

A sandwich with segment convexity ☆



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ABSTRACT

The aim of this note is to give a sufficient condition for pairs of functions to have a convex separator when the underlying structure is a Cartan–Hadamard manifold, or more generally: a reduced Birkhoff system. Some exotic behavior of convex hulls are also studied.

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1. Introduction

As it is well-known, separation theorems play a crucial role in many fields of Analysis and Geometry, and they can be interesting on their own right. Let us quote here the convex separation theorem of Baron, Matkowski, and Nikodem [1], one of our main motivations:

Theorem. *Let D be a convex subset of a real vector space X , and let $f, g: D \rightarrow \mathbb{R}$ be given functions. There exists a convex separator for f and g if and only if*

$$f\left(\sum_{k=0}^n t_k x_k\right) \leq \sum_{k=0}^n t_k g(x_k) \quad (1)$$

holds for all $n \in \mathbb{N}$, $x_0, \dots, x_n \in D$ and $t_0, \dots, t_n \in [0, 1]$ with $t_0 + \dots + t_n = 1$. Moreover, if X is finite-dimensional, then the length of the involved combinations can be restricted to $n \leq \dim(X)$.

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The necessity part of the statement is a straightforward calculation in both cases. To prove sufficiency in the dimension-free case, the convex envelope of g has to be used. Surprisingly, the most delicate issue is sufficiency in the finite-dimensional setting: An important tool of Convex Geometry, the classical Carathéodory Theorem [5] has to be applied.

The convex separation theorem above still motivates researchers. In a recent paper [13], the authors present an extension for functions defined on complete Riemannian manifolds. Unfortunately, their generalization is false: As it can easily be seen, the two-dimensional cases of the main results of [1] and [13] do not coincide.

The authors in [13] construct a set as the union of segments joining pairs of points of an epigraph. Then they claim (without explanation) its convexity (page 164, line 7, displayed formula). Clearly, such a construction, in general, does not result in a convex set. Thus the original intent of [13] remains a nice and nontrivial challenge: Extend the convex separation theorem of [1] to Riemannian manifolds.

In this challenge, one has to face two crucial problems. Firstly: What kind of structures should be used to have convexity without convex combinations? Secondly: What is the corresponding form of (1) in lack of algebraic manipulations? The proper choice to the structure turns out to be Birkhoff systems, the generalizations of Cartan–Hadamard manifolds. Inequality (1) has to be replaced by another one, in order that an iteration process can be applied.

2. Convex separation in Birkhoff systems

The precise axiomatic discussion of Euclidean geometry is due to Hilbert [7]; a nice and simplified presentation can be found in the book of Hartshorne [6]. Later Birkhoff initiated [3] and then together with Beatley elaborated [4] an elegant and didactic approach which is based on the ruler and the protractor. In what follows, we shall need some of their notions and axioms.

Assume that X is the set of *points* with at least two elements. Consider a family \mathcal{L} of subsets of X whose elements are termed *lines*. Let further $d: X^2 \rightarrow \mathbb{R}$ be a given function called a *metric*. We require two axioms: The postulate of incidence and the postulate of the ruler.

- Any two distinct points determine a unique line containing them.
- For each $\ell \in \mathcal{L}$ there exists a bijection $c: \mathbb{R} \rightarrow \ell$ such that $d(c(t), c(s)) = |t - s|$.

In this case, we say that (X, \mathcal{L}, d) is a *reduced Birkhoff system*. A bijection $c: \mathbb{R} \rightarrow \ell$ satisfying the condition in the second postulate is called a *ruler* for ℓ . The postulate of the ruler implies immediately that the cardinality of each line is continuum.

Moreover, we can introduce a ternary relation called *betweenness* on X : the point b is between the points a and c if a, b, c are three different collinear points, and $d(a, c) = d(a, b) + d(b, c)$. Using the abbreviation (abc) to this fact, one can prove that the axioms of abstract betweenness are satisfied:

- If (abc) , then a, b, c are pairwise distinct and collinear; further, (cba) .
- For distinct points a, b , there exists c such that (abc) .
- If (abc) , then (acb) and (bac) do not hold.

Using betweenness, the notion of *line segment* $[a, b]$ spanned by the points a, b can be defined in the following way. If $a = b$, then $[a, b] := \{a\}$; otherwise,

$$[a, b] := \{t \in X \mid (atb)\} \cup \{a, b\}.$$

If $a \neq b$, and ℓ is the unique line passing through them, then let $c: \mathbb{R} \rightarrow \ell$ be a ruler for ℓ such that $c(\alpha) = a$ and $c(\beta) = b$. Then we call the bijection

$$\tilde{c}: [0, 1] \rightarrow [a, b], \quad \tilde{c}(t) := c((\beta - \alpha)t + \alpha)$$

the *standard parametrization* of the segment $[a, b]$. Clearly, $\tilde{c}(0) = a$, and $\tilde{c}(1) = b$. When there is no risk of confusion, we shall also denote a standard parametrization simply by c , without tilde. Note that, unlike a ruler, a standard parametrization does not need to be distance preserving (unless $d(a, b) = 1$).

Once having segments, we have convexity concepts. A set $K \subseteq X$ is *convex* if $[a, b] \subseteq K$ holds for all $a, b \in K$. The family of convex sets is denoted by $\mathcal{C}(X)$. It turns out that $\mathcal{C}(X)$ is a convex structure indeed in the abstract sense of van de Vel [14]. That is,

- X and \emptyset are convex sets;
- the intersection of convex sets is convex;
- the union of nested convex sets is convex.

The *convex hull* of $H \subseteq X$, as usual, is the smallest convex set that contains H :

$$\text{conv}(H) := \bigcap \{K \in \mathcal{C}(X) \mid H \subseteq K\}.$$

It can be proved that segments are convex, and a set H is convex if and only if $\text{conv}(H) = H$. Moreover, the mapping $\text{conv}: \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is a hull operator, that is, an idempotent, monotone and extensive map. For further precise details, we refer to the paper [2] or to the excellent monograph [14]. Convex hulls are finitely inner represented in the following sense:

Lemma 1. *If (X, \mathcal{L}, d) is a reduced Birkhoff system and $H \subseteq X$, then*

$$\text{conv}(H) = \bigcup \{\text{conv}(F) \mid F \subseteq H, \text{card}(F) < \infty\}.$$

Proof. Denote the right-hand side of the formula above by K . Then $H \subseteq K$ holds evidently; moreover, for each finite set $F \subseteq H$, we have that $\text{conv}(F) \subseteq \text{conv}(H)$. Thus $K \subseteq \text{conv}(H)$. To complete the proof we have to show that K is convex.

If $a, b \in K$, then $a \in \text{conv}(F_a)$ and $b \in \text{conv}(F_b)$ with suitable finite subsets F_a and F_b of H . The set $F = F_a \cup F_b$ is finite; furthermore, $\text{conv}(F_a) \subseteq \text{conv}(F)$ and $\text{conv}(F_b) \subseteq \text{conv}(F)$ hold. Thus $[a, b] \subseteq \text{conv}(F) \subseteq K$, which was to be proved. \square

Note that convex hulls are finitely inner represented in *any* convex structure. The proof of this fact is based on transfinite methods, and can be found in the monograph [14]. When convexity is defined via segments, the presented elementary approach can also be followed.

Unfortunately, neither the definition, nor Lemma 1 provides a constructive method for finding the convex hull of a concrete set. Especially for finite sets, the formula of Lemma 1 terminates in a ‘circulus vitiosus’. Therefore the fixed point theorem of Kantorovich [9] will play a distinguished role for us. Its iteration process is a constructive method to approximate convex hulls.

Lemma 2. *Let (X, \mathcal{L}, d) be a reduced Birkhoff system, and let H be an arbitrary subset of X . Consider the Kantorovich iteration*

$$H_1 := H, \quad H_{n+1} := \bigcup \{[x, y] \mid x, y \in H_n\}.$$

Then,

$$\text{conv}(H) = \bigcup_{n \in \mathbb{N}} H_n.$$

Proof. Clearly, $\mathcal{A} := \{H_n \mid n \in \mathbb{N}\}$ is an increasing chain, and $H \subseteq \bigcup \mathcal{A}$. The iteration process guarantees that $\bigcup \mathcal{A}$ is a convex set. Let $C \subseteq X$ be a convex set such that $H_1 = H \subseteq C$. Then $H_2 \subseteq C$ by the convexity of C . By induction, $H_n \subseteq C$ for all $n \in \mathbb{N}$. Thus $\bigcup \mathcal{A} \subseteq C$. In other words, $\bigcup \mathcal{A}$ is the smallest convex set containing H , and the proof is completed. \square

Assume that (X, \mathcal{L}, d) is a reduced Birkhoff system, and let $X^* := X \times \mathbb{R}$. For arbitrary elements (x_0, y_0) and (x_1, y_1) of X^* , the first projections determine a line ℓ in \mathcal{L} provided that $x_0 \neq x_1$. Let $c: \mathbb{R} \rightarrow \ell$ be a ruler for ℓ such that $c(s_0) = x_0$ and $c(s_1) = x_1$ hold, and define $c^*: \mathbb{R} \rightarrow X^*$ by

$$c^*(t) = (c(at(s_1 - s_0) + s_0), at(y_1 - y_0) + y_0),$$

where

$$a := \frac{1}{\sqrt{(s_0 - s_1)^2 + (y_0 - y_1)^2}}.$$

Then $\ell^* := \{c^*(t) \mid t \in \mathbb{R}\}$ is called the *line* connecting (x_0, y_0) and (x_1, y_1) . If $x_0 = x_1$, and $y_0 \neq y_1$, then let $c: \mathbb{R} \rightarrow X$ be the constant mapping given by $c(t) = x_0$, and then define the line connecting (x_0, y_0) and (x_1, y_1) by the same formulae as above. In this way, we can specify the *lines* of X^* denoted by \mathcal{L}^* .

Finally, define the *metric* on X^* by

$$d^*((x_0, y_0), (x_1, y_1)) := \sqrt{d^2(x_0, x_1) + (y_0 - y_1)^2}.$$

The triple $(X^*, \mathcal{L}^*, d^*)$ obtained in this way will be called the *vertical extension* of the reduced Birkhoff system (X, \mathcal{L}, d) . The most important property of vertical extensions is subsumed by the following lemma.

Lemma 3. *The vertical extension of a reduced Birkhoff system is a reduced Birkhoff system.*

Proof. It can easily be checked that the postulate of incidence is valid in the vertical extension. Keeping the previous notations, consider a line ℓ^* determined by (x_0, y_0) and (x_1, y_1) with distinct first projections. We claim that c^* serves as a ruler for ℓ^* . Indeed, since c is a ruler for ℓ , we arrive at

$$\begin{aligned} d^*(c^*(t), c^*(s)) &= \sqrt{a^2(t-s)^2(s_1-s_0)^2 + a^2(t-s)^2(y_1-y_0)^2} \\ &= |t-s|a\sqrt{(s_0-s_1)^2 + (y_0-y_1)^2} = |t-s|. \end{aligned}$$

If $x_0 = x_1$, the case of vertical lines, can be handled similarly. \square

Assume that D is a nonempty convex subset in a reduced Birkhoff system. We say that a function $\varphi: D \rightarrow \mathbb{R}$ is *segment convex*, or simply: *convex*, if

$$\varphi(c(t)) \leq (1-t)\varphi(x_0) + t\varphi(x_1)$$

holds for all $x_0, x_1 \in D$ and for all $t \in [0, 1]$, where $c: [0, 1] \rightarrow D$ is the unique line determined by the properties $c(0) = x_0$ and $c(1) = x_1$.

Our main result gives a sufficient condition for the existence of a convex separator. To formulate it, we need the following concept. We say that a reduced Birkhoff system (X, \mathcal{L}, d) is *drop complete* if, for each convex set $K \subseteq X$ and for all $x_0 \in X$, the usual drop representation holds:

$$\text{conv}(\{x_0\} \cup K) = \bigcup \{[x_0, x] \mid x \in K\}.$$

Now we can formulate our main result, the adequate version of the Baron–Matkowski–Nikodem convex separation theorem. We point out that it is not a direct copy of the classical result: contrary to (1), both functions have to be involved in the upper estimation. The mixed form enables us to iterate our inequality recursively.

Theorem 1. *Let D be a convex set in a reduced Birkhoff system (X, \mathcal{L}, d) whose vertical extension is drop complete. If, for all $n \in \mathbb{N}$, $x_0, \dots, x_n \in D$, $x \in \text{conv}\{x_1, \dots, x_n\}$, and for all $t \in [0, 1]$, the functions $f, g: D \rightarrow \mathbb{R}$ satisfy the inequality*

$$f(c(t)) \leq (1-t)g(x_0) + tf(x), \quad (2)$$

where $c: [0, 1] \rightarrow D$ is the segment joining $x_0 = c(0)$ and $x = c(1)$, then there exists a convex function $\varphi: D \rightarrow \mathbb{R}$ fulfilling $f \leq \varphi \leq g$.

Proof. Let $E := \text{conv}(\text{epi}(g))$. First we show that $f(x) \leq y$ whenever $(x, y) \in E$. By Lemma 1, each point of E belongs to the convex hull of some finite subset of $\text{epi}(g)$. If (x, y) belongs to a singleton, then $f(x) \leq y$ holds trivially. Assume that the desired inequality holds if (x, y) belongs to the convex hull of any n element subset of $\text{epi}(g)$. Consider the case when

$$(x, y) \in \text{conv}\{(x_0, y_0), \dots, (x_n, y_n)\} \quad \text{and} \quad g(x_0) \leq y_0, \dots, g(x_n) \leq y_n.$$

The vertical extension is a drop complete reduced Birkhoff system, thus there exists a point (x^*, y^*) and $t \in [0, 1]$ such that

$$(x^*, y^*) \in \text{conv}\{(x_1, y_1), \dots, (x_n, y_n)\} \quad \text{and} \quad (x, y) = (c(t), (1-t)y_0 + ty^*),$$

where $c: [0, 1] \rightarrow D$ is the segment fulfilling $c(0) = x_0$ and $c(1) = x^*$. Using the inductive assumption and (2), we arrive at

$$f(x) = f(c(t)) \leq (1-t)g(x_0) + tf(x^*) \leq (1-t)y_0 + ty^* = y,$$

which was our claim. This property ensures that the formula

$$\varphi(x) := \inf\{y \in \mathbb{R} \mid (x, y) \in E\}$$

defines a function $\varphi: D \rightarrow \mathbb{R}$. Clearly, $f \leq \varphi \leq g$. Finally we prove that φ is convex. Let $x_0, x_1 \in D$ be arbitrary and choose $y_0, y_1 \in \mathbb{R}$ such that (x_0, y_0) and (x_1, y_1) belong to E . Since E is convex,

$$(c(t), (1-t)y_0 + ty_1) \in E$$

holds for all $t \in [0, 1]$, where $c: [0, 1] \rightarrow D$ is the segment fulfilling $c(0) = x_0$ and $c(1) = x_1$. By the definition of φ , we have that $\varphi(c(t)) \leq (1-t)y_0 + ty_1$. Taking the infimum at y_0 and y_1 , we get the convexity of φ , and this completes the proof. \square

Assume that the underlying reduced Birkhoff system is a vector space. Then, using induction, one can check easily that (2) implies (1). Unfortunately, the converse implication is not valid. Thus our main results (the previous and the next theorems) are only sufficient conditions for the existence of a convex separator. However, under this generality, a full characterization cannot be expected.

Consider a reduced Birkhoff system (X, \mathcal{L}, d) , and let

$$K := \{k \in \mathbb{N} \mid \forall H \subseteq X, \forall x \in \text{conv}(H) : \exists F \subseteq H : x \in \text{conv}(F), \text{card}(F) \leq k\}.$$

If K is not empty, then $\kappa := \min K$ is called the *Carathéodory number* of the system; and if K is empty, then the Carathéodory number is defined to be $+\infty$. Equivalently, the Carathéodory number can be defined as the least κ such that the representation of Lemma 1 remains valid if we allow only the convex hulls of sets with at most κ elements on the right-hand side, or as $+\infty$ if there is no finite κ with this property. If the Carathéodory number of the vertical extension is known, we can strengthen the statement of Theorem 1. The proof is essentially the same, thus we omit it.

Theorem 2. *Keeping the conditions of the previous theorem, assume that the Carathéodory number of the vertical extension is κ . Then the size of the involved convex hull can be reduced to $n \leq \kappa$.*

If X is a finite-dimensional vector space, the classical separation result of [1] restricts the length of the involved combination to $\dim(X) + 1$, while the Carathéodory number of the vertical extension is $\kappa = \dim(X) + 2$. In other words, the reduction of Theorem 2 can be improved in the finite-dimensional setting.

The drop completeness of the vertical extensions is required both in Theorem 1 and Theorem 2. Clearly, this assumption makes the underlying reduced Birkhoff system drop complete as well. Thus the question arises, quite evidently: *Does the extension inherit drop completeness from the original system?* In order to justify the conditions of the main results, we will give a negative answer in the last section.

3. Convex separation in Cartan–Hadamard manifolds

The celebrated theorem of Hopf states that *each simply connected, complete Riemannian manifold of positive sectional curvature is compact*. In contrast to this behavior, nonpositive curvature results in an opposite feature according to the theorem of Cartan and Hadamard:

Theorem. *The exponential map at any point of a simply connected, complete Riemannian manifold of non-positive sectional curvature is a global diffeomorphism between the tangent space at the given point and the manifold.*

These manifolds are called *Cartan–Hadamard manifolds*. In particular, by this theorem, each Cartan–Hadamard manifold is homeomorphic to a Euclidean space. Moreover, geodesics can be parametrized along the entire set of reals, and two geodesics can have at most one common point. This means that the postulate of incidence and the postulate of ruler are satisfied, and we can formulate the next statement.

Lemma 4. *Each Cartan–Hadamard manifold is a reduced Birkhoff system.*

By Lemma 3 and Lemma 4, the vertical extension of a Cartan–Hadamard manifold is a reduced Birkhoff system. Moreover, now the vertical extension has a very close relation with the product Riemannian metric. In fact, exactly this relation (formulated in the next lemma) has motivated the notion of vertical extensions. For the technical background of the proof, we refer to the monograph of Sakai [12].

Lemma 5. *If M is a Cartan–Hadamard manifold, then $M \times \mathbb{R}$ is also a Cartan–Hadamard manifold with respect to the product Riemannian structure, and the induced Birkhoff structure coincides with the vertical extension of the Birkhoff structure of M .*

Proof. Let d be the dimension of M , and denote the components of the metric tensor by g_{ij} . Then the metric tensor and its inverse of the product manifold $M \times \mathbb{R}$ are represented as

$$(G_{ij}) = \begin{pmatrix} (g_{ij}) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad (G^{ij}) = \begin{pmatrix} (g^{ij}) & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly, the product manifold is a simply connected and complete Riemannian manifold. Moreover, using the Koszul formulae

$$\Gamma_{ij}^k = \frac{1}{2} G^{kl} \left(\frac{\partial G_{jl}}{\partial x^i} + \frac{\partial G_{li}}{\partial x^j} - \frac{\partial G_{ij}}{\partial x^l} \right)$$

for the Christoffel symbols of $M \times \mathbb{R}$, we can conclude that $\Gamma_{ij}^k = 0$ whenever $(d+1) \in \{i, j, k\}$. Consider now a geodesic c^* in the product manifold $M \times \mathbb{R}$. Since its coordinate functions satisfy the second-order differential equations

$$c^{*k''} + (\Gamma_{ij}^k \circ c^*) c^{*i'} c^{*j'} = 0$$

and the Christoffel symbols have the previously mentioned properties, $c^{*(d+1)''} = 0$ follows. Thus the last component of c^* is affine. Furthermore, by the behavior of the Christoffel symbols and by the geodesic differential equation again, the first projection of c^* results in a geodesic of M . Therefore any geodesic c^* connecting the points (x_0, y_0) and (x_1, y_1) of the product manifold $M \times \mathbb{R}$ can be parametrized as

$$c^*(t) = (c(t), (1-t)y_0 + ty_1),$$

where the geodesic c of M connects the points $c(0) = x_0$ and $c(1) = x_1$. Note also that this parametrization is a global one in case of Cartan–Hadamard manifolds. This shows that the product manifold $M \times \mathbb{R}$ is the vertical extension of M .

In particular, the vertical extension is a simply connected and complete manifold, as well. Now we show that its sectional curvature is nonpositive. Let $\sigma \subset T_{(x,y)}(M \times \mathbb{R}) = T_x M \oplus \mathbb{R}$ be an arbitrary plane, and let b_1, b_2 be a base in σ . If $T_x M \cap \sigma = \{0\}$, then $\dim(T_{(x,y)}(M \times \mathbb{R})) = d+2$, which is a contradiction. Thus $\dim(T_x M \cap \sigma) \geq 1$, and we may assume that the second direct component of b_2 is zero. Then b_1 has a direct decomposition $b_1 = b_{11} + b_{12}$. Recall that the sign of the sectional curvature depends only on the sign of $\mathcal{R}(b_1, b_2, b_2, b_1)$, where \mathcal{R} is the Riemannian curvature tensor. Since its components can be obtained by

$$R_{ijkl} = \left(\frac{\partial \Gamma_{jk}^m}{\partial x^i} - \frac{\partial \Gamma_{ik}^m}{\partial x^j} + \Gamma_{jk}^r \Gamma_{ir}^m - \Gamma_{ik}^r \Gamma_{jr}^m \right) G_{lm},$$

we arrive at $R_{ijkl} = 0$ provided that $(d+1) \in \{i, j, k, l\}$. Thus \mathcal{R} vanishes if one of the arguments contains b_{12} . Therefore,

$$\begin{aligned} \mathcal{R}(b_1, b_2, b_2, b_1) &= \mathcal{R}(b_{11} + b_{12}, b_2, b_2, b_{11} + b_{12}) = \\ &= \mathcal{R}(b_{11}, b_2, b_2, b_{11}) + \mathcal{R}(b_{12}, b_2, b_2, b_{11}) + \\ &+ \mathcal{R}(b_{11}, b_2, b_2, b_{12}) + \mathcal{R}(b_{12}, b_2, b_2, b_{12}) = \mathcal{R}(b_{11}, b_2, b_2, b_{11}) \leq 0, \end{aligned}$$

since the second direct components of b_{11} and b_2 are zero and the sectional curvature of M is nonpositive. This completes the proof. \square

For more details on Cartan–Hadamard manifolds, we recommend the book of Jost [8]. As direct consequences of Theorem 1 and Theorem 2, we can formulate the next two corollaries.

Corollary 1. *Let D be a convex set in a Cartan–Hadamard manifold M whose vertical extension is drop complete. If, for all $n \in \mathbb{N}$, $x_0, \dots, x_n \in D$, $x \in \text{conv}\{x_1, \dots, x_n\}$, and for all $t \in [0, 1]$, the functions $f, g: D \rightarrow \mathbb{R}$ satisfy (2) where $c: [0, 1] \rightarrow \mathbb{R}$ is the geodesic segment joining $x_0 = c(0)$ and $x = c(1)$, then there exists a convex function $\varphi: D \rightarrow \mathbb{R}$ fulfilling $f \leq \varphi \leq g$.*

Corollary 2. *Keeping the conditions of the previous theorem, assume that the Carathéodory number of the vertical extension is κ . Then the size of the involved convex hull can be reduced to $n \leq \kappa$.*

Cartan–Hadamard manifolds and Euclidean spaces are quite “close” relatives. Hence one may expect that the Carathéodory number of a Cartan–Hadamard manifold M , accordingly to the Euclidean case, can be expressed as $\kappa = \dim(M) + 1$. However, as far as we know, this is still an open problem posed by Ledyae, Treiman, and Zhu [10].

4. The exotic behavior of convex hulls

The aim of this section is to prove that drop completeness of the vertical extension cannot be changed to drop completeness of the underlying system in Theorem 1 and Theorem 2. The main reason is the exotic behavior of convex hulls: It may occur that the convex hull of three points in a Cartan–Hadamard manifold is not contained in a two-dimensional submanifold. To construct such an example, let us recall here some basic facts in hyperbolic geometry. For references, see the book of Ratcliffe [11].

The hyperbolic plane, denoted by \mathbb{H}^2 in the forthcoming, is a two-dimensional Cartan–Hadamard manifold with constant sectional curvature -1 . We will use two of its several models. The first one is the Beltrami–Klein model (known also as the Cayley–Klein model): Here the plane is the open unit disc, and the lines are its Euclidean chord segments. The distance of this model will not be used.

The second model is the Poincaré half-plane model. The plane is the upper open Cartesian half-plane $\mathbb{R} \times \mathbb{R}_+$; lines are either circles with center on the boundary line or vertical Euclidean half-lines. Its metric plays a key role in our investigation. The distance of $a = (a_1, a_2)$ and $b = (b_1, b_2)$ is given by

$$d(a, b) = 2 \ln \frac{\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} + \sqrt{(a_1 - b_1)^2 + (a_2 + b_2)^2}}{2\sqrt{a_2 b_2}}. \quad (3)$$

In particular, if $a_1 = b_1$, this formula can be simplified (which will be quite convenient for us):

$$d(a, b) = |\ln a_2 - \ln b_2|. \quad (4)$$

If (x_0, y_0) and (x_1, y_1) are points of the vertical extension $\mathbb{H}^2 \times \mathbb{R}$, and $c: [0, 1] \rightarrow \mathbb{H}^2$ is the unique geodesic fulfilling $c(0) = x_0$ and $c(1) = x_1$, then the unique geodesic segment $c^*: [0, 1] \rightarrow \mathbb{H}^2 \times \mathbb{R}$ which connects (x_0, y_0) and (x_1, y_1) is given by

$$c^*(t) = (c(t), (1-t)y_0 + ty_1).$$

Since geodesics are of constant speed, we have $d(x_0, x) = td(x_0, x_1)$ for all $t \in [0, 1]$, where $x = c(t)$. Thus we can reconstruct the points (x, y) between (x_0, y_0) and (x_1, y_1) from x as

$$(x, y) = \left(x, y_0 + \frac{d(x_0, x)}{d(x_0, x_1)}(y_1 - y_0) \right). \quad (5)$$

Theorem 3. *The hyperbolic plane is drop complete, while its vertical extension is not.*

Proof. The Beltrami–Klein model shows that the convex structure of \mathbb{H}^2 can be identified with the convex structure of the open disc inherited from \mathbb{R}^2 . In particular, the drop representation is valid in \mathbb{H}^2 .

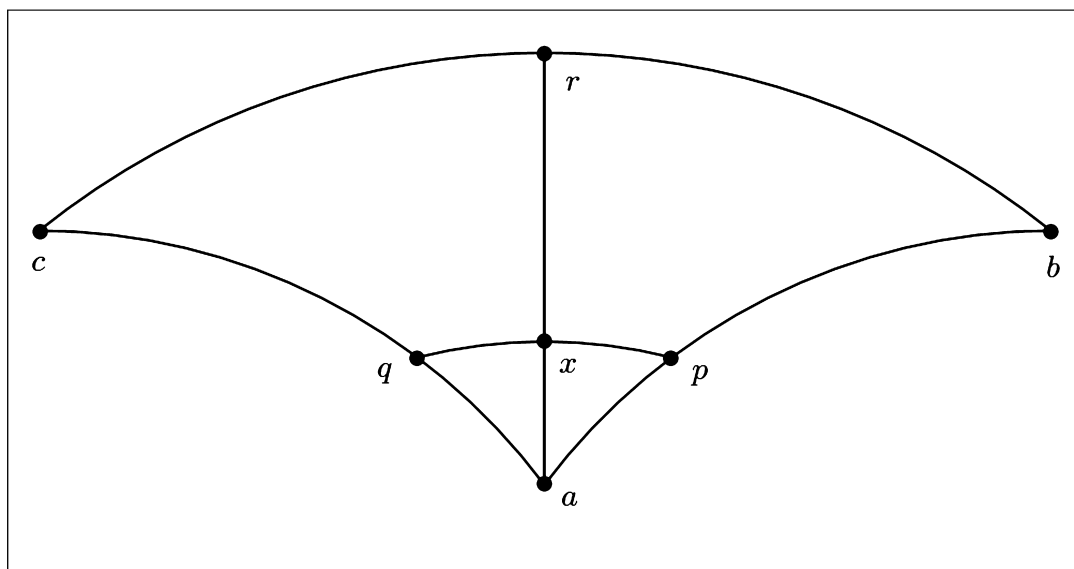
Now we prove that the vertical extension $\mathbb{H}^2 \times \mathbb{R}$ is not drop complete. We will illustrate it using the convex hull of the points

$$A = ((0, 3), 0); \quad B = ((4, 5), 1); \quad C = ((-4, 5), 1).$$

Consider their projections a, b, c and the additional points p, q, r on \mathbb{H}^2 :

$$a = (0, 3), \quad b = (4, 5), \quad c = (-4, 5); \quad p = (1, 4), \quad q = (-1, 4), \quad r = (0, \sqrt{41}).$$

It is immediate to check that $p \in [a, b]$ and $q \in [a, c]$, furthermore $r \in [b, c]$ (the segments here are meant in the hyperbolic geodesic sense: they are arcs of circles). Finally, we need the intersection of $[p, q]$ and $[a, r]$, which turns out to be $x = (0, \sqrt{17})$. The next figure shows these choices:



Let P and Q be the points on $[A, B]$ and $[A, C]$ in the vertical extension, whose first projections are p and q , respectively. Now we reconstruct their last coordinates from the projections. By the distance formula (3),

$$d(a, b) = 2 \ln \frac{\sqrt{20} + \sqrt{80}}{2\sqrt{15}} = \ln 3 \quad \text{and} \quad d(a, p) = 2 \ln \frac{\sqrt{2} + \sqrt{50}}{2\sqrt{12}} = \ln 3 - \ln 2.$$

Thus the second common projection of P and Q is obtained via (5) as

$$\varepsilon_1 := \frac{d(a, p)}{d(a, b)} = 1 - \frac{\ln 2}{\ln 3} < \frac{2}{5}.$$

The estimation above can be checked even by hand. Moreover, the points of the segment $[P, Q]$ share this common last component. Therefore,

$$X_1 := ((0, \sqrt{17}), \varepsilon_1) \in [P, Q] \subseteq \text{conv}\{A, B, C\}.$$

Clearly, $R = ((0, \sqrt{41}), 1) \in [B, C]$. Now we determine the point of the vertical extension whose first projection is x and belongs to the segment $[A, R]$. Using (4) and (5), its last component turns out to be

$$\varepsilon_2 := \frac{\ln \sqrt{17} - \ln 3}{\ln \sqrt{41} - \ln 3} = \frac{\ln 17 - 2 \ln 3}{\ln 41 - 2 \ln 3} > \frac{2}{5}.$$

This estimation, with a bit more effort, can also be checked by hand. Thus,

$$X_2 := ((0, \sqrt{17}), \varepsilon_2) \in [A, R] \subseteq \bigcup \{[A, D] \mid D \in [B, C]\}.$$

Since $X_1 \neq X_2$, we can conclude that the drop representation involving $\{A\}$ and $[B, C]$ does not cover the entire convex hull of A, B, C , which was to be proved. \square

The Beltrami–Klein open sphere model and the first part of the argument show that *the geodesic convex structure of hyperbolic space is compatible with the Euclidean convex structure of the open ball*. In particular, the hyperbolic space is drop complete in any dimension, and its combinatorial invariants coincide with the standard Euclidean ones. Using the Cartan–Hadamard theorem or the results of [2], it can be proved that these properties are also true for two-dimensional Cartan–Hadamard manifolds.

As we have already mentioned, the greatest advantage of Lemma 2 is that it can be implemented. In fact, the theorem above was conjectured via a computer algorithm. In what follows, we sketch briefly its pseudo code.

We shall need two functions. The first one, **geod** calculates a point of a geodesic between two points:

$$\text{geod}: (\mathbb{H}^2 \times \mathbb{R}) \times (\mathbb{H}^2 \times \mathbb{R}) \times \mathbb{R} \rightarrow \mathbb{H}^2 \times \mathbb{R}$$

so that $\text{geod}(A, B, 0) = A$ and $\text{geod}(A, B, 1) = B$ hold. The function **dist** calculates the hyperbolic distance of two points in \mathbb{H}^2 :

$$\text{dist}(a, b): \mathbb{H}^2 \times \mathbb{H}^2 \rightarrow \mathbb{R}.$$

The list **ConvexHull** collects the points of the convex hull as an ordered list. Initially we put the points of the set whose convex hull is to be computed into **ConvexHull**. **ConvexHull**[i] is the i th element in **ConvexHull**. Indexing starts with 0. The parameter **iterations** is the number of iterations. Finally, the parameter **res** is the hyperbolic distance between points to be calculated along geodesics.

The algorithm takes two points A and B from **ConvexHull** and adds the points of the geodesic from A to B with hyperbolic distance **res** from each other to **ConvexHull**. The point B is chosen so that repetitions are avoided.

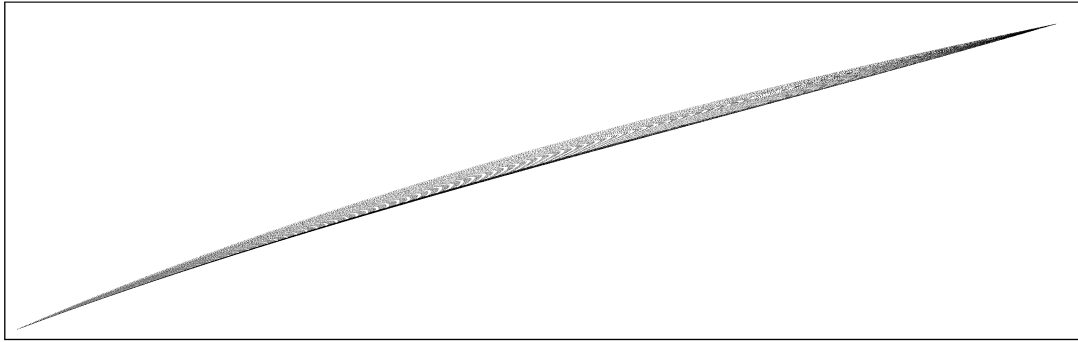
- The variables **size** and **previoussize** keep track of the number of points calculated in the convex hull. Initially
 - **size** := |**ConvexHull**|.
 - **previoussize** := 0.
- The following loop is to be performed **iterations** many times.
 - For $i = 0, \dots, \text{size} - 2$, perform the following:
 - * For $j = \max(\text{previoussize}, i + 1), \dots, \text{size} - 1$, perform the following:
 - $A := \text{ConvexHull}[i]$.
 - $B := \text{ConvexHull}[j]$.


```

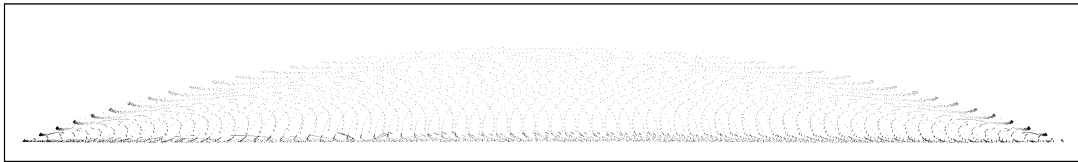
·  $d := \text{dist}(A, B)$ .
· For  $k = 1, 2, \dots$  while  $k \cdot \text{res} < d$ ,
  add  $\text{geod}(A, B, k \cdot \text{res}/d)$  to  $\text{ConvexHull}$ .
-  $\text{previousSize} := \text{size}$ ;
-  $\text{size} := |\text{ConvexHull}|$ .

```

Using our algorithm, we can illustrate some points of $\text{conv}\{A, B, C\}$ in the proof of Theorem 3. The first figure is the intersection of the approximation of the convex hull with the plane $[a, r] \times \mathbb{R}$:



The second figure is the intersection of the approximation with $[p, q] \times \mathbb{R}$:



In each case, $\text{iterations} = 2$ and $\text{res} = 0.006$. In the concrete implementation, points are plotted whose distance from the planes is at most 0.01.

The program is available and freely downloadable from the homepage below:

<http://shrek.unideb.hu/~ftzydk/convex/>

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