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Continuous Beams with Unilateral Elastic Supports[1]

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We analyse the problem of continuous elastic beam on unilateral elastic supports. The study is done in the appropriate analytical framework by using the functional analysis methods and the variational inequalities theory results.

Key Words: unilateral problems; beams; variational inequalities.

1. INTRODUCTION

In this paper we shall study a nonlinear problem of the theory of structures: the continuous elastic beam on unilateral elastic supports. We analyse, for the sake of simplicity, the two-spawn scheme.

The beam, of linearly elastic material, covers the interval $\Omega =]a, c[$ of the real axis. The supported cross-sections have abscissa a, b, c with $a < b < c$.

The cross section is variable: its bending stiffness B ($B \geq b_0 \in]0, +\infty[$) is essentially bounded on Ω.

The loading on the beam consists of distributed forces $q_0 \in L^2(\Omega)$ (positive downward). Furthermore, at abscissas $x_1, \dots, x_m$ ($a = x_1 < \cdots < x_m = c$) concentrated forces (positive downward) and couples (positive clockwise) act upon, with intensities $F_1, \dots, F_m$ and $\mathscr{M}_1, \dots, \mathscr{M}_m$ (with $F_i^2 + \mathscr{M}_i^2 > 0 \ \forall i \in \{2, \dots, m-1\}$).

We assume that the bending moment is a linear function of the linearised curvature.

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Let us denote by $u(x)$ the vertical displacement of the cross-section whose abscissa is x (positive downward) and by q the element of $(H^2(\Omega))'$

$$\langle q, v\rangle = \int_\Omega q_0 v\, dx + \sum_{i=1}^{m} F_i v(x_i) + \sum_{i=1}^{m} \mathscr{M}_i v'(x_i) \quad \forall v \in H^2(\Omega).$$

The reacting behaviour of the elastic unilateral support can be described by Winkler's model

$$R_i = -k_i u^+(i) \ (i \in \{a, b, c\}),$$

where elastic constant k_i of the support i ($i \in \{a, b, c\}$) is an element of $]0, +\infty[$.

The problem is to find the beam's elastic line u, i.e., Problem (1).[2]

Find $u \in H^2(\Omega)$ such that

(1.1) $(-Bu'')'' + q - k_b u^+(b)\delta_b = 0$ on Ω in the distribution sense

(1.2) $(-Bu'')(a^+) = \mathscr{M}_1$

(1.3) $(-Bu'')'(a^+) = k_a u^+(a) - F_1$

(1.4) $(-Bu'')(c^-) = -\mathscr{M}_m$

(1.5) $(-Bu'')'(c^-) = F_m - k_c u^+(c).$

In Section 2 we prove (Theorem 1) the equivalence of Problem (1) with a variational equation. We also prove (Theorem 2) that Problem (1) is equivalent to a variational inequality and to minimize the energy functional.

In Section 3 we analyse existence and uniqueness of the solution. We give a necessary condition (Theorem 3) for Problem (1) to have a solution. In Theorem 4 we obtain the existence and uniqueness results.

2. SOME EQUIVALENT FORMULATION

Let us consider the virtual works' equation (Problem (2))

$$u \in H^2(\Omega)\colon \int_a^c - Bu''v''\, dx + \langle q, v\rangle - k_a u^+(a) v(a)$$
$$-k_b u^+(b) v(b) - k_c u^+(c) v(c) = 0 \quad \forall v \in H^2(\Omega).$$

[2] Let us denote $\forall y \in \mathbf{R}$ by δ_y the Dirac distribution relative to y.

We have

THEOREM 1. *The following statements are equivalent*

(i) *u is a solution to Problem* (1)

(ii) *u is a solution to Problem* (2).

Proof. (i) ⇒ (ii). First of all, let us observe that $\forall\varphi \in C_0^\infty(\Omega)^3$

$$\langle q, \varphi\rangle = \int_a^c q_0\,\varphi\,dx - \sum_{i=1}^m F_i \int_a^c H_{x_i}(x)\,\varphi'(x)\,dx - \sum_{i=1}^m \mathscr{M}_i \int_a^c H_{x_i}(x)\,\varphi''(x)\,dx$$

$$\langle k_b u^+(b)\,\delta_b, \varphi\rangle = k_b u^+(b)\,\varphi(b) = -k_b u^+(b) \int_a^c H_b(x)\,\varphi'(x)\,dx.$$

Therefore from (1.1) we get

$$D^2 T_{(-Bu'') - \sum_{i=1}^m \mathscr{M}_i H_{x_i}(x)} + DT_{\sum_{i=1}^m F_i H_{x_i}(x) - k_b u^+(b) H_b(x)} = -T_{q_0}$$

from which

$$(3)\quad (-Bu'')(x) = \sum_{i=1}^m \mathscr{M}_i H_{x_i}(x) + \sum_{i=1}^m F_i H_{x_i}(x)(x_i - x) + k_b u^+(b) H_b(x)(x-b) - \int_a^x \left(\int_a^t q_0(z)\,dz \right) dt + h_1(x-a) + h_2 \text{ a.e. on } \Omega,$$

where $h_1, h_2 \in \mathbf{R}$.

Now let us put $\forall x \in \Omega - \{x_2, \dots, x_{m-1}\}$

$$f(x) = \sum_{i=1}^m \mathscr{M}_i H_{x_i}(x) + \sum_{i=1}^m F_i H_{x_i}(x_i - x) + k_b u^+(b) H_b(x)(x-b) - \int_a^x \left(\int_a^t q_0(z)\,dz \right) dt + h_1(x-a) + h_2$$

and observe that, because of (3), we have

$$(-Bu'')(x) = f(x) \text{ a.e. on } \Omega.$$

[3] Let us put $\forall (y, z) \in \mathbf{R}^2$

$$H_z(y) = \begin{cases} 1 & \text{if } y \ge z \\ 0 & \text{if } y < z. \end{cases}$$

Obviously, one of the following possibilities can occur

(4) $$\exists j \in \{1, \dots, m-1\} : b \in]x_j, x_{j+1}[\text{ or}$$

(5) $$\exists j \in \{2, \dots, m-1\} : b = x_j.$$

In the (4) case we have $\forall v \in H^2(\Omega)$

$$\int_\Omega - Bu'' v'' \, dx = \int_{]a,b[} fv'' \, dx + \int_{]b,x_2[} fv'' \, dx + \sum_{i=2}^{m-1} \int_{]x_i, x_{i+1}[} fv'' \, dx \quad \text{if } j = 1$$

$$\int_\Omega - Bu'' v'' \, dx = \sum_{i=1}^{j-1} \int_{]x_i, x_{i+1}[} fv'' \, dx + \int_{]x_j, b[} fv'' \, dx + \int_{]b, x_{j+1}[} fv'' \, dx + \sum_{i=j+1}^{m-1} \int_{]x_i, x_{i+1}[} fv'' \, dx \quad \text{if } j \in \{2, \dots, m-2\}.$$

$$\int_\Omega - Bu'' v'' \, dx = \sum_{i=1}^{m-2} \int_{]x_i, x_{i+1}[} fv'' \, dx + \int_{]x_{m-1}, b[} fv'' \, dx + \int_{]b,c[} fv'' \, dx \quad \text{if } j = m-1.$$

In the (5) case we have $\forall v \in H^2(\Omega)$

$$\int_\Omega - Bu'' v'' \, dx = \int_\Omega fv'' \, dx = \sum_{i=1}^{m-1} \int_{]x_i, x_{i+1}[} fv'' \, dx.$$

Consequently, by part integration, in either (4) or (5) cases we have

$$\begin{aligned}
&\int_\Omega - Bu'' v'' \, dx \\
&\quad = h_1 v(a) - h_2 v'(a) \\
&\qquad + \Bigg(\sum_{i=1}^{m} \mathscr{M}_i + \sum_{i=1}^{m-1} F_i (x_i - c) + k_b u^+(b)(c-b) \\
&\qquad\qquad - \int_a^c \left(\int_a^t q_0(z) \, dz \right) dt + h_1 (c-a) + h_2 \Bigg) v'(c) \\
&\qquad + \left(\sum_{i=1}^{m} F_i - k_b u^+(b) + \int_a^c q_0(x) \, dx - h_1 \right) v(c) \\
&\qquad - \sum_{i=1}^{m} \mathscr{M}_i v'(x_i) - \sum_{i=1}^{m} F_i v(x_i) - \int_\Omega q_0 v dx + k_b u^+(b) v(b)
\end{aligned}$$

from which

$$
\begin{aligned}
(6)\quad & \int_\Omega - Bu'' v''\,dx + \langle q, v\rangle - k_a u^+(a) v(a) \\
&\quad - k_b u^+(b) v(b) - k_c u^+(c) v(c) \\
&\qquad = -h_2 v'(a) + \big(h_1 - k_a u^+(a)\big) v(a) \\
&\qquad\quad + \Bigg(\sum_{i=1}^{m} \mathscr{M}_i + \sum_{i=1}^{m-1} F_i (x_i - c) + k_b u^+(b)(c-b) \\
&\qquad\qquad - \int_a^c \left(\int_a^t q_0(z)\,dz \right) dt + h_1 (c-a) + h_2 \Bigg) v'(c) \\
&\qquad\quad + \left(\sum_{i=1}^{m} F_i - k_b u^+(b) + \int_a^c q_0(x)\,dx - h_1 - k_c u^+(c) \right) v(c).
\end{aligned}
$$

Because of (3), (1.2), (1.3), (1.4), and (1.5) we have

$$
h_2 = 0
$$

$$
h_1 = k_a u^+(a)
$$

$$
\begin{aligned}
& \sum_{i=1}^{m-1} \mathscr{M}_i + \sum_{i=1}^{m-1} F_i (x_i - c) + k_b (c-b) u^+(b) - \int_a^c \left(\int_a^t q_0(z)\,dz \right) dt \\
&\quad + h_1 (c-a) + h_2 = -\mathscr{M}_m \\
&\quad - \sum_{i=1}^{m-1} F_i + k_b u^+(b) - \int_a^c q_0(x)\,dx + h_1 \\
&\qquad = F_m - k_c u^+(c).
\end{aligned}
$$

Therefore (ii) is true.

(ii) ⇒ (i). Because u is solution to variational equation, $\forall \varphi \in C_0^\infty(\Omega)$ we have

$$
\begin{aligned}
& \int_a^c - Bu'' \varphi''\,dx + \langle q, \varphi\rangle - k_a u^+(a)\varphi(a) \\
&\qquad - k_b u^+(b)\varphi(b) - k_c u^+(c)\varphi(c) = 0
\end{aligned}
$$

so that (1.1) is proven.

After that let us proceed as earlier in the proof. Then from (1.1) we get (3) and (6). By using (6) we have $\forall v \in H^2(\Omega)$

$$\begin{aligned}
&-h_2 v'(a) + (h_1 - k_a u^+(a))v(a) \\
&\quad + \left(\sum_{i=1}^{m} \mathscr{M}_i + \sum_{i=1}^{m-1} F_i(x_i - c) + k_b u^+(b)(c-b) \right. \\
&\qquad \left. - \int_a^c \left(\int_a^t q_0(z)\,dz \right) dt + h_1(c-a) + h_2 \right) v'(c) \\
&\quad + \left(\sum_{i=1}^{m} F_i - k_b u^+(b) + \int_a^c q_0(x)\,dx - h_1 - k_c u^+(c) \right) v(c) = 0.
\end{aligned}$$

From this, for suitable choices of v, we have

$h_2 = 0$ (choosing v: $v(a) = v'(c) = v(c) = 0$, $v'(a) \neq 0$);

$h_1 = k_a u^+(a)$ (choosing v: $v'(a) = v'(c) = v(c) = 0$, $v(a) \neq 0$);

$$\begin{aligned}
&\sum_{i=1}^{m} \mathscr{M}_i + \sum_{i=1}^{m-1} F_i(x_i - c) + k_b u^+(b)(c-b) \\
&\qquad - \int_a^c \left(\int_a^t q_0(z)\,dz \right) dt + h_1(c-a) + h_2 = 0
\end{aligned}$$

(choosing v: $v(a) = v'(a) = v(c) = 0$, $v'(c) \neq 0$);

$$\sum_{i=1}^{m} F_i - k_b u^+(b) + \int_a^c q_0(x)\,dx - h_1 - k_c u^+(c) = 0$$

(choosing v: $v(a) = v'(a) = v'(c) = 0$, $v(c) \neq 0$).

Consequently, because of (3), (1.2), (1.3), (1.4), and (1.5) are true. The thesis follows. ∎

Let us now consider the variational inequality

$$\begin{aligned}
(7) \qquad & u \in H^2(\Omega): \int_a^c B u''(v'' - u'')\,dx - \langle q, v - u \rangle \\
&\quad + \frac{1}{2}\left(k_a (v^+(a))^2 + k_b (v^+(b))^2 + k_c (v^+(c))^2 \right) \\
&\quad - \frac{1}{2}\left(k_a (u^+(a))^2 + k_b (u^+(b))^2 + k_c (u^+(c))^2 \right) \geq 0 \\
&\qquad \forall v \in H^2(\Omega).
\end{aligned}$$

Furthermore, let us consider the functional energy

$$J: v \in H^2(\Omega) \to \frac{1}{2}\int_\Omega B(v'')^2\,dx - \langle q, v\rangle + \frac{1}{2}\left(k_a(v^+(a))^2 + k_b(v^+(b))^2 + k_c(v^+(c))^2\right)$$

and the minimum problem of the functional J

$$u \in H^2(\Omega): J(u) \leq J(v) \quad \forall v \in H^2(\Omega). \tag{8}$$

Let us now show

THEOREM 2. *The following statements are equivalent*

(9) *u is a solution to variational equation* (2)

(10) *u is a solution to variational inequality* (7)

(11) *u is a solution to Problem* (8)

Proof. We proceed in a similar way as in [9].

3. EXISTENCE AND UNIQUENESS RESULTS

Let us now denote by P_1 the subspace of $H^2(\Omega)$ given by not greater than first degree polynomials[4] and let

$$\forall x \in \mathbf{R}\ \mathbf{1}(x) = 1, \quad \forall (x,y) \in \mathbf{R}^2\ p_y(x) = x - y.$$

Moreover, let us denote by P and M_0 the resultant and the moment with respect to the origin of the loads applied to the beam

$$P = \int_\Omega q_0\,dx + \sum_{i=1}^m F_i$$

$$M_0 = \int_\Omega q_0 x\,dx + \sum_{i=1}^m F_i x_i + \sum_{i=1}^m \mathscr{M}_i.$$

Let us now show

THEOREM 3. *A necessary condition for Problem* (1) *to have a solution is* $P \geq 0$. *If* $P = 0$, $M_0 = 0$ *is a necessary condition so that Problem* (1) *admits a solution.*

[4] P_1 can be used to describe the compatible rigid body displacements of the structure.

If $P > 0$, *a necessary condition so that Problem* (1) *admits a solution is that the point* $\xi = M_0/P$ *belongs to* $[a, c]$.

Proof. Let us suppose that Problem (1) admits a solution. For $v = -1$, we obtain from variational inequality (2)

$$\langle q, \mathbf{1} \rangle = k_a u^+(a) + k_b u^+(b) + k_c u^+(c) \geq 0;$$

i.e., $P \geq 0$.

Let us now deal with the case $P = 0$. We have

$$\langle q, p_0 \rangle = k_a u^+(a)a + k_b u^+(b)b + k_c u^+(c)c.$$

On the other hand, because

$$P = \langle q, \mathbf{1} \rangle = k_a u^+(a) + k_b u^+(b) + k_c u^+(c) = 0$$

we have $u^+(a) = u^+(b) = u^+(c) = 0$, so that

$$M_0 = \langle q, p_0 \rangle = 0.$$

Finally, let us analyse the case $P > 0$.

Proceeding by contradiction, let us assume

$$\xi = \frac{M_0}{P} < a \quad \left[\text{resp. } \xi = \frac{M_0}{P} > c\right].$$

Consequently a real number $z \in]\xi, a[$ [resp. $z \in]c, \xi[$] exists and we have

$$\langle q, p_z \rangle = (\xi - z)\langle q, \mathbf{1} \rangle < 0 \quad [\text{resp. } \langle q, p_z \rangle = (\xi - z) < q, \mathbf{1} \gg 0].$$

On the other hand, choosing $v = p_z$, from variational inequality (2) we obtain

$$\langle q, p_z \rangle = k_a u^+(a)(a - z) + k_b u^+(b)(b - z) + k_c u^+(c)(c - z)$$

from which

$$\langle q, p_z \rangle \geq 0 \quad [\text{resp. } \langle q, p_z \rangle \leq 0].$$

The thesis follows. ∎

By Theorem 3, for Problem (1) to have a solution, one of the following possibilities must occur

(12) $P = 0,\ M_0 = 0$

(13) $P > 0,\ \xi = M_0/P \in \{a, c\}$

(14) $P > 0,\ \xi = M_0/P \in]a, c[$.

The following theorem holds

THEOREM 4. *In Cases* (12) *and* (13), *Problem* (1) *has infinite solutions*; *their set is a subset of an element of* $H^2(\Omega)/P_1$.

Furthermore in Case (13) *the following results*

(15) *every solution u to Problem* (1) *is such that*

$$\xi = a \Rightarrow u(a) = \frac{P}{k_a},\ u(b) \leq 0,\ u(c) \leq 0$$

$$\xi = c \Rightarrow u(a) \leq 0,\ u(b) \leq 0,\ u(c) = \frac{P}{k_c}$$

(16) *each and every solution to Problem* (1) *is of the type*

$$u_0 + h(x - \xi),$$

where u_0 *is a whatever solution to Problem* (1) *and*

$$\xi = a \Rightarrow h \leq \min\left\{ -\frac{u_0(b)}{b-a}, \frac{u_0(c)}{c-a} \right\}$$

$$\xi = c \Rightarrow h \geq \max\left\{ -\frac{u_0(b)}{b-c}, \frac{u_0(a)}{a-c} \right\}.$$

In Case (14) *Problem* (1) *has at least one solution. If* $\xi \neq b$, *the solution to Problem* (1) *is unique*.

Proof. At first let us study the problem

$$(17) \qquad u \in H^2(\Omega)\colon \int_\Omega - Bu'' v''\, dx + \langle q, v \rangle = 0 \quad \forall v \in H^2(\Omega).$$

To this aim we consider the bilinear form

$$a\colon ([w],[v]) \in H^2(\Omega)/P_1 \to \int_\Omega Bw'' v''\, dx$$

and the linear form

$$g\colon [v] \in H^2(\Omega)/P_1 \to \langle q, v \rangle.$$

Now let us notice that

$$\exists c_1, c_2 \in]0, +\infty[\colon \quad \forall v \in H^2(\Omega)\ c_1 \|v''\|_{L^2(\Omega)} \leq \|[v]\|_{H^2(\Omega)/P_1} \leq c_2 \|v''\|_{L^2(\Omega)}.$$

Then, by using the same method as in [8], we prove that g is continuous, $a([w],[v])$ is continuous and coercive.

Obviously $H^2(\Omega)/P_1$ is an Hilbert space. Thus for the Lax–Milgram theorem the variational equation

$$(18)\quad [u] \in H^2(\Omega)/P_1\colon\ a([u],[v]) = \langle g, [v]\rangle \quad \forall [v] \in H^2(\Omega)/P_1$$

has a unique solution, which we denote with the symbol $[\tilde{u}]$.

Obviously every $u^* \in [\tilde{u}]$ is such that

$$\int_\Omega - Bu^{*\prime\prime} v'' \, dx = -\langle q, v\rangle \quad \forall v \in H^2(\Omega)$$

and, consequently, is solution to (17).

Thus, $\forall u^* \in [\tilde{u}]$ the function

$$u^* - \max\{u^*(a), u^*(b), u^*(c)\}$$

is solution to the variational equation (2), i.e., of Problem (1).

Therefore in Case (12), Problem (1) has infinite solutions. Let us now prove that, if u_1, u_2 are solutions to Problem (1), then $u_1 - u_2 \in P_1$.

Because $\langle q, \mathbf{1}\rangle = 0$, we have

$$\int_a^c - Bu_1''(u_1'' - u_2'')\, dx + \langle q, u_1 - u_2\rangle = 0$$

$$\int_a^c - Bu_2''(u_1'' - u_2'')\, dx + \langle q, u_1 - u_2\rangle = 0$$

from which

$$\int_a^c B(u_1'' - u_2'')^2 \, dx = 0$$

from which

$$\|u_1'' - u_2''\|_{L^2(\Omega)} = 0$$

and consequently $u_1 - u_2 \in P_1$.

Let us now consider Case (13).

Let us assume $\xi = a$ [resp. $\xi = c$] and consider the set

$$H = \{v \in H^2(\Omega)\colon v(a) = 0,\ v'(a) = 0\}$$
$$\left[\text{resp. } H = \{v \in H^2(\Omega)\colon v(c) = 0,\ v'(c) = 0\}\right].$$

Clearly H is a subspace of $H^2(\Omega)$. Let H be equipped with the norm

$$\|\bullet\|_H\colon v \in H \to \|v''\|_{L^2(\Omega)}.$$

Because $\forall v \in H$

$$|v(x)| \le \|v'\|_{L^2(\Omega)}(c-a)^{1/2} \text{ a.e. on }]a,c[$$

$$|v'(x)| \le \|v''\|_{L^2(\Omega)}(c-a)^{1/2} \text{ a.e. on }]a,c[,$$

we have

$$\|v\|^2_{H^2(\Omega)} \le \big((c-a)^4 + (c-a)^2 + 1\big)\|v''\|_{L^2(\Omega)}.$$

Thus the norm $\|\bullet\|_H$ is equivalent to the norm of $H^2(\Omega)$.

On the other hand, H, equipped with the inner product

$$(\bullet,\bullet)_H : (u,v) \in H \times H \to \int_\Omega u'' v''\, dx$$

is a Hilbert space.

Moreover,

$$a(\bullet,\bullet) : (u,v) \in H \times H \to \int_a^c B u'' v''\, dx$$

is a bilinear form on H, continuous and coercive; q is a form on H linear and continuous.

Thus for the Lax–Milgram theorem there exists an unique $\tilde{u} \in H$ such that

$$\langle q, v\rangle = \int_a^c B\tilde{u}'' v''\, dx \quad \forall v \in H.$$

Let us now note with u the function $\tilde{u} + \alpha x + \beta$, where

$$\alpha \le \min\left\{\left(-\frac{P}{k_a} - \tilde{u}(b)\right)\frac{1}{b-a}, \left(-\frac{P}{k_a} - \tilde{u}(c)\right)\frac{1}{c-a}\right\}$$

$$\left[\text{resp. } \alpha \ge \max\left\{\left(-\frac{P}{k_c} - \tilde{u}(b)\right)\frac{1}{b-c}, \left(-\frac{P}{k_c} - \tilde{u}(a)\right)\frac{1}{a-c}\right\}\right]$$

and

$$\beta = -\alpha a + \frac{P}{k_a} \quad \left[\text{resp. } \beta = -\alpha c + \frac{P}{k_c}\right].$$

We will now verify that u is solution to variational inequality (2) and consequently to Problem (1).

In fact we have

$$u(a) = \frac{P}{k_a} \quad [\text{resp. } u(a) \le 0]$$

$$u(b) \le 0$$

$$u(c) \le 0 \quad \left[\text{resp. } u(c) = \frac{P}{k_c}\right],$$

from which

$$u^+(a) = \frac{P}{k_a} \quad [\text{resp. } u^+(a) = 0],\ u^+(b) = 0,\ u^+(c) = 0$$
$$\left[\text{resp. } u^+(c) = \frac{P}{k_c}\right].$$

Moreover $\forall v \in H^2(\Omega)$, putting $\tilde{v} = v - v'(a)p_a - v(a)$ [resp. $\tilde{v} = v - v'(c)p_c - v(c)$], it results $\tilde{v}(a) = \tilde{v}'(a) = 0$ [resp. $\tilde{v}(c) = \tilde{v}'(c) = 0$]; i.e., $\tilde{v} \in H$.

Thus we have

$$\int_a^c - Bu''v''\,dx + \langle q, v\rangle$$
$$= \int_a^c - B\tilde{u}''\tilde{v}''\,dx + \langle q, \tilde{v}\rangle + v'(a)(\langle q, p_0\rangle - a\langle q, \mathbf{1}\rangle) + v(a)\langle q, \mathbf{1}\rangle$$
$$= v(a)P \quad \left[\text{resp. } \int_a^c - Bu''v''\,dx + \langle q, v\rangle = v(c)P\right].$$

and from this

$$\int_a^c - Bu''v''\,dx + \langle q, v\rangle$$
$$= k_a u^+(a)v(a) + k_b u^+(b)v(b) + k_c u^+(c)v(c).$$

Therefore Problem (1) has infinite solutions.

Let u_0, u be solutions to Problem (1), with u_0 such that

$$\xi = a \Rightarrow u_0(a) = \frac{P}{k_a}, \quad u_0(b) \le 0, \quad u_0(c) \le 0$$

$$\xi = c \Rightarrow u_0(a) \le 0, \quad u_0(b) \le 0, \quad u_0(c) = \frac{P}{k_c}.$$

We have

$$\int_a^c -Bu_0''(u_0''-u'')\,dx + \langle q, u_0 - u\rangle$$
$$= k_a u_0^+(a)\big(u_0(a)-u(a)\big) + k_b u_0^+(b)\big(u_0(b)-u(b)\big)$$
$$+ k_c u_0^+(c)\big(u_0(c)-u(c)\big),$$

$$\int_a^c -Bu''(u_0''-u'')\,dx + \langle q, u_0 - u\rangle$$
$$= k_a u^+(a)\big(u_0(a)-u(a)\big) + k_b u^+(b)\big(u_0(b)-u(b)\big)$$
$$+ k_c u^+(c)\big(u_0(c)-u(c)\big);$$

hence

$$\int_a^c B(u_0''-u'')^2\,dx + k_a\big(u_0^+(a)-u^+(a)\big)\big(u_0(a)-u(a)\big)$$
$$+ k_b\big(u_0^+(b)-u^+(b)\big)\big(u_0(b)-u(b)\big)$$
$$+ k_c\big(u_0^+(c)-u^+(c)\big)\big(u_0(c)-u(c)\big) = 0.$$

This implies that

$$u_0 - u \in P_1,$$
$$u_0(a) - u(a) = 0 \quad \text{or} \quad u_0(a) \le 0,\ u(a) \le 0,$$
$$u_0(b) - u(b) = 0 \quad \text{or} \quad u_0(b) \le 0,\ u(b) \le 0,$$
$$u_0(c) - u(c) = 0 \quad \text{or} \quad u_0(c) \le 0,\ u(c) \le 0.$$

Therefore we have

$$\xi = a \Rightarrow u(a) = \frac{P}{k_a},\ u(b) \le 0,\ u(c) \le 0$$

$$\xi = c \Rightarrow u(a) \le 0,\ u(b) \le 0,\ u(c) = \frac{P}{k_c}.$$

Let us now observe that, because u_0 is a solution to variational equation (2), for $v = u_0 - u$ we have

$$\langle q, u_0 - u\rangle = 0.$$

Hence by putting $u_0(x) - u(x) = hx + f \quad \forall x \in \;]a, c[$, where $h, f \in \mathbf{R}$, we have

$$0 = \langle q, u_0 - u\rangle = h\langle q, p_\xi\rangle + (h\xi + f)\langle q, \mathbf{1}\rangle = \big(u_0(\xi) - u(\xi)\big)\langle q, \mathbf{1}\rangle$$

from which it follows directly $u_0(\xi) = u(\xi)$.

Therefore, we have proven that

$$u(x) - u_0(x) = h(x - \xi) \quad \forall x \in]a, c[.$$

Let us now observe that, if $\xi = a$, because $u(b) = u_0(b) + h(b - a) \leq 0$, $u(c) = u_0(c) + h(c - a) \leq 0$, we have

$$h \leq \min\left\{ -\frac{u_0(b)}{b - a}, -\frac{u_0(c)}{c - a} \right\}, \tag{19}$$

if $\xi = c$, because $u(a) = u_0(a) + h(a - c) \leq 0$, $u(b) = u_0(b) + h(b - c) \leq 0$, we have

$$h \geq \max\left\{ -\frac{u_0(b)}{b - c}, -\frac{u_0(a)}{a - c} \right\}. \tag{20}$$

In the above discussion we have proven that, if u_0 is a whatever solution to Problem (1), every other solution is sum of u_0 and of the polynomial hp_ξ, where h is a convenable real number.

Let us now establish that, if u_0 is a particular solution to Problem (1) and hp_ξ is a polynomial such that h satisfies (19) [resp. (20)], if $\xi = a$ [resp. $\xi = c$], then function $u = u_0 + hp_\xi$ is a solution to Problem (1).

Indeed we have

$$\begin{aligned} u^+(a) &= u_0^+(a) && [\text{resp. } u^+(a) = u_0^+(a) = 0], \\ u^+(b) &= u_0^+(b) = 0 && [\text{resp. } u^+(b) = u_0^+(b) = 0], \\ u^+(c) &= u_0^+(c) = 0 && [\text{resp. } u^+(c) = u_0^+(c)]; \end{aligned}$$

therefore u is a solution to variational equation (2).

Finally, we are also concerned here with the proof that in Case (14), Problem (1) has a solution (unique if $\xi \neq b$).

To this aim let us consider variational inequality (7). It is a known result that, to prove that variational inequality (7) admits at least a solution, it is sufficient establish that

$$\begin{aligned} &\exists r \in R^+: \quad \forall v \in H^2(\Omega) \text{ with } \|v\|_{H^2(\Omega)} = r \\ &\int_a^c B(v'')^2\, dx - \langle q, v \rangle + \frac{1}{2}k_a(v^+(a))^2 \\ &\qquad + \frac{1}{2}k_b(v^+(b))^2 + \frac{1}{2}k_c(v^+(c))^2 > 0. \end{aligned} \tag{21}$$

Using a proof by contradiction, let us assume (21) to be false. Thus a sequence $\{v_n\}$ of elements of $H^2(\Omega)$ exists such that

$$\forall n \in \mathbf{N}\ \|v_n\|_{H^2(\Omega)} = n \quad \forall n \in \mathbf{N} \int_a^c B(v_n'')^2\,dx + \frac{1}{2}k_a\big(v_n^+(a)\big)^2 + \frac{1}{2}k_b\big(v_n^+(b)\big)^2 + \frac{1}{2}k_c\big(v_n^+(c)\big)^2 \le \langle q, v_n\rangle$$

or, in terms of the notation $w_n = \dfrac{v_n}{n}$,

$$\|w_n\|_{H^2(\Omega)} = 1 \quad \forall n \in \mathbf{N} \tag{22}$$

$$\int_a^c B(w_n'')^2\,dx + \frac{1}{2}k_a\big(w_n^+(a)\big)^2 + \frac{1}{2}k_b\big(w_n^+(b)\big)^2 + \frac{1}{2}k_c\big(w_n^+(c)\big)^2 \le \frac{1}{n}\langle q, w_n\rangle \quad \forall n \in \mathbf{N}.$$

Therefore we get

$$b_0\|w_n''\|_{L^2(\Omega)}^2 \le \frac{1}{n}\|q\|_{(H^2(\Omega))'}.$$

which implies

$$\lim_{n\to+\infty} \|w_n''\|_{L^2(\Omega)} = 0. \tag{23}$$

On the other hand, since the first of (22), it is possible to extract from $\{w_n\}$ a subsequence (which we indicate by the same index) weakly convergent in $H^2(\Omega)$ (and thus strongly in $H^1(\Omega)$) to an element w. Thus,

$$\lim_{n\to+\infty} \|w_n - w\|_{H^1(\Omega)} = 0. \tag{24}$$

Let us consider now the functional

$$\Phi\colon v \in H^2(\Omega) \to \int_a^c B(v'')^2 dx + \frac{1}{2}k_a^+(v^+(a))^2 + \frac{1}{2}k_b^+(v^+(b))^2 + \frac{1}{2}k_c^+(v^+(c))^2.$$

It is easy to verify that Φ is weakly lower semicontinuous.
Therefore

$$\Phi(w) \le \lim_{n\to+\infty}{}' \Phi(w_n)$$

and from (22) we get

$$\int_a^c B(w'')^2\,dx + k_a(w^+(a))^2 + k_b(w^+(b))^2 + k_c(w^+(c))^2 = 0.$$

Hence $w'' = 0$ a.e. on $]a,c[$, $w(a) \leq 0$, $w(b) \leq 0$, $w(c) \leq 0$.

From this and taking into account (23) and (24) we have

$$\lim_{n \to +\infty} w_n = w \text{ strongly in } H^2(\Omega).$$

Moreover because the first of (22), $w \neq 0$ a.e. on $]a,c[$.

Let us prove that $\forall x \in]a,c[\; w(x) < 0$.

By absurd, let us suppose that

$$\exists x_0 \in]a,c[: w(x_0) \geq 0.$$

Because $w'' = 0$ a.e. on $]a,c[$, w is not greater than a first degree polynomial; i.e., $\exists l, m \in \mathbf{R}$ such that

$$w(x) = mx + l \quad \forall x \in]a,c[.$$

Clearly w is monotonic. As a consequence, if $w(x_0) > 0$, because $w(a) \leq 0$ and $w(c) \leq 0$, we have an absurd result. If $w(x_0) = 0$, we get $w = 0$ on $[a,c]$, which contradict the condition $w \neq 0$ a.e. on $]a,c[$.

Finally we observe that

$$\begin{aligned}\langle q,w \rangle &= m\langle q, p_\xi \rangle + (m\xi + 1)\langle q, \mathbf{1} \rangle \\ &= (m\xi + l)\langle q, \mathbf{1} \rangle = w(\xi)\langle q, \mathbf{1} \rangle < 0.\end{aligned}$$

On the other hand, because of (22) we get

$$\langle q,w \rangle = \lim_{n \to +\infty} \langle q, w_n \rangle \geq 0.$$

This absurd proves that variational inequality (7) (and by Theorem (2), Problem (1)) has at least a solution.

We must now study the uniqueness of the solution.

Let u_1, u_2 be solutions to Problem (1).

By proceeding in a similar way as in Case (13), we have

$$u_2 - u_1 \in P_1,\; u_2(\xi) = u_1(\xi).$$

Moreover, because of the hypothesis $P = \langle q, \mathbf{1} \rangle > 0$, we get

$$\exists\, s \in \{a,b,c\} : u_1(s) = u_2(s).$$

Therefore, if $\xi \neq b$, because $u_2 - u_1$ belongs to P_1 and has the distinct zeros s, ξ we get

$$u_2 - u_1 = 0$$

and the thesis follows.

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