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Oscillation Criteria for First Order Neutral Differential Equations with Positive and Negative Coefficients

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Sufficient conditions for the oscillation of the equation $\frac{d}{dt}[x(t) - R(t)x(t-r)] + P(t)x(t-\tau) - Q(t)x(t-\delta) = 0$, where $P, Q, R \in C([t_0, \infty), R^+)$, $r \in (0, \infty)$, $\tau, \delta \in R^+$, $\tau \geq \delta$, are obtained where $R(t) + \int_{t-\tau+\delta}^{t} Q(u)\,du - 1$ is allowed to oscillate and the condition $\int_{t_0}^{\infty} s[P(s) - Q(s-\tau+\delta)] \int_s^{\infty} [P(u) - Q(u-\tau+\delta)]\,du\,ds = \infty$ is not necessary. Some examples are given, which show that the results here are almost sharp. © 2001 Academic Press

Key Words: neutral equation; oscillation.

1. INTRODUCTION

Consider the neutral differential equation with positive and negative coefficients

$$\frac{d}{dt}[x(t) - R(t)x(t-r)] + P(t)x(t-\tau) - Q(t)x(t-\delta) = 0, \quad (1)$$

where

$$\begin{aligned} &P, Q, R \in C([t_0, \infty), R^+), \qquad r \in (0, \infty), \tau, \delta \in R^+, \tau \geq \delta, \\ &\bar{P}(t) \equiv P(t) - Q(t-\tau+\delta) \geq 0 \text{ and not identically zero.} \end{aligned} \quad (2)$$

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The oscillation of Eq. (1) has recently been investigated by many authors including Chuanxi and Ladas [2], Farrall *et al.* [4], Lalli and Zhang [6], Ruan [7], Yu [10], Yu and Wang [11], and Yu and Yan [12]. For an excellent survey, we refer to Erbe *et al.* [3], and Gyori and Ladas [5]. Recently, Yu and Yan [12] studied the oscillation of Eq. (1), where the condition

$$\int_{t_0}^{\infty} s\bar{P}(s)\int_{s}^{\infty}\bar{P}(u)\,du\,ds = \infty \tag{3}$$

was used. However, all the known results were obtained in the following three cases. (i) $R(t) + \int_{t-\tau+\delta}^{t} Q(u)\,du \le 1$; (ii) $R(t) + \int_{t-\tau+\delta}^{t} Q(u)\,du \equiv 1$; (iii) $R(t) + \int_{t-\tau+\delta}^{t} Q(u)\,du \ge 1$.

In this paper, by some new technique, we investigate the oscillation of Eq. (1) where $R(t) + \int_{t-\tau+\delta}^{t} Q(u)\,du - 1$ is allowed to oscillate and condition (3) is not necessary. When $\tau = \delta$, our results present some new oscillation criteria for the neutral equation without negative coefficient

$$\frac{d}{dt}\left[x(t) - R(t)x(t-r)\right] + \bar{P}(t)x(t-\tau) = 0, \tag{$*$}$$

which has been studied in many papers (see [1, 3, 5, 8, 9]). Some examples are given to illustrate the advantage of our results. The examples also show that the results here are almost sharp.

Let $\rho = \max\{r, \tau, \delta\}$. By a solution of Eq. (1) we mean a function $x(t) \in C([\bar{t} - \rho, \infty), R)$, for some $\bar{t} \ge t_0$ such that $x(t) - R(t)x(t-r)$ is continuously differentiable on $[\bar{t}, \infty)$ and such that Eq. (1) is satisfied for $t \ge \bar{t}$.

As usual, a solution of Eq. (1) is said to be oscillatory if it has arbitrarily large zeros and nonoscillatory if it is either eventually positive or eventually negative.

In the sequel, when we write a functional inequality without specifying its domain of validity, we assume that it holds for all sufficiently large t. Also we always assume that Eq. (1) satisfies condition (2) without further declaration. Throughout this paper, we define $E[t_1, t_2] = \bigcup_{i=0}^{\infty}[t_1 + ir, t_2 + ir]$, where $t_2 > t_1 \ge t_0$.

2. BASIC LEMMAS

The following Lemma 3 and Lemma 4 generalize some important lemmas in the literature. To prove these two lemmas we need the following Lemma 1 and Lemma 2.

LEMMA 1. *Assume that* $A \geq 0$, $r > 0$, $a < b < \infty$, *and that* $f \in C([a,b], R^+)$, $g \in C[R^+, R^+]$. *If* $y \in C([a-r, b], R^+)$ *satisfies*

$$y(s) \leq g[y(s-r)] + A + \int_a^s f(u)y(u)\,du, \qquad a \leq s \leq b, \tag{4}$$

then

$$y(s) \leq g[y(s-r)] + e^{F(s)}\left[A + \int_a^s f(u)g[y(u-r)]\,du\right], \qquad a \leq s \leq b, \tag{5}$$

where $F(s) = \int_a^s f(u)\,du$.

Proof. First we claim that (5) holds for $s \in [a, a+r]$. In fact, since $C^1[a, a+r]$ is dense in $C[a, a+r]$, we can choose, for any $\epsilon > 0$, $g_1 \in C^1([a, a+r], R^+)$ such that $|g[y(s-r)] - g_1(s)| < \epsilon$, $a \leq s \leq a+r$. From (4) it follows that

$$y(s) \leq g_1(s) + A + \epsilon + \int_a^s f(u)y(u)\,du, \qquad a \leq s \leq a+r.$$

By the Gronwall inequality

$$\begin{aligned} y(s) &\leq e^{F(s)}\left[g_1(a) + A + \epsilon + \int_a^s g_1'(u)e^{-F(u)\,du}\right] \\ &\leq g_1(s) + e^{F(s)}\left[A + \epsilon + \int_a^s f(u)g_1(u)\,du\right] \\ &\leq g[y(s-r)] + e^{F(s)}\left[A + \int_a^s f(u)g[y(u-r)]\,du\right] \\ &\quad + \epsilon + \epsilon e^{F(a+r)}\left[1 + \int_a^{a+r} f(u)\,du\right], \qquad a \leq s \leq a+r, \end{aligned}$$

which, by the fact that $\epsilon > 0$ can be arbitrarily small, implies that (5) holds for $s \in [a, a+r]$.

For $a + r \leq s \leq a + 2r$, rewrite (4) into the inequality

$$y(s) \leq g[y(s-r)] + A + \int_a^{a+r} f(u)y(u)\,du + \int_{a+r}^s f(u)y(u)\,du, \qquad a+r \leq s \leq a+2r.$$

By applying the conclusion obtained above to this inequality on $[a + r, a + 2r]$, we find

$$y(s) \le g[y(s-r)] + e^{F_1(s)}\left[A + \int_a^{a+r} f(u)y(u)\,du + \int_{a+r}^{s} f(u)g[y(u-r)]\,du\right], \tag{6}$$

where $F_1(s) = \int_{a+r}^{s} f(u)\,du$. Noticing that (5) holds for $a \le s \le a + r$, we obtain

$$\int_a^{a+r} f(u)y(u)\,du \le (e^{F(a+r)} - 1)A + e^{F(a+r)}\int_a^{a+r} f(u)g[y(u-r)]\,du,$$

which, combining with (6), implies that

$$y(s) \le g[y(s-r)] + e^{F(s)}\left[A + \int_a^{s} f(u)g[y(u-r)]\,du\right],$$

$$a + r \le s \le a + 2r.$$

That is, (5) holds for $s \in [a + r, a + 2r]$. Set $N_1 = [\frac{b-a}{r}] + 1$. Repeating the above process N_1 times, we can get that (5) holds on $[a, b]$. The proof is complete.

Let $N \in \{0, 1, 2, \ldots\}$ be the smallest number such that $\delta + Nr \ge \tau$. Denote

$$Q_0(t) = Q(t), \qquad Q_k(t) = Q_{k-1}(t)R(t - \delta - (k-1)r),$$

$$k = 1, 2, \ldots, N.$$

LEMMA 2. *Assume that there exists* $\beta_k > 0$, $k = 0, 1, 2, \ldots, N-1$, *such that eventually*

$$\begin{aligned} &Q_k(t) \le \beta_k, \qquad k = 0, 1, \ldots, N-1, \text{ if } N \ge 1, \\ &\bar{P}(t) \ge (b_{N-1} - b_{-1})Q(t - \tau + \delta) + b_{N-1}Q_N(t - \tau + \delta + Nr), \end{aligned} \tag{7}$$

where $b_{-1} = \operatorname{sgn}(\tau - \delta)$, *and* $b_k = b_{k-1}e^{(\tau-\delta)b_{k-1}\beta_k}$, $k = 0, 1, 2, \ldots, N-1$. *Let* $x(t)$ *be an eventually positive solution of Eq.* (1) *and*

$$z(t) = x(t) - R(t)x(t-r) - \int_{t-\tau+\delta}^{t} Q(s)x(s-\delta)\,ds. \tag{8}$$

Then eventually

$$z(t) > 0 \quad \text{or} \quad x(t) \le -\alpha + R(t)x(t-r) \tag{9}$$

for some $\alpha > 0$.

Proof. Suppose that $z(t) \le 0$. We will prove (9) by showing that there exist $\alpha > 0$ and $\bar{t} \ge t_0$ such that

$$x(t) \le -\alpha + R(t)x(t-r), \qquad t \ge \bar{t}. \tag{10}$$

From (1), we have eventually

$$z'(t) = -\bar{P}(t)x(t-\tau) \le 0, \tag{11}$$

which, by (2), implies that there exists a positive number $\alpha > 0$ such that eventually

$$x(t) > 0 \quad \text{and} \quad z(t) \le -\alpha. \tag{12}$$

Choose $\bar{t} \ge t_0 + 2(\delta + Nr)$ such that (2), (7), (11), and (12) hold for $t \ge \bar{t} - 2(\delta + Nr)$. For convenience we denote

$$\begin{aligned}
A(t) &= \int_{t-\delta-Nr}^{t} Q(u-\tau+\delta)\,du,\\
x_k(t) &= x(t-\delta-kr),\\
q_k(s) &= \int_{t-\tau+\delta}^{s} Q_k(u)\,du, \qquad t-\tau+\delta \le s \le t,\ k = 0,1,\dots,N.
\end{aligned}$$

For $t \ge \bar{t}$, integrating (11) on $[t-\delta-Nr, t]$, and by (12), we get for $t \ge \bar{t}$,

$$\begin{aligned}
x(t) \le -\alpha + R(t)x(t-r) &- \int_{t-\delta-Nr}^{t} \bar{P}(u)x(t-\tau)\,du\\
&+ b_{-1}\int_{t-\tau+\delta}^{t} Q_0(u)x_0(u)\,du.
\end{aligned} \tag{13}$$

For $t-\tau+\delta \le s \le t$, by (13)

$$\begin{aligned}
x_0(s) &\le R(s-\delta)x_1(s) + b_{-1}\int_{t-2\tau+\delta}^{s} Q_0(u)x_0(u)\,du\\
&\le R(s-\delta)x_1(s) + b_{-1}A(t) + b_{-1}\int_{t-\tau+\delta}^{s} Q_0(u)x_0(u)\,du,
\end{aligned}$$

which, by Lemma 1, yields that for $t - \tau + \delta \le s \le t$

$$x_0(s) \le R(s-\delta)x_1(s) + b_{-1}e^{b_{-1}q_0(s)}\left[A(t) + \int_{t-\tau+\delta}^{s} Q_1(u)x_1(u)\,du\right], \tag{14}$$

from which, we have

$$x_1(s) \le R(s-\delta-r)x_2(s) + b_0 A(t) + b_0\int_{t-\tau+\delta}^{s} Q_1(u)x_1(u)\,du. \tag{14$'$}$$

By applying Lemma 1 to (14)′ and repeating this process, we can get for $k = 1, 2, \ldots, N$

$$\begin{aligned} x_{k-1}(s) \le {} & R(s-\delta-(k-1)r)x_k(s) \\ & + b_{k-2}e^{b_{k-2}q_{k-1}(s)}\left[A(t) + \int_{t-\tau+\delta}^{s} Q_k(u)x_k(u)\,du\right], \end{aligned} \tag{15}$$

and

$$\begin{aligned} x_k(s) \le {} & R(s-\delta-kr)x_{k+1}(s) \\ & + b_{k-1}A(t) + b_{k-1}\int_{t-\tau+\delta}^{s} Q_k(u)x_k(u)\,du. \end{aligned} \tag{15$'$}$$

By (15) with $k = 1$, one can get

$$\begin{aligned} b_{-1}\int_{t-\tau+\delta}^{t} Q_0(u)x_0(u)\,du \le {} & b_0\int_{t-\tau+\delta}^{t} Q_1(u)x_1(u)\,du \\ & + (b_0 - b_{-1})A(t). \end{aligned}$$

Generally, by (15), we find for $k = 1, 2, \ldots, N$

$$\begin{aligned} b_{k-2}\int_{t-\tau+\delta}^{t} Q_{k-1}(u)x_{k-1}(u)\,du \le {} & b_{k-1}\int_{t-\tau+\delta}^{t} Q_k(u)x_k(u)\,du \\ & + (b_{k-1} - b_{k-2})A(t). \end{aligned} \tag{16}$$

Adding (16) from $k = 1$ to $k = N$, we get

$$\begin{aligned} b_{-1}\int_{t-\tau+\delta}^{t} Q_0(u)x_0(u)\,du \le {} & b_{N-1}\int_{t-\tau+\delta}^{t} Q_N(u)x_N(u)\,du \\ & + (b_{N-1} - b_{-1})A(t), \end{aligned}$$

which, combining with (13) and (7) and by noticing that $t - \delta - Nr + \tau \leq t$, we find for $t \geq \bar{t}$,

$$
\begin{aligned}
x(t) \leq & -\alpha + R(t)x(t-r) \\
& \quad - \int_{t-Nr-\delta}^{t} \big[\bar{P}(u) - (b_N - b_{-1}) Q(u - \tau + \delta) \\
& \qquad\qquad - b_N Q_N(u - \tau + \delta + Nr) \big] x(u-\tau)\, du \\
\leq & -\alpha + R(t)x(t-r).
\end{aligned}
$$

So (10) holds and the proof is complete.

LEMMA 3. *Assume that* (7) *holds and that there exists* $t^* \geq t_0$ *such that*

$$R(t^* + ir) \leq 1, \qquad i = 1, 2, \dots . \tag{17}$$

Let $x(t)$ *be an eventually positive solution of Eq.* (1) *and* $z(t)$ *be defined by* (8). *Then we have eventually*

$$z'(t) \leq 0, \qquad z(t) > 0. \tag{18}$$

Proof. If (18) does not hold, by Lemma 1, we have eventually

$$x(t) \leq -\alpha + R(t)x(t-r). \tag{19}$$

The rest of the proof is similar to that of [1, Lemma 1] and we omit it. The proof is complete.

LEMMA 4. *Assume that* (7) *holds and that there exists* $s_0 \geq t_0$ *such that* $R(t) > 0$ *for* $t \geq s_0$ *and*

$$\sum_{k=1}^{\infty} \big[R(s_1) R(s_2) \cdots R(s_k) \big]^{-1} = \infty, \tag{20}$$

where $s_k = s_0 + kr$, $k = 1, 2, \dots$. *Let* $x(t)$ *be an eventually positive solution of Eq.* (1) *and* $z(t)$ *be defined by* (8). *Then eventually* (18) *holds.*

Proof. If (18) does not hold, by Lemma 2, we have (19). The rest of the proof is similar to that of [8, Lemma 2], and thus, is omitted. The proof is complete.

3. MAIN RESULTS

THEOREM 1. *Assume that* (7) *and* (17) *hold and that there exists a nonincreasing function* $\gamma \in C([t_0, \infty), R^+)$ *such that*

$$R(t) \geq 1 + \gamma(t), \qquad t \in E[t_1, t_2], \tag{21}$$

and

$$\int_{E[t_1+\tau, t_2+\tau]} \bar{P}(t)[1+\gamma(t)]^{t/r}\, dt = \infty. \tag{22}$$

Then every solution of Eq. (1) *oscillates.*

Proof. Assume, otherwise, that Eq. (1) has an eventually positive solution $x(t)$. Let $z(t)$ be defined by (8). By Lemma 3, there exists $T \geq t_0$ such that

$$z'(t) \leq 0, \qquad z(t) > 0, \qquad x(t-\rho) > 0, \qquad t \geq T, \tag{23}$$

which implies that

$$x(t) \geq R(t)x(t-r), \qquad t \geq T. \tag{24}$$

Now choose a positive integer N^* such that $t_1 + N^*r \geq T$. Then by (21) and (24), we get

$$x(t) \geq (1+\gamma(t))(t-r), \qquad t \in E[t_1 + N^*r, t_2 + N^*r].$$

Let $M = \min\{x(t) : t_1 + (N^* - 1)r \leq t \leq t_2 + (N^* - 1)r\}$; then $M > 0$. By the decreasing nature of $\gamma(\cdot)$, we have

$$x(t) \geq (1+\gamma(t))M, \qquad t \in [t_1 + N^*r, t_2 + N^*r]$$

and

$$x(t) \geq (1+\gamma(t))^2 M, \qquad t \in [t_1 + (N^*+1)r, t_2 + (N^*+1)r].$$

By induction we can obtain for $t \in [t_1 + (N^*+i)r, t_2 + (N^*+i)r]$, $i = 0, 1, 2, \ldots,$

$$x(t) \geq (1+\gamma(t))^{i+1} M.$$

Noticing that $t \in [t_1 + (N^*+i)r, t_2 + (N^*+i)r]$ implies $i \geq \frac{1}{r}(t - t_2) - N^*$, we have for $t \in [t_1 + (N^*+i)r, t_2 + (N^*+i)r]$, $i = 0, 1, 2, \ldots,$

$$x(t) > (1+\gamma(t))^{(1/r)(t-t_2)-N^*+1} M,$$

which implies that

$$\begin{aligned} x(t-\tau) &> (1+\gamma(t-\tau))^{(1/r)(t-\tau-t_2)-N^*+1} M \\ &\geq (1+\gamma(t))^{t/r-\beta} M, \\ &\qquad t \in E[t_1 + \tau + N^*r, t_2 + \tau + N^*r], \end{aligned} \tag{25}$$

where $\beta = (t_2 + \tau)/r + N^* - 1$.

Next, for any $t \geq t_1 + \tau + N^* r$, denote $A_t[t_1, t_2] = [t_1 + \tau + N^* r, t] \cap E[t_1 + \tau + N^* r, t_2 + \tau + N^* r]$. Then, by (22) and the fact that $1 + \gamma(t)$ is bounded on $[t_0, \infty)$, we have

$$\lim_{t \to \infty} \int_{A_t[t_1, t_2]} \overline{P}(s)(1 + \gamma(s))^{s/r - \beta} \, ds = \infty. \tag{26}$$

Now integrating (10) on $[t_1 + \tau + N^* r, t]$, we get by (25)

$$\begin{aligned} z(t) - z(t_1 + \tau + N^* r) &\leq -\int_{t_1 + \tau + N^* r}^{t} \overline{P}(s) x(s - \tau) \, ds \\ &\leq -\int_{A_1[t_1, t_2]} \overline{P}(s) x(s - \tau) \, ds \\ &\leq -M \int_{A_t[t_1, t_2]} \overline{P}(s)(1 + \gamma(s))^{t/r - \beta} \, ds, \end{aligned}$$

which, together with (26), implies that $z(t) \to -\infty$ as $t \to \infty$. This contradicts (23). The proof is complete.

THEOREM 2. *Assume that* (7) *and* (20)–(22) *hold. Then every solution of Eq.* (1) *oscillates.*

Proof. Otherwise, let $x(t)$ be an eventually positive solution and $z(t)$ be defined by (8). By Lemma 4, we have eventually $z(t) > 0$. The rest of the proof is similar to that of Theorem 1, and we omitted it. The proof is complete.

Remark 1. When $\tau = \delta$, Eq. (1) reduces to the neutral equation without negative coefficient

$$\frac{d}{dt}[x(t) - R(t)x(t - r)] + \overline{P}(t)x(t - \tau) = 0, \tag{$*$}$$

which has been studied in many papers (see [1, 3, 5, 8, 9]). In this case $b_k = 1$ for $k = -1, 0, 1, \ldots, N$ and condition (7) is always satisfied, so we get the following new oscillation criteria for Eq. $(*)$, in which $R(t) - 1$ is allowed to be oscillatory and condition (3) is not necessary (see Example 1 and Remark 2).

THEOREM 3. *Assume that* (17) (*or* (20)), (21), *and* (22) *hold. Then every solution of Eq.* $(*)$ *oscillates.*

COROLLARY 1. *Assume that* (17) (*or* (20)), (21), *and* (22) *hold and that*

$$Q_k(t) = o(1) \ (t \to \infty), \qquad k = 0, 1, 2, \ldots, N - 1, \ \textit{if } N \geq 1. \tag{27}$$

Further suppose that there exists $\alpha > 0$ *such that*

$$\bar{P}(t) \geq \alpha Q(t - \tau + \delta) + (1 + \alpha) Q_N(t - \tau + \delta + Nr). \tag{28}$$

Then every solution of Eq. (1) *oscillates.*

Proof. Choose $\beta > 0$ sufficiently small such that $b_N < 1 + \alpha$, where $b_{-1} = \operatorname{sgn}(\tau - \delta)$ and $b_k = b_{k-1} e^{(\tau - \delta) b_{k-1} \beta}$, $k = 0, 1, 2, \ldots, N - 1$. From (27), for sufficiently large t, $Q_k(t) < \beta$, $k = 0, 1, \ldots, N - 1$, which, combining with (28), implies that (7) holds with $\beta_k = \beta$, for $k = 0, 1, 2, \ldots, N - 1$. By Theorem 1 (or Theorem 2) every solution of Eq. (1) oscillates and the proof is complete.

COROLLARY 2. *Assume that* (17) (*or* (20)), (21), *and* (22) *hold and that* $R(t)$ *is bounded on* $[t_0, \infty)$. *If*

$$Q(t) = o(1) \qquad \textit{and} \qquad \max_{t - \tau + \delta \leq s \leq t + r} Q(s) = o(P(t)) \qquad (t \to \infty), \tag{29}$$

then every solution of Eq. (1) *oscillates.*

Proof. It is clear that (29) implies that (27) and (28) hold. By Corollary 1 every solution of Eq. (1) oscillates and the proof is complete.

EXAMPLE 1. Consider the neutral equation

$$\frac{d}{dt}\left[x(t) - \left(1 - \frac{\cos t}{1 + t^{\alpha}}\right) x(t - 2\pi)\right] + C_1 e^{-\eta t^{1-\alpha}} x(t - 2\pi) - C_2 e^{-\sigma t} x(t) = 0, \tag{30}$$

$t \geq 0$, where $0 < \alpha < 1$, $\eta < \frac{1}{2\pi}$, $\sigma > \max\{0, \eta\}$, $C_1 > 0$, and $C_2 \geq 0$. Choose $\eta < \eta_1 < \min\{\frac{1}{2\pi}, \sigma\}$ and set $c = e^{\eta/\eta_1}$. Then we can rewrite Eq. (30) into the equation

$$\frac{d}{dt}\left[x(t) - \left(1 - \frac{\cos t}{1 + t^{\alpha}}\right) x(t - 2\pi)\right] + C_1 c^{-\eta_1 t^{1-\alpha}} x(t - 2\pi) - C_2 e^{-\sigma t} x(t) = 0, \tag{31}$$

$t \geq 0$, which satisfies (17) with $t^* \in [\frac{3}{2}\pi, \frac{5}{2}\pi]$ and (29). Further choose $0 < \zeta < 1$ such that $\eta_1 < \frac{\zeta}{2\pi}$ and set $t_1 = 2\pi - \arccos \zeta$, $t_2 = 2\pi + \arccos \zeta$. Noticing that $c < e$, one can verify that (21) and (22) are also satisfied with $\gamma(t) = \zeta/(1 + t^{\alpha})$. By Corollary 2, every solution of (31), and so (30), oscillates.

Remark 2. Equation (30) does not satisfy (3) and (i)–(iii) mentioned in Section 1, which are demanded in previous results. When $C_2 = 0$ and $0 < \eta < \frac{1}{2\pi}$, the oscillation of the equation cannot be determined by the theorems in the literature including [9]. In addition, in the case that $C_2 = 0$ and $\eta > \frac{1}{2\pi}$, by [9, Theorem 4], one can show that Eq. (30) has an eventually positive solution $x(t)$, so our results are almost sharp.

EXAMPLE 2. Consider the neutral equation

$$\frac{d}{dt}\left[x(t) - (1 - t^{-2})x(t-1)\right] + \frac{1}{(t-1)^{\alpha}}x(t-1) - \frac{1}{t^3}x(t) = 0, \qquad t \geq 2, \quad (32)$$

where $\alpha < 1$. It is easy to see (20) and other conditions in Corollary 2 are satisfied. So by Corollary 2, every solution of Eq. (32) oscillates.

Remark 3. Equation (32) does not satisfy the conditions of the theorems in [12].

REFERENCES

1. Ming-Po Chen, J. S. Yu, and L. H. Huang, Oscillations of first order neutral differential equations with variable coefficients, *J. Math. Anal. Appl.* **185** (1994), 288–301.
2. Q. Chuanxi and G. Ladas, Oscillation in differential equations with positive and negative coefficients, *Canad. Math. Bull.* **33** (1990), 442–450.
3. L. H. Erbe, Q. K. Kong, and B. G. Zhang, "Oscillation Theory for Functional Differential Equations," Dekker, New York, 1995.
4. K. Farrall, E. A. Grove, and G. Ladas, Neutral delay differential equations with positive and negative coefficients, *Appl. Anal.* **27** (1988), 181–197.
5. I. Gyori and G. Ladas, "Oscillation Theory of Delay Differential Equations with Applications," Oxford Univ. Press, London/New York, 1991.
6. B. S. Lalli and B. G. Zhang, Oscillation of first order neutral differential equations, *Appl. Anal.* **39** (1990), 265–274.
7. S. G. Ruan, Oscillations for first order neutral differential equations with variable coefficients, *Bull. Austral. Math. Soc.* **43** (1991), 147–152.
8. J. H. Shen, New oscillation criteria for odd order neutral equations, *J. Math. Anal. Appl.* **201** (1996), 387–395.
9. X. H. Tang and J. S. Yu, Oscillation and existence of positive solution for odd-order neutral equations with "integrally small" coefficients, *J. Math. Anal. Appl.* **222** (1998), 286–296.
10. J. S. Yu, Neutral differential equations with positive and negative coefficients, *Acta Math. Sinica* **34** (1991), 517–523.
11. J. S. Yu and Z. C. Wang, Some further results on oscillation of neutral differential equations, *Bull. Austral. Math. Soc.* **46** (1992), 149–157.
12. J. S. Yu and J. Yan, Oscillation in first order neutral differential equations with "integrally small" coefficients, *J. Math. Anal. Appl.* **187** (1994), 361–370.