{2}\int_0^t\int_\Omega e^v a_t^2 + \frac{1}{2}\int_\Omega e^{v(\cdot,t)}|\nabla a(\cdot,t)|^2$$

$$\le \frac{1}{2}\int_\Omega e^{v_0}|\nabla a_0|^2 + c_1\int_0^t\int_\Omega v^2 + c_2\int_0^t\int_\Omega z^2 + \mu^2 c_3\int_0^t\int_\Omega (1 - ae^v)^2$$

$$\le \frac{1}{2}\int_\Omega e^{v_0}|\nabla a_0|^2 + c_1\int_0^\infty\int_\Omega v^2 + c_2\int_0^\infty\int_\Omega z^2 + \mu^2 c_3\int_0^\infty\int_\Omega (1 - ae^v)^2$$

$$\le \frac{1}{2}\int_\Omega e^{v_0}|\nabla a_0|^2 + c_1 c_4 + c_2 c_5 + \mu^2 c_3 c_6 \qquad \text{for all } t > 0$$

due to (4.14) and (4.15), which immediately entails (4.11) and (4.12) because $e^v \ge 1$. $\quad\square$

Now using Lemma 4.4 in conjunction with the compactness properties implied by Lemma 4.5 we can claim the following L^2 stabilization feature of a.

Lemma 4.6. *If $\beta < 1$, then we have*

$$\|a(\cdot,t) - 1\|_{L^2(\Omega)} \to 0 \qquad \text{as } t \to \infty. \tag{4.21}$$

Proof. Combining Lemma 4.4 with Lemma 3.4 and Corollary 4.3 we readily see that

$$\int_0^\infty\int_\Omega (a - 1)^2 < \infty.$$

Relying on this basic decay information together with the relative compactness properties implied by (4.11) and (4.12), we use a straightforward argument by contradiction to achieve (4.21); we may refer to [17, Lemma 3.5] for details in a quite similar setting. $\quad\square$

A simple interpolation yields a slightly stronger convergence statement in the sense:

Lemma 4.7. *Let $\beta < 1$. Then the solution of (1.1) has the property*

$$u(\cdot, t) \to 1 \quad in \ L^p(\Omega) \quad for \ all \ p \in (2, \infty) \qquad as \ t \to \infty. \tag{4.22}$$

Proof. Since

$$\begin{aligned}
\|u - 1\|_{L^p(\Omega)} &= \|u - 1\|_{L^\infty(\Omega)}^{\frac{p-2}{p}} \|u - 1\|_{L^2(\Omega)}^{\frac{2}{p}} \\
&= \|e^v a - 1\|_{L^\infty(\Omega)}^{\frac{p-2}{p}} \|e^v a - 1\|_{L^2(\Omega)}^{\frac{2}{p}} \\
&\leq \left\{ e^{\|v_0\|_{L^\infty(\Omega)}} \|a\|_{L^\infty(\Omega)} + 1 \right\}^{\frac{p-2}{p}} \cdot \left\{ \|a - 1\|_{L^2(\Omega)} + \|e^v a(1 - e^{-v})\|_{L^2(\Omega)} \right\}^{\frac{2}{p}} \\
&\leq \left\{ e^{\|v_0\|_{L^\infty(\Omega)}} \|a\|_{L^\infty(\Omega)} + 1 \right\}^{\frac{p-2}{p}} \cdot \left\{ \|a - 1\|_{L^2(\Omega)} + e^{\|v_0\|_{L^\infty(\Omega)}} \|a\|_{L^\infty(\Omega)} \|v\|_{L^2(\Omega)} \right\}^{\frac{2}{p}}
\end{aligned}$$

for all $t > 0$ due to the Hölder inequality, (3.1), (2.1) and the basic inequality $1 - e^{-s} \leq s$ for all $s \geq 0$, this straightly results from Lemma 4.6, Corollary 4.3 and Lemma 3.4. $\quad\square$

Up to now, we have achieved all goals in Theorem 1.1.

Proof of Theorem 1.1. This immediately is the outcomes of Lemma 2.1, Lemma 3.4, Lemma 4.1, Corollary 4.3 and Lemma 4.7. $\quad\square$

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